Homework 10. Due April 18

1. **(5 pts)** Consider the heat equation in the unit square $[0,1]^2$

$$u_t = \Delta u, \quad (x,y) \in [0,1]^2. \tag{1}$$

Impose the homogeneous Dirichlet boundary conditions: $u = 0$ on $\partial\Omega$. Consider the uniform square mesh $(J+1)\times(J+1)$ in the unit square $[0,1]^2$ and discretize Laplace's operator to this mesh using central differences and taking the boundary conditions into account. The resulting "method of lines" equation will be of the form

$$\frac{d}{dt}U = AU, \tag{2}$$

where A is the discretized operator Δ, an $(J-1)^2 \times (J-1)^2$ matrix.

   (a) Verify that the eigenvectors of A are

$$v_{k_x,k_y}(x_r, y_s) = \sin(k_x x_r)\sin(k_y y_s), \quad k_x, k_y = \pi, 2\pi, \ldots, (J-1)\pi, \tag{3}$$

   where (x_r, y_s) are the mesh points. Find the corresponding eigenvalues.

   (b) Suppose we are applying the forward Euler time discretization to (2). Find the relationship between the time step Δt and the mesh step h such that this method is stable for all J.

2. **(8 pts)** Read Section 9.8.2 in R. LeVeque, *Finite difference methods for ordinary and partial differential equations*. This book is available online via the UMD library.

Consider the heat equation

$$u_t = \Delta u + f(x,y,t), \quad (x,y) \in \Omega = [-1,1]^2, \quad u(x,y,0) = 0, \tag{4}$$

with homogeneous *Neumann* boundary conditions (insulation) and the source term

$$f(x,y,t) = \exp\left[-10\left((x - 0.6\cos(2\pi t))^2 + (y - \sin(2\pi t))^2\right)\right]. \tag{5}$$

Solve this IBVP (initial-boundary value problem) on the time interval $0 \le t \le 1$ using the ADI scheme. Plot the numerical solution as a heat map at time 0, 0.2, 0.4, 0.6, 0.8, and 1.

Link your code to your solution.

3. **(8 pts)** Read Section 9 in Remarks around 50 lines of Matlab: short finite element implementation.

Consider the following Initial and Boundary Value Problem (IBVP) in 2D:

$$u_t = \Delta u + 1, \quad (x, y) \in \Omega = \{(x, y) \in \mathbb{R}^2 \mid 1 < r < 2\}, \tag{6}$$
$$u|_{t=0} = r + \cos(\phi), \tag{7}$$
$$u|_{r=1} = u|_{r=2} = 0, \tag{8}$$

where r and ϕ are the polar coordinates. Solve this problem using the finite element method and a scheme based on the trapezoidal rule:

$$u_{n+1} = u_n + \tfrac{1}{2}\Delta t \left(\Delta u_{n+1} + \Delta u_n\right) + \Delta t.$$

(a) Derive equations for the weak and the FEM solutions of the IBVP (6)-(8) analogous to Eq. (13) and the two unnumbered equations right below it in Section 9 on page 127 in *Remarks around 50 lines of Matlab: short finite element implementation*. Use time step `dt = 0.01`.

(b) Make your program plot the following figures:

- with the computed solution at $t = 0.1$ (use `trisurf`);
- with the computed solution at $t = 1$ (use `trisurf`);
- with the computed solution at time $t = 1$ as a function of r. You can do it e.g., as follows:

```
u = U(:,N+1); % N+1 corresponds to t=1.
r = sqrt(coordinates(:,1).^2 + coordinates(:,2).^2);
[rsort,isort] = sort(r,'ascend');
usort = u(isort);
plot(rsort,usort,'Linewidth',2);
```

At $t = 1$, the function u will virtually reach the stationary solution $\Delta u + 1 = 0$ satisfying the BC (8). This stationary solution can be found exactly:

$$u(r) = \frac{1 - r^2}{4} + \frac{3\log(r)}{4\log 2}. \tag{9}$$

Plot the graph of the exact stationary solution (9) in the same figure.

Link your code to your solution.

Hint: You might find helpful my codes MyFEMheat.m (Matlab) or MyFEMheat.ipynb (Python) implementing the Backward Euler time integrator described in "Remarks around 50 lines of Matlab: ...".