Scientific Computing HW 1

Ryan Chen

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1. Pick $T < t^*$ where $t^* := t_0 + \frac{1}{y_0}$. Fix $r \in \mathbb{R}$. Then $f(t, y) := y^2$ is continuous on the cylinder $Q := \{t_0 \leq t \leq T, \ |y - y_0| \leq r\}$. Now for all $(t, y) \in Q$,

   $$|y - y_0| \leq r \implies |y| = |y - y_0 + y_0| \leq |y - y_0| + |y_0| \leq r + |y_0| \implies |f(t,y)| = |y^2| = |y|^2 \leq (r + |y_0|)^2$$

   i.e. $|f| \leq M$ on Q where $M := (r + |y_0|)^2$. Then by theorem 1, the IVP has a solution for $0 \leq t - t_0 \leq \min(\frac{r}{M}, T - t_0)$.

2. Pick $0 \leq T < \infty$. Fix $r \in \mathbb{R}$. Then $f(t, y) := 2y^{1/2}$ is continuous on the cylinder $Q := \{t_0 \leq t \leq T, \ |y| \leq r\}$. Now for all $(t, y) \in Q$ with $y \geq 0$, using similar arguments as in P1,

   $$|y| \leq r \implies |f(t,y)| = |2y^{1/2}| = 2y^{1/2} \leq 2r^{1/2}$$

   i.e. $|f| \leq M$ on Q where $M := 2r^{1/2}$. Then by theorem 1, the IVP has a solution for $0 \leq t \leq \min(\frac{r}{M}, T)$.

   To see that $f(y) := 2y^{1/2}$ is not Lipschitz at $y = 0$, suppose there exists $C > 0$ such that $|f(x) - f(0)| \leq C|x - 0|$ for all $x \geq 0$, i.e. $2x^{1/2} \leq Cx$. But if we set $x = \frac{1}{C^2}$, then $2 \leq Cx^{1/2} = C\frac{1}{C} = 1$, a contradiction.

3. Using the Taylor expansion of y at t_n,

   $$\tau_{n+1} = y + hy' + \frac{1}{2}h^2 y'' + \frac{1}{6}h^3 y''' + O(h^4) - y$$
   $$- h \left[\frac{23}{12}y' - \frac{4}{3}\left(y' - hy'' + \frac{1}{2}h^2 y''' + O(h^3)\right) + \frac{5}{12}\left(y' - 2hy'' + 2h^2 y''' + O(h^3)\right) \right]$$

   Collect coefficients of the following terms:

   $$\begin{aligned} y: \quad & 1 - 1 = 0 \\ hy': \quad & 1 - \frac{23}{12} + \frac{4}{3} - \frac{5}{12} = \frac{1}{12}(12 - 23 + 16 - 5) = 0 \\ h^2 y'': \quad & \frac{1}{2} - \frac{4}{3} + \frac{5}{6} = \frac{1}{6}(3 - 8 + 5) = 0 \\ h^3 y''': \quad & \frac{1}{6} + \frac{2}{3} - \frac{5}{6} = \frac{1}{6}(1 + 4 - 5) = 0 \end{aligned}$$

   Thus $\tau_{n+1} = O(h^4)$, i.e. the method is consistent of order 3.

4. Substituting k into the recurrence,

   $$u_{n+1} = u_n + hf\left(t_n + \frac{1}{2}h, u_n + \frac{1}{2}hk\right)$$

   Using the Taylor expansion of y at $t_n + \frac{1}{2}h$,

   $$\tau_{n+1} = y + hy' + \frac{1}{2}h^2 y'' + O(h^3) - y - h\left[y' + \frac{1}{2}y'' + O(h^2)\right]$$

Collect coefficients of the following terms:

$$
\begin{aligned}
y: & \quad 1 - 1 = 0 \\
hy': & \quad 1 - 1 = 0 \\
h^2 y'': & \quad \frac{1}{2} - \frac{1}{2} = 0
\end{aligned}
$$

Thus $\tau_{n+1} = O(h^3)$, i.e. the method is consistent of order 2.

5. (a) Since we have four undetermined coefficients, we take a third order Taylor expansion. Any higher order expansion would give an overconstrained system.

$$
\tau_{n+1} = y + hy' + \frac{1}{2}h^2 y'' + \frac{1}{6}h^3 y''' + O(h^4) - a_0 y - a_1 \left[y' - hy' + \frac{1}{2}h^2 y'' - \frac{1}{6}h^3 y''' + O(h^4) \right]
$$
$$
- h \left[b_0 y' + b_1 \left(y' - hy'' + \frac{1}{2}h^2 y''' + O(h^3) \right) \right]
$$

Collect coefficients of the following terms. To seek consistency of the highest possible order (in this case order 3), set each sum equal to 0:

$$
\begin{aligned}
y: & \quad 1 - a_0 - a_1 = 0 \\
hy': & \quad 1 + a_1 - b_0 - b_1 = 0 \\
h^2 y'': & \quad \frac{1}{2} - \frac{1}{2}a_1 + b_1 = 0 \\
h^3 y''': & \quad \frac{1}{6} + \frac{1}{6}a_1 - \frac{1}{2}b_1 = 0
\end{aligned}
$$

The solution of this system (using WolframAlpha) is $a_0 = -4$, $a_1 = 5$, $b_0 = 4$, $b_1 = 2$.

(b) **Pf.** Applying the method (with coefficients from the last part) to $y' = 0 =: f(t, y)$ with initial condition $y(0) = a$,
$$
u_{n+1} + 4u_n - 5u_{n-1} = 0
$$

Using the ansatz solution r^n,

$$
0 = r^2 + 4r - 5 = (r + 5)(r - 1) \implies r = 1, -5 \implies u_n = A + B(-5)^n
$$

The exact solution of the IVP is $y(t) = a$. Perturb the values of the first two iterates, say $u_0 = a + \delta_0$ and $u_1 = a + \delta_1$. Then

$$
a + \delta_0 = A + B, \ a + \delta_1 = A - 5B \implies \delta_0 - \delta_1 = 6B
$$

If $\delta_0 \neq \delta_1$ then $B \neq 0$, in which case the solution blows up. Thus the method is unstable.

(c) https://github.com/RokettoJanpu/Scientific-Computing-2/blob/main/hw1.ipynb