Take-home Final exam. Problem 1. Due Thursday, May 16, 11:59 PM

- **All codes must be written by you from scratch. All codes must be linked to the pdf file.**
- **Every student must work independently.**
- **You should submit a single pdf file with your solutions and link your codes to it.**
- **You should type your solutions. I will deduct 10% of your score for a handwritten solution.**
- **You can use any course materials and any textbooks. You are not allowed to use help of other people and of Chat GPT.**

1. **(10 pts)** Consider the van der Pol oscillator

$$\frac{d^2x}{dt^2} - \mu(1-x^2)\frac{dx}{dt} + x = 0. \tag{1}$$

Introduce $y = \frac{dx}{dt}$ and rewrite it as a system of two ODEs for x and y. This oscillator is famous due to its stable limit cycle. As the parameter μ increases, the ODE system becomes more and more stiff. You will need to use the DIRK2 method for integrating this ODE system: $\begin{array}{c|cc} \gamma & \gamma & 0 \\ 1 & 1-\gamma & \gamma \\ \hline & 1-\gamma & \gamma \end{array}$ where $\gamma = 1 - \frac{1}{\sqrt{2}}$. Then DIRK2 is L-stable and 2nd-order accurate.

In all cases, use $(x = 2,\ y = 0)$ as the initial condition. At large enough μ, this point lies close enough to the limit cycle.

(a) Set $\mu = 10^2$, $T_{\max} = 200$, and choose time step $h = 10^{-3}$. Implement the DIRK2 method and integrate the ODE. Measure the CPU time. Plot x and y versus T in Figure 1 and y versus x in Figure 2.

Try to increase μ to 10^3 while leaving all other settings unchanged and plot the same graphs. It should be clear that the numerical solution with a constant time step that is small enough so that the figure in the xy-plane looks nice and for a time long enough so that the whole period is displayed will take too long.

(b) Make the time step adaptive using the approach illustrated on page 22 in `ODEsolvers.pdf`. Set `atol = rtol = 1e-5`, propose one more vector b in the Butcher array that makes the method 1st order, introduce the automatic error estimate e and the step acceptance criterion

$$||e|| < \texttt{atol} + \texttt{rtol} \cdot ||[x, y]||. \tag{2}$$

Propose an algorithm that increases, leaves the same, or decreases the time step depending on the relationship between $||e||$ and $\texttt{atol} + \texttt{rtol} \cdot ||[x, y]||$. Use the resulting solver to integrate the ODE with $\mu = 10^6$ on the time interval $[0, 2 \cdot 10^6]$. Measure the CPU time. Plot x and y versus T in Figure 1 and y versus x in Figure 3.

2. **(10 pts)** Consider the following Boundary Value Problem (BVP) in 2D:

$$\nabla \cdot \left(e^{-\beta V(x,y)} M(x,y) \nabla u \right) = 0, \quad (x,y) \in \Omega \tag{3}$$

$$\frac{\partial u}{\partial n} = 0, \quad u \in \Gamma_N \tag{4}$$

$$u(\partial A) = 0, \quad u(\partial B) = 1, \tag{5}$$

The domain Ω is the square $[-1,1]^2$ with two removed circles A and B of radii 0.2 centered at $(-0.5, -0.1\sin(0.5\pi))$ and $(0.5, 0.1\sin(0.5\pi))$ respectively. The scalar function $V(x,y)$ and the matrix function $M(x,y)$ are given, respectively, by

$$V(x,y) = \cos(2\pi x) + (y - 0.1\sin(\pi x))^2,$$

$$M(x,y) = \begin{bmatrix} 1 + 0.5\cos(\pi x) & 0.5\sin(\pi x) \\ 0.5\sin(\pi x) & 1 + 0.5\cos(\pi x) \end{bmatrix}.$$

The boundary Γ_N is the outer boundary of the square. Set $\beta = 1$.

(a) Derive an integral equation formulation for the problem (3)–(5) similar to Eq. (66) in `elliptic.pdf`.

(b) Triangulate the domain Ω and plot your triangulation.

(c) Adapt the FEM solver for this problem and solve it. Explain how you compute the entries of the stiffness matrix. Plot the FEM solution using `plt.tricontourf` in Python or `trisurf` and `view(2)` in Matlab.

3. **(10 pts)** Consider the following traffic model

$$\rho_t + [\rho v(\rho)]_x = 0, \quad \text{where} \quad v(\rho) = 1 - \rho^2. \tag{6}$$

The function $\rho(t,x)$ is the density of the traffic, and the function $v(\rho(t,x))$ is the velocity of the traffic. The initial density is given by

$$\rho(0,x) = \rho_0(x) = \begin{cases} 0.1, & x < 0 \\ 0.1 + 0.8x, & 0 \le x \le 1 \\ 0.9, & x > 1 \end{cases}. \tag{7}$$

(a) Find the equation of characteristics and plot them starting at $x_0 = -2, -1.9, -1.8..., 2$.

(b) Find the time at which the shock first forms.

(c) Find the shock speed at large times.

(d) Solve the equation numerically using Godunov's method on the computational domain $x \in [-2,2]$ for time $t \in [0,10]$. You must choose an appropriate numerical flux function. Write out this flux function in your pdf file. Plot the numerical solution at time $t = 1, 2, ..., 10$.