Homework 5. Due March 7

Please upload a single pdf file on ELMS. Link your codes to your pdf (i.e., put your codes to dropbox, Github, google drive, etc. and place links to them in your pdf file with your solutions.

1. **(5 pts)** Consider the general two-step linear method of the form

$$u_{n+1} + \alpha_0 u_n + \alpha_1 u_{n-1} = h\left(\beta_{-1} f_{n+1} + \beta_0 f_n + \beta_1 f_{n-1}\right). \tag{1}$$

   Use the main theorem (Theorem 7 in `ODEsolvers.pdf`) and the boundary locus technique (Section 7.7 `ODEsolvers.pdf`) to do the tasks below.

   (a) Check that the method is consistent if and only if

$$1 + \alpha_0 + \alpha_1 = 0 \quad \text{and} \quad 2 + \alpha_0 = \beta_{-1} + \beta_0 + \beta_1. \tag{2}$$

   (b) Check that the method (1) is 3rd order consistent if and only if

$$2\beta_{-1} + \beta_0 = 2 + \frac{\alpha_0}{2} \quad \text{and} \quad 2\beta_{-1} + \frac{\beta_0}{2} = \frac{4}{3} + \frac{\alpha_0}{6}. \tag{3}$$

   (c) Check that the method is stable if and only if $\alpha_0 \in (-2, 0]$.

   (d) Observe that the 3rd order consistency conditions together with the stability condition define a one-parameter family of two-step 3rd-order methods. This family can be paramertized by $\alpha_0 \in (-2, 0]$. Write a code that plots the boundaries of the RAS for $\alpha = -1.8, -1.7, \ldots, -1.1$ in one figure and the boundaries of the RAS for $\alpha = -1.0, -0.9, \ldots, -0.1$ in another figure. Include legend in each figure.

   (e) Note that in all cases the boundaries of the RAS are simple closed curves. This means that the RAS is either outside or inside the contour. Therefore, it is sufficient to check the Root Condition at a single point inside each contour to conclude whether the RAS lies inside or outside the contour. Do it for each contour and say where is the RAS.

2. **(10 pts)** Solve exercise 2 in my lecture notes `SymplecticMethods.pdf`.

3. **(10 pts)** Consider the motion in the gravitational field with the Hamiltonian

$$H(u, v, x, y) = \frac{1}{2}u^2 + \frac{1}{2}v^2 - \frac{1}{\sqrt{x^2 + y^2}}, \tag{4}$$

   where $x,\ y$ are the coordinates and $u,\ v$ are the momenta in the reduced units.

(a) **(2 pts)** Write the Hamiltonian equations of motion. Set the initial conditions to $u(0) = 0,\ v(0) = \frac{1}{2},\ x(0) = 2,\ y(0) = 0$. Check that the total energy is negative, hence the motion will follow an elliptic trajectory.

The exact motion according to these Hamiltonian equations with the initial conditions $u(0) = 0,\ v(0) = \frac{1}{2},\ x(0) = 2,\ y(0) = 0$ occurs by the elliptical orbit with one focus at the origin, the major semiaxis $a = 4/3$, eccentricity $e = 1/2$, and the exact period of revolution $T = 2\pi a^{3/2} = 9.673596609249161$. Hence, for the exact orbit, $x_{\max} = a(1+e) = 2$, and $x_{\min} = -a(1-e) = -2/3$.

(b) **(4 pts)** Integrate the system for 10 revolutions using the implicit midpoint rule. Proceed as follows. Define

$$z := \begin{bmatrix} u \\ v \\ x \\ y \end{bmatrix}, \quad f(z) := \frac{dz}{dt};$$

then

$$k = f(z_n + \tfrac{h}{2}k), \quad z_{n+1} = z_n + hk.$$

At each time step, you will need to solve the 4D nonlinear system

$$F(k) := k - f(z_n + \tfrac{h}{2}k) = 0. \tag{5}$$

Find the initial approximation for k by linearizing f and solving the linear system:

$$k = f(z_n) + \tfrac{h}{2}Df(z_n)k,$$

where $Df(z_n)$ is the Jacobian matrix of f evaluated at z_n. Then find the solution of Eq. (5) using Newton's iteration. Plot x and y components of your numerical solutions on the same xy-plane. Plot the Hamiltonian versus time for your numerical solution. Do this task with time steps such that there are 100, 1000, and 10000 steps per period. You should generate a total of 6 figures.

(c) **(4 pts)** Integrate the same system using the Stoermer-Verlet method described in the previous problem using the same time steps and generate the same plots.