AMSC661, Spring 2024 Maria Cameron

Homework 7. Due March 28

1. **(5 pts)** Consider the following boundary-value problem for the Poisson equation:

$$-\Delta u = f(x,y), \quad (x,y) \in \Omega = (0,1)^2, \quad u = 0, \quad (x,y) \in \partial\Omega. \tag{1}$$

Suppose that we are planning to solve it numerically on a uniform square mesh with step $h = 1/N$ in both x and y directions using the finite difference method with a 5-point stencil:

$$\frac{1}{h^2}\left(4U_P - U_W - U_E - U_N - U_S\right) = f_P, \quad P \in \{(i,j)\}_{i,j=1}^{N-1}. \tag{2}$$

The total number of mesh points where u is to be determined is $(N-1)^2$.

Verify that the mesh functions

$$v_{k_x,k_y}(x_i,y_j) = \sin\left(k_x x_i\right)\sin\left(k_y y_j\right), \quad x_i = hi, \quad y_j = hj,$$

are eigenvectors of the matrix A in the left-hand side of (2) for all

$$k_x, k_y = \pi, 2\pi, \ldots, (N-1)\pi.$$

Determine the corresponding eigenvalues. *Hint: you do not need to write out the matrix A explicitly. Instead, plug $v_{k_x,k_y}(x_i,y_j)$ into the left-hand side of* (2) *and use trigonometric formulas to show that*

$$(Av_{k_x,k_y})(x_i,y_j) = \lambda(k_x,k_y)v_{k_x,k_y}(x_i,y_j).$$

Then the constant $\lambda(k_x,k_y)$ will be the corresponding eigenvalue.

Use the approximation $\sin a \approx a$ for small a to estimate the smallest and the largest eigenvalue of A. Find the condition number of A $\kappa(A) = \|A\|_2\|A^{-1}\|_2$.

2. **(5 pts)** Consider a 1D boundary-value problem

$$-\frac{d}{dx}\left(k(x)\frac{d}{dx}u\right) = 0, \quad a < x < b, \quad u(a) = u_a, \quad u(b) = u_b, \tag{3}$$

where the heat conductance coefficient $k(x)$ is the following piecewise constant function

$$k(x) = \begin{cases} k_1, & a \le x < c \\ k_2, & c < x \le b \end{cases}. \tag{4}$$

It follows from the integral form of Fourier's law that the temperature u and the heat flux $k(x)u_x$ must be continuous at $x = c$.

(a) Find the exact solution to this problem analytically.

(b) Set $u_a = 0$, $u_b = 10$, $k_1 = 10$, $k_2 = 1$, $a = 0$, $b = 10$, $c = 4$. Choose $h = 1.0$. Use the finite difference scheme

$$L_h U_P = -\frac{1}{h^2}\left(U_W k_w + U_E k_e - U_P(k_e + k_w)\right) \tag{5}$$

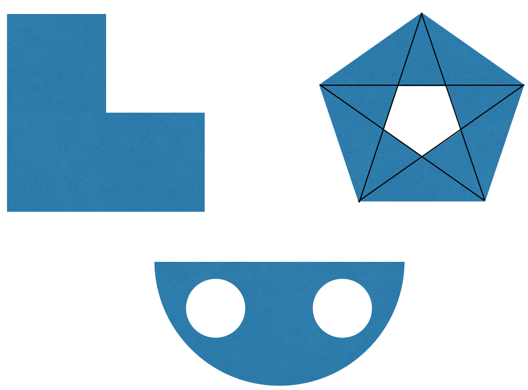
where W is to the left of P, E is to the right of P, and e and w are the midpoints between E and P and W and P respectively. You need to evaluate $k(x)$ at these midpoints. In this case, it is not a problem that $k(x)$ is not defined at c. Compute the numerical solution U using this scheme. Plot it and also plot the exact solution in the same figure. You should see that these solutions exactly coincide.

3. **(5 pts)** *Triangular mesh generation using* `distmesh2d.m` *by P.-O. Persson.* Read [1], at least its first 12 pages (at least up to Section 6 "Mesh Generation in Higher Dimensions").

If you prefer Matlab, download the distmesh package `distmesh.zip` available at http://persson.berkeley.edu/distmesh/.

If you prefer Python, you can download my Python version of P.-O. Persson's code available at GitHub, user `mar1akc`, package `transition_path_theory_FEM_distmesh`, file `distmesh.py`.

Mesh the shapes in the Figure below using distmesh2d.m following examples in [1].



You can pick arbitrary sizes as soon as topologies are preserved, and you can do uniform meshing.

4. **5 pts** Consider the following BVP in 1D:

$$-u_{xx} = f(x), \quad 0 < x < 1, \quad u(0) = u(1) = 1.$$

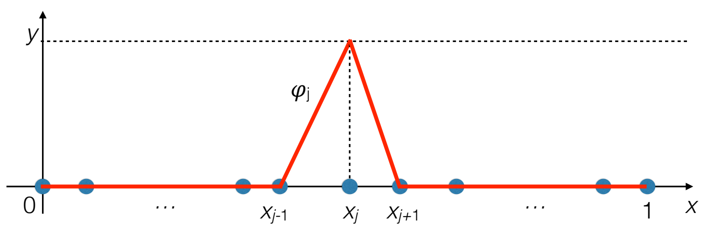
Work out all steps of the FEM on it.

(a) Let $w(x)$ be a twice continuously differentiable function on $(0, 1)$ such that $w(0) = w(1) = 0$. Use integration by parts to reduce the BVP to an integral equation.

(b) Partition the interval $[0, 1]$:

$$0 = x_0 < x_1 < \ldots < x_N < x_{N+1} = 1.$$

Define the basis functions $\phi_i(x)$, $1 \leq i \leq N$ as shown in the figure below ($\phi_i(x_i) = 0$, $\phi_i(x_j) = 0$, $j \neq i$, ϕ_i is piecewise linear).



$$\phi_i(x) = \begin{cases} 0, & x \leq x_{i-1},\ x \geq x_{i+1}, \\ \frac{x - x_{i-1}}{x_i - x_{i-1}}, & x_{i-1} < x \leq x_i, \\ \frac{x_{i+1} - x}{x_{i+1} - x_i}, & x_i < x < x_{i+1}. \end{cases}$$

Calculate the stiffness matrix and the load vector.

(c) In what case the FEM solution would coincide with the finite difference solution using the central difference scheme?

References

[1] P.-O. Persson, G. Strang, A Simple Mesh Generator in MATLAB. SIAM Review, Volume 46 (2), pp. 329-345, June 2004 (PDF)

[2] Jochen Alberty, Carsten Carstensen and Stefan A. Funken, Remarks around 50 lines of Matlab: short finite element implementation