

Scientific Computing HW 6

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- In the following cases, the system $Au = f$ is shown on the left, and the block structure of A (for a general $J \times J$ mesh) is compactly rewritten on the right.

(a)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
-4	2				1									
1	-4	1				1								
3	1	-4	1				1							
4		1	-4	1				1						
5			2	-4					1					
6	1				-4	2				1				
7		1			1	-4	1			1				
8			1		1	-4	1				1			
9				1	1	-4	1					1		
10					1	2	-4						1	
11						1	-4	2						
12							1	-4	1					
13								1	-4	1				
14									1	-4	1			
15										2	-4			

A

$\bullet u_8 = 0$

u_1	-1
u_2	-1
u_3	-1
u_4	-1
u_5	-1
u_6	0
u_7	0
u_8	0
u_9	0
u_{10}	0
u_{11}	0
u_{12}	0
u_{13}	0
u_{14}	0
u_{15}	0

f

$A = \begin{bmatrix} (\mathcal{T}) \times (\mathcal{T}+1) \text{ blocks} & \\ \mathbf{T} & \mathbf{I} & & \\ \mathbf{I} & \mathbf{T} & \mathbf{I} & & \\ & \ddots & \ddots & \ddots & \\ & & \mathbf{T} & \mathbf{I} & & \\ & & \mathbf{I} & \mathbf{T} & & \\ & & & & & \end{bmatrix}$

$\mathbf{T} = \begin{bmatrix} -4 & 2 & & & \\ 1 & -4 & 1 & & \\ & 1 & -4 & 1 & & \\ & & 1 & -4 & 1 & & \\ & & & 1 & -4 & 1 & & \\ & & & & 1 & -4 & 1 & & \\ & & & & & 2 & -4 & & \end{bmatrix}$

(b)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
-4	1	2												
1	-4	1	2											
3	1	-4	1	2										
4	1		-4	1	1									
5	1		1	-4	1	1								
6	1		1	1	-4	1								
7		1			-4	1	1							
8			1		1	-4	1	1						
9				1	1	-4	1		1					
10					1	1	-4	1	1					
11						1	1	-4	1	1				
12							2	1	-4	1				
13							2	1	1	-4	1			
14							2	1	1	-4	1			
15								2	1	1	-4			

A

$\bullet u_8 = 0$

u_1	-1
u_2	0
u_3	0
u_4	-1
u_5	0
u_6	0
u_7	-1
u_8	0
u_9	0
u_{10}	-1
u_{11}	0
u_{12}	0
u_{13}	-1
u_{14}	0
u_{15}	0

f

$A = \begin{bmatrix} (\mathcal{T}-1) \times (\mathcal{T}-1) \text{ blocks} & \\ \mathbf{T} & 2\mathbf{I} & & \\ \mathbf{I} & \mathbf{T} & \mathbf{I} & & \\ & \mathbf{I} & \mathbf{T} & \mathbf{I} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \mathbf{T} & \mathbf{I} & & \\ & & & & \mathbf{I} & \mathbf{T} & \mathbf{I} & & \\ & & & & & & & 2\mathbf{I} & \mathbf{T} & & \\ & & & & & & & & & & \end{bmatrix}$

$\mathbf{T} = \begin{bmatrix} -4 & 1 & & & & & & & \\ 1 & -4 & 1 & & & & & & \\ & 1 & -4 & 1 & & & & & \\ & & 1 & -4 & 1 & & & & \\ & & & 1 & -4 & 1 & & & \\ & & & & 1 & -4 & 1 & & & \\ & & & & & 2 & -4 & & & \\ & & & & & 2 & 1 & -4 & 1 & & \\ & & & & & 2 & 1 & 1 & -4 & 1 & & \\ & & & & & & 2 & 1 & 1 & -4 & 1 & & \\ & & & & & & & 2 & 1 & 1 & -4 & 1 & & \\ & & & & & & & & 2 & 1 & 1 & -4 & 1 & & \\ & & & & & & & & & 2 & 1 & 1 & -4 & 1 & & \\ & & & & & & & & & & 2 & 1 & 1 & -4 & 1 & & \\ & & & & & & & & & & & 2 & 1 & 1 & -4 & 1 & & \\ & & & & & & & & & & & & 2 & -4 & & & & \end{bmatrix}$

$$\begin{array}{c|cccc|cccc|cccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
 \hline
 1 & -4 & 1 & & & 1 & & & & & & & \\
 2 & 1 & -4 & 1 & & 1 & & & & & & & \\
 3 & & 1 & -4 & 1 & & 1 & & & & & & \\
 4 & & & 1 & -4 & 1 & & & & & & & \\
 5 & 1 & & & & -4 & 1 & 1 & 1 & & & & \\
 6 & & 1 & & & 1 & -4 & 1 & & 1 & & & \\
 7 & & & 1 & & 1 & -4 & 1 & & & 1 & & \\
 8 & & & & 1 & 1 & -4 & 1 & & & & 1 & \\
 9 & & & & & & & -4 & 1 & 1 & & & \\
 10 & & & & & & & 1 & -4 & 1 & & & \\
 11 & & & & & & & 1 & 1 & -4 & 1 & & \\
 12 & & & & & & & 1 & 1 & 1 & -4 & 1 & \\
 \end{array} \cdot \begin{array}{c|c}
 u_1 & -1 \\
 u_2 & -1 \\
 u_3 & -1 \\
 u_4 & -1 \\
 u_5 & 0 \\
 u_6 & 0 \\
 u_7 & 0 \\
 u_8 & 0 \\
 u_9 & 0 \\
 u_{10} & 0 \\
 u_{11} & 0 \\
 u_{12} & 0
 \end{array} = \begin{array}{c|cc|c}
 \text{$J \times J$ blocks} & & & \\
 \begin{matrix} T & I \\ I & T & I \\ I & & \ddots & \\ \ddots & T & I \\ I & & T \end{matrix} & & \begin{matrix} -4 & 1 & 1 \\ 1 & -4 & 1 \\ 1 & 1 & -4 \end{matrix} \\
 A = & & T = & \\
 & & & \\
 & & &
 \end{array}$$

(c)

$$\begin{array}{c|cccc|cccc|cccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
 \hline
 1 & -4 & 1 & & & 1 & & & & 1 & & & \\
 2 & 1 & -4 & 1 & & 1 & & & & 1 & & & \\
 3 & & 1 & -4 & 1 & & 1 & & & & 1 & & \\
 4 & & & 1 & -4 & 1 & & & & & & & \\
 5 & 1 & & & & 1 & -4 & 1 & 1 & & & & \\
 6 & & 1 & & & 1 & -4 & 1 & 1 & & & & \\
 7 & & & 1 & & 1 & -4 & 1 & 1 & & & & \\
 8 & & & & 1 & 1 & -4 & 1 & 1 & & & & \\
 9 & & & & & 1 & 1 & -4 & 1 & 1 & & & \\
 10 & 1 & & & & & 1 & -4 & 1 & 1 & & & \\
 11 & & 1 & & & & 1 & 1 & -4 & 1 & & & \\
 12 & & & 1 & & & 1 & 1 & 1 & -4 & 1 & & \\
 \end{array} \cdot \begin{array}{c|c}
 u_1 & 0 \\
 u_2 & 0 \\
 u_3 & -1 \\
 u_4 & 0 \\
 u_5 & 0 \\
 u_6 & -1 \\
 u_7 & 0 \\
 u_8 & 0 \\
 u_9 & -1 \\
 u_{10} & 0 \\
 u_{11} & 0 \\
 u_{12} & -1
 \end{array} = \begin{array}{c|cc|c}
 \text{$(J-1) \times (J-1)$ blocks} & & & \\
 \begin{matrix} T & I & I \\ I & T & I \\ I & & \ddots & \\ \ddots & T & I \\ I & & T \end{matrix} & & \begin{matrix} -4 & 1 & & \\ 1 & -4 & 1 & \\ 1 & 1 & \ddots & \\ & \ddots & -4 & 1 \\ & & 1 & -4 \end{matrix} \\
 A = & & T = & \\
 & & &
 \end{array}$$

(d)

(e) In this case we do not compactly rewrite the block structure of A , but it is still clear.

$$\begin{array}{c|cccc|cccc|cccc|cccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\
 \hline
 1 & -4 & 1 & & & 1 & & & & & & & & & & & \\
 2 & 1 & -4 & 1 & & 1 & & & & & & & & & & & \\
 3 & & 1 & -4 & 1 & & 1 & & & & & & & & & & \\
 4 & & & 1 & -4 & 1 & & & & & & & & & & & \\
 5 & 1 & & & & 1 & -4 & 1 & & 1 & & & & & & & \\
 6 & & 1 & & & & 1 & -4 & 1 & & 1 & & & & & & \\
 7 & & & 1 & & 1 & -4 & 1 & & 1 & & & & & & & \\
 8 & & & & 1 & 1 & -4 & 1 & & 1 & & & & & & & \\
 9 & & & & & 1 & 1 & -4 & 1 & 1 & & & & & & & \\
 10 & 1 & & & & & 1 & 1 & -4 & 1 & 1 & & & & & & \\
 11 & & & & & & & 1 & -4 & 1 & 1 & & & & & & \\
 12 & & & & & & & 1 & 1 & -4 & 1 & 1 & & & & & \\
 13 & & & & & & & & 1 & -4 & 1 & 1 & & & & & \\
 14 & & & & & & & & & 1 & -4 & 1 & 1 & & & & \\
 15 & & & & & & & & & & 1 & -4 & 1 & 1 & & & \\
 16 & & & & & & & & & & & 1 & -4 & 1 & 1 & &
 \end{array} \cdot \begin{array}{c|c}
 u_1 & 0 \\
 u_2 & 0 \\
 u_3 & 0 \\
 u_4 & 0 \\
 u_5 & 0 \\
 u_6 & -1 \\
 u_7 & -1 \\
 u_8 & 0 \\
 u_9 & 0 \\
 u_{10} & 0 \\
 u_{11} & -1 \\
 u_{12} & 0 \\
 u_{13} & -1 \\
 u_{14} & 0 \\
 u_{15} & -1 \\
 u_{16} & 0
 \end{array} = \begin{array}{c|cc|c}
 & & & \\
 & & &
 \end{array}$$

2. (a) The BVP on the domain $\Omega := [-\pi, \pi] \times [0, 2]$ is

$$u_{xx} + u_{yy} = g(x) := \begin{cases} -\cos x, & -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ 0, & \text{else} \end{cases} \quad (2.1)$$

with BCs

$$u \Big|_{x=\pi} = u \Big|_{x=-\pi}, \quad u_x \Big|_{x=\pi} = u_x \Big|_{x=-\pi}, \quad u \Big|_{y=0} = 0, \quad u_y \Big|_{y=2} = 0$$

(b) Fix $J \in \mathbb{N}$ and set mesh steps in the x and y axes,

$$h_x := \frac{2\pi}{J}, \quad h_y := \frac{2}{J}$$

Then

$$\begin{aligned} u_{xx}(x, y) &= \frac{1}{h_x^2} [u(x + h_x, y) - 2u(x, y) + u(x - h_x, y)] + O(h_x^2) \\ u_{yy}(x, y) &= \frac{1}{h_y^2} [u(x, y + h_y) - 2u(x, y) + u(x, y - h_y)] + O(h_y^2) \end{aligned}$$

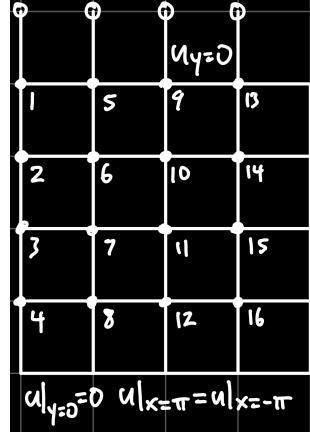
Plug these expressions into (2.1) and use the compass direction notation from lecture.

$$-2u_P \left[\frac{1}{h_x^2} + \frac{1}{h_y^2} \right] + \frac{1}{h_x^2}[u_E + u_W] + \frac{1}{h_y^2}[u_N + u_S] = g_P$$

Set $a := \frac{1}{h_x^2}$, $b := \frac{1}{h_y^2}$, $c := a + b$, so that

$$-2cu_P + a[u_E + u_W] + b[u_N + u_S] = g_P \quad (2.2)$$

To explore the appropriate numerical method, take $J = 4$. The mesh steps create a mesh from Ω with $(J+1)^2 = 25$ points. Using the BCs, it is enough to solve for the values of 16 points, labeled below.



Using the fact $u|_{y=0} = 0$, the values of the points on the bottom edge are all 0. Using the fact $u_y|_{y=2} = 0$, the values of the “ghost points” above the mesh from left to right are, respectively, u_2, u_6, u_{10}, u_{14} . Using the fact $u|_{x=-\pi} = u|_{x=\pi}$, the following pairs of points are treated as adjacent: 1 and 13, 2 and 14, 3 and 15, 4 and 16. Apply (2.2) to each point in the mesh to obtain a system $Au = f$, shown below on the left, with the block structure of A compactly rewritten on the right.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	
1	-2c	2b		a					a				U ₁	$g(-\pi + \frac{2}{n})$			
2	b	-2c	b		a					a			U ₂	$g(-\pi + \frac{2}{n})$			
3	b	-2c	b		a					a			U ₃	$g(-\pi + \frac{2}{n})$			
4	b	-2c		a							a		U ₄	$g(-\pi + \frac{2}{n})$			
5	a		-2c	2b		a							U ₅	$g(-\pi + \frac{1}{n})$			
6	a		b	-2c	b		a						U ₆	$g(-\pi + \frac{1}{n})$			
7	a		b	-2c	b		a						U ₇	$g(-\pi + \frac{1}{n})$			
8	a		b	-2c		a							U ₈	$g(-\pi + \frac{1}{n})$			
9		a		-2c	2b		a						U ₉	$g(-\pi + \frac{2}{n})$			
10		a		b	-2c	b		a					U ₁₀	$g(-\pi + \frac{2}{n})$			
11		a		b	-2c	b		a					U ₁₁	$g(-\pi + \frac{2}{n})$			
12		a		b	-2c		a						U ₁₂	$g(-\pi + \frac{2}{n})$			
13	a			a		-2c	2b						U ₁₃	$g(-\pi + \frac{2}{n})$			
14	a			a		b	-2c	b					U ₁₄	$g(-\pi + \frac{2}{n})$			
15	a			a		b	-2c	b					U ₁₅	$g(-\pi + \frac{2}{n})$			
16	a			a		a	b	-2c					U ₁₆	$g(-\pi + \frac{2}{n})$			

(c) The stationary heat distribution is solved for $J = 100$ and plotted below.

Code: <https://github.com/RokettoJanpu/scientific-computing-2-redux/blob/main/hw6.ipynb>

