AMSC661, Spring 2024 Maria Cameron

Homework 12. Due May 7.

1. **(5 pts)** Consider the Greenberg traffic model

$$\rho_t + [-\rho \log(\rho)]_x = 0, \quad \rho(x,0) = \rho_0(x). \tag{1}$$

Here, ρ is the density of cars, and the velocity v depends on the density according to $v(\rho) = v_{\max} \log\left(\frac{\rho_{\max}}{\rho}\right)$, where $v_{\max}$ and $\rho_{\max}$ are set to be 1 for convenience.

   (a) Find the formula for the characteristic $x(t)$ of Eq. (1) starting at the point $(x = x_0, t = 0)$ (the curve $x(t)$ passing through $(x = x_0, t = 0)$ along which ρ is constant, i.e., $\frac{d}{dt}\rho(x(t), t) = 0$).

   (b) Plot the characteristics and the shock line on the xt-plane for the Riemann problem

$$\rho_0(x) = \begin{cases} 0.1, & x < 0, \\ 0.9, & x > 0. \end{cases}$$

   (c) Suppose

$$\rho_0(x) = 0.5 + \frac{0.9}{\pi} \arctan(x). \tag{2}$$

   Find the time when the shock appears. Then find the eventual shock speed.

2. **(5 pt)**

   (a) Show that MacCormack's method

$$\begin{aligned} U_j^* &= U_j^n - \frac{k}{h}\left[f(U_{j+1}^n) - f(U_j^n)\right], \\ U_j^{n+1} &= \frac{1}{2}\left(U_j^n + U_j^*\right) - \frac{k}{2h}\left[f(U_j^*) - f(U_{j-1}^*)\right], \end{aligned} \tag{3}$$

   reduces to the Lax-Wendroff method for $f(u) \equiv au$.

   (b) Show that MacCormack's method is second-order consistent on smooth solutions.

   (c) Determine a numerical flux function for MacCormack's method that allows us to rewrite it in the conservative form. Rewrite it in the conservative form. Show that the method is consistent.

3. **(5 pts)** Consider Godunov's method for solving $u_t + [f(u)]_x = 0$. In class we have established that if $f(u)$ is convex (if f is twice differentiable then $f''(u) > 0$), the following four cases exhaust all possibilities:

(a) $f'(u_L) \ge 0$ and $f'(u_R) \ge 0$. Then $u^* = u_L$.

(b) $f'(u_L) \le 0$ and $f'(u_R) \le 0$. Then $u^* = u_R$.

(c) $f'(u_L) \ge 0 \ge f'(u_R)$. Then

$$u^* = \begin{cases} u_L, & \text{if } \frac{f(u_L)-f(u_R)}{u_L-u_R} > 0, \\ u_R, & \text{if } \frac{f(u_L)-f(u_R)}{u_L-u_R} < 0. \end{cases} \tag{4}$$

(d) $f'(u_L) < 0 < f'(u_R)$. Then $u^* = u_s$ (transonic rarefaction), where the value u_s is such that $f'(u_s) = 0$. It is called the *sonic point*. For example, for the Burgers equation $u_t + [u^2/2]_x = 0$, $u_s = 0$.

In the first three cases, the value u^* is either u_L and u_R, and it can be simply determined by Eq. (4). Note that in Cases 1 and 2, u^* is the same whether the physically correct weak solution to the Riemann problem is a shock wave or a rarefaction. Only in Case 4, the transonic rarefaction, the value of u^* differs from the one determined by Eq. (4). This is the value of u for which the characteristic speed is zero.

Verify that the numerical flux determined by Cases 1 - 4 can be rewritten more compactly as

$$F(u_L, u_R) = \begin{cases} \min_{u_L \le u \le u_R} f(u), & \text{if } u_L \le u_R, \\ \max_{u_R \le u \le u_L} f(u), & \text{if } u_L > u_R. \end{cases} \tag{5}$$

Remark: It was proven that the numerical flux given by Eq. (5) gives the physically correct flux for scalar conservation laws even if $f(u)$ is non-convex.

4. **(5 pts)** Consider the Burgers equation $u_t + [\frac{1}{2}u^2]_x = 0$ with the initial condition $u_0(x) = 1$ on $[0, 1]$ and $u_0(x) = 0$ otherwise. Implement the following methods for conservation laws: Lax-Friedrichs, Richtmyer, MacCormack, and Godunov and apply them to the problem above. Compute the numerical solution by each of the methods with the same time step and plot it at times $t = 0, 1, 2, 3, 4, 5, 6$. Plot the exact solution as well. It is found in `Hyperbolic.pdf`.

5. **5 pts** Read an article on the Kuramoto-Sivashinsky equation available at http://people.maths.ox.ac.uk/trefethen/pdectb/kuramoto2.pdf. A detailed description of the method can be found in Kassam&Trefethen (2005).

   Solve the equation

$$u_t + u_{xxxx} + u_{xx} + \tfrac{1}{2}(u^2)_x = 0, \quad u(x, 0) = \cos(x/16)(1 + \sin(x/16)) \tag{6}$$

   on the interval $[0, 32\pi]$ with periodic boundary condition. Proceed as follows. Assume first that you need to solve

$$u_t = -u_{xxxx} - u_{xx} := Lu. \tag{7}$$

Write

$$u(x,t) = \sum_{k=-\infty}^{\infty} u_k(t) e^{ikx/16}.$$

Plug this into the equation and obtain an exact solution $u(x,t)$ of Eq. (7). Define the solution operator e^{tL} so that $u(x,t) = e^{tL}u(x,0)$. Now return to Eq. (6). Note that $u_t = Lu + N(u)$ where $N(u) := -\frac{1}{2}(u^2)_x$. Define a new unknown function $v(x,t)$ by $u(x,t) = e^{tL}v(x,t)$. Plug this into $u_t = Lu + N(u)$ and obtain the following equation for $v(x,t)$:

$$v_t = e^{-tL}N(e^{tL}v). \tag{8}$$

Solve Eq. (8) using 4th order Runge-Kutta method on the time interval [0,200]. Plot the surface $u(x,t)$ using the command `imagesc`. Compare it with the one in the article above.

Hint: modify the program `KdVrkm.m` *that solves the Korteweg-de Vries equation*

$$u_t + u_{xxx} + \frac{1}{2}(u^2)_x = 0$$

using the proposed approach.

References

[1] R. J. LeVeque, Numerical Methods for Conservation Laws, Second Edition, Birkh auser, Basel, Boston, Berlin, 1992

[2] M. Cameron's notes `burgers.pdf` available on ELMS.