

## Scientific Computing HW 6

Ryan Chen

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1. In the following cases, the system  $Au = f$  is shown on the left, and the block structure of  $A$  (for a general  $J \times J$  mesh) is compactly rewritten on the right.

|    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 |
| 1  | -4 | 2  |    | 1  |    |    |    |    |    |    |    |    |    |    |
| 2  | 1  | -4 | 1  |    | 1  |    |    |    |    |    |    |    |    |    |
| 3  |    | 1  | -4 | 1  |    | 1  |    |    |    |    |    |    |    |    |
| 4  |    |    | 1  | -4 | 1  |    | 1  |    |    |    |    |    |    |    |
| 5  |    |    |    | 2  | -4 |    |    | 1  |    |    |    |    |    |    |
| 6  | 1  |    |    | -4 | 2  |    | 1  |    |    |    |    |    |    |    |
| 7  | 1  |    |    | 1  | -4 | 1  |    | 1  |    |    |    |    |    |    |
| 8  |    | 1  |    |    | 1  | -4 | 1  |    | 1  |    |    |    |    |    |
| 9  |    |    | 1  |    |    | 1  | -4 | 1  |    | 1  |    |    |    |    |
| 10 |    |    |    | 1  |    |    | 2  | -4 |    | 1  |    |    |    |    |
| 11 |    |    |    |    | 1  |    |    | -4 | 2  |    |    |    |    |    |
| 12 |    |    |    |    |    | 1  |    | 1  | -4 | 1  |    |    |    |    |
| 13 |    |    |    |    |    |    | 1  |    | 1  | -4 | 1  |    |    |    |
| 14 |    |    |    |    |    |    |    | 1  |    | 1  | -4 | 1  |    |    |
| 15 |    |    |    |    |    |    |    |    | 1  |    | 2  | -4 |    |    |

(a)  $A$        $U$        $f$

|    | 1  | 2  | 3  | 4  | 5  | 6 | 7  | 8  | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|----|----|----|----|----|----|---|----|----|---|----|----|----|----|----|----|
| 1  | -4 | 1  |    | 2  |    |   |    |    |   |    |    |    |    |    |    |
| 2  | 1  | -4 | 1  |    | 2  |   |    |    |   |    |    |    |    |    |    |
| 3  |    | 1  | -4 |    |    | 2 |    |    |   |    |    |    |    |    |    |
| 4  | 1  |    |    | -4 | 1  |   | 1  |    |   |    |    |    |    |    |    |
| 5  |    | 1  |    | 1  | -4 | 1 |    | 1  |   |    |    |    |    |    |    |
| 6  |    |    | 1  | 1  | -4 |   |    | 1  |   |    |    |    |    |    |    |
| 7  |    |    |    | 1  | -4 | 1 |    |    | 1 |    |    |    |    |    |    |
| 8  |    |    |    |    | 1  | 1 | -4 | 1  |   | 1  |    |    |    |    |    |
| 9  |    |    |    |    |    | 1 | 1  | -4 |   |    | 1  |    |    |    |    |
| 10 |    |    |    |    |    |   | 1  | -4 | 1 |    | 1  |    |    |    |    |
| 11 |    |    |    |    |    |   |    | 1  | 1 | -4 | 1  |    |    |    |    |
| 12 |    |    |    |    |    |   |    |    | 1 | 1  | -4 |    | 1  |    |    |
| 13 |    |    |    |    |    |   |    |    |   | 2  | -4 | 1  |    |    |    |
| 14 |    |    |    |    |    |   |    |    |   |    | 2  | 1  | -4 | 1  |    |
| 15 |    |    |    |    |    |   |    |    |   |    |    | 2  | 1  | -4 |    |

$$\begin{array}{c|cccc|cccc|cccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
 \hline
 1 & -4 & 1 & & & 1 & & & & & & & \\
 2 & 1 & -4 & 1 & & 1 & & & & & & & \\
 3 & & 1 & -4 & 1 & & 1 & & & & & & \\
 4 & & & 1 & -4 & 1 & & & & & & & \\
 5 & 1 & & & & -4 & 1 & 1 & 1 & & & & \\
 6 & & 1 & & & 1 & -4 & 1 & & 1 & & & \\
 7 & & & 1 & & 1 & -4 & 1 & & & 1 & & \\
 8 & & & & 1 & 1 & -4 & 1 & & & & 1 & \\
 9 & & & & & & & -4 & 1 & 1 & & & \\
 10 & & & & & & & 1 & -4 & 1 & & & \\
 11 & & & & & & & 1 & 1 & -4 & 1 & & \\
 12 & & & & & & & 1 & 1 & 1 & -4 & 1 & \\
 \end{array} \cdot \begin{array}{c|c}
 u_1 & -1 \\
 u_2 & -1 \\
 u_3 & -1 \\
 u_4 & -1 \\
 u_5 & 0 \\
 u_6 & 0 \\
 u_7 & 0 \\
 u_8 & 0 \\
 u_9 & 0 \\
 u_{10} & 0 \\
 u_{11} & 0 \\
 u_{12} & 0
 \end{array} = \begin{array}{c|cc|c}
 \text{$J \times J$ blocks} & & & \\
 \begin{matrix} T & I \\ I & T & I \\ I & & \ddots & \\ \ddots & T & I \\ I & & T \end{matrix} & & \begin{matrix} -4 & 1 & 1 \\ 1 & -4 & 1 \\ 1 & 1 & -4 \end{matrix} \\
 A = & & T = & \\
 & & & \\
 & & & 
 \end{array}$$

(c)

$$\begin{array}{c|cccc|cccc|cccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
 \hline
 1 & -4 & 1 & & & 1 & & & & 1 & & & \\
 2 & 1 & -4 & 1 & & 1 & & & & 1 & & & \\
 3 & & 1 & -4 & 1 & & 1 & & & & 1 & & \\
 4 & & & 1 & -4 & 1 & & & & & & & \\
 5 & 1 & & & & 1 & -4 & 1 & 1 & & & & \\
 6 & & 1 & & & 1 & -4 & 1 & 1 & & & & \\
 7 & & & 1 & & 1 & -4 & 1 & 1 & & & & \\
 8 & & & & 1 & 1 & -4 & 1 & 1 & & & & \\
 9 & & & & & 1 & 1 & -4 & 1 & 1 & & & \\
 10 & 1 & & & & & 1 & -4 & 1 & 1 & & & \\
 11 & & 1 & & & & 1 & 1 & -4 & 1 & & & \\
 12 & & & 1 & & & 1 & 1 & 1 & -4 & 1 & & \\
 \end{array} \cdot \begin{array}{c|c}
 u_1 & 0 \\
 u_2 & 0 \\
 u_3 & -1 \\
 u_4 & 0 \\
 u_5 & 0 \\
 u_6 & -1 \\
 u_7 & 0 \\
 u_8 & 0 \\
 u_9 & -1 \\
 u_{10} & 0 \\
 u_{11} & 0 \\
 u_{12} & -1
 \end{array} = \begin{array}{c|cc|c}
 \text{$(J-1) \times (J-1)$ blocks} & & & \\
 \begin{matrix} T & I & I \\ I & T & I \\ I & & \ddots & \\ \ddots & T & I \\ I & & T \end{matrix} & & \begin{matrix} -4 & 1 & & \\ 1 & -4 & 1 & \\ 1 & 1 & \ddots & \\ & \ddots & -4 & 1 \\ & & 1 & -4 \end{matrix} \\
 A = & & T = & \\
 & & & 
 \end{array}$$

(d)

(e) In this case we do not compactly rewrite the block structure of  $A$ , but it is still clear.

$$\begin{array}{c|cccc|cccc|cccc|cccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\
 \hline
 1 & -4 & 1 & & & 1 & & & & & & & & & & & \\
 2 & 1 & -4 & 1 & & 1 & & & & & & & & & & & \\
 3 & & 1 & -4 & 1 & & 1 & & & & & & & & & & \\
 4 & & & 1 & -4 & 1 & & & & & & & & & & & \\
 5 & 1 & & & & 1 & -4 & 1 & & 1 & & & & & & & \\
 6 & & 1 & & & & 1 & -4 & 1 & & 1 & & & & & & \\
 7 & & & 1 & & 1 & -4 & 1 & & 1 & & & & & & & \\
 8 & & & & 1 & 1 & -4 & 1 & & 1 & & & & & & & \\
 9 & & & & & 1 & 1 & -4 & 1 & 1 & & & & & & & \\
 10 & 1 & & & & & 1 & 1 & -4 & 1 & 1 & & & & & & \\
 11 & & & & & & & 1 & -4 & 1 & 1 & & & & & & \\
 12 & & & & & & & 1 & 1 & -4 & 1 & 1 & & & & & \\
 13 & & & & & & & & 1 & -4 & 1 & 1 & & & & & \\
 14 & & & & & & & & & 1 & -4 & 1 & 1 & & & & \\
 15 & & & & & & & & & & 1 & -4 & 1 & 1 & & & \\
 16 & & & & & & & & & & & 1 & -4 & 1 & 1 & & 
 \end{array} \cdot \begin{array}{c|c}
 u_1 & 0 \\
 u_2 & 0 \\
 u_3 & 0 \\
 u_4 & 0 \\
 u_5 & 0 \\
 u_6 & -1 \\
 u_7 & -1 \\
 u_8 & 0 \\
 u_9 & 0 \\
 u_{10} & 0 \\
 u_{11} & -1 \\
 u_{12} & 0 \\
 u_{13} & -1 \\
 u_{14} & 0 \\
 u_{15} & -1 \\
 u_{16} & 0
 \end{array} = \begin{array}{c|cc|c}
 & & & \\
 & & & 
 \end{array}$$

2. (a) The BVP on the domain  $\Omega := [-\pi, \pi] \times [0, 2]$  is

$$u_{xx} + u_{yy} = g(x) := \begin{cases} -\cos x, & -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ 0, & \text{else} \end{cases} \quad (2.1)$$

with BCs

$$u \Big|_{x=\pi} = u \Big|_{x=-\pi}, \quad u_x \Big|_{x=\pi} = u_x \Big|_{x=-\pi}, \quad u \Big|_{y=0} = 0, \quad u_y \Big|_{y=2} = 0$$

(b) Fix  $J \in \mathbb{N}$  and set mesh steps in the  $x$  and  $y$  axes,

$$h_x := \frac{2\pi}{J}, \quad h_y := \frac{2}{J}$$

Then

$$\begin{aligned} u_{xx}(x, y) &= \frac{1}{h_x^2} [u(x + h_x, y) - 2u(x, y) + u(x - h_x, y)] + O(h_x^2) \\ u_{yy}(x, y) &= \frac{1}{h_y^2} [u(x, y + h_y) - 2u(x, y) + u(x, y - h_y)] + O(h_y^2) \end{aligned}$$

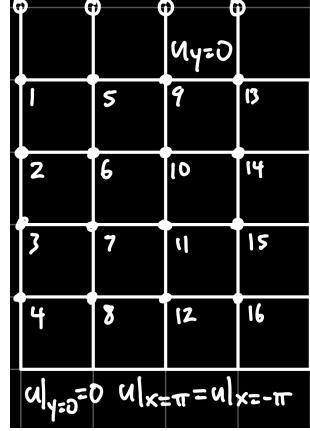
Plug these expressions into (2.1) and use the compass direction notation from lecture.

$$-2u_P \left[ \frac{1}{h_x^2} + \frac{1}{h_y^2} \right] + \frac{1}{h_x^2}[u_E + u_W] + \frac{1}{h_y^2}[u_N + u_S] = g_P$$

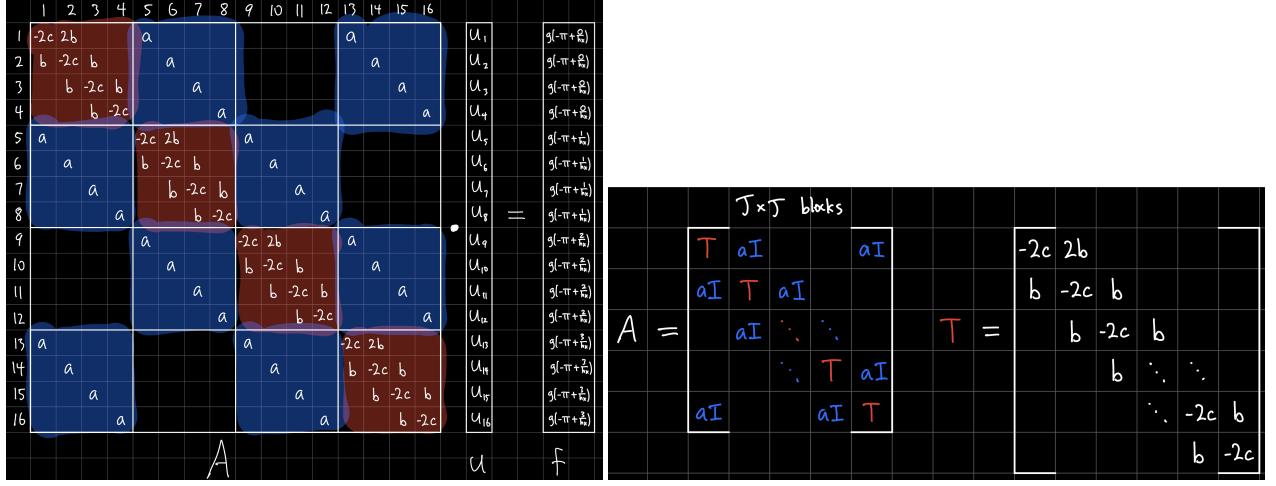
Set  $a := \frac{1}{h_x^2}$ ,  $b := \frac{1}{h_y^2}$ ,  $c := a + b$ , so that

$$-2cu_P + a[u_E + u_W] + b[u_N + u_S] = g_P \quad (2.2)$$

To explore the appropriate numerical method, take  $J = 4$ . The mesh steps create a mesh from  $\Omega$  with  $(J+1)^2 = 25$  points. Using the BCs, it is enough to solve for the values of 16 points, labeled below.



Using the fact  $u|_{y=0} = 0$ , the values of the points on the bottom edge are all 0. Using the fact  $u_y|_{y=2} = 0$ , the values of the “ghost points” above the mesh from left to right are, respectively,  $u_2, u_6, u_{10}, u_{14}$ . Using the fact  $u|_{x=-\pi} = u|_{x=\pi}$ , the following pairs of points are treated as adjacent: 1 and 13, 2 and 14, 3 and 15, 4 and 16. Apply (2.2) to each point in the mesh to obtain a system  $Au = f$ , shown below on the left, with the block structure of  $A$  compactly rewritten on the right.



(c) The stationary heat distribution is solved for  $J = 100$ .

Code: <https://github.com/RokettoJanpu/scientific-computing-2-redux/blob/main/hw6.ipynb>

```

import numpy as np
import matplotlib.pyplot as plt
from scipy.sparse import spdiags

# right hand side of Laplace's equation
def g(x):
    return -np.cos(x) if -np.pi/2 <= x <= np.pi/2 else 0

# set mesh size
J = 100
hx = 2*np.pi/J
hy = 2/J
a = 1/hx**2
b = 1/hy**2
c = a + b

# define block structure of A
v = b*np.ones(J)
v[1] = 2*b
T = spdiags([b*np.ones(J), -2*c*np.ones(J), v], [-1, 0, 1]).toarray()
S = spdiags([np.ones(J), np.ones(J), np.ones(J)], [-J+1, -1, 1, J-1]).toarray()
A = np.kron(np.identity(J), T) + np.kron(S, a*np.identity(J))

# define f
f = np.zeros(J**2)
for j in range(J**2):
    f[j] = g(-np.pi + int(j/J)*hx)

# solve Au=f for u, convert u to 2d array, fill in BCs
u = np.linalg.solve(A, f)
u = np.reshape(u, (J, J), order = 'F')
u = np.pad(u, (0, 1))
for j in range(J):
    u[j][-1] = u[j][0]

# plot u
plt.title('stationary heat distribution for J = {}'.format(J))
plt.xlabel('x')
plt.ylabel('y')
im = plt.imshow(u, cmap = plt.cm.magma, extent = [-np.pi, np.pi, 0, 2])
plt.colorbar(im)
plt.savefig('hw6_2_plot.png')
plt.show()

```

Stationary heat distribution:

