Scientific Computing HW 12

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1. (a) The PDE is

$$\rho_t + \partial_x f(\rho) = 0, \quad f(\rho) := -\rho \ln \rho$$

which becomes

$$\rho_t + f'(\rho)\rho_x = 0, \quad f'(\rho) = -\ln \rho - \rho\frac{1}{\rho} = -\ln \rho - 1$$

Consider the curve Γ given by $x(t)$ satisfying

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f'(\rho(x(t), t)), \quad x(0) = x_0$$

We see that Γ is a characteristic of the PDE since

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(x(t), t) = \rho_t + \rho_x\frac{\mathrm{d}x}{\mathrm{d}t} = \rho_t + \rho_x f'(\rho(x(t), t)) = 0$$

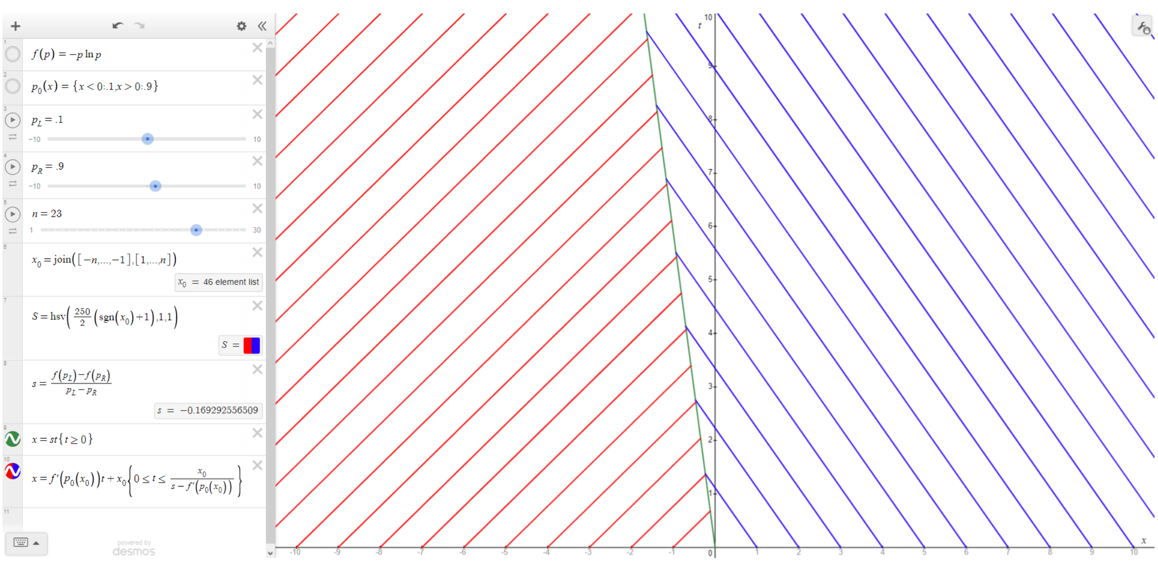
hence ρ is constant on Γ. In particular,

$$\rho(x(t), t) = \rho(x(0), 0) = \rho_0(x_0)$$

Thus Γ is given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f'(\rho_0(x_0)) \implies x(t) = f'(\rho_0(x_0))t + x_0 = [-\ln \rho_0(x_0) - 1]t + x_0$$

(b) Some characteristics are plotted below. Those with $x_0 < 0$ are red, those with $x_0 > 0$ are blue, and the shock line is green.



(c) In this part the initial density is

$$\rho_0(x) = \frac{1}{2} + \frac{9}{10\pi}\arctan x$$

First compute

$$f''(\rho) = -\rho^{-1}, \quad \rho_0'(x) = \frac{9}{10\pi}(x^2+1)^{-1}$$

The equation of a characteristic starting at a point $(x_0, 0)$, considered as function of t and x_0, is

$$x = f'(\rho_0(x_0))t + x_0$$

Then the shock appears at time t_s when $\partial_{x_0} x = 0$, i.e.

$$f''(\rho_0(x_0))\rho_0'(x_0)t_s + 1 = 0$$

$$\implies t_s = -\left[f''(\rho_0(x_0))\rho_0'(x_0)\right]^{-1} = -\left[-\left(\frac{1}{2} + \frac{9}{10\pi}\arctan x_0\right)^{-1} \frac{9}{10\pi}(x_0^2+1)^{-1}\right]^{-1}$$

$$= \left(\frac{1}{2} + \frac{9}{10\pi}\arctan x_0\right)\frac{10\pi}{9}(x_0^2+1) = \left(\frac{5\pi}{9} + \arctan x_0\right)(x_0^2+1)$$

Now we find

$$\lim_{x\to\pm\infty} \rho_0(x) = \frac{1}{2} + \frac{9}{10\pi}\left(\pm\frac{\pi}{2}\right) = \frac{1}{2} \pm \frac{9}{20} \implies \rho_L = \frac{1}{20}, \quad \rho_R = \frac{19}{20}$$

Then compute

$$f(\rho_L) = -\frac{1}{20}\ln\frac{1}{20} = \frac{1}{20}\ln 20$$

$$f(\rho_R) = -\frac{19}{20}\ln\frac{19}{20} = \frac{19}{20}\ln\frac{20}{19}$$

Thus the eventual shock speed is

$$s = \frac{f(\rho_L) - f(\rho_R)}{\rho_L - \rho_R} = \frac{\frac{1}{20}\ln 20 - \frac{19}{20}\ln\frac{20}{19}}{\frac{1}{20} - \frac{19}{20}} = \frac{\ln 20 - 19\ln 20 + 19\ln 19}{1 - 19} = \ln 20 - \frac{19}{18}\ln 19 \approx -0.112$$

2. (a) Using $f(u) = au$,

$$u_j^* = u_j^n - \frac{ak}{h}(u_{j+1}^n - u_j^n)$$

In turn,

$$u_j^{n+1} = \frac{1}{2}\left[u_j^n + u_j^n - \frac{ak}{h}(u_{j+1}^n - u_j^n)\right] - \frac{ak}{2h}\left[u_j^n - \frac{ak}{h}(u_{j+1} - u_j^n) - u_{j-1}^n + \frac{ak}{h}(u_j^n - u_{j-1}^n)\right]$$

$$= u_j^n - \frac{ak}{2h}(u_{j+1}^n - u_j^n + u_j^n - u_{j-1}^n) + \frac{a^2k^2}{2h^2}(u_{j+1}^n - u_j^n - u_j^n + u_{j-1}^n)$$

$$= u_j^n - \frac{ak}{2h}(u_{j+1}^n - u_{j-1}^n) + \frac{a^2k^2}{2h^2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$$

Which coincides with Lax–Wendroff.

3. (a) Recall that $F(u_L, u_R) = f(u^*(u_L, u_R))$. From the fact $f'' > 0$, we know f' is increasing. For now, suppose $u_L \le u_R$.

- In case 1, from $f'(u_L) \geq 0$ we have $f' \geq 0$ hence f is increasing, thus

$$F(u_L, u_R) = f(u_L) = \min_{u_L \leq u \leq u_R} f(u)$$

- In case 2, from $f'(u_R) \leq 0$, we have $f' \leq 0$ hence f is decreasing, thus

$$F(u_L, u_R) = f(u_R) = \min_{u_L \leq u \leq u_R} f(u)$$

- In case 3, from $f'(u_L) \geq 0$ we have $f' \geq 0$ hence f is increasing. This along with $u_L \leq u_R$ gives $f(u_L) < f(u_R)$, which implies

$$\frac{f(u_L) - f(u_R)}{u_L - u_R} > 0$$

  From this we have $u^* = u_L$, which along with f being increasing implies

$$F(u_L, u_R) = f(u_L) = \min_{u_L \leq u \leq u_R} f(u)$$

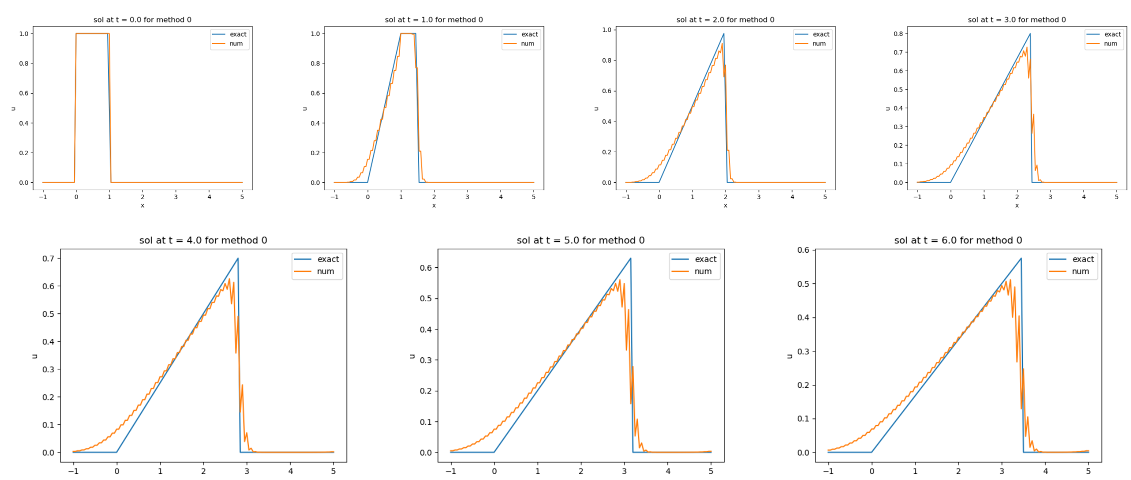
- In case 4, from $f'(u_s) = 0$ and $f'' > 0$, we know f attains a global minimum at u_s, thus

$$F(u_L, u_R) = f(u_s) = \min_{u_L \leq u \leq u_R} f(u)$$

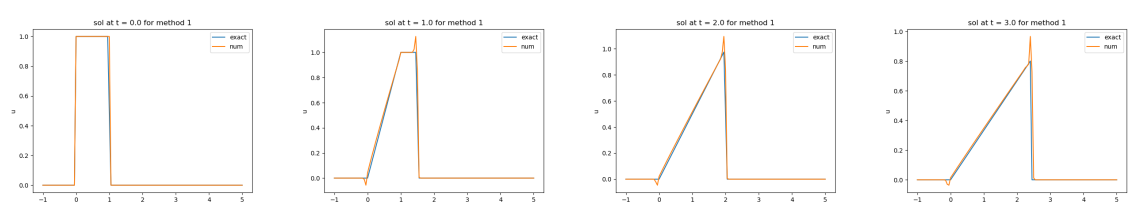
Similar arguments can be made for when we suppose $u_L > u_R$.

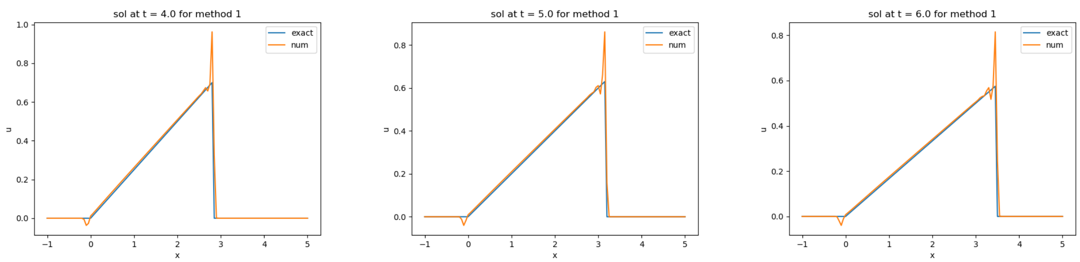
4. Code: `https://github.com/RokettoJanpu/scientific-computing-2-redux/blob/main/hw13q3.ipynb`

Lax–Friedrichs:

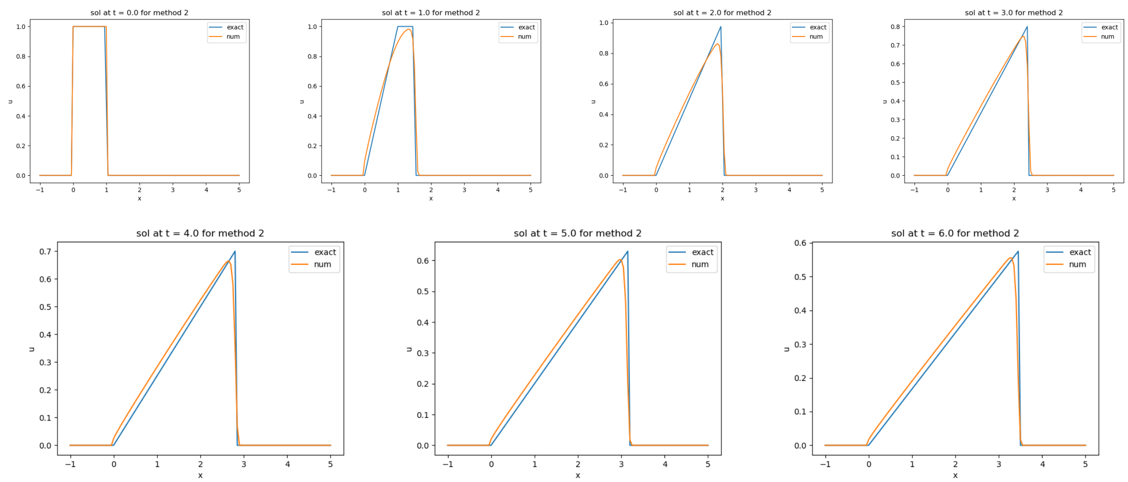


Richtmyer:





MacCormack:



5. Write the PDE as

$$u_t = \underbrace{-u_{xxxx} - u_{xx}}_{=:Lu} \underbrace{-\frac{1}{2}(u^2)_x}_{=:N(u)}$$

First solving $u_t = Lu$, write

$$u(x,t) = \sum_{k=-\infty}^{\infty} u_k(t) e^{ikx/16}$$

so that

$$u_t = \sum_{k=-\infty}^{\infty} u_k'(t) e^{ikx/16}, \quad u_{xx} = \sum_{k=-\infty}^{\infty} u_k(t) \left(-\left(\frac{k}{16}\right)^2 \right) e^{ikx/16}, \quad u_{xxxx} = \sum_{k=-\infty}^{\infty} u_k(t) \left(\frac{k}{16}\right)^4 e^{ikx/16}$$

Plugging in these derivatives and using the fact that the basis functions $e^{ikx/16}$ are linearly independent,

$$u_k'(t) = \left[\left(\frac{k}{16}\right)^2 - \left(\frac{k}{16}\right)^4 \right] u_k(t) \implies u_k(t) = u_k(0) e^{[(k/16)^2 - (k/16)^4]t}$$

giving the solution

$$u(x,t) = \sum_{k=-\infty}^{\infty} u_k(0) e^{ikx/16} e^{[(k/16)^2 - (k/16)^4]t}, \quad u_k(0) = \frac{1}{32\pi} \int_0^{32\pi} u(x,0) e^{-ikx/16} dx$$

Define the solution operator e^{tL} by specifying its action on the basis functions $e^{ikx/16}$.

$$e^{tL}(e^{ikx/16}) := e^{ikx/16}e^{[(k/16)^2-(k/16)^4]t}$$

We check that we can rewrite the solution as

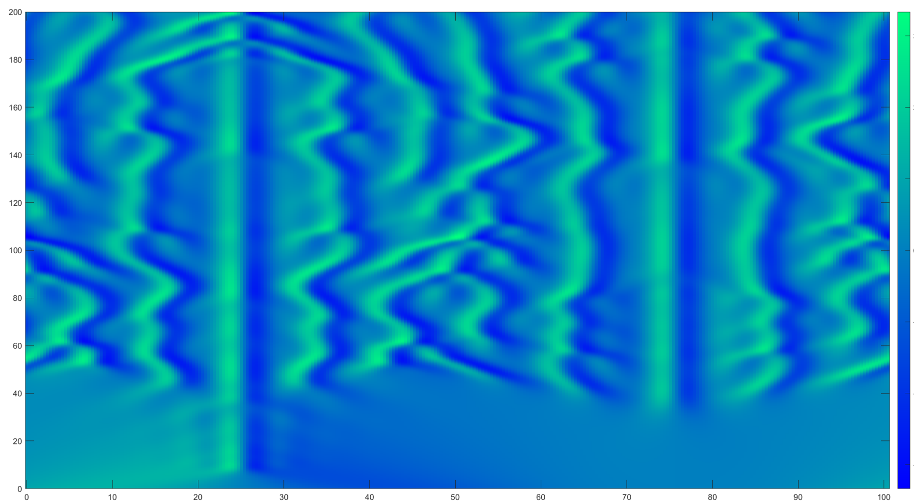
$$u(x,t) = \sum_{k=-\infty}^{\infty} u_k(0)e^{tL}(e^{ikx/16}) = e^{tL}\sum_{k=-\infty}^{\infty} u_k(0)e^{ikx/16} = e^{tL}u(x,0)$$

Let v satisfy $u = e^{tL}v$. Plugging into the equation $u_t = Lu + N(u)$, we obtain an equation for v.

$$Le^{tL}v + e^{tL}v_t = Le^{tL}v + N(e^{tL}v) \implies e^{tL}v_t = N(e^{tL}v) \implies v_t = e^{-tL}N(e^{tL}v)$$

The file KdVrkm.m was modified to solve this PDE:

`https://github.com/RokettoJanpu/scientific-computing-2-redux/blob/main/KdVrkm.m`



The plot is very similar to the one in the linked article. The initial data does not vary significantly, but it splits into many high frequency waves, giving rise to a complex looking solution. On the other hand, characteristic lines appear, upon which solutions appear stationary, and moreover nearby solutions tend toward these lines.