Scientific Computing HW 4

Ryan Chen

February 29, 2024

1. The characteristic polynomial is

$$\rho(z) = z^k + \sum_{j=0}^{k-1} \alpha_j z^{k-1-j}$$

and the recurrence is

$$u_{n+1} + \sum_{j=0}^{k-1} \alpha_j u_{n-j} = 0$$

Define a linear operator $D := r\frac{\mathrm{d}}{\mathrm{d}r}$. Then for all n and l,

$$D[n^l r^n] = r\frac{\mathrm{d}}{\mathrm{d}r} n^l r^n = rn^l nr^{n-1} = n^{l+1}r^n$$

so that in particular,

$$n^l r^n = D[n^{l-1}r^n] = D^2[n^{l-2}r^n] = \ldots = D^l[r^n] \quad (1.1)$$

Plug $u_n = n^l r^n$ into the LHS of the recurrence, apply (1.1) to each term, then use the linearity of D.

$$u_{n+1} + \sum_{j=0}^{k-1} \alpha_j u_{n-j} = (n+1)^l r^{n+1} + \sum_{j=0}^{k-1} \alpha_j (n-j)^l r^{n-j} = D^l[r^{n+1}] + \sum_{j=0}^{k-1} \alpha_j D^l[r^{n-j}] = D^l \left[r^{n+1} + \sum_{j=0}^{k-1} \alpha_j r^{n-j} \right]$$

$$= D^l \left[r^{n-k+1} \left(r^k + \sum_{j=0}^{k-1} \alpha_j r^{k-1-j} \right) \right] = D^l[r^{n-k+1}\rho(r)] = D^l[0] = 0$$

2. We first establish a lemma. For the $(k-1) \times k$ matrix

$$\begin{bmatrix} \lambda & -1 & & & \\ & \lambda & -1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & -1 \end{bmatrix}$$

The submatrix formed by deleting the jth column has determinant $(-1)^{k-j}\lambda^{j-1}$. To see this, write the submatrix in block form.

$$M := \begin{bmatrix} \lambda & -1 & & & & & & & \\ & \lambda & -1 & & & & & & \\ & & \ddots & \ddots & & & 0_{(j-1)\times(k-j)} & & \\ & & & \lambda & -1 & & & & \\ & & & & \lambda & & & & \\ & & & & & -1 & & & \\ & & & & & \lambda & -1 & & \\ & 0_{(k-j)\times(j-1)} & & & & & \ddots & \ddots & \\ & & & & & & & \lambda & -1 \end{bmatrix}$$

Using the fact that

$$\det \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} = \det(B)\det(C)$$

we have $\det M = \lambda^{j-1}(-1)^{k-j}$.

We now aim to show $\det(\lambda I - A) = \rho(\lambda)$.

$$\lambda I - A = \begin{bmatrix} \lambda & -1 & & & \\ & \lambda & -1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & -1 \\ \alpha_{k-1} & \alpha_{k-2} & \dots & \alpha_1 & \lambda + \alpha_0 \end{bmatrix}$$

Expanding over row k, starting at column k, and using the above lemma,

$$\det(\lambda I - A) = (\lambda + \alpha_0)(-1)^{k-k}\lambda^{k-1} + \sum_{j=1}^{k-1}(-1)^{k+k-j}\alpha_j(-1)^{k-k+j}\lambda^{k-j-1} = \lambda^k + \alpha_0\lambda^{k-1} + \sum_{j=1}^{k-1}(-1)^{2k}\alpha_j\lambda^{k-j-1}$$

$$= \lambda^k + \alpha_0\lambda^{k-1} + \sum_{j=1}^{k-1}\alpha_j\lambda^{k-j-1} = \lambda^k + \sum_{j=0}^{k-1}\alpha_j\lambda^{k-j-1} = \rho(\lambda)$$

3. Note: For explicit matrix norm computations we will use the Euclidean norm

$$\|A\| = \left[\sum_{i,j} A_{ij}^2\right]^{1/2}$$

Pf. Write the Jordan form of A.

$$A = SJS^{-1}, \quad J = \mathrm{diag}(J_1, J_2, \dots, J_s), \quad J_q = r_q I_{m_q} + N_q, \quad N_q \overset{m_q \times m_q}{=} \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \end{bmatrix}$$

For all p,

$$A^p = SJ^pS^{-1}, \quad J^p = \mathrm{diag}(J_1^p, J_2^p, \dots, J_s^p)$$

so we seek a bound on $\|J^p\|$, where $J = rI + N$ is a Jordan block for an eigenvalue r with multiplicity m (we temporarily use J to denote a single block instead of the whole matrix).

If $m = 1$ then $J = [r]$, and since ρ satisfies the root condition, $|r| \leq 1$, giving

$$\|J^p\| = |r|^p \leq 1$$

Now say $m > 1$, so that $|r| < 1$ by the root condition. If $r = 0$ then $J = N$, hence $\|J^p\| = 0$ for all $p \geq m$. Now say $r \neq 0$. For $p \geq m$,

$$J^p = (rI + N)^p = \sum_{j=0}^{p}\binom{p}{j} r^{p-j}N^j = \sum_{j=0}^{m}\binom{p}{j} r^{p-j}N^j$$

$$\implies \|J^p\| \leq \sum_{j=0}^{p}\binom{p}{j}|r|^{p-j}\|N^j\|$$

Fix j with $0 \le j \le m$. We establish a bound on $\binom{p}{j}|r|^{p-j}$. Note

$$\binom{p}{j}|r|^{p-j} = \frac{p!}{j!(p-j)!}|r|^p|r|^{-j} = \frac{1}{|r|^j j!}|r|^p \prod_{i=0}^{j-1}(p-i) = \frac{1}{|r|^j j!} \cdot \frac{\prod_{i=0}^{j-1}(p-i)}{|1/r|^p}$$

Considering the fraction in the RHS as $p \to \infty$, the denominator, an exponential in p with base $|1/r| > 1$, grows faster than the numerator, a polynomial in p. This means $\binom{p}{j}|r|^{p-j}$ tends to 0 as $p \to \infty$, hence it has a bound M_j independent of p.

Writing small powers of N,

$$N^0 = I = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \ddots \\ & & & 0 & 1 \end{bmatrix}, \quad N^2 = \begin{bmatrix} 0 & 0 & 1 & \\ & 0 & 0 & 1 \\ & & 0 & \ddots & 1 \\ & & & 0 & 0 \end{bmatrix}, \quad N^3 = \begin{bmatrix} 0 & 0 & 0 & 1 & \\ & 0 & 0 & 0 & 1 \\ & & 0 & 0 & 0 \\ & & & 0 & 0 \end{bmatrix}$$

we see that for $0 \le j \le m$,

$$\|N^j\| = \left[\sum_{i,j}(N^j)_{ij}^2\right]^{1/2} = [m-j]^{1/2} \le m^{1/2}$$

We have a bound on $\|J^p\|$ independent of p. Here we re–introduce the subscript q.

$$\|J_q^p\| \le \max\left[\|J_q^1\|, \|J_q^2\|, \ldots, \|J_q^{m_q-1}\|, \sum_{j=0}^{m_q} M_{q,j} m_q^{1/2}\right] =: K_q$$

Now looking at the whole Jordan matrix J,

$$\|J^p\| \le \sum_{q=1}^{s} \|J_q^p\| \le \sum_{q=1}^{s} K_q$$

Thus

$$\|A^p\| \le \|S\|\|S^{-1}\| \sum_{q=1}^{s} K_q$$

4. Plug the supposed solution U_n into the LHS of the recurrence.

$$U_{n+1} = A^{n-k+2}U_{k-1} + h\sum_{j=0}^{n+1-k} A^j G_{n-j} = A^{n-k+2}U_{k-1} + hG_n + h\sum_{j=1}^{n+1-k} A^j G_{n-j}$$

$$= A^{n-k+2}U_{k-1} + hG_n + h\sum_{j=0}^{n-k} A^{j+1}G_{n-j-1}$$

Plug the solution into the RHS.

$$AU_n + hG_n = A^{n-k+2}U_{k-1} + h\sum_{j=0}^{n-k} A^{j+1}G_{n-j-1} + hG_n$$

The two expressions are equal, so U_n indeed solves the recurrence.

5. (a) For BDF2, the interpolant is

$$p(t) = y_{n+1} + y[t_{n+1}, t_n](t - t_{n+1}) + y[t_{n+1}, t_n, t_{n-1}](t - t_{n+1})(t - t_n)$$

The derivative of the last term at t_{n+1} is

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=t_{n+1}} (t - t_{n+1})(t - t_n) = (t - t_n) + (t - t_{n+1})\bigg|_{t=t_{n+1}} = 2t_{n+1} - t_n - t_{n+1} = t_{n+1} - t_n = h_n$$

and so

$$p'(t_{n+1}) = y[t_{n+1}, t_n] + y[t_{n+1}, t_n, t_{n-1}]h_n$$

Equating this with $f_{n+1} := f(t_{n+1}, u_{n+1})$ gives the BDF2 method.

$$\frac{u_{n+1} - u_n}{h_n} + \frac{\frac{u_{n+1}-u_n}{h_n} - \frac{u_n - u_{n-1}}{h_{n-1}}}{h_n + h_{n-1}} h_n = f_{n+1}$$

From $h_n = t_{n+1} - t_n$ and $\omega = \frac{h_n}{h_{n-1}}$,

$$1 + \omega = 1 + \frac{h_n}{h_{n-1}} = \frac{h_{n-1} + h_n}{h_{n-1}} \implies (1 + \omega)^2 = \frac{1}{h_{n-1}^2}\left[h_{n-1}^2 + h_n^2 + 2h_{n-1}h_n\right] \quad (5.1)$$

$$1 + 2\omega = 1 + 2\frac{h_n}{h_{n-1}} = \frac{h_{n-1} + 2h_n}{h_{n-1}} \quad (5.2)$$

From (5.1) and (5.2),

$$\frac{(1+\omega)^2}{1 + 2\omega} = \frac{h_{n-1}}{h_{n-1} + 2h_n}\frac{1}{h_{n-1}^2}\left[h_{n-1}^2 + h_n^2 + 2h_{n-1}h_n\right] = \frac{1}{h_{n-1} + 2h_n}\left[h_{n-1} + \frac{h_n^2}{h_{n-1}} + 2h_n\right]$$

$$= h_{n-1}(1 + 2\omega)\left[h_{n-1} + \frac{h_n^2}{h_{n-1}} + 2h_n\right] \quad (5.3)$$

Now multiply the method by $h_n(h_n + h_{n-1})$.

$$(u_{n+1} - u_n)(h_n + h_{n-1}) + \left[\frac{u_{n+1} - u_n}{h_n} - \frac{u_n - u_{n-1}}{h_{n-1}}\right] h_n^2 = h_n(h_n + h_{n-1})f_{n+1}$$

Collect coefficients of the following terms. Rewrite them using (5.2) and (5.3).

$$u_{n+1}: \quad h_n + h_{n-1} + \frac{h_n^2}{h_n} = 2h_n + h_{n-1} \overset{(5.2)}{=} h_{n-1}(1 + 2\omega)$$

$$u_n: \quad -(h_n + h_{n-1}) + \left[-\frac{1}{h_n} - \frac{1}{h_{n-1}}\right] h_n^2 = -\left[2h_n + h_{n-1} + \frac{h_n^2}{h_{n-1}}\right] \overset{(5.3)}{=} -h_{n-1}(1 + 2\omega)\frac{(1+\omega)^2}{1 + 2\omega}$$

$$u_{n-1}: \quad \frac{h_n^2}{h_{n-1}} = h_{n-1}\omega^2$$

Putting these together, the method is

$$h_{n-1}(1 + 2\omega)u_{n+1} - h_{n-1}(1 + 2\omega)\frac{(1+\omega)^2}{1 + 2\omega}u_n + h_{n-1}\omega^2 = h_n(h_n + h_{n-1})f_{n+1}$$

Divide by $h_{n-1}(1 + 2\omega)$.

$$u_{n+1} - \frac{(1+\omega)^2}{1 + 2\omega}u_n + \frac{\omega^2}{1 + 2\omega}u_{n-1} = h_n\frac{\frac{h_n}{h_{n-1}} + 1}{1 + 2\omega}f_{n+1} = h_n\frac{1 + \omega}{1 + 2\omega}f_{n+1}$$

(b) From the LHS of the method, define

$$\rho(z) := z^2 - \frac{(1+\omega)^2}{1+2\omega} z + \frac{\omega^2}{1+2\omega}$$

By Vieta's formulas, the roots r, s of ρ satisfy

$$r + s = \frac{(1+\omega)^2}{1+2\omega} = \frac{1+\omega^2+2\omega}{1+2\omega} = \frac{\omega^2}{1+2\omega} + 1$$

$$rs = \frac{\omega^2}{1+2\omega}$$

From these we see

$$r = \frac{\omega^2}{1+2\omega}, \quad s = 1$$

Since $\omega < 1 + \sqrt{2}$,

$$w^2 < (1+\sqrt{2})^2 = 1 + 2 + 2 \cdot \sqrt{2} = 1 + 2(1+\sqrt{2}) = 1 + 2\omega \implies 0 \leq r = \frac{\omega^2}{1+2\omega} < 1 \implies |r| < 1$$

This leaves $s = 1$ as the only root with modulus 1, and it has multiplicity 1. Thus ρ satisfies the root condition, hence the method is stable.