Scientific Computing HW 7

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1. We must compute (we will temporarily drop the dependence of v on k_x, k_y)

$$Av = \frac{1}{h^2}(4U_P v - U_W v - U_E v - U_N v - U_S v)$$

Using the identity $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$,

$$\begin{aligned}
U_P v &= \sin k_x hi \sin k_y hj \\
U_W v &= \sin k_x h(i-1) \sin k_y hj = [\sin k_x hi \cos k_x h - \cos k_x hi \sin k_x h] \sin k_y hj \\
U_E v &= \sin k_x h(i+1) \sin k_y hj = [\sin k_x hi \cos k_x h + \cos k_x hi \sin k_x h] \sin k_y hj \\
U_N v &= \sin k_x hi \sin k_y h(j+1) = \sin k_x hi[\sin k_y hj \cos k_y h + \cos k_y hj \sin k_y h] \\
U_S v &= \sin k_x hi \sin k_y h(j-1) = \sin k_x hi[\sin k_y hj \cos k_y h - \cos k_y hj \sin k_y h]
\end{aligned}$$

Thus

$$\begin{aligned}
U_W v + U_E v + U_N v + U_S v &= [\sin k_x hi \cos k_x h - \cos k_x hi \sin k_x h] \sin k_y hj \\
&- [\sin k_x hi \cos k_x h + \cos k_x hi \sin k_x h] \sin k_y hj \\
&- \sin k_x hi[\sin k_y hj \cos k_y h + \cos k_y hj \sin k_y h] \\
&- \sin k_x hi[\sin k_y hj \cos k_y h - \cos k_y hj \sin k_y h] \\
&= -2 \sin k_x hi \sin k_y hj \cos k_x h - 2 \sin k_x hi \sin k_y hj \cos k_y h \\
&= -2 \sin k_x hi \sin k_y hj(\cos k_x h + \cos k_y h)
\end{aligned}$$

At last we find the eigenvalues $\lambda = \lambda(k_x, k_y)$.

$$Av_{k_x,k_y} = \underbrace{\frac{1}{h^2}[4 - 2(\cos k_x h + \cos k_y h)]}_{\lambda(k_x,k_y)} \sin k_x hi \sin k_y hj = \lambda(k_x, k_y) v_{k_x,k_y}$$

To estimate the smallest and largest eigenvalues of A, first write

$$\lambda = N^2 \left[4 - 2\left(\cos\frac{n\pi}{N} + \cos\frac{m\pi}{N}\right)\right]$$

We see λ is maximized at $n, m = N-1$, and $\cos\frac{(N-1)\pi}{N} \sim 1$, hence $\lambda_{max} \sim 8N^2$. We also see λ is minimized at $n, m = 1$, and using the fact $\cos x \sim 1 - \frac{1}{2}x^2$ for $x \sim 0$, we have $\cos\frac{\pi}{N} \sim 1 - \frac{\pi^2}{N^2}$, hence $\lambda_{min} \sim N^2\left[4 - 4\left(1 - \frac{\pi^2}{N^2}\right)\right] = 4\pi^2$. Lastly, the condition number is

$$\kappa(A) = \left|\frac{\lambda_{max}}{\lambda_{min}}\right| \sim \frac{4N^2}{\pi^2}$$

2. (a) Multiplying the BVP by -1 and integrating it, we find $k(x)u' = M$ for some constant M, i.e. $u' = \frac{M}{k(x)}$. The solution is then

$$u(x) = u_a + \int_a^x \frac{M}{k(s)} ds$$

If $x \le c$ then

$$u(x) = u_a + \int_a^x \frac{M}{k_1} ds = u_a + \frac{M}{k_1}(x-a)$$

If $x > c$ then

$$u(x) = u_a + \int_a^c \frac{M}{k(s)} ds + \int_c^x \frac{M}{k(s)} ds = u_a + \int_a^c \frac{M}{k_1} ds + \int_c^x \frac{M}{k_2} ds = u_a + \frac{M}{k_1}(c-a) + \frac{M}{k_2}(x-c)$$

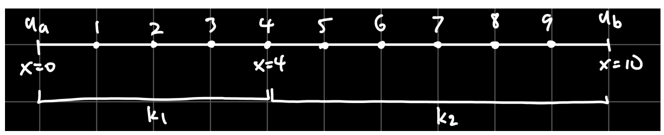
Apply BCs.

$$u_b = u(b) = u_a + \frac{M}{k_1}(c-a) + \frac{M}{k_2}(b-c) = u_a + M\left[\frac{c-a}{k_1} + \frac{b-c}{k_2}\right] \implies M = \frac{u_b - u_a}{\frac{c-a}{k_1} + \frac{b-c}{k_2}}$$

In summary, the solution is

$$u(x) = \begin{cases} u_a + \frac{M}{k_1}(x-a), & x \le c \\ u_a + \frac{M}{k_1}(c-a) + \frac{M}{k_2}(x-c), & x > c \end{cases} \qquad \text{where} \qquad M = \frac{u_b - u_a}{\frac{c-a}{k_1} + \frac{b-c}{k_2}}$$

(b) Given the parameters, we solve for the values of the 9 mesh points shown below.



The finite difference scheme is

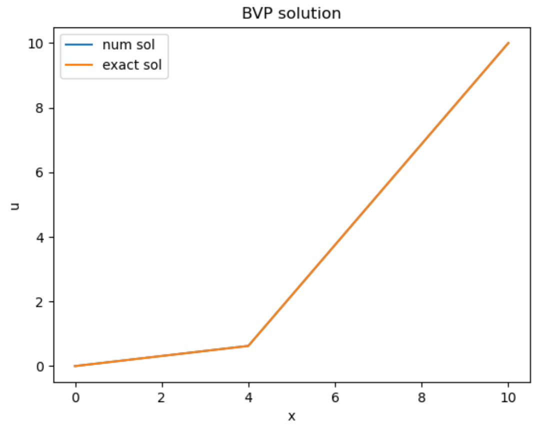
$$L_h u_P = -\frac{1}{h^2}\left[k_w u_W + k_e u_E - (k_e + k_w) u_P\right] = 0 \implies -(k_w + k_e) u_P + k_w u_W + k_e u_E = 0$$

Applying the scheme to each mesh point, we obtain a linear system.

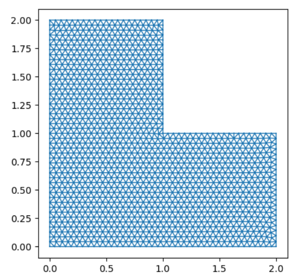
$$\begin{array}{c|ccccccccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\hline
1 & -2k_1 & k_1 & & & & & & & \\
2 & k_1 & -2k_1 & k_1 & & & & & & \\
3 & & k_1 & -2k_1 & k_1 & & & & & \\
4 & & & k_1 & -k_1-k_2 & k_2 & & & & \\
5 & & & & k_2 & -2k_2 & k_2 & & & \\
6 & & & & & k_2 & -2k_2 & k_2 & & \\
7 & & & & & & k_2 & -2k_2 & k_2 & \\
8 & & & & & & & k_2 & -2k_2 & k_2 \\
9 & & & & & & & & k_2 & -2k_2 \\
\hline
\end{array}
\cdot
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{bmatrix}
=
\begin{bmatrix} -k_1 u_a \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -k_2 u_b \end{bmatrix}$$

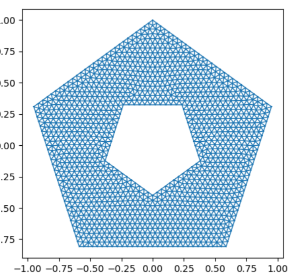
We solve it and plot the numerical solution u along with the exact solution from part (a). In this case the solutions agree exactly.

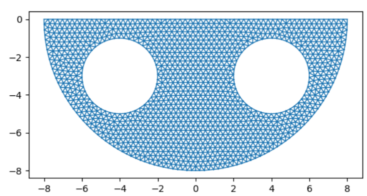
Code: `https://github.com/RokettoJanpu/scientific-computing-2-redux/blob/main/hw7q2.ipynb`



3. Code: `https://github.com/RokettoJanpu/Scientific-Computing-2/blob/main/hw7q2.ipynb`







4. (a) Multiply the BVP by w and integrate over $[0, 1]$.

$$-\int_0^1 w(x)u''(x)dx = \int_0^1 w(x)f(x)dx$$

Integrating by parts and using the fact $w(0) = w(1) = 0$, the LHS is

$$-\int_0^1 w(x)u''(x)dx = -w(x)u(x)\Big|_0^1 + \int_0^1 w'(x)u(x)dx = \int_0^1 w'(x)u(x)dx$$

Thus we obtain an integral equation for u.

$$\int_0^1 w'(x)u(x)dx = \int_0^1 w(x)f(x)dx$$

(b) First define other basis functions

$$\varphi_0(x) := \begin{cases} 0, & x \geq x_1 \\ \frac{x_1 - x}{x_1}, & x < x_1 \end{cases}, \quad \varphi_{N+1}(x) := \begin{cases} 0, & x \leq x_N \\ \frac{x - x_N}{x_{N+1} - x_N}, & x > x_N \end{cases}$$

When computing the stiffness matrix (with rows and columns starting at 0 instead of 1)

$$A_{ij} = \int_0^1 \varphi_i'(x)\varphi_j'(x)dx$$

we will use the fact it is symmetric. First compute

$$\varphi_i'(x) = \begin{cases} 0, & x < x_{i-1} \text{ or } x > x_{i+1} \\ \frac{1}{x_i - x_{i-1}}, & x_{i-1} < x < x_i \\ -\frac{1}{x_{i+1}-x_i}, & x_i < x < x_{i+1} \end{cases}, \quad \varphi_0'(x) = \begin{cases} 0, & x > x_1 \\ -\frac{1}{x_1}, & x < x_1 \end{cases}, \quad \varphi_{N+1}'(x) = \begin{cases} 0, & x < x_N \\ \frac{1}{x_{N+1}-x_N}, & x > x_N \end{cases}$$

Fix $1 \leq i \leq N$ and examine cases of the value of j.

- If $j = i$ then

  $$\varphi_i'(x)\varphi_j'(x) = \begin{cases} 0, & x < x_{i-1} \text{ or } x > x_{i+1} \\ \frac{1}{(x_i - x_{i-1})^2}, & x_{i-1} < x < x_i \\ \frac{1}{(x_{i+1}-x_i)^2}, & x_i < x < x_{i+1} \end{cases}$$

  hence

  $$A_{ij} = \frac{x_i - x_{i-1}}{(x_i - x_{i-1})^2} + \frac{x_{i+1} - x_i}{(x_{i+1} - x_i)^2} = \frac{1}{x_i - x_{i-1}} + \frac{1}{x_{i+1} - x_i}$$

- If $j = i + 1$ then

  $$\varphi_i'(x)\varphi_j'(x) = \begin{cases} 0, & x < x_i \text{ or } x > x_{i+1} \\ -\frac{1}{(x_{i+1}-x_i)^2}, & x_i < x < x_{i+1} \end{cases}$$

  hence

  $$A_{ij} = -\frac{x_{i+1} - x_i}{(x_{i+1} - x_i)^2} = -\frac{1}{x_{i+1} - x_i}$$

  In particular, if $j = i - 1$ then $i = j + 1$, so that by symmetry

  $$A_{ij} = A_{ji} = -\frac{1}{x_{j+1} - x_j} = -\frac{1}{x_i - x_{i-1}}$$

- $j \geq i + 2$ then $\varphi_i'\varphi_j' = 0$ hence $A_{ij} = 0$. In particular, if $j \leq i - 2$ then $i \geq j + 2$, so that by symmetry $A_{ij} = A_{ji} = 0$.

Now examine cases for $i = 0$.

- If $j = 0$,

  $$\varphi_0'(x)^2 = \begin{cases} 0, & x > x_1 \\ \frac{1}{x_1^2}, & x < x_1 \end{cases}$$

  hence $A_{00} = \frac{1}{x_1}$.
- If $j = 1$, by symmetry $A_{01} = A_{10} = -\frac{1}{x_1 - x_0} = -\frac{1}{x_1}$.
- If $2 \leq j \leq N + 1$ then $\varphi_0'\varphi_j' = 0$, hence $A_{0j} = 0$.

Examine cases for $i = N + 1$.

- If $0 \leq j \leq N - 1$ then $\varphi_{N+1}'\varphi_j' = 0$, hence $A_{N+1,j} = 0$.
- If $j = N$, by symmetry $A_{N+1,N} = A_{N,N+1} = -\frac{1}{x_{N+1} - x_N}$.
- If $j = N + 1$,

  $$\varphi_{N+1}'(x)^2 = \begin{cases} 0, & x < x_N \\ \frac{1}{(x_{N+1}-x_N)^2}, & x > x_N \end{cases}$$

  hence $A_{N+1,N+1} = \frac{1}{x_{N+1} - x_N}$.

To summarize, for $1 \le i \le N$,

$$A_{ij} = \begin{cases} \frac{1}{x_i - x_{i-1}} + \frac{1}{x_{i+1} - x_i}, & j = i \\ -\frac{1}{x_{i+1} - x_i}, & j = i+1 \\ -\frac{1}{x_i - x_{i-1}}, & j = i-1 \\ 0, & j \le i-2 \text{ or } j \ge i+2 \end{cases}$$

For $i = 0$,

$$A_{0j} = \begin{cases} \frac{1}{x_1}, & j = 0 \\ -\frac{1}{x_1}, & j = 1 \\ 0, & 2 \le j \le N+1 \end{cases}$$

For $i = N+1$,

$$A_{N+1,j} = \begin{cases} 0, & 0 \le j \le N-1 \\ -\frac{1}{x_{N+1} - x_N}, & j = N \\ \frac{1}{x_{N+1} - x_N}, & j = N+1 \end{cases}$$

We now compute the load vector b. For $1 \le i \le N$, we approximate

$$\int_0^1 \varphi_i(x) f(x) dx = \int_{x_{i-1}}^{x_i} \varphi_i f dx + \int_{x_i}^{x_{i+1}} \varphi_i f dx \approx f\left(\frac{x_{i-1} + x_i}{2}\right) \frac{x_i - x_{i-1}}{2} + f\left(\frac{x_i + x_{i+1}}{2}\right) \frac{x_{i+1} - x_i}{2}$$

From the BCs, set the vector u_D with components

$$(u_D)_i := \begin{cases} 1, & i = 0 \text{ or } i = N+1 \\ 0, & 1 \le i \le N \end{cases}$$

so that

$$Au_D = \begin{bmatrix} \frac{1}{x_1} \\ -\frac{1}{x_1} \\ 0 \\ \vdots \\ 0 \\ -\frac{1}{x_{N+1} - x_N} \\ \frac{1}{x_{N+1} - x_N} \end{bmatrix}$$

Thus for $1 \le i \le N$,

$$b_i = -(Au_D)_i + \int_0^1 \varphi_i f dx \approx \begin{cases} \frac{1}{x_1} + f\left(\frac{x_1}{2}\right) \frac{x_1}{2} + f\left(\frac{x_1 + x_2}{2}\right) \frac{x_2 - x_1}{2}, & i = 1 \\ f\left(\frac{x_{i-1} + x_i}{2}\right) \frac{x_i - x_{i-1}}{2} + f\left(\frac{x_i + x_{i+1}}{2}\right) \frac{x_{i+1} - x_i}{2}, & 2 \le i \le N-1 \\ \frac{1}{x_{N+1} - x_N} + f\left(\frac{x_{N-1} + x_N}{2}\right) \frac{x_N - x_{N-1}}{2} + f\left(\frac{x_N + x_{N+1}}{2}\right) \frac{x_{N+1} - x_N}{2}, & i = N \end{cases}$$

(c) Solutions for FEM and FDM coincide if the mesh stepsize is constant.