

Scientific Computing HW 7

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1. We must compute (we will temporarily drop the dependence of v on k_x, k_y)

$$Av = \frac{1}{h^2}(4U_P v - U_W v - U_E v - U_N v - U_S v)$$

Using the identity $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$,

$$\begin{aligned} U_P v &= \sin k_x h i \sin k_y h j \\ U_W v &= \sin k_x h (i - 1) \sin k_y h j = [\sin k_x h i \cos k_x h - \cos k_x h i \sin k_x h] \sin k_y h j \\ U_E v &= \sin k_x h (i + 1) \sin k_y h j = [\sin k_x h i \cos k_x h + \cos k_x h i \sin k_x h] \sin k_y h j \\ U_N v &= \sin k_x h i \sin k_y h (j + 1) = \sin k_x h i [\sin k_y h j \cos k_y h + \cos k_y h j \sin k_y h] \\ U_S v &= \sin k_x h i \sin k_y h (j - 1) = \sin k_x h i [\sin k_y h j \cos k_y h - \cos k_y h j \sin k_y h] \end{aligned}$$

Thus

$$\begin{aligned} U_W v + U_E v + U_N v + U_S v &= [\sin k_x h i \cos k_x h - \cos k_x h i \sin k_x h] \sin k_y h j \\ &\quad - [\sin k_x h i \cos k_x h + \cos k_x h i \sin k_x h] \sin k_y h j \\ &\quad - \sin k_x h i [\sin k_y h j \cos k_y h + \cos k_y h j \sin k_y h] \\ &\quad - \sin k_x h i [\sin k_y h j \cos k_y h - \cos k_y h j \sin k_y h] \\ &= -2 \sin k_x h i \sin k_y h j \cos k_x h - 2 \sin k_x h i \sin k_y h j \cos k_y h \\ &= -2 \sin k_x h i \sin k_y h j (\cos k_x h + \cos k_y h) \end{aligned}$$

At last we find the eigenvalues $\lambda = \lambda(k_x, k_y)$.

$$Av_{k_x, k_y} = \frac{1}{h^2} \underbrace{[4 - 2(\cos k_x h + \cos k_y h)]}_{\lambda(k_x, k_y)} \sin k_x h i \sin k_y h j = \lambda(k_x, k_y) v_{k_x, k_y}$$

To estimate the smallest and largest eigenvalues of A , first write

$$\lambda = N^2 \left[4 - 2 \left(\cos \frac{n\pi}{N} + \cos \frac{m\pi}{N} \right) \right]$$

We see λ is maximized at $n, m = N - 1$, and $\cos \frac{(N-1)\pi}{N} \sim 1$, hence $\lambda_{max} \sim 8N^2$. We also see λ is minimized at $n, m = 1$, and using the fact $\cos x \sim 1 - \frac{1}{2}x^2$ for $x \sim 0$, we have $\cos \frac{\pi}{N} \sim 1 - \frac{\pi^2}{N^2}$, hence $\lambda_{min} \sim N^2 \left[4 - 4 \left(1 - \frac{\pi^2}{N^2} \right) \right] = 4\pi^2$. Lastly, the condition number is

$$\kappa(A) = \left| \frac{\lambda_{max}}{\lambda_{min}} \right| \sim \frac{4N^2}{\pi^2}$$

2. (a) Multiplying the BVP by -1 and integrating it, we find $k(x)u' = M$ for some constant M , i.e. $u' = \frac{M}{k(x)}$. The solution is then

$$u(x) = u_a + \int_a^x \frac{M}{k(s)} ds$$

If $x \leq c$ then

$$u(x) = u_a + \int_a^x \frac{M}{k_1} ds = u_a + \frac{M}{k_1}(x - a)$$

If $x > c$ then

$$u(x) = u_a + \int_a^c \frac{M}{k(s)} ds + \int_c^x \frac{M}{k(s)} ds = u_a + \int_a^c \frac{M}{k_1} ds + \int_c^x \frac{M}{k_2} ds = u_a + \frac{M}{k_1}(c - a) + \frac{M}{k_2}(x - c)$$

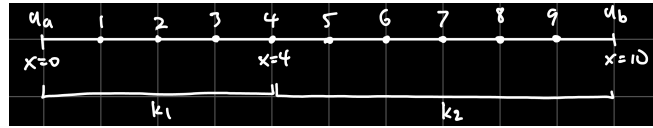
Apply BCs.

$$u_b = u(b) = u_a + \frac{M}{k_1}(c - a) + \frac{M}{k_2}(b - c) = u_a + M \left[\frac{c - a}{k_1} + \frac{b - c}{k_2} \right] \implies M = \frac{u_b - u_a}{\frac{c - a}{k_1} + \frac{b - c}{k_2}}$$

In summary, the solution is

$$u(x) = \begin{cases} u_a + \frac{M}{k_1}(x - a), & x \leq c \\ u_a + \frac{M}{k_1}(c - a) + \frac{M}{k_2}(x - c), & x > c \end{cases} \quad \text{where} \quad M = \frac{u_b - u_a}{\frac{c - a}{k_1} + \frac{b - c}{k_2}}$$

- (b) Given the parameters, we solve for the values of the 9 mesh points shown below.



The finite difference scheme is

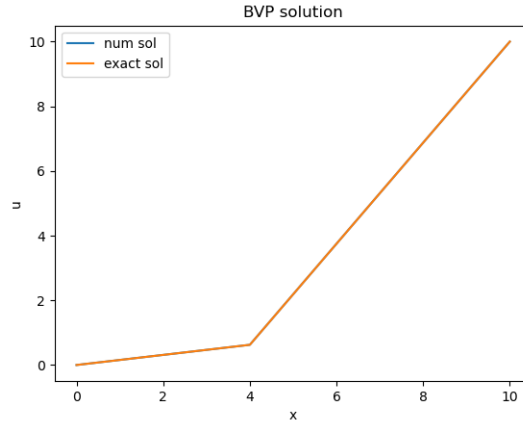
$$L_h u_P = -\frac{1}{h^2} [k_w u_W + k_e u_E - (k_e + k_w) u_P] = 0 \implies -(k_w + k_e) u_P + k_w u_W + k_e u_E = 0$$

Applying the scheme to each mesh point, we obtain a linear system.

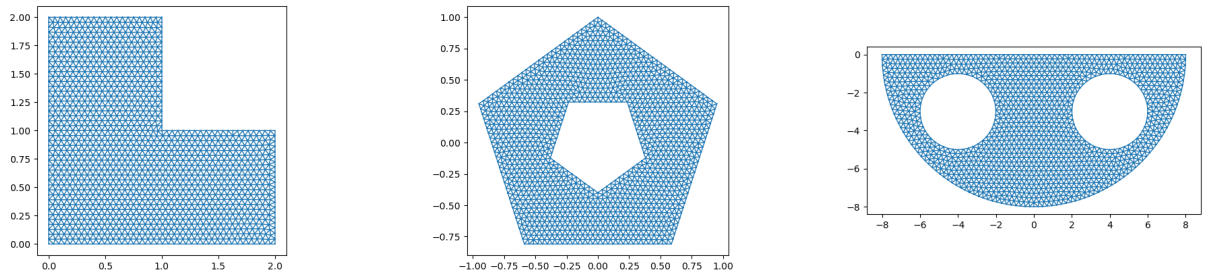
	1	2	3	4	5	6	7	8	9			
1	$-2k_1$	k_1								u_1		$-k_1 u_a$
2	k_1	$-2k_1$	k_1							u_2		0
3		k_1	$-2k_1$	k_1						u_3		0
4			k_1	$-k_1 - k_2$	k_2					u_4		0
5				k_2	$-2k_2$	k_2				u_5	=	0
6					k_2	$-2k_2$	k_2			u_6		0
7						k_2	$-2k_2$	k_2		u_7		0
8							k_2	$-2k_2$	k_2	u_8		0
9								k_2	$-2k_2$	u_9		$-k_2 u_b$

We solve it and plot the numerical solution u along with the exact solution from part (a). In this case the solutions agree exactly.

Code: <https://github.com/RokettoJanpu/scientific-computing-2-redux/blob/main/hw7q2.ipynb>



3. Code: <https://github.com/RokettoJanpu/scientific-computing-2-redux/blob/main/hw7q3.ipynb>



4. (a) Multiply the BVP by w and integrate over $[0, 1]$.

$$-\int_0^1 w(x)u''(x)dx = \int_0^1 w(x)f(x)dx$$

Integrating by parts and using the fact $w(0) = w(1) = 0$, the LHS is

$$-\int_0^1 w(x)u''(x)dx = -w(x)u(x)\Big|_0^1 + \int_0^1 w'(x)u(x)dx = \int_0^1 w'(x)u(x)dx$$

Thus we obtain an integral equation for u .

$$\int_0^1 w'(x)u(x)dx = \int_0^1 w(x)f(x)dx$$

(b) First define other basis functions

$$\varphi_0(x) := \begin{cases} 0, & x \geq x_1 \\ \frac{x_1-x}{x_1}, & x < x_1 \end{cases}, \quad \varphi_{N+1}(x) := \begin{cases} 0, & x \leq x_N \\ \frac{x-x_N}{x_{N+1}-x_N}, & x > x_N \end{cases}$$

When computing the stiffness matrix (with rows and columns starting at 0 instead of 1)

$$A_{ij} = \int_0^1 \varphi'_i(x)\varphi'_j(x)dx$$

we will use the fact it is symmetric. First compute

$$\varphi'_i(x) = \begin{cases} 0, & x < x_{i-1} \text{ or } x > x_{i+1} \\ \frac{1}{x_i - x_{i-1}}, & x_{i-1} < x < x_i \\ -\frac{1}{x_{i+1} - x_i}, & x_i < x < x_{i+1} \end{cases}, \quad \varphi'_0(x) = \begin{cases} 0, & x > x_1 \\ -\frac{1}{x_1}, & x < x_1 \end{cases}, \quad \varphi'_{N+1}(x) = \begin{cases} 0, & x < x_N \\ \frac{1}{x_{N+1} - x_N}, & x > x_N \end{cases}$$

Fix $1 \leq i \leq N$ and examine cases of the value of j .

- If $j = i$ then

$$\varphi'_i(x)\varphi'_j(x) = \begin{cases} 0, & x < x_{i-1} \text{ or } x > x_{i+1} \\ \frac{1}{(x_i - x_{i-1})^2}, & x_{i-1} < x < x_i \\ \frac{1}{(x_{i+1} - x_i)^2}, & x_i < x < x_{i+1} \end{cases}$$

hence

$$A_{ij} = \frac{x_i - x_{i-1}}{(x_i - x_{i-1})^2} + \frac{x_{i+1} - x_i}{(x_{i+1} - x_i)^2} = \frac{1}{x_i - x_{i-1}} + \frac{1}{x_{i+1} - x_i}$$

- If $j = i + 1$ then

$$\varphi'_i(x)\varphi'_j(x) = \begin{cases} 0, & x < x_i \text{ or } x > x_{i+1} \\ -\frac{1}{(x_{i+1} - x_i)^2}, & x_i < x < x_{i+1} \end{cases}$$

hence

$$A_{ij} = -\frac{x_{i+1} - x_i}{(x_{i+1} - x_i)^2} = -\frac{1}{x_{i+1} - x_i}$$

In particular, if $j = i - 1$ then $i = j + 1$, so that by symmetry

$$A_{ij} = A_{ji} = -\frac{1}{x_{j+1} - x_j} = -\frac{1}{x_i - x_{i-1}}$$

- $j \geq i + 2$ then $\varphi'_i\varphi'_j = 0$ hence $A_{ij} = 0$. In particular, if $j \leq i - 2$ then $i \geq j + 2$, so that by symmetry $A_{ij} = A_{ji} = 0$.

Now examine cases for $i = 0$.

- If $j = 0$,

$$\varphi'_0(x)^2 = \begin{cases} 0, & x > x_1 \\ \frac{1}{x_1^2}, & x < x_1 \end{cases}$$

hence $A_{00} = \frac{1}{x_1}$.

- If $j = 1$, by symmetry $A_{01} = A_{10} = -\frac{1}{x_1 - x_0} = -\frac{1}{x_1}$.
- If $2 \leq j \leq N + 1$ then $\varphi'_0\varphi'_j = 0$, hence $A_{0j} = 0$.

Examine cases for $i = N + 1$.

- If $0 \leq j \leq N - 1$ then $\varphi'_{N+1}\varphi'_j = 0$, hence $A_{N+1,j} = 0$.
- If $j = N$, by symmetry $A_{N+1,N} = A_{N,N+1} = -\frac{1}{x_{N+1} - x_N}$.
- If $j = N + 1$,

$$\varphi'_{N+1}(x)^2 = \begin{cases} 0, & x < x_N \\ \frac{1}{(x_{N+1} - x_N)^2}, & x > x_N \end{cases}$$

hence $A_{N+1,N+1} = \frac{1}{x_{N+1} - x_N}$.

To summarize, for $1 \leq i \leq N$,

$$A_{ij} = \begin{cases} \frac{1}{x_i - x_{i-1}} + \frac{1}{x_{i+1} - x_i}, & j = i \\ -\frac{1}{x_{i+1} - x_i}, & j = i + 1 \\ -\frac{1}{x_i - x_{i-1}}, & j = i - 1 \\ 0, & j \leq i - 2 \text{ or } j \geq i + 2 \end{cases}$$

For $i = 0$,

$$A_{0j} = \begin{cases} \frac{1}{x_1}, & j = 0 \\ -\frac{1}{x_1}, & j = 1 \\ 0, & 2 \leq j \leq N + 1 \end{cases}$$

For $i = N + 1$,

$$A_{N+1,j} = \begin{cases} 0, & 0 \leq j \leq N - 1 \\ -\frac{1}{x_{N+1} - x_N}, & j = N \\ \frac{1}{x_{N+1} - x_N}, & j = N + 1 \end{cases}$$

We now compute the load vector b . For $1 \leq i \leq N$, we approximate

$$\int_0^1 \varphi_i(x) f(x) dx = \int_{x_{i-1}}^{x_i} \varphi_i f dx + \int_{x_i}^{x_{i+1}} \varphi_i f dx \approx f\left(\frac{x_{i-1} + x_i}{2}\right) \frac{x_i - x_{i-1}}{2} + f\left(\frac{x_i + x_{i+1}}{2}\right) \frac{x_{i+1} - x_i}{2}$$

From the BCs, set the vector u_D with components

$$(u_D)_i := \begin{cases} 1, & i = 0 \text{ or } i = N + 1 \\ 0, & 1 \leq i \leq N \end{cases}$$

so that

$$Au_D = \begin{bmatrix} \frac{1}{x_1} \\ -\frac{1}{x_1} \\ 0 \\ \vdots \\ 0 \\ -\frac{1}{x_{N+1} - x_N} \\ \frac{1}{x_{N+1} - x_N} \end{bmatrix}$$

Thus for $1 \leq i \leq N$,

$$b_i = -(Au_D)_i + \int_0^1 \varphi_i f dx \approx \begin{cases} \frac{1}{x_1} + f\left(\frac{x_1}{2}\right) \frac{x_1}{2} + f\left(\frac{x_1 + x_2}{2}\right) \frac{x_2 - x_1}{2}, & i = 1 \\ f\left(\frac{x_{i-1} + x_i}{2}\right) \frac{x_i - x_{i-1}}{2} + f\left(\frac{x_i + x_{i+1}}{2}\right) \frac{x_{i+1} - x_i}{2}, & 2 \leq i \leq N - 1 \\ \frac{1}{x_{N+1} - x_N} + f\left(\frac{x_{N-1} + x_N}{2}\right) \frac{x_N - x_{N-1}}{2} + f\left(\frac{x_N + x_{N+1}}{2}\right) \frac{x_{N+1} - x_N}{2}, & i = N \end{cases}$$

(c) Solutions for FEM and FDM coincide if the mesh stepsize is constant.