Scientific Computing HW 8

Ryan Chen

April 1, 2024

1. (a) The matrix G is given by

$$G = A^{-1} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A := \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

We find A^{-1} by its adjugate. By cofactor expansion over the first row,

$$D := \det(A) = \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} - \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = x_2y_3 - x_3y_2 - x_1y_3 + x_3y_1 + x_1y_2 - x_2y_1$$

The matrix of cofactors is

$$\operatorname{cof}(A) = \begin{bmatrix} x_2y_3 - x_3y_2 & -x_1y_3 + x_3y_1 & x_1y_2 - x_2y_1 \\ -y_3 + y_2 & y_3 - y_1 & -y_2 + y_1 \\ x_3 - x_2 & -x_3 + x_1 & x_2 - x_1 \end{bmatrix}$$

The adjugate of A is

$$\operatorname{adj}(A) = \operatorname{cof}(A)^T = \begin{bmatrix} x_2y_3 - x_3y_2 & -y_3 + y_2 & x_3 - x_2 \\ -x_1y_3 + x_3y_1 & y_3 - y_1 & -x_3 + x_1 \\ x_1y_2 - x_2y_1 & -y_2 + y_1 & x_2 - x_1 \end{bmatrix}$$

Thus

$$G = \frac{1}{D}\operatorname{adj}(A) \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{D} \begin{bmatrix} y_2 - y_3 & x_3 - x_2 \\ y_3 - y_1 & x_1 - x_3 \\ y_1 - y_2 & x_2 - x_1 \end{bmatrix}$$

Fix an even permutation (i, j, k) of $1, 2, 3$ (i.e. one of $(1, 2, 3)$, $(2, 3, 1)$, $(3, 1, 2)$).

$$\eta_i(x, y) = \frac{\begin{vmatrix} 1 & x & y \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix}}{\begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix}}$$

Since the permutation (i, j, k) is even, the denominator is $\det\left(A^T\right) = \det(A) = D$. By cofactor expansion over the first row, the numerator is $x_jy_k - x_ky_j - (y_k - y_j)x + (x_k - x_j)y$, so that

$$\partial_x\eta_i(x, y) = \frac{1}{D}(y_j - y_k), \quad \partial_y\eta_i(x, y) = \frac{1}{D}(x_k - x_j)$$

Thus

$$\begin{bmatrix} \partial_x\eta_1(x, y) & \partial_y\eta_1(x, y) \\ \partial_x\eta_2(x, y) & \partial_y\eta_2(x, y) \\ \partial_x\eta_3(x, y) & \partial_y\eta_3(x, y) \end{bmatrix} = \frac{1}{D} \begin{bmatrix} y_2 - y_3 & x_3 - x_2 \\ y_3 - y_1 & x_1 - x_3 \\ y_1 - y_2 & x_2 - x_1 \end{bmatrix} = G$$

(b) Using the functions η_j from (a) and the fact $\boldsymbol{\nabla}\eta_j$ is the jth row of G,

$$u(x,y) = \sum_{j=1}^{3} u_j \eta_j(x,y) \implies \boldsymbol{\nabla} u(x,y) = \sum_{j=1}^{3} u_j \boldsymbol{\nabla}\eta_j(x,y) = \frac{1}{D}\left(u_1 \begin{bmatrix} y_2 - y_3 \\ x_3 - x_2 \end{bmatrix} + u_2 \begin{bmatrix} y_3 - y_1 \\ x_1 - x_3 \end{bmatrix} + u_3 \begin{bmatrix} y_1 - y_2 \\ x_2 - x_1 \end{bmatrix} \right)$$

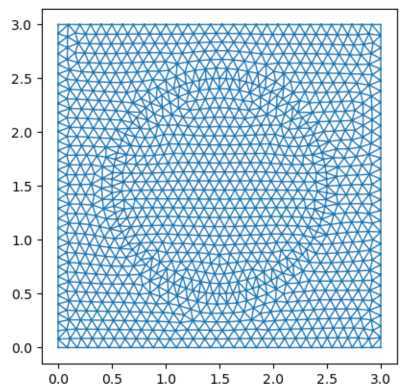
$$\implies \boldsymbol{\nabla} u(x,y) = (x_2y_3 - x_3y_2 - x_1y_3 + x_3y_1 + x_1y_2 - x_2y_1)^{-1} \begin{bmatrix} u_1(y_2 - y_3) + u_2(y_3 - y_1) + u_3(y_1 - y_2) \\ u_1(x_3 - x_2) + u_2(x_1 - x_3) + u_3(x_2 - x_1) \end{bmatrix}$$

2. Code: `https://github.com/RokettoJanpu/scientific-computing-2-redux/blob/main/hw8.ipynb`

These functions were written:

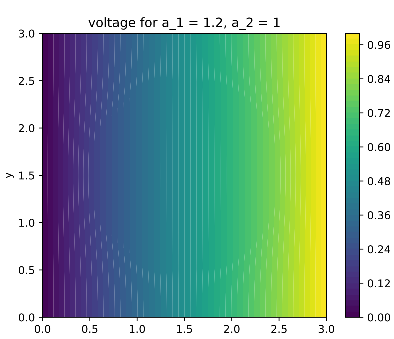
- a_func: Determine the conductivity at a point.
- find_vert_line_pts: Find the mesh points along a given vertical line
- FEM_hw_solver: FEM solver for the BVP in the HW.

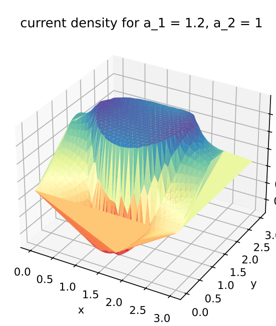
We use the function distmesh2D to generate the mesh. For the fixed points, we pick the corners of the square and 100 evenly spaced points on the circle with center $(1.5, 1.5)$ and radius 1.



We solve for the voltage with the function FEM_hw_solver. We then plot the voltage and the current density.

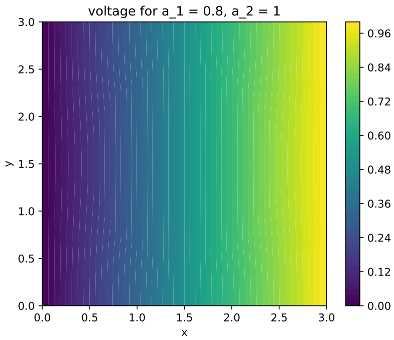
- For $a_1 = 1.2$ and $a_2 = 1$:

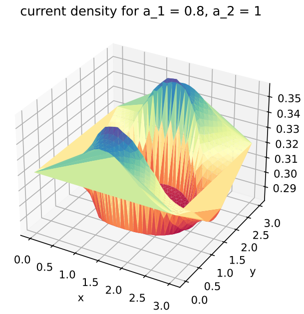




Note that in this case $a_1 > a_2$, which visually corresponds to the fact that the current density is greater on the inside of the circle than the outside.

- For $a_1 = 0.8$ and $a_2 = 1$:





Note that in this case $a_1 < a_2$, which visually corresponds to the fact that the current density is greater on the outside of the circle than the inside.

These cases illustrate the principle of "the path of least resistance", in the sense that current flow is higher in regions of greater electrical conductivity.

3. (a) Write

$$u(x) = \int_0^1 G(x,y)f(y)dy = \int_0^x G(x,y)f(y)dy + \int_x^1 G(x,y)f(y)dy$$

$$= (1-x)\int_0^x yf(y)dy + x\int_x^1 (1-y)f(y)dy = (1-x)\int_0^x yf(y)dy - x\int_1^x (1-y)f(y)dy$$

Then compute

$$u'(x) = -\int_0^x yf(y)dy + (1-x)xf(x) - \int_1^x (1-y)f(y)dy - x(1-x)f(x)$$

$$= -\int_0^x yf(y)dy - \int_1^x f(y)dy + \int_1^x yf(y)dy = -\int_1^x f(y)dy$$

$$\implies u''(x) = -f(x)$$

(b) Compute

$$\partial_y G(x,y) = \begin{cases} 1-x, & y < x \\ -x, & y > x \end{cases}$$

Then

$$\int_0^1 v'(y)\partial_y G(x,y)dy = \int_0^x v'(y)\partial_y G(x,y)dy + \int_x^1 v'(y)\partial_y G(x,y)dy$$

$$= (1-x)\int_0^x v'(y)dy - x\int_x^1 v'(y)dy = \int_0^x v'(y)dy - x\int_0^x v'(y)dy - x\int_x^1 v'(y)dy = \int_0^x v'(y)dy - x\int_0^1 v'(y)dy$$

$$= v(x) - v(0) - x[v(1) - v(0)] = v(x) - 0 - x[0-0] = v(x)$$

Since $u \in H_0^1([0,1])$, the identity in particular holds for u.

(c) The linear interpolant $I_h u$, as a linear combination of the basis functions ϕ_j, is

$$I_h u = \sum_{j=1}^n u(x_j)\phi_j$$

(d) The function

$$\psi_j(x) := G(x_j, y) = \begin{cases} (1 - x_j)y, & y \le x_j \\ x_j(1 - y), & y > x_j \end{cases}$$

satisfies $\psi_j(0) = \psi_j(1) = 0$ and is linear on each interval $[x_i, x_{i+1}]$ for $0 \le i \le n$, so $\psi_j \in S_h$. Also note that ψ_j is not differentiable at x_j, but is differentiable at all other points.

Suppose the functions ψ_j are linearly dependent, i.e. there exist scalars c_j, not all 0, so that

$$\sum_{j=1}^{n} c_j \psi_j = 0$$

In particular the linear combination

$$\sum_{j=1}^{n} c_j \psi_j$$

is differentiable. From the assumption we can pick i so that $c_i \ne 0$. Since the functions ψ_j, for $j \ne i$, are differentiable at x_i, the linear combination

$$\sum_{\substack{j=1 \\ j \ne i}}^{n} c_j \psi_j$$

is differentiable at x_i. Also, since ψ_i is not differentiable at x_i and $c_i \ne 0$, the function $c_i \psi_i$ is not differentiable at x_i. This means the linear combination $\sum_{j=1}^{n} c_j \psi_j$ is not differentiable at x_i, a contradiction. Thus the functions ψ_j are linearly independent.

Lastly, since $\dim(S_h) = n$, the n linearly independent functions ψ_j form a basis of S_h.

(e) Recall that

$$\partial_y G(x, y) = \begin{cases} 1 - x, & y < x \\ -x, & y > x \end{cases}, \quad \phi_i'(x) = \begin{cases} 0, & x < x_{i-1} \text{ or } x > x_{i+1} \\ \frac{1}{x_i - x_{i-1}}, & x_{i-1} < x < x_i \\ -\frac{1}{x_{i+1} - x_i}, & x_i < x < x_{i+1} \end{cases}$$

First write

$$\int_0^1 [I_h u]'(y) \partial_y G(x_j, y) dy = \int_0^1 \left[\sum_{i=1}^{n} u(x_i) \phi_i'(y) \right] \partial_y G(x_j, y) dy$$

$$= \sum_{i=1}^{n} u(x_i) \int_0^1 \phi_i'(y) \partial_y G(x_j, y) dy = \sum_{i=1}^{n} u(x_i) \underbrace{\int_{x_{i-1}}^{x_{i+1}} \phi_i'(y) \partial_y G(x_j, y) dy}_{=: a_{ij}}$$

Now examine cases.

- If $i = j$,

$$a_{ij} = \int_{x_{j-1}}^{x_j} \frac{1 - x_j}{x_j - x_{j-1}} dy + \int_{x_j}^{x_{j+1}} \frac{-x_j}{-(x_{j+1} - x_j)} dy = 1 - x_j + x_j = 1$$

- If $i \ge j + 1$,

$$a_{ij} = \int_{x_{i-1}}^{x_i} \frac{-x_j}{x_i - x_{i-1}} dy + \int_{x_i}^{x_{i+1}} \frac{-x_j}{-(x_{i+1} - x_i)} dy = -x_j + x_j = 0$$

- If $i \leq j-1$,

$$a_{ij} = \int_{x_{i-1}}^{x_i} \frac{1-x_j}{x_i - x_{i-1}} dy + \int_{x_i}^{x_{i+1}} \frac{1-x_j}{-(x_{i+1}-x_i)} dy = 1 - x_j - (1-x_j) = 0$$

This means $a_{ij} = \delta_{ij}$, thus

$$\int_0^1 [I_h u]'(y) \partial_y G(x_j, y) dy = \sum_{i=1}^n u(x_i)\delta_{ij} = u(x_j)$$

4. (a) Using the identity $\boldsymbol{\nabla} \cdot (fv) = \boldsymbol{\nabla} f \cdot v + f \boldsymbol{\nabla} \cdot v$ for any scalar function f and vector field v,

$$\boldsymbol{\nabla} \cdot \left(e^{-\beta V(x)} \boldsymbol{\nabla} u\right) + \beta e^{-\beta V(x)} f(x) \cdot \boldsymbol{\nabla} u = e^{-\beta V(x)} \Delta u - \beta e^{-\beta V(x)} \boldsymbol{\nabla} V(x) \cdot \boldsymbol{\nabla} u + \beta e^{-\beta V(x)} f(x) \cdot \boldsymbol{\nabla} u$$

$$= \beta e^{-\beta V(x)} \left[\beta^{-1} \Delta u - \boldsymbol{\nabla} V(x) \cdot \boldsymbol{\nabla} u + f(x) \cdot \boldsymbol{\nabla} u\right] = \beta e^{-\beta V(x)} \left[\beta^{-1} \Delta u + b(x) \cdot \boldsymbol{\nabla} u\right]$$

and since $\beta e^{-\beta V(x)} \neq 0$ for all x,

$$\beta^{-1} \Delta u + b(x) \cdot \boldsymbol{\nabla} u = 0 \iff \boldsymbol{\nabla} \cdot \left(e^{-\beta V(x)} \boldsymbol{\nabla} u\right) + \beta e^{-\beta V(x)} f(x) \cdot \boldsymbol{\nabla} u = 0$$

hence the two BVPs are equivalent.

(b) Decompose

$$b(x,y) = \begin{bmatrix} x - x^3 - 10xy^2 \\ -y - x^2 y \end{bmatrix} = \underbrace{\begin{bmatrix} x - x^3 - xy^2 \\ -y - 3y^3 - x^2 y \end{bmatrix}}_{=:F(x,y)} + \underbrace{\begin{bmatrix} -9xy^2 \\ 3y^3 \end{bmatrix}}_{=:f(x,y)}$$

We check

$$\boldsymbol{\nabla} \times F = \partial_x(-y - 3y^3 - x^2 y) + \partial_y(x - x^3 - xy^2) = -2xy + 2xy = 0$$
$$\boldsymbol{\nabla} \cdot f = \partial_x(-9xy^2) + \partial_y(3y^3) = -9y^2 + 9y^2 = 0$$

so the decomposition is as desired. For the vector field

$$-F(x,y) = \begin{bmatrix} -x + x^3 + xy^2 \\ y + 3y^3 + x^2 y \end{bmatrix}$$

a potential V is found by

$$V(x,y) = \int (-x + x^3 + xy^2) dx = -\frac{1}{2}x^2 + \frac{1}{4}x^4 + \frac{1}{2}x^2 y^2 + g(y)$$

for some function g, and

$$V(x,y) = \int (y + 3y^3 + x^2 y) dy = \frac{1}{2}y^2 + \frac{3}{4}y^4 + \frac{1}{2}x^2 y^2 + h(x)$$

for some function h. Putting the calculations together,

$$V(x,y) = -\frac{1}{2}x^2 + \frac{1}{4}x^4 + \frac{1}{2}x^2 y^2 + \frac{1}{2}y^2 + \frac{3}{4}y^4$$