UNIVERSITY OF TEXAS AT AUSTIN

Methods of Applied Mathematics Problem Set 1

Qi Lei

November 1, 2018

1 Exercise 6.1

Compute he Fourier transform of $e^{-|x|}$ for $x \in \mathbb{R}$.
Solution:

$$\begin{aligned}
\hat{f}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|} e^{-ix\xi} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-x-ix\xi} dx + \int_{-\infty}^{0} e^{x-ix\xi} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} (e^{-x-ix\xi} - e^{-x+ix\xi}) dx \\
&= \frac{1}{\sqrt{2\pi}} [\frac{1}{-(1+i\xi)}(-1) - \frac{1}{-1+i\xi}(-1)] \\
&= \frac{1}{\sqrt{2\pi}} [\frac{1-i\xi}{1+\xi^2} + \frac{-(1+i\xi)}{1+\xi^2}] \\
&= \frac{1}{\sqrt{2\pi}} \frac{-2i\xi}{1+\xi^2} \\
&= -\sqrt{\frac{2}{\pi}} \frac{i\xi}{1+\xi^2}
\end{aligned}$$

2 Exercise 6.2

Compute the Fourier transform of $e^{-a|x|^2},\ a>0$, directly, where $x\in\mathbb{R}$.
Solution:

$$
\begin{aligned}
\hat{f}(\xi) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|^2}e^{-ix\xi}dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a(x+\frac{i\xi}{2a})^2+\frac{-\xi^2}{4a}}dx \qquad x' \doteq x+\frac{i\xi}{2a} \\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{\xi^2}{4a}}\int_{-\infty}^{\infty} e^{-ax^2}dx \\
&= \frac{e^{-\frac{\xi^2}{4a}}}{2a}
\end{aligned}
$$

3 Exercise 6.4

$f\in L_1(\mathbb{R}^d)$, and $f(x)=g(|x|)$ for some g, show that $\tilde{f}(\xi)=h(|\xi|)$ for some h. Solution:

$$
\begin{aligned}
\hat{f}(\xi) &= \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} f(x)e^{-ix\xi}dx \qquad \text{(polar coordinate transformation)} \\
&= \frac{1}{(2\pi)^{d/2}}\int_0^{\infty}\int_{\partial S^d} f(r\alpha)e^{-ir\alpha\cdot\xi}r^{d-1}d\alpha dr \\
&= \frac{1}{(2\pi)^{d/2}}\int_0^{\infty} g(r)r^{d-1}\int_{\partial S^d} e^{-ir\alpha\cdot\xi}d\alpha dr
\end{aligned} \tag{3.1}
$$

For any ξ, $\exists$ rotation Q, such that $\xi=|\xi|Qe_1$, so that $\alpha\cdot\xi=\alpha^T\xi=\alpha^T(Qe_1)|\xi|=|\xi|(Q^T\alpha)^Te_1$

$$
\begin{aligned}
\therefore\ (3.1) &= \frac{1}{(2\pi)^{d/2}}\int_0^{\infty} g(r)r^{d-1}\int_{\partial S^d} e^{-ir(Q^T\alpha)\cdot e_1|\xi|}d\alpha dr \\
&= \frac{1}{(2\pi)^{d/2}}\int_0^{\infty} g(r)r^{d-1}\int_{\partial S^d} e^{-ir(Q^T\alpha)\cdot e_1|\xi|}d(Q^T\alpha)dr \quad \beta\doteq Q^T\alpha \\
&= \frac{1}{(2\pi)^{d/2}}\int_0^{\infty} g(r)r^{d-1}\int_{\partial S^d} e^{-ir\beta_1|\xi|}d\beta dr \\
&= h(|\xi|)
\end{aligned}
$$

only depends on $|\xi|$

4 Exercise 6.10

Let the field be complex and define $T: L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)$ by

$$Tf(x)=\int e^{-|x-y|^2/2}f(y)dy$$

Show that T is positive, injective operator, but not surjective.
Solution:

$$\begin{aligned} \text{Positive} &\iff \langle Tf(x), f(x)\rangle \geq 0, \forall f \in L^2(\mathbb{R}^d) \\ &\iff \int\int e^{-|x-y|^2/2} f(y)\,dy\, f(x)\,dx \geq 0, \forall f \in L^2(\mathbb{R}^d) \end{aligned}$$