Numerical Solution of Partial Differential Equations

Classification of second order PDEs:

The general second order linear PDE in 2 independent variables x, y is of the form

$$A u_{xx} + B u_{xy} + C u_{yy} + D u_x + E u_y + F u = 0 \quad \longrightarrow (1)$$

where A, B, C, D, E, F are functions of x, y.

Eqⁿ (1) is parabolic if $B^2 - 4AC = 0$

Elliptic if $B^2 - 4AC < 0$

Hyperbolic if $B^2 - 4AC > 0$.

We will study the numerical solⁿ of the following PDE:

(1) one dimensional Wave Equation:

Eqⁿ: $c^2 u_{xx} - u_{tt} = 0$ Nature: Hyperbolic

(2) one dimensional Heat Equation:

Eqⁿ: $c^2 u_{xx} - u_t = 0$ Nature: Parabolic

(3) Two dimensional Laplace Equation:

Eqⁿ $u_{xx} + u_{yy} = 0$ Nature: Elliptic

Numerical Solution of one dimensional Wave Equation :-

Working rule

1) Consider $c^2 u_{xx} = u_{tt}$ subject to the boundary conditions $u(0,t) = 0$, $u(l,t) = 0$ and the initial conditions $u(x,0) = f(x)$, $u_t(x,0) = 0$.

2) Given the value of h, find $\boxed{k = h/c}$, where h, k are step sizes.

3) The points of division for x are: $x_0, x_1 = x_0+h, x_2 = x_0+2h, \ldots\ldots x_n = x_0+nh$ and that of t are: $t_0, t_1 = t_0+k, t_2 = t_0+2k, \ldots\ldots t_n = t_0+nk$.

4) form the following basic informative table :

$t \backslash x$		x_0	x_1	x_2		x_n
		0	1	2	------	n
t_0	0	$u_{0,0} = 0$	$u_{1,0}$	$u_{2,0}$	---	$u_{n,0} = 0$
t_1	1	$u_{0,1} = 0$	$u_{1,1}$	$u_{2,1}$	---	$u_{n,1} = 0$
t_2	2	$u_{0,2} = 0$	$u_{1,2}$	$u_{2,2}$	---	$u_{n,2} = 0$
$\vdots$		-	- -	- -	- -	-
t_n	n	$u_{0,n} = 0$	$u_{1,n}$	$u_{2,n}$	----	$u_{n,n} = 0$

5) The values along the first row are calculated using the condition $u(x,0) = f(x)$.

6) The values along the second row are calculated using the relation $u_{i,1} = \frac{1}{2}\left[u_{i-1,0} + u_{i+1,0}\right]$

7) Remaining rows are found using the relation :

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1} \quad \text{(Explicit formula)}.$$

Numerical Solution of one dimensional Heat Equation :-

Solution of Heat Eqn by Schmidt Explicit formula :

Working rule for Bendre - Schmidt formula :-

1) Consider the heat equation $u_t = c^2 u_{xx}$ subject to the boundary conditions $u(0,t) = 0$, $u(l,t) = 0$ and the initial condition $u(x,0) = f(x)$.

2) Given the value of h, find $\boxed{k = \frac{h^2}{2c^2}}$, h, k are step sizes.

3) The points of division for x are :

$x_0, x_1 = x_0 + h, x_2 = x_0 + 2h, \ldots x_n = x_0 + nh$, and that of t are : $t_0, t_1 = t_0 + k, t_2 = t_0 + 2k, \ldots t_n = t_0 + nk$.

4) Form the following basic informative table:

t \ x		x_0	x_1	x_2	---	x_n
		0	1	2	----	n
t_0	0	$u_{0,0} = 0$	$u_{1,0}$	$u_{2,0}$	---	$u_{n,0} = 0$
t_1	1	$u_{0,1} = 0$	$u_{1,1}$	$u_{2,1}$	---	$u_{n,1} = 0$
t_2	2	$u_{0,2} = 0$	$u_{1,2}$	$u_{2,2}$	---	$u_{n,2} = 0$
--	--	--	--	--	---	---
t_n	n	$u_{0,n} = 0$	$u_{1,n}$	$u_{2,n}$	----	$u_{n,n} = 0$

5) The values along the first row are calculated using the condition $u(x,0) = f(x)$.

6) The values along the second row and the remaining rows are calculated using the relation

$$u_{i,j+1} = \frac{1}{2}\left[u_{i-1,j} + u_{i+1,j}\right]$$

Known as Bendre - Schmidt formula (A deduction from Schmidt Explicit formula).

Note:- Schmidt Explicit formula is given by

$$u_{i,j+1} = \alpha u_{i-1,j} + (1-2\alpha) u_{i,j} + \alpha u_{i+1,j} \quad \text{where } \alpha = \frac{kc^2}{h^2}$$

This formula will be used when both h and k are specified in the given data.

Solution of Heat Equation by Crank-Nicholson Method:

Crank-Nicholson Explicit formula is given by

$$-\lambda u_{i-1,j+1} + (2+2\lambda) u_{i,j+1} - \lambda u_{i+1,j+1} =$$
$$\lambda u_{i-1,j} + (2-2\lambda) u_{i,j} + \lambda u_{i+1,j}$$

$$\text{where } \lambda = \frac{k}{ch^2}$$

Solution of Laplace's Equation in two dimensions

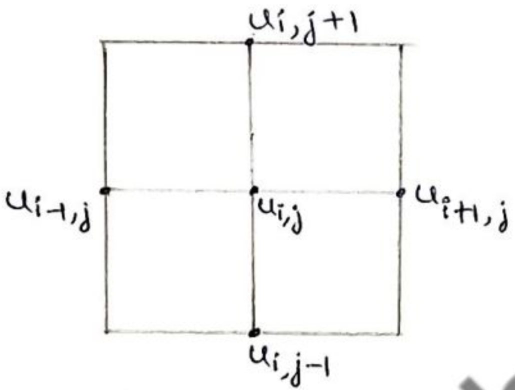
Laplace's equation in two dimensions is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

* Standard five point formula is given by

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1}\right]$$

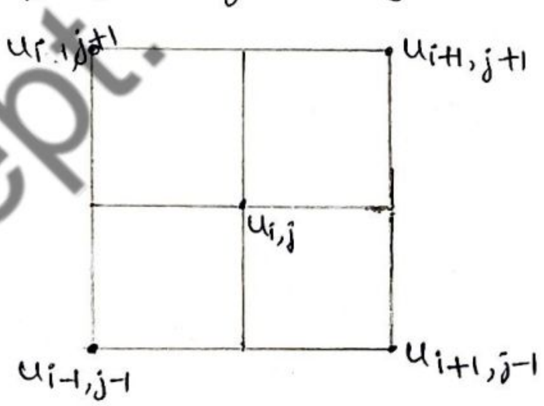
Observe that value of $u_{i,j}$ at any interior mesh point is the average of its values at four neighbouring points as exhibited below:



* Diagonal five point formula is given by

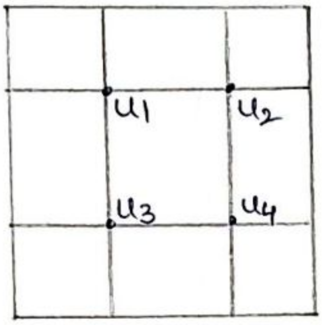
$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j+1} + u_{i-1,j-1} + u_{i+1,j+1} + u_{i+1,j-1}\right]$$

Observe that the value of $u_{i,j}$ is the average of its values at four neighbouring diagonal mesh points as exhibited below:



Working rule :

Type 1 : odd number of squares are formed (Generally 9 squares)



To find u_1, u_2, u_3, u_4 – Standard five point formula is used, Since the boundaries will be known.

Type 2 : Even number of squares are formed (Generally 16 squares)

* Given the boundary values, first find u_5 using standard five point formula.
* Compute u_1, u_3, u_7, u_9 using Diagonal five point formula.
* Lastly compute u_2, u_4, u_6, u_8 using standard five point formula.

Note :- Abbreviations used : i) SF for Standard five point formula
ii) DF for diagonal five point formula.

Problems on classification of II order PDE:

1) Classify the following partial differential equations:

(i) $u_{xx} + 4u_{xy} + 4u_{yy} - u_x + 2u_y = 0$.

(ii) $x^2 u_{xx} + (1-y^2)u_{yy} = 0, -1 < y < 1$.

(iii) $(1+x^2)u_{xx} + (5+2x^2)u_{xt} + (4+x^2)u_{tt} = 0$

(iv) $y^2 u_{xx} - 2y u_{xy} + u_{yy} - u_y = 8y$

Sol^n:-

(i) Given $u_{xx} + 4u_{xy} + 4u_{yy} - u_x + 2u_y = 0$

Comparing with $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0$

we get $A = 1, B = 4, C = 4$

$\therefore B^2 - 4AC = 4^2 - 4\times1\times4 = 16 - 16 = 0$

Since $B^2 - 4AC = 0$, given PDE is parabolic.

(ii) Given $x^2 u_{xx} + (1-y^2)u_{yy} = 0, \quad -1 < y < 1$

Comparing, $A = x^2, B = 0, C = 1-y^2$

$$\therefore B^2 - 4AC = 0 - 4x^2(1-y^2)$$
$$= -4x^2(1-y^2) < 0$$

Since $B^2 - 4AC < 0$, given PDE is elliptic.

(iii) Given $(1+x^2)u_{xx} + (5+2x^2)u_{xt} + (4+x^2)u_{tt} = 0$

Comparing, $A = 1+x^2, B = 5+2x^2, C = 4+x^2$

$$\therefore B^2 - 4AC = (5+2x^2)^2 - 4(1+x^2)(4+x^2)$$
$$= 25 + 4x^4 + 20x^2 - 4(4+5x^2+x^4)$$
$$= 25 + 4x^4 + 20x^2 - 16 - 20x^2 - 4x^4$$
$$= 9 > 0$$

Since $B^2 - 4AC > 0$, given PDE is hyperbolic.

(iv) Given $y^2 u_{xx} - 2y u_{xy} + u_{yy} - u_y - 8y = 0$.

comparing $A = y^2$, $B = -2y$, $C = 1$.

$\therefore B^2 - 4AC = (-2y)^2 - 4y^2 \times 1 = 4y^2 - 4y^2 = 0$.

Since $B^2 - 4AC = 0$, given PDE is parabolic.

Problems on 1-D Wave Equation :-

1) Solve the wave equation $\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}$ subject to

$u(0,t)=0$, $u(4,t)=0$, $u_t(x,0)=0$, $u(x,0)=x(4-x)$

by taking $h=1$, $k=0.5$ upto 4 steps.

Sol^n:- Since $0 \le x \le 4$, the points of division for x are $0, 1, 2, 3, 4$ (x_0 to x_4).

Since $k=0.5$, the values corresponding to t upto 4 steps are $0, 0.5, 1, 1.5, 2$ (t_0 to t_4).

Consider the following table (sol^n) :-

t \ x		x_0 = 0	x_1 = 1	x_2 = 2	x_3 = 3	x_4 = 4
t_0	0	$u_{0,0}=0$	$u_{1,0}=3$	$u_{2,0}=4$	$u_{3,0}=3$	$u_{4,0}=0$
t_1	0.5	$u_{0,1}=0$	$u_{1,1}=2$	$u_{2,1}=3$	$u_{3,1}=2$	$u_{4,1}=0$
t_2	1	$u_{0,2}=0$	$u_{1,2}=0$	$u_{2,2}=0$	$u_{3,2}=0$	$u_{4,2}=0$
t_3	1.5	$u_{0,3}=0$	$u_{1,3}=-2$	$u_{2,3}=-3$	$u_{3,3}=-2$	$u_{4,3}=0$
t_4	2	$u_{0,4}=0$	$u_{1,4}=-3$	$u_{2,4}=-4$	$u_{3,4}=-3$	$u_{4,4}=0$

The values of 1st and last column are zero since $u(0,t)=0$, $u(4,t)=0$.

I row :- we have $u(x,0)=x(4-x)$

$\therefore u_{1,0} = 1\times 3 = 3$; $u_{2,0} = 2\times 2 = 4$, $u_{3,0} = 3\times 1 = 3$

II row :- $u_{i,1} = \frac{1}{2}\left[u_{i-1,0} + u_{i+1,0}\right]$

$\therefore u_{1,1} = \frac{1}{2}\left[u_{0,0} + u_{2,0}\right] = \frac{0+4}{2} = 2$

$$u_{2,1} = \frac{1}{2}\left[u_{1,0} + u_{3,0}\right] = \frac{3+3}{2} = 3$$ ②

$$u_{3,1} = \frac{1}{2}\left[u_{2,0} + u_{4,0}\right] = \frac{4+0}{2} = 2.$$

III row to V row: $u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$

III row: $u_{1,2} = u_{0,1} + u_{2,1} - u_{1,0} = 0+3-3 = 0.$

III^{ly} $u_{2,2} = 2+2-4 = 0$

$u_{3,2} = 3+0-3 = 0.$

IV row: $u_{1,3} = 0+0-2 = -2$

$u_{2,3} = 0+0-3 = -3$

$u_{3,3} = 0+0-2 = -2$

V row: $u_{1,4} = 0-3-0 = -3$

$u_{2,4} = -2-2-0 = -4$

$u_{3,4} = -3-0-0 = -3$

∴ Required solⁿ is the table itself.

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2) Evaluate the pivotal values of the eqⁿ $u_{tt} = 16u_{xx}$ taking $h=1$ upto $t=1.25$. The boundary conditions are $u(0,t) = u(5,t) = 0$, $u_t(x,0) = 0$ and $u(x,0) = x^2(5-x)$.

Solⁿ:- Given $u_{tt} = 16u_{xx}$

$\Rightarrow c^2 = 16$ (or) $\boxed{c=4}$

Since $\boxed{h=1}$, we have $k = h/c = 1/4 \Rightarrow \boxed{k = 0.25}$

Step size of x : $h = 1$ where $0 \le x \le 5$

Step size of t ; $k = 0.25$ where $0 \le t \le 1.25$

Consider the following table (sol^n) :

$t \backslash x$		x_0	x_1	x_2	x_3	x_4	x_5
		0	1	2	3	4	5
t_0	0	$u_{0,0}=0$	$u_{1,0}=4$	$u_{2,0}=12$	$u_{3,0}=18$	$u_{4,0}=16$	$u_{5,0}=0$
t_1	0.25	$u_{0,1}=0$	$u_{1,1}=6$	$u_{2,1}=11$	$u_{3,1}=14$	$u_{4,1}=9$	$u_{5,1}=0$
t_2	0.5	$u_{0,2}=0$	$u_{1,2}=7$	$u_{2,2}=8$	$u_{3,2}=2$	$u_{4,2}=-2$	$u_{5,2}=0$
t_3	0.75	$u_{0,3}=0$	$u_{1,3}=2$	$u_{2,3}=-2$	$u_{3,3}=-8$	$u_{4,3}=-7$	$u_{5,3}=0$
t_4	1	$u_{0,4}=0$	$u_{1,4}=-9$	$u_{2,4}=-14$	$u_{3,4}=-11$	$u_{4,4}=-6$	$u_{5,4}=0$
t_5	1.25	$u_{0,5}=0$	$u_{1,5}=-16$	$u_{2,5}=-18$	$u_{3,5}=-12$	$u_{4,5}=-4$	$u_{5,5}=0$

The values in the 1st & the last column are zero since $u(0,t)=0,\ u(5,t)=0$.

I row : we have $u(x,0) = x^2(5-x)$

$\therefore u_{1,0} = 1\times 4 = 4,\ u_{2,0} = 4\times 3 = 12,\ u_{3,0} = 9\times 2 = 18,$

$u_{4,0} = 16\times 1 = 16$

II row : $u_{i,1} = \frac{1}{2}\left[u_{i-1,0} + u_{i+1,0}\right]$

$\therefore u_{1,1} = \frac{1}{2}\left[u_{0,0} + u_{2,0}\right] = \frac{0+12}{2} = 6.$

$u_{2,1} = \frac{1}{2}\left[u_{1,0} + u_{3,0}\right] = \frac{4+18}{2} = 11.$

$u_{3,1} = \frac{12+16}{2} = 14$, $u_{4,1} = \frac{18+0}{2} = 9.$

III row to VI row :- $u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$

III row : $u_{1,2} = 0+11-4 = 7;\ u_{2,2} = 6+14-12 = 8$

$u_{3,2} = 11+9-18 = 2;\ u_{4,2} = 14+0-16 = -2$

IV row: $u_{1,3} = 0+8-6 = 2$; $u_{2,3} = 7+2-11 = -2$

$u_{3,3} = 8-2-14 = -8$; $u_{4,3} = 2+0-9 = -7$

V row:- $u_{1,4} = 0-2-7 = -9$; $u_{2,4} = 2-8-8 = -14$

$u_{3,4} = -2-7-2 = -11$; $u_{4,4} = -8+0+2 = -6$.

VI row: $u_{1,5} = 0-14-2 = -16$; $u_{2,5} = -9-11+2 = -18$

$u_{3,5} = -14-6+8 = -12$; $u_{4,5} = -11+0+7 = -4$

Required solⁿ is the table itself.

3) The transverse displacement u of a point at a distance x from one end and at any time t of a vibrating string satisfies the equation $u_{tt} = 25 u_{xx}$ with the boundary conditions $u(0,t) = u(5,t) = 0$ and the initial conditions

$$u(x,0) = \begin{cases} 20x, & 0 \le x \le 1 \\ 5(5-x), & 1 \le x \le 5 \end{cases}, \text{ and } u_t(x,0) = 0.$$

Solve this equation numerically upto $t=1$ taking $h=1$, $k=0.2$.

Solⁿ:- comparing $u_{tt} = 25 u_{xx}$ with $u_{tt} = c^2 u_{xx}$, we have

$c^2 = 25 \Rightarrow \boxed{c=5}$

Since $h=1$; the values of x in $0 \le x \le 5$ are $0,1,2,3,4,5$.

Since $k=0.2$; the values of t upto $t=1$ are $0, 0.2, 0.4, 0.6, 0.8, 1$.

The values in the 1st and the last column are zero, since $u(0,t) = 0 = u(5,t)$.

Consider the following table (sol^n)

t \ x		x_0 = 0	x_1 = 1	x_2 = 2	x_3 = 3	x_4 = 4	x_5 = 5
t_0	0	$u_{0,0}=0$	$u_{1,0}=20$	$u_{2,0}=15$	$u_{3,0}=10$	$u_{4,0}=5$	$u_{5,0}=0$
t_1	0.2	$u_{0,1}=0$	$u_{1,1}=7.5$	$u_{2,1}=15$	$u_{3,1}=10$	$u_{4,1}=5$	$u_{5,1}=0$
t_2	0.4	$u_{0,2}=0$	$u_{1,2}=-5$	$u_{2,2}=2.5$	$u_{3,2}=10$	$u_{4,2}=5$	$u_{5,2}=0$
t_3	0.6	$u_{0,3}=0$	$u_{1,3}=-5$	$u_{2,3}=-10$	$u_{3,3}=-2.5$	$u_{4,3}=5$	$u_{5,3}=0$
t_4	0.8	$u_{0,4}=0$	$u_{1,4}=-5$	$u_{2,4}=-10$	$u_{3,4}=-15$	$u_{4,4}=-7.5$	$u_{5,4}=0$
t_5	1	$u_{0,5}=0$	$u_{1,5}=-5$	$u_{2,5}=-10$	$u_{3,5}=-15$	$u_{4,5}=-20$	$u_{5,5}=0$

I row :- $u(x,0) = \begin{cases} 20x & , 0 \le x \le 1 \\ 5(5-x) & , 1 \le x \le 5. \end{cases}$

$\therefore \quad u_{1,0} = 20\times 1 = 20 \quad ; \quad u_{2,0} = 5\times 3 = 15 \, ;$

$u_{3,0} = 5\times 2 = 10 \quad ; \quad u_{4,0} = 5\times 1 = 5$

II row :- $u_{i,1} = \frac{1}{2}\left[u_{i-1,0} + u_{i+1,0}\right]$

$u_{1,1} = \frac{1}{2}(0+15) = 7.5 \quad ; \quad u_{2,1} = \frac{1}{2}(20+10) = 15$

$u_{3,1} = \frac{1}{2}(15+5) = 10 \quad ; \quad u_{4,1} = \frac{1}{2}(10+0) = 5.$

III row to VI row : $u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$ (Explicit formula)

III row :- $u_{1,2} = 0+15-20 = -5 \quad ; \quad u_{2,2} = 7.5+10-15 = 2.5$

$u_{3,2} = 15+5-10 = 10 \quad ; \quad u_{4,2} = 10+0-5 = 5$

IV row :- $u_{1,3} = 0+2.5-7.5 = -5 \, ; \quad u_{2,3} = -5+10-15 = -10$

$u_{3,3} = 2.5+5-10 = -2.5 \quad ; \quad u_{4,3} = 10+0-5 = 5$

<u>V row</u> :- $u_{1,4} = 0-10+5 = -5$; $u_{2,4} = -5-2.5-2.5 = -10$

$u_{3,4} = -10+5-10 = -15$; $u_{4,4} = -2.5+0-5 = -7.5$

<u>VI row</u> :- $u_{1,5} = 0-10+5 = -5$; $u_{2,5} = -5-15+10 = -10$

$u_{3,5} = -10-7.5+2.5 = -15$; $u_{4,5} = -15+0-5 = -20$

∴ Required solⁿ is the table itself.

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4) Solve $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ given that $u(x,0)=0$; $u(0,t)=0$;

$u_t(x,0) = 0$ and $u(1,t) = 100\sin(\pi t)$ in the range $0 \le t \le 1$

by taking $h = 1/4$.

<u>Solⁿ</u> :- (2 condⁿˢ are different $u(x,0)$ & $u(1,t)$)

Comparing $u_{tt} = u_{xx}$ with $u_{tt} = c^2 u_{xx}$, we have

$c^2 = 1 \Rightarrow \boxed{c=1}$

Given $\boxed{h = 1/4}$ ∴ $k = h/c = \frac{1/4}{1} \Rightarrow \boxed{k = 1/4}$

Since $h = 1/4$, Points of division of x : $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$.

Since $k = 1/4$, Points of division of t : $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$.

Table here

Given $u(x,0) = 0 \Rightarrow$ values of u along the 1st row are zero.

Given $u(0,t) = 0 \Rightarrow$ values of u along the 1st column are zero.

Also given that $u(1,t) = 100\sin(\pi t)$ i.e. $u(x_4, t) = 100\sin(\pi t)$

(applicable for last column)

$\therefore u(x_4, t_1) = u(1, 1/4) = 100\sin(\pi/4) = 70.7$

$u(x_4, t_2) = u(1, 1/2) = 100\sin(\pi/2) = 100$

$u(x_4, t_3) = u(1, 3/4) = 100\sin(3\pi/4) = 70.7$

$u(x_4, t_4) = u(1, 1) = 100\sin\pi = 0$.

Consider the following table (solⁿ)

t \ x		x_0	x_1	x_2	x_3	x_4
		0	1/4	1/2	3/4	1
t_0	0	$u_{0,0}=0$	$u_{1,0}=0$	$u_{2,0}=0$	$u_{3,0}=0$	$u_{4,0}=0$
t_1	1/4	$u_{0,1}=0$	$u_{1,1}=0$	$u_{2,1}=0$	$u_{3,1}=0$	$u_{4,1}=70.7$
t_2	1/2	$u_{0,2}=0$	$u_{1,2}=0$	$u_{2,2}=0$	$u_{3,2}=70.7$	$u_{4,2}=100$
t_3	3/4	$u_{0,3}=0$	$u_{1,3}=0$	$u_{2,3}=70.7$	$u_{3,3}=100$	$u_{4,3}=70.7$
t_4	1	$u_{0,4}=0$	$u_{1,4}=70.7$	$u_{2,4}=100$	$u_{3,4}=70.7$	$u_{4,4}=0$

II row :- $u_{i,1} = \frac{1}{2}\left[u_{i-1,0} + u_{i+1,0}\right]$

$u_{1,1} = \frac{1}{2}(0+0) = 0$; $u_{2,1} = \frac{1}{2}(0+0) = 0$; $u_{3,1} = \frac{1}{2}(0+0) = 0$

III row to V row :- $u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$

III row :- $u_{1,2} = 0+0-0 = 0$; $u_{2,2} = 0+0-0 = 0$.

$u_{3,2} = 0+70.7-0 = 70.7$;

IV row :- $u_{1,3} = 0+0-0 = 0$; $u_{2,3} = 0+70.7-0 = 70.7$

$u_{3,3} = 0+100-0 = 100$;

V row :- $u_{1,4} = 0+70.7-0 = 70.7$; $u_{2,4} = 0+100-0 = 100$

$u_{3,4} = 70.7+70.7-70.7 = 70.7$.

∴ Required solⁿ is the table solⁿ.

5) Solve the wave eqⁿ $4\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$ given $u(0,t) = u(5,t) = 0$, $t \geq 0$. $u(x,0) = x(5-x)$, $\frac{\partial}{\partial t}u(x,0) = 0$, $0 < x < 5$

Find u at $t=2$ given $h=1$, $k=0.5$.

Ans :- Required values of u at $t=2$ are : $-3, -5, -5, -3$.

Problems on 1-D Heat Equation (Schmidt Formula)

1) Find the numerical soln of the parabolic equation $\frac{\partial^2 u}{\partial x^2} = 2\frac{\partial u}{\partial t}$, when $u(0,t) = 0 = u(4,t)$ and $u(x,0) = x(4-x)$ by taking $h=1$. find the values upto $t=5$.

Soln:- Given $U_{xx} = 2U_t$

$$\Rightarrow \quad U_t = \frac{1}{2}U_{xx}.$$

comparing this with standard form $U_t = c^2 U_{xx}$, we have $c^2 = 1/2$.

Since $\boxed{h=1}$, $K = \frac{h^2}{2c^2} = \frac{1}{2\times\frac{1}{2}} \Rightarrow \boxed{K=1}$

Since $h=1$, the values of x in $0 \le x \le 4$ are $0,1,2,3,4$.

Since $K=1$, the values of t in $0 \le t \le 5$ are $0,1,2,3,4,5$.

Consider the following table (soln):

t \ x		x_0	x_1	x_2	x_3	x_4
		0	1	2	3	4
t_0	0	$u_{0,0}=0$	$u_{1,0}=3$	$u_{2,0}=4$	$u_{3,0}=3$	$u_{4,0}=0$
t_1	1	$u_{0,1}=0$	$u_{1,1}=2$	$u_{2,1}=3$	$u_{3,1}=2$	$u_{4,1}=0$
t_2	2	$u_{0,2}=0$	$u_{1,2}=1.5$	$u_{2,2}=2$	$u_{3,2}=1.5$	$u_{4,2}=0$
t_3	3	$u_{0,3}=0$	$u_{1,3}=1$	$u_{2,3}=1.5$	$u_{3,3}=1$	$u_{4,3}=0$
t_4	4	$u_{0,4}=0$	$u_{1,4}=0.75$	$u_{2,4}=1$	$u_{3,4}=0.75$	$u_{4,4}=0$
t_5	5	$u_{0,5}=0$	$u_{1,5}=0.5$	$u_{2,5}=0.75$	$u_{3,5}=0.5$	$u_{4,5}=0$

The values in the first & last column are zero since $u(0,t)=0$, $u(4,t)=0$.

I row :- $u(x,0) = x(4-x)$

$u_{1,0} = 1\cdot(4-1) = 1\times 3 = 3$; $u_{3,0} = 3\times 1 = 3$

$u_{2,0} = 2\times 2 = 4$

formula for II row onwards :

$$u_{i,j+1} = \frac{1}{2}\left[u_{i-1,j} + u_{i+1,j}\right]$$

II row : $u_{i,1} = \frac{1}{2}\left[u_{i-1,0} + u_{i+1,0}\right]$

$\therefore u_{1,1} = \frac{1}{2}\left[u_{0,0} + u_{2,0}\right] = \frac{1}{2}[0+4] = 2$.

$u_{2,1} = \frac{1}{2}\left[u_{1,0} + u_{3,0}\right] = \frac{1}{2}[3+3] = 3$.

$u_{3,1} = \frac{1}{2}\left[u_{2,0} + u_{4,0}\right] = \frac{1}{2}[4+0] = 2$.

III row : $u_{i,2} = \frac{1}{2}\left[u_{i-1,1} + u_{i+1,1}\right]$

$\therefore u_{1,2} = \frac{1}{2}(0+3) = 1.5$

$u_{2,2} = \frac{1}{2}(2+2) = 2$; $u_{3,2} = \frac{1}{2}(3+0) = 1.5$

IV row : $u_{i,3} = \frac{1}{2}\left[u_{i-1,2} + u_{i+1,2}\right]$

$\therefore u_{1,3} = \frac{1}{2}(0+2) = 1$, $u_{2,3} = \frac{1}{2}(1.5+1.5) = 1.5$

$u_{3,3} = \frac{1}{2}(2+0) = 1$.

V row : $u_{i,4} = \frac{1}{2}\left[u_{i-1,3} + u_{i+1,3}\right]$

$\therefore u_{1,4} = \frac{1}{2}(0+1.5) = 0.75$, $u_{2,4} = \frac{1}{2}(1+1) = 1$.

$u_{3,4} = \frac{1}{2}(1.5+0) = 0.75$

VI row :- $u_{i,5} = \frac{1}{2}\left[u_{i-1,4} + u_{i+1,4}\right]$

$\therefore u_{1,5} = \frac{1}{2}(0+1) = 0.5$, $u_{2,5} = \frac{1}{2}(0.75+0.75) = 0.75$

$u_{3,5} = \frac{1}{2}(1+0) = 0.5$.

Above table is the required solution.

2) solve $U_{xx} = 32 U_t$ subject to the conditions $u(0,t) = 0$, $u(1,t) = t$ and $u(x,0) = 0$. find the values of u up to $t = 5$ by schmidt's process taking $h = 1/4$.

sol:- Given $U_{xx} = 32 U_t$

$\Rightarrow \frac{1}{32} U_{xx} = U_t$; comparing with $c^2 U_{xx} = U_t$, we get

$\boxed{c^2 = 1/32}$ and given $\boxed{h = 1/4}$

$\therefore \quad K = \frac{h^2}{2c^2} = \frac{1/16}{1/16} \Rightarrow \boxed{K = 1}$

since $h = 1/4$, the values of x in $0 \le x \le 1$ are $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$.

since $K = 1$, the values of t in $0 \le t \le 5$ are $0, 1, 2, 3, 4, 5$.

Consider the following table (soln);

$t \backslash x$		x_0	x_1	x_2	x_3	x_4
		0	1/4	1/2	3/4	1
t_0	0	$u_{0,0} = 0$	$u_{1,0} = 0$	$u_{2,0} = 0$	$u_{3,0} = 0$	$u_{4,0} = 0$
t_1	1	$u_{0,1} = 0$	$u_{1,1} = 0$	$u_{2,1} = 0$	$u_{3,1} = 0$	$u_{4,1} = 1$
t_2	2	$u_{0,2} = 0$	$u_{1,2} = 0$	$u_{2,2} = 0$	$u_{3,2} = 0.5$	$u_{4,2} = 2$
t_3	3	$u_{0,3} = 0$	$u_{1,3} = 0$	$u_{2,3} = 0.25$	$u_{3,3} = 1$	$u_{4,3} = 3$
t_4	4	$u_{0,4} = 0$	$u_{1,4} = 0.125$	$u_{2,4} = 0.5$	$u_{3,4} = 1.625$	$u_{4,4} = 4$
t_5	5	$u_{0,5} = 0$	$u_{1,5} = 0.25$	$u_{2,5} = 0.875$	$u_{3,5} = 2.25$	$u_{4,5} = 5$

Given: $u(0,t) = 0 \Rightarrow$ the values in the 1st column are zero.

$u(x,0) = 0 \Rightarrow$ the values in the 1st row are zero

Also $u(1,t) = t$ ie $u(x_4, t) = t$ is applicable to last column.

$\therefore \; u(x_4, t_1) = t_1 = 1$; $u(x_4, t_2) = t_2 = 2$ etc formula for II row onwards:

$$u_{i,j+1} = \frac{1}{2}\left[u_{i-1,j} + u_{i+1,j}\right]$$

II row: $u_{i,1} = \frac{1}{2}\left[u_{i-1,0} + u_{i+1,0}\right]$

$\therefore\ u_{1,1} = \frac{1}{2}(0+0) = 0$, $u_{2,1} = \frac{1}{2}(0+0) = 0$, $u_{3,1} = \frac{1}{2}(0+0) = 0$

Similarly for the remaining rows.

Required solution is the table itself.

---

3) find the values of $u(x,t)$ satisfying the parabolic equation $u_t = 4u_{xx}$ and the boundary conditions $u(0,t) = 0 = u(8,t)$, $u(x,0) = 4x - \frac{x^2}{2}$ at $x = i$; $i = 0,1,2 \ldots 8$ and $t = j/8$; $j = 0,1, \ldots 4$.

Sol^n :- Given $u_t = 4u_{xx}$; comparing with $u_t = c^2 u_{xx}$,

we get $c^2 = 4 \Rightarrow \boxed{c=2}$

Given $x = i \Rightarrow x = 0,1,2, \ldots 8 \Rightarrow \boxed{h=1}$

and $t = j/8 \Rightarrow t = 0, \frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{4}{8} \Rightarrow \boxed{k = 1/8}$

Consider the following table (sol^n) :-

$t \backslash x$		x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
		0	1	2	3	4	5	6	7	8
t_0	0	$u_{0,0} =$ 0	$u_{1,0} =$ 3.5	$u_{2,0} =$ 6	$u_{3,0} =$ 7.5	$u_{4,0} =$ 8	$u_{5,0} =$ 7.5	$u_{6,0} =$ 6	$u_{7,0} =$ 3.5	$u_{8,0} =$ 0
t_1	$\frac{1}{8}$	$u_{0,1} =$ 0	$u_{1,1} =$ 3	$u_{2,1} =$ 5.5	$u_{3,1} =$ 7	$u_{4,1} =$ 7.5	$u_{5,1} =$ 7	$u_{6,1} =$ 5.5	$u_{7,1} =$ 3	$u_{8,1} =$ 0
t_2	$\frac{2}{8}$	$u_{0,2} =$ 0	$u_{1,2} =$ 2.75	$u_{2,2} =$ 5	$u_{3,2} =$ 6.5	$u_{4,2} =$ 7	$u_{5,2} =$ 6.5	$u_{6,2} =$ 5	$u_{7,2} =$ 2.75	$u_{8,2} =$ 0
t_3	$\frac{3}{8}$	$u_{0,3} =$ 0	$u_{1,3} =$ 2.5	$u_{2,3} =$ 4.625	$u_{3,3} =$ 6	$u_{4,3} =$ 6.5	$u_{5,3} =$ 6	$u_{6,3} =$ 4.625	$u_{7,3} =$ 2.5	$u_{8,3} =$ 0
t_4	$\frac{4}{8}$	$u_{0,4} =$ 0	$u_{1,4} =$ 2.3125	$u_{2,4} =$ 4.25	$u_{3,4} =$ 5.5625	$u_{4,4} =$ 6	$u_{5,4} =$ 5.5625	$u_{6,4} =$ 4.25	$u_{7,4} =$ 2.3125	$u_{8,4} =$ 0

The values in the first & last column are zero since $u(0,t) = 0$, $u(8,t) = 0$.

I row: $u(x,0) = 4x - \frac{x^2}{2}$

$\therefore u_{1,0} = 4 - \frac{1}{2} = 3.5$; $u_{2,0} = 8 - 2 = 6$

$u_{3,0} = 12 - \frac{9}{2} = 7.5$, $u_{4,0} = 16 - 8 = 8$, $u_{5,0} = 20 - \frac{25}{2} = 7.5$

$u_{6,0} = 24 - \frac{36}{2} = 6$, $u_{7,0} = 28 - \frac{49}{2} = 3.5$, $u_{8,0} = 32 - 32 = 0$.

formula for II row onwards

$$u_{i,j+1} = \frac{1}{2}\left[u_{i-1,j} + u_{i+1,j}\right]$$

II row: $u_{i,1} = \frac{1}{2}\left[u_{i-1,0} + u_{i+1,0}\right]$

$u_{1,1} = \frac{1}{2}(0+6) = 3$; $u_{2,1} = \frac{1}{2}(3.5+7.5) = 5.5$

$u_{3,1} = \frac{1}{2}(6+8) = 7$; $u_{4,1} = \frac{1}{2}(7.5+7.5) = 7.5$

$u_{5,1} = \frac{1}{2}(8+6) = 7$; $u_{6,1} = \frac{1}{2}(7.5+3.5) = 5.5$

$u_{7,1} = \frac{1}{2}(6+0) = 3$

Similarly, remaining row values are calculated & tabulated.

Required solution is the table itself.

P.T.O

4) Solve numerically the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ subject to the conditions $u(0,t) = 0 = u(1,t)$, $t \geq 0$ and $u(x,0) = \sin \pi x$, $0 \leq x \leq 1$. Carry out computations for 2 levels taking $h = 1/3$ and $k = 1/36$.

Soln :- Given $U_t = U_{xx}$. Comparing with $c^2 U_{xx} = U_t$, we get $c^2 = 1$. Since $h = 1/3$, $k = 1/36$, we see that

$$k = \frac{h^2}{2c^2} = \frac{1/9}{2} = \frac{1}{18} \text{ is not satisfied.}$$

∴ we use schmidt explicit formula in the form,

$$u_{i,j+1} = a\,u_{i-1,j} + (1-2a)\,u_{i,j} + a\,u_{i+1,j}$$

where $a = \frac{kc^2}{h^2} = \frac{1/36 \times 1}{1/9} = \frac{9}{36} \Rightarrow \boxed{a = 1/4}$

$$\therefore\; u_{i,j+1} = \frac{1}{4} u_{i-1,j} + \frac{1}{2} u_{i,j} + \frac{1}{4} u_{i+1,j}$$

$$(\text{or})\quad u_{i,j+1} = \frac{1}{4}\left[u_{i-1,j} + 2u_{i,j} + u_{i+1,j}\right] \longrightarrow (1)$$

Since $h = 1/3$, the values of x in $0 \leq x \leq 1$ are $0, \frac{1}{3}, \frac{2}{3}, 1$.

t \ x		x_0	x_1	x_2	x_3	
		0	$1/3$	$2/3$	1	
t_0	0	$u_{0,0} = 0$	$u_{1,0} = 0.866$	$u_{2,0} = 0.866$	$u_{3,0} = 0$	
t_1	$1/36$	$u_{0,1} = 0$	$u_{1,1} = 0.6495$	$u_{2,1} = 0.6495$	$u_{3,1} = 0$	first level
t_2	$2/36$	$u_{0,2} = 0$	$u_{1,2} = 0.4871$	$u_{2,2} = 0.4871$	$u_{3,2} = 0$	second level

The values in the first & last column are zero since $u(0,t) = 0$, $u(1,t) = 0$.

P.T.O

I row: $u(x,0) = \sin \pi x$

$$u(x_1, 0) = u_{1,0} = \sin \pi/3 \qquad (\because \; x_1 = 1/3)$$
$$= 0.8660$$

$$u(x_2, 0) = u(2/3, 0) = \sin \frac{2\pi}{3} = 0.8660$$

To compute 2^{nd} & 3^{rd} row using eqn (1).

ie to compute $u_{1,1}$, $u_{2,1}$ and $u_{1,2}$, $u_{2,2}$.

$$u_{1,1} = \frac{1}{4}\left[u_{0,0} + 2u_{1,0} + u_{2,0}\right] = 0.6495$$

$$u_{2,1} = \frac{1}{4}\left[u_{1,0} + 2u_{2,0} + u_{3,0}\right] = 0.6495$$

$$u_{1,2} = \frac{1}{4}\left[u_{0,1} + 2u_{1,1} + u_{2,1}\right] = 0.4871$$

$$u_{2,2} = \frac{1}{4}\left[u_{1,1} + 2u_{2,1} + u_{3,1}\right] = 0.4871.$$

Required soln is the table itself upto 2 levels.

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5) Solve $u_t = u_{xx}$ subject to the conditions $u(0,t) = 0$, $u(1,t) = 0$, $u(x,0) = \sin \pi x$ for $0 \le t \le 0.1$ by taking $h = 0.2$.

Problems on 1-D Heat Equation (Cranck-Nicholson formula)

1) Solve the heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $0 \le x \le 1$, $t \geq 0$ subjected to $u(x,0) = \sin \pi x$, $0 \le x \le 1$ and $u(0,t) = 0 = u(1,t)$ $t \geq 0$ by Crank-Nicholson scheme taking $h = \frac{1}{3}$, $K = \frac{1}{18}$. find the solution at the first time level.

Soln:- Given $U_t = U_{xx}$; comparing with $U_t = c^2 U_{xx}$, we have $c^2 = 1 \Rightarrow \boxed{c=1}$

Since $h = 1/3$ and $0 \le x \le 1$, we have $x : 0, \frac{1}{3}, \frac{2}{3}, 1$

Since $K = 1/18$ and $t \geq 0$, solution is required at the 1st time level, we have $t : 0, 1/18$

$$\therefore \lambda = \frac{K}{c^2 h^2} = \frac{1/18}{(1/3)^3} \Rightarrow \lambda = \frac{1/18}{1/9} \Rightarrow \boxed{\lambda = \frac{1}{2}}$$

Consider the following table (soln):

t \ x		x_0	x_1	x_2	x_3
		0	1/3	2/3	1
t_0	0	$u_{0,0} = 0$	$u_{1,0} = 0.8660$	$u_{2,0} = 0.8660$	$u_{3,0} = 0$
t_1	1/18	$u_{0,1} = 0$	$u_{1,1} = 0.5196$	$u_{2,1} = 0.5196$	$u_{3,1} = 0$

Given: $u(0,t) = 0 = u(1,t)$ $\Rightarrow$ the values in the first & last column are zero.

I row: $u(x,0) = \sin \pi x$

$u_{1,0} = u(x_1, 0) = u\left(\frac{1}{3}, 0\right) = \sin \pi/3 = \frac{\sqrt{3}}{2} = 0.8660$

$u_{2,0} = u(x_2, 0) = u\left(\frac{2}{3}, 0\right) = \sin \frac{2\pi}{3} = 0.8660$

To evaluate $u_{1,1}$ and $u_{2,1}$ using Crank-Nicholson scheme given by $-\lambda u_{i-1,j+1} + (2+2\lambda) u_{i,j+1} - \lambda u_{i+1,j+1} =$

$$\lambda u_{i-1,j} + (2-2\lambda) u_{i,j} + \lambda u_{i+1,j}$$

with $\lambda = 1/2$

$$-\frac{1}{2}u_{i-1,j+1} + 3u_{i,j+1} - \frac{1}{2}u_{i+1,j+1} = \frac{1}{2}u_{i-1,j} + u_{i,j} + \frac{1}{2}u_{i+1,j} \rightarrow (1)$$

Put $j=0$, $i=1,2$ (for solution at 1st time level)

$j=0$, $i=1$ in eqn (1),

$$-\frac{1}{2}u_{0,1} + 3u_{1,1} - \frac{1}{2}u_{2,1} = \frac{1}{2}u_{0,0} + u_{1,0} + \frac{1}{2}u_{2,0}$$

$$\Rightarrow \cancel{-\frac{1}{2}\times 0}^{\to 0} + 3u_{1,1} - \frac{1}{2}u_{2,1} = \cancel{\frac{1}{2}\times 0}^{\to 0} + 0.8660 + \frac{1}{2}\times 0.8660$$

$$\Rightarrow 3u_{1,1} - \frac{1}{2}u_{2,1} = 1.299$$

$$\Rightarrow 6u_{1,1} - u_{2,1} = 2.598 \longrightarrow (2)$$

$j=0$, $i=2$ in eqn (1)

$$-\frac{1}{2}u_{1,1} + 3u_{2,1} - \frac{1}{2}u_{3,1} = \frac{1}{2}u_{1,0} + u_{2,0} + \frac{1}{2}u_{3,0}$$

$$\Rightarrow -\frac{1}{2}u_{1,1} + 3u_{2,1} \cancel{-\frac{1}{2}\times 0}^{\to 0} = \frac{1}{2}\times 0.8660 + 0.8660 + \cancel{\frac{1}{2}\times 0}^{\to 0}$$

$$\Rightarrow -\frac{1}{2}u_{1,1} + 3u_{2,1} = 1.299$$

$$\Rightarrow -u_{1,1} + 6u_{2,1} = 2.598 \longrightarrow (3)$$

Solving (2) & (3) simultaneously,

$u_{1,1} = 0.5196$, $u_{2,1} = 0.5196$

Required soln is the table itself.

P.T.O

2) Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $0 \le x \le 1$, $t \ge 0$ subjected to

$u(x,0) = \sin \pi x$, $0 \le x \le 1$ and $u(0,t) = 0 = u(1,t)$, $t \ge 0$

using Crank-Nicholson scheme. Take $h = \frac{1}{4}$, $k = \frac{1}{32}$ and find the solution at the first time level.

Sol^n :- Given $U_t = U_{xx} \Rightarrow \boxed{c^2 = 1}$

Since $h = \frac{1}{4}$, the values of x in $0 \le x \le 1$ are $0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1$.

Since $k = \frac{1}{32}$, $t \ge 0$ and sol^n is required upto 1st time level,

we have $t = 0, 1/32$.

$\therefore \lambda = \frac{k}{c^2h^2} = \frac{1/32}{1/16} = \frac{1}{2}$ ie $\boxed{\lambda = 1/2}$

Consider the following table (sol^n):

t \ x		x_0	x_1	x_2	x_3	x_4
		0	1/4	2/4	3/4	1
t_0	0	$u_{0,0} = 0$	$u_{1,0} = 0.7071$	$u_{2,0} = 1$	$u_{3,0} = 0.7071$	$u_{4,0} = 0$
t_1	$\frac{1}{32}$	$u_{0,1} = 0$	$u_{1,1} = 0.5265$	$u_{2,1} = 0.7445$	$u_{3,1} = 0.5265$	$u_{4,1} = 0$

Given: $u(0,t) = 0 = u(1,t) \Rightarrow$ the values in the first & last column are zero.

I row: $u(x,0) = \sin \pi x$

$\therefore u_{1,0} = u(\frac{1}{4}, 0) = \sin \pi/4 = \frac{1}{\sqrt{2}}$; $u_{2,0} = u(\frac{2}{4}, 0) = \sin \frac{2\pi}{4} = 1$

$u_{3,0} = u(\frac{3}{4}, 0) = \sin \frac{3\pi}{4} = 1/\sqrt{2} = 0.7071$

To evaluate $u_{1,1}$, $u_{2,1}$ and $u_{3,1}$ using Crank-Nicholson formula:

$$-\lambda u_{i-1,j+1} + (2+2\lambda) u_{i,j+1} - \lambda u_{i+1,j+1} = \lambda u_{i-1,j} + (2-2\lambda) u_{i,j} + \lambda u_{i+1,j}$$

with $\lambda = 1/2$, we get

$$-\frac{1}{2}u_{i-1,j+1} + 3u_{i,j+1} - \frac{1}{2}u_{i+1,j+1} = \frac{1}{2}u_{i-1,j} + u_{i,j} + \frac{1}{2}u_{i+1,j} \rightarrow (1)$$

Put $j=0$, $i=1,2,3$ (for solution at first time level)

<u>$j=0, i=1$</u> in eq^n (1),

$$-\frac{1}{2}\cancel{u_{0,1}}^{\to 0} + 3u_{1,1} - \frac{1}{2}u_{2,1} = \frac{1}{2}\cancel{u_{0,0}}^{\to 0} + u_{1,0} + \frac{1}{2}u_{2,0}$$

$$\Rightarrow 3u_{1,1} - \frac{1}{2}u_{2,1} = 0.7071 + \frac{1}{2}\times 1 = 1.2071$$

$$\Rightarrow \frac{6u_{1,1} - u_{2,1}}{2} = 1.2071$$

$$\text{(or)} \quad 6u_{1,1} - u_{2,1} = 2.4142 \longrightarrow (2)$$

<u>$j=0, i=2$</u> in eq^n (1)

$$-\frac{1}{2}u_{1,1} + 3u_{2,1} - \frac{1}{2}u_{3,1} = \frac{1}{2}u_{1,0} + u_{2,0} + \frac{1}{2}u_{3,0}$$

$$-\frac{1}{2}u_{1,1} + 3u_{2,1} - \frac{1}{2}u_{3,1} = \frac{0.7071}{2} + 1 + \frac{0.7071}{2}$$

$$\Rightarrow \frac{-u_{1,1} + 6u_{2,1} - u_{3,1}}{2} = 1.7071$$

$$\Rightarrow -u_{1,1} + 6u_{2,1} - u_{3,1} = 3.4142 \longrightarrow (3)$$

<u>$j=0, i=3$</u> in eq^n (1),

$$-\frac{1}{2}u_{2,1} + 3u_{3,1} - \frac{1}{2}\cancel{u_{4,1}}^{\to 0} = \frac{1}{2}u_{2,0} + u_{3,0} + \frac{1}{2}\cancel{u_{4,0}}^{\to 0}$$

$$\Rightarrow -\frac{1}{2}u_{2,1} + 3u_{3,1} = \frac{1}{2} + 0.7071$$

$$\Rightarrow -u_{2,1} + 6u_{3,1} = 2(1.2071)$$

$$\Rightarrow -u_{2,1} + 6u_{3,1} = 2.4142 \longrightarrow (4)$$

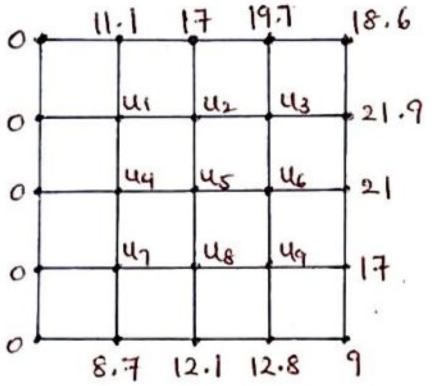
Solving (2), (3), (4) simultaneously, we get

$u_{1,1} = 0.5265$, $u_{2,1} = 0.7445$, $u_{3,1} = 0.5265$.

Required sol^n is the table itself.

Problems on 2-D Laplace's Equation

1) Solve Laplace's eqn $u_{xx} + u_{yy} = 0$ for the following square mesh with boundary values as shown in the following figure :



Soln :- To find u_5 using S.F :

$$u_5 = \frac{1}{4}(0+21+17+12.1) = 12.525$$

To find u_1, u_3, u_9, u_7 using D.F :

$$u_1 = \frac{1}{4}(0+17+0+12.525) = 7.3813$$

$$u_3 = \frac{1}{4}(17+18.6+12.525+21) = 17.2813$$

$$u_9 = \frac{1}{4}(12.525+21+12.1+9) = 13.6563$$

$$u_7 = \frac{1}{4}(0+12.525+0+12.1) = 6.1563$$

To find u_2, u_4, u_6, u_8 using S.F :-

$$u_2 = \frac{1}{4}(17+12.525+7.3813+17.2813) = 13.5469$$

$$u_4 = \frac{1}{4}(0+12.525+7.3813+6.1563) = 6.5157$$

$$u_6 = \frac{1}{4}(12.525+21+17.2813+13.6563) = 16.1157$$

$$u_8 = \frac{1}{4}(12.525+12.1+6.1563+13.6563) = 11.1094$$

Thus the required values of $u(x, y)$ at the interior mesh points, correct to 2 decimal places are : $u_1 = 7.381$, $u_2 = 13.55$, $u_3 = 17.28$, $u_4 = 6.52$, $u_5 = 12.53$, $u_6 = 16.12$, $u_7 = 6.16$, $u_8 = 11.11$, $u_9 = 13.66$.

2) Solve $\nabla^2 u = 0$ in the square region bounded by the Co-ordinate axes and the lines $x = 4$, $y = 4$ with the boundary Conditions given by the analytical expressions,

(i) $u(0,y) = 0$ for $0 \le y \le 4$ (ii) $u(4,y) = 12+y$ for $0 \le y \le 4$

(iii) $u(x,0) = 3x$ for $0 \le x \le 4$ (iv) $u(x,4) = x^2$ for $0 \le x \le 4$.

Soln :- In 2 dimensions, $\nabla^2 u = 0$ is $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

Dividing the square region into 16 squares of side one unit.

To write boundary conditions from the given expression:

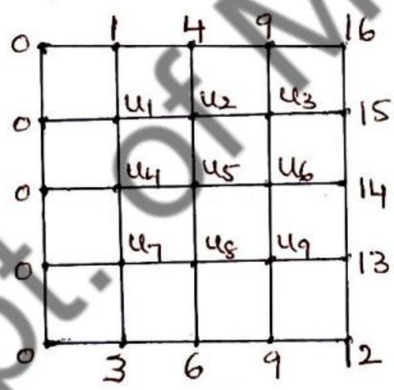
(i) $u(0,y) = 0 \Rightarrow u(0,1) = 0 = u(0,2) = u(0,3) = u(0,4) = u(0,0)$.

(ii) $u(4,y) = 12+y \Rightarrow u(4,0) = 12,\ u(4,1) = 13,\ u(4,2) = 14,$
$u(4,3) = 15,\ u(4,4) = 16$.

(iv) $u(x,0) = 3x \Rightarrow u(0,0) = 0,\ u(1,0) = 3,\ u(2,0) = 6,$
$u(3,0) = 9,\ u(4,0) = 12$.

(v) $u(x,4) = x^2 \Rightarrow u(0,4) = 0,\ u(1,4) = 1,\ u(2,4) = 4,$
$u(3,4) = 9,\ u(4,4) = 16$.

Representing these on a square region :-



To find u_5 using S.F :-

$$u_5 = \frac{1}{4}(0+14+4+6) = 6$$

To find u_1, u_3, u_7, u_9 using D.F :-

$$u_1 = \frac{1}{4}(0+4+0+6) = 2.5$$

$$u_3 = \frac{1}{4}(4+16+6+14) = 10$$

$$u_7 = \frac{1}{4}(0+6+0+6) = 3$$

$$u_9 = \frac{1}{4}(6+14+6+12) = 9.5$$

To find u_2, u_4, u_6, u_8 using S.F :-

$$u_2 = \frac{1}{4}(4+6+2.5+10) = 5.625$$

$$u_4 = \frac{1}{4}(0+6+2.5+3) = 2.875$$

$$u_6 = \frac{1}{4}(6+14+10+9.5) = 9.875$$

$$u_8 = \frac{1}{4}(6+6+3+9.5) = 6.125$$

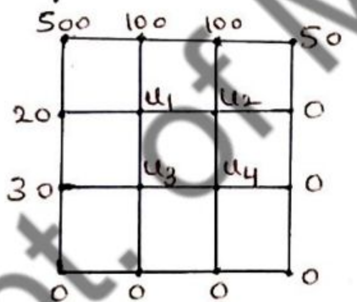
Thus the required values of $u(x,y)$ are:

$u_1 = 2.5$, $u_2 = 5.625$, $u_3 = 10$, $u_4 = 2.875$, $u_5 = 6$,

$u_6 = 9.875$, $u_7 = 3$, $u_8 = 6.125$, $u_9 = 9.5$.

---

3) Solve $U_{xx} + U_{yy} = 0$ in the following square region with the boundary conditions as indicated in the figure :



Soln: To find u_1, u_2, u_3, u_4 using S.F :-

$$u_1 = \frac{1}{4}(20 + u_2 + 100 + u_3) \Rightarrow 4u_1 - u_2 - u_3 = 120 \rightarrow (1)$$

$$u_2 = \frac{1}{4}(u_1 + 0 + 100 + u_4) \Rightarrow 4u_2 - u_1 - u_4 = 100 \rightarrow (2)$$

$$u_3 = \frac{1}{4}(u_1 + 0 + 30 + u_4) \Rightarrow 4u_3 - u_1 - u_4 = 30 \rightarrow (3)$$

$$u_4 = \frac{1}{4}(u_2 + 0 + u_3 + 0) \Rightarrow 4u_4 - u_2 - u_3 = 0 \rightarrow (4)$$

To eliminate u_4 from (2), (3), (4),

from (2), $u_4 = 4u_2 - u_1 - 100 \rightarrow$ (5)

Sub in (3), $4u_3 - u_1 - (4u_2 - u_1 - 100) = 30$

$\Rightarrow 4u_3 - \cancel{u_1} - 4u_2 + \cancel{u_1} + 100 = 30$

$\Rightarrow -4u_2 + 4u_3 = -70 \rightarrow$ (6)

Sub in (4), $4(4u_2 - u_1 - 100) - u_2 - u_3 = 0$

$\Rightarrow 16u_2 - 4u_1 - 400 - u_2 - u_3 = 0$

$\Rightarrow -4u_1 + 15u_2 - u_3 = 400 \rightarrow$ (7)

Solving (1), (6), (7),

ie., $4u_1 - u_2 - u_3 = 120$

$-4u_2 + 4u_3 = -70$

$-4u_1 + 15u_2 - u_3 = 400$

Solving $u_1 = 45.83$, $u_2 = 40.42$, $u_3 = 22.92$

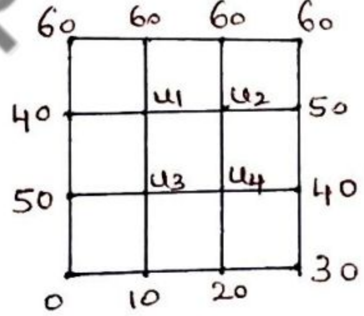
Sub in (5), $u_4 = 15.85$

Thus required values of $u(x, y)$ are:

$u_1 = 45.83$, $u_2 = 40.42$, $u_3 = 22.92$, $u_4 = 15.85$

---

4) Evaluate the function $u(x,y)$ satisfying the Laplace eq^n $u_{xx} + u_{yy} = 0$ at the pivotal points of the figure:



P.T.O.

Sol^n: To find u_1, u_2, u_3, u_4 using s.f :-

$$u_1 = \frac{1}{4}(40 + u_2 + 60 + u_3) \Rightarrow 4u_1 - u_2 - u_3 = 100 \rightarrow (1)$$

$$u_2 = \frac{1}{4}(u_1 + 50 + 60 + u_4) \Rightarrow 4u_2 - u_1 - u_4 = 110 \rightarrow (2)$$

$$u_3 = \frac{1}{4}(u_1 + 10 + 50 + u_4) \Rightarrow 4u_3 - u_1 - u_4 = 60 \rightarrow (3)$$

$$u_4 = \frac{1}{4}(u_2 + 20 + u_3 + 40) \Rightarrow 4u_4 - u_2 - u_3 = 60 \rightarrow (4)$$

To eliminate u_4 from (2), (3), (4),

from (2), $u_4 = 4u_2 - u_1 - 110 \rightarrow (5)$

Sub in (3), $4u_3 - u_1 - (4u_2 - u_1 - 110) = 60$

$$\Rightarrow 4u_3 - \cancel{u_1} - 4u_2 + \cancel{u_1} + 110 = 60$$

$$\Rightarrow -4u_2 + 4u_3 = -50 \rightarrow (6)$$

Sub in (4), $4(4u_2 - u_1 - 110) - u_2 - u_3 = 60$

$$\Rightarrow 16u_2 - 4u_1 - 440 - u_2 - u_3 = 60$$

$$\Rightarrow -4u_1 + 15u_2 - u_3 = 500 \rightarrow (7)$$

Solving (1), (6), (7),

i.e., $4u_1 - u_2 - u_3 = 100$

$$-4u_2 + 4u_3 = -50$$

$$-4u_1 + 15u_2 - u_3 = 500$$

Solving $u_1 = 45.83$, $u_2 = 47.92$, $u_3 = 35.42$

Sub in (5), $u_4 = 35.85$.

Thus the required values of $u(x,y)$ are:

$u_1 = 45.83$, $u_2 = 47.92$, $u_3 = 35.42$, $u_4 = 35.85$