Chapter 3

Linear Systems of Equations

1 Introductory notes

1.1 Matrix form and augmented matrix of a system of equations

Suppose you model an engineering problem, and obtain an algebraic system of n linear equations and n unknowns x_i, $i = 1, 2, ..., n$. It can have the following general form

$$\left\{ \sum_{j=1}^{n} a_{ij}\, x_j = b_i \right\}_{i=1,2,\ldots,n} \iff \begin{cases} a_{11}\, x_1 + a_{12}\, x_2 + \cdots + a_{1n}\, x_n = b_1 \\ a_{21}\, x_1 + a_{22}\, x_2 + \cdots + a_{2n}\, x_n = b_2 \\ \vdots \\ a_{n1}\, x_1 + a_{n2}\, x_2 + \cdots + a_{nn}\, x_n = b_n. \end{cases}$$

We can write this system in the matrix form

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{3n} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}}_{\mathbf{b}}, \tag{3.1}$$

or more compactly as a *matrix equation* as

$$A\,\mathbf{x} = \mathbf{b}.$$

The equation is *homogeneous* if all $b_i = 0$, otherwise *inhomogeneous*.
A is the so-called *coefficient matrix*.
The matrix

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{3n} & b_n \end{array}\right), \tag{3.2}$$

denoted $A \mid \mathbf{b}$ or $[A\mathbf{b}]$ is called the *augmented coefficient matrix*.
We can also write the system as a *vector equation* as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b} \iff \sum_{j=1}^{n} x_j\, \mathbf{a}_j = \mathbf{b},$$

where $\mathbf{a}_j$ is the column vector

$$\mathbf{a}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix}.$$

• **Exercise:** The prices of a bare computer and of its operating system make together CFA 150,000. The computer is 9 times costly than the OS.

1. Formulate this problem as a set of linear equations.
2. Write it down into the matrix form.
3. Deduce the coefficient matrix and the augmented coefficient matrix.

Note that you might obtain different but equivalent answers.

1.2 Reduced row echelon form

A matrix is in the *reduced row echelon form* (RREF) when it satisfies the following conditions:
(i) it is in the row echelon form.
(ii) all leading coefficients are 1.
(iii) each column containing a leading 1 has zeros everywhere.

Example: The following matrice are in the reduced row echelon form:

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\quad \begin{bmatrix}1&0&-1&0\\0&1&5&0\\0&0&0&1\end{bmatrix},\quad \begin{bmatrix}1&0&-1&0&2\\0&1&5&0&9\\0&0&0&1&3\end{bmatrix}$$

RREF vs REF: In the rref,
- the leading coefficients are 1 (they can be different from 1 in the ref)
- in a column with a leading 1, all other entries are zero (only entries below the leading coefficient are zero in a ref).

2 Inverse matrix method

This method is used for nonsingular quadratic systems. Such systems have square and invertible matrices (they have nonzero determinant). Suppose we want to solve

$$A\mathbf{x} = \mathbf{b}.$$

Since A is invertible, there exists a matrix A^{-1} such that

$$A^{-1}A = AA^{-1} = I.$$

From where we can write

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

It follows that the solution vector can be found by simple multiplication of the inverse matrix by the vector of free terms in the second phase of the method. The determination of the inverse matrix constitutes an essential and most difficult problem to be solved in the first stage.

Solved exercise. Solve the following system of equations using the method of inverse matrix.

$$\begin{bmatrix}1&-2&3\\-1&1&2\\2&-1&-1\end{bmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}12\\8\\4\end{pmatrix}$$

Phase 1: compute the inverse matrix. The determinant is

$$\begin{vmatrix}1&-2&3\\-1&1&2\\2&-1&-1\end{vmatrix} = 1(-1+2)-(-2)(1-4)+3(1-2) = -8.$$

The comatrix and its transpose are

$$\begin{bmatrix}1&+3&-1\\-5&-7&-3\\-7&-5&-1\end{bmatrix} \Longrightarrow \begin{bmatrix}1&-5&-7\\3&-7&-5\\-1&-3&-1\end{bmatrix}.$$

Hence the inverse matrix is

$$A^{-1} = \frac{1}{8}\begin{bmatrix}-1&5&7\\-3&7&5\\1&3&1\end{bmatrix}.$$

Phase 2: Matrix multiplication

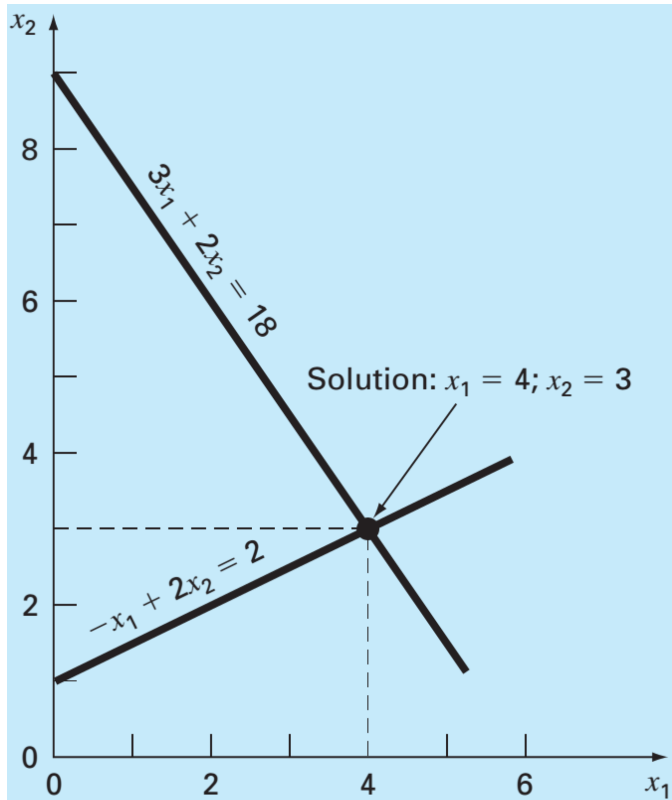
$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{8}\begin{bmatrix}-1&5&7\\-3&7&5\\1&3&1\end{bmatrix}\begin{pmatrix}12\\8\\4\end{pmatrix} = \frac{1}{8}\begin{pmatrix}56\\40\\40\end{pmatrix} \Longrightarrow \begin{cases}x_1 = 7\\x_2 = 5\\x_3 = 5\end{cases}.$$

3 Solving a small set of equations: the Cramer's rule

- **Exercise.** Solve the system

$$3x_1 + 2x_2 = 18$$
$$-x_1 + 2x_2 = 2$$

3.1 The graphical method



This method is generally used for a set of 2 equations. Because we are dealing with linear systems, each equation is a straight line. Hence the two equations by plotting them on Cartesian coordinates with one axis corresponding to x_1 and the other to x_2. The solution is obtained as the intersection of the two lines.

- Single solution $\Longrightarrow$ the two lines intersects.
- No solution $\Longrightarrow$ the two lines are parallel (and not merged).
- Infinitely many solutions $\Longrightarrow$ the two lines are merged (they coincide).

3.2 Substitution or unknown elimination method

The elimination of unknowns is an algebraic approach which is done by combining equations in such a way that one of the unknowns will be eliminated when two equations are combined.

For the above system, we have

$$\begin{aligned}
-x_1 + 2x_2 = 2 &\Longrightarrow x_1 = 2x_2 - 2 \\
3x_1 + 2x_2 = 18 &\Longrightarrow 3(2x_2 - 2) + 2x_2 = 18 \\
&\Longrightarrow 8x_2 = 24 \\
&\Longrightarrow x_2 = 3 \\
&\Longrightarrow x_1 = 2(3) - 2 = 4.
\end{aligned}$$

3.3 Cramer's rule

For solving the above system

$$3x_1 + 2x_2 = 18$$
$$-x_1 + 2x_2 = 2,$$

we consider the following determinants

$$D = \begin{vmatrix} 3 & 2 \\ -1 & 2 \end{vmatrix} = 8, \quad D_1 = \begin{vmatrix} 18 & 2 \\ 2 & 2 \end{vmatrix} = 32, \quad D_2 = \begin{vmatrix} 3 & 18 \\ -1 & 2 \end{vmatrix} = 24.$$

D is the determinant of the system. We define the other determinants, D_1 and D_2, by replacing columns 1 and 2, respectively, by the constants in the rhs of the system of equations. Then Cramer's rule states that the solution should reads

$$x_1 \equiv \frac{D_1}{D} = \frac{32}{8} = 4, \quad x_2 \equiv \frac{D_2}{D} = \frac{24}{8} = 3.$$

Cramer's rule is a determinant-based method for solving a matrix equation of n linear equations $A\mathbf{x} = \mathbf{b}$. It is used when $\det A \neq 0$. Cramer's rule is not very suitable for numerical simulations as it involves the computation of determinants which is very time consuming at higher order. It is best suited to small numbers of equations.

Cramer's rule states that each unknown in a system of linear algebraic equations may be expressed as a fraction of two determinants with the denominator being $\det A$ and with the numerator obtained from $\det A$ by replacing the column of coefficients of the unknown in question by the constants in $\mathbf{b}$.

We have

$$x_i = \frac{D_i}{D}, \text{ e.g., } x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} & \cdots & a_{1n} \\ a_{21} & b_2 & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & b_n & a_{n3} & \cdots & a_{nn} \end{vmatrix}}{|A|}$$

- Single solution $\Longrightarrow D \neq 0$.
- No solution $\Longrightarrow D = 0$ and at least one $D_i \neq 0$. Note that for a 2×2 system, it suffices to have $D = 0$ and $\mathbf{b} \neq \mathbf{0}$.
- Infinitely many solutions $\Longrightarrow D = 0$ and at least one $D_i = 0$ (and no $D_i \neq 0$). Note that when $\mathbf{b} = \mathbf{0}$, all $D_i = 0$.

Example 1: Using Cramer's rule, solve the following set of equations

$$\begin{cases} x - y - 2z = -6 \\ x - 2y + z = 4 \\ 2x - y - z = -16 \end{cases}$$

Example 2: Using Cramer's rule, find the values of α for which each of the systems has (a) a single solution, (b) no solution, (c) infinitely many solutions.

$$\begin{cases} x + \alpha y = 0 \\ \alpha x + y = 0 \end{cases}, \quad \begin{cases} x + \alpha y = 1 \\ \alpha x + y = 1 \end{cases}, \quad \begin{cases} x + \alpha y = 0 \\ \alpha x + y = 1 \end{cases}, \quad \begin{cases} x + \alpha y = 1 \\ \alpha x + y = 2 \end{cases}$$

4 Elimination methods

Gaussian elimination and Gauss-Jordan elimination methods use elementary row operations to put the system matrix into a final form that allows solving the system. For **Gaussian elimination**, the end matrix has the *row echelon form* (generally upper triangular), and the solution is obtained via regressive (or back) substitution. As for the **Gauss-Jordan elimination**, it turns the end matrix into a *reduced row echelon form* (generally the identity matrix), and the solutions are obtained directly.

4.1 Gaussian elimination

By elementary row operations, the system is transformed into echelon form. From that form the unknowns are determined by backward substitution. Of these there are two groups:

1. Basic or leading variables: corresponding to pivots (or to leading coefficients in the ref or leading ones in the rref).
2. Free variables: the others.

Note: The matrix obtained in the end of Gaussian elimination is in the row echelon form. Therefore, Gaussian elimination is used for putting a matrix into ref, i.e., $[A|b] \sim [A'|b']$, where $A' = REF(A)$. If A is square, then $A' \equiv U$, an upper triangular matrix. Leading and free variables can be easily seen after putting the matrix into the ref.

Example 1: Using Gaussian elimination, solve the following set of equations

$$\begin{cases} x-y-2z+u=0 \\ x-2y+z-u=-2 \\ 2x-y-z+3u=2 \end{cases} \Longrightarrow \begin{cases} x-y-2z+u=0 \\ -y+3z-2u=-2 \\ y+3z+u=2 \end{cases} \Longrightarrow \begin{cases} x-y-2z+u=0 \\ -y+3z-2u=-2 \\ 6z-u=0 \end{cases} \Longrightarrow \begin{cases} z=u/6 \\ y=2-3u/2 \\ x=2-13u/6 \end{cases}$$

(x, y, z are basic variables, u is a free variable)

Example 2. Using Gaussian elimination, obtain the row echelon form of the matrix

$$M = \begin{bmatrix} 3 & 2 & -1 \\ 6 & -6 & 7 \\ 3 & -4 & 4 \end{bmatrix}.$$

Note that the row echelon form of a matrix is not unique. A Gaussian elimination where no zero pivot is expected to exist is sometimes referred to as the *naive Gaussian elimination.*

• Gaussian elimination with row permutation.

Whenever a zero pivot occurs in the course of Gaussian elimination, a row exchange (permutation) is necessary to complete the elimination. Rows exchange do not change the solution of a linear system of equations. Gaussian elimination

Example. Using Gaussian elimination, solve the matrix equation $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 1 \\ -3 & 0 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix}.$$

4.2 Gauss-Jordan elimination

By elementary row operations, the system is transformed into reduced row echelon form. For this we have to take forward the Gaussian row elimination procedure so that all the pivots are one, and all the entries above and below the pivots are eliminated. From the rref, the unknowns are determined directed from the reduced vector of free terms.

Note: The matrix obtained in the end of Gauss-Jordan elimination is in the reduced row echelon form, i.e., $[A|b] \sim [A'|b']$, where $A' = RREF(A)$. If A is square, then $A' \equiv I$, the identity matrix, and $b' = x$, the solution vector. Therefore, Gauss-Jordan elimination is a method for putting a matrix into rref.

Example: Using Gauss-Jordan elimination, solve the following set of equations

$$\begin{cases} x-y-2z=-6 \\ x-2y+z=4 \\ 2x-y-z=-16 \end{cases}$$

Using Gauss-Jordan reduction to bring an invertible matrix A into reduced row echelon form, which the identity matrix, we can compute the matrix inverse. Elementary rows operations are applied on an augmented matrix as follows

$$[AI] \sim [IB].$$

The resulting matrix B is the inverse of A.

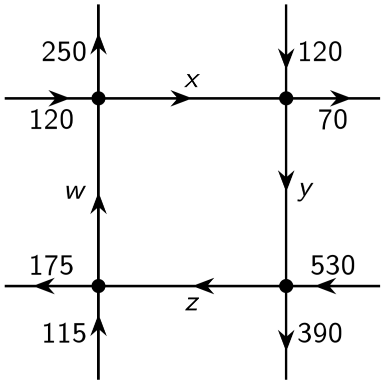
Exercise 1. Put the following matrices into reduced row echelon form and state which columns are pivot columns:

$$A = \begin{bmatrix} 3 & -7 & -2 & -7 \\ -3 & 5 & 1 & 5 \\ 6 & -4 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 1 \\ 3 & 6 & 2 \end{bmatrix}$$

Exercise 2. Using Gauss-Jordan reduction, compute the inverse of

$$C = \begin{bmatrix} 3 & -7 & 2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix}.$$

Exercise 3. *Civil Engineering:* Consider the following network of one-way roads, see graph. The traffic flow is expressed in cars/hour. How much traffic flows through the four labeled segments?



5 Matrix decompositions

Decompostition methods for solving linear systems of equations are sometimes called *factorization* or *triangularization methods*.

5.1 LU decomposition and application

a) Overview

This method is useful when solving multiple systems of equations with the same coefficient matrix A, but different right-hand-side constants b_i. Unlike Gaussian elimination, LU decomposition methods separates the time-consuming elimination of the matrix A from the manipulations of the right-hand side $\mathbf{b}$. Hence, once A has been decomposed, multiple right-hand-side vectors can be evaluated efficiently. The method has two stages:

- The *decomposition stage* requires to write the matrix A as a product of lower and upper triangular matrices such that $A = LU$, where

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}, \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}.$$

- The *substitution stage* is a two-step determination of the solution $\mathbf{x}$ for arbitrary $\mathbf{b}$'s as follows:

$$A\mathbf{x} = \mathbf{b} \Longleftrightarrow \begin{cases} L\mathbf{y} = \mathbf{b}, \\ U\mathbf{x} = \mathbf{y} \end{cases}$$

In a first step, we solve the first equation by forward substitution to get the intermediate solution $\mathbf{y}$. The second step consists in using $\mathbf{y}$ into the second equation, and apply backward substitution to get the solution $\mathbf{x}$.

b) With nonzero pivots

This is the version of LU decomposition that uses Gaussian elimination. When pivots used during Gauss elimination are all nonzero, the coefficient matrix A can be straightforwardly written as a product of matrices L and U, where U is the end matrix of the elimination phase, and L the matrix of elimination factors with ones in the diagonal. For a 3-by-3 matrix, that reads

$$A = LU, \text{ with } U = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 \\ f_{21} & 1 & 0 \\ f_{31} & f_{32} & 1 \end{bmatrix}.$$

Proof:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \underbrace{a_{21} - f_{21}a_{11}}_{0} & \underbrace{a_{22} - f_{21}a_{12}}_{a'_{22}} & \underbrace{a_{23} - f_{21}a_{13}}_{a'_{23}} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \equiv A' \Longrightarrow A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ f_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{L_1} A'.$$

$$A' \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ \underbrace{a_{31} - f_{31}a_{11}}_{0} & \underbrace{a_{32} - f_{31}a_{12}}_{a'_{32}} & \underbrace{a_{33} - f_{31}a_{13}}_{a'_{33}} \end{bmatrix} \equiv A'' \Longrightarrow A' = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ f_{31} & 0 & 1 \end{bmatrix}}_{L_2} A'' \Longrightarrow A = L_1 L_2 A''.$$

$$A'' \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & \underbrace{a'_{32} - f_{31}a_{12}}_{0} & \underbrace{a'_{33} - f_{31}a_{13}}_{a''_{33}} \end{bmatrix} \equiv A''' = U \Longrightarrow A'' = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & f_{32} & 1 \end{bmatrix}}_{L_2} A''' \Longrightarrow A = L_1 L_2 L_3 U.$$

Hence

$$L = L_1 L_2 L_3 = \begin{bmatrix} 1 & 0 & 0 \\ f_{21} & 1 & 0 \\ f_{31} & f_{32} & 1 \end{bmatrix}.$$

Note that inverting L simply changes the signs of the f_{ij} entries.

Example: Reconsider the system below, and solve using LU decomposition.

$$\begin{aligned} 0.3i_1 + 0.52i_2 + i_3 &= -0.01 \\ 0.5i_1 + i_2 + 1.9i_3 &= 0.67 \\ 0.1i_1 + 0.3i_2 + 0.5i_3 &= -0.44 \end{aligned}$$

Decomposition stage: Gaussian elimination led to the end matrix

$$U = \begin{bmatrix} 0.3 & 0.52 & 1 \\ 0 & 0.4/3 & 0.7/3 \\ 0 & 0 & -0.055 \end{bmatrix} \tag{3.3}$$

with elimination coefficients $f_{21} = +\frac{0.5}{0.3}$, $f_{31} = +\frac{0.1}{0.3}$, $f_{32} = +\frac{0.38}{0.4}$. Hence

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 5/3 & 1 & 0 \\ 1/3 & 0.95 & 1 \end{bmatrix} \tag{3.4}$$

Forward substitution:

$$L\mathbf{y} = \mathbf{b} \Longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 5/3 & 1 & 0 \\ 1/3 & 0.95 & 1 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -0.01 \\ 0.67 \\ -0.44 \end{pmatrix} \tag{3.5}$$

$$y_1 = -0.01, \quad (5/3)(-0.01) + y_2 = 0.67 \Longrightarrow y_2 = 2.06/3.$$

$$(1/3)(-0.01) + 0.95(2.06/3) + y_3 = -0.44 \Longrightarrow y_3 = -1.089.$$

Back substitution:

$$U\mathbf{x} = \mathbf{y} \Longrightarrow \begin{bmatrix} 0.3 & 0.52 & 1 \\ 0 & 0.4/3 & 0.7/3 \\ 0 & 0 & -0.055 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -0.01 \\ 2.06/3 \\ -1.089 \end{pmatrix} \tag{3.6}$$

$$\begin{aligned} x_3 &= 1.089/0.055 = 19.8, \\ (0.4/3)x_2 + (0.7/3)(19.8) &= 2.06/3 \Longrightarrow x_2 = -29.5, \\ 0.3x_1 + 0.52(-29.5) + 19.8 &= -0.01 \Longrightarrow x_1 = -14.9. \end{aligned}$$

Note: That particular LU decomposition where L has 1's on the diagonal is formally referred to as a *Doolittle decomposition*. There exists an alternative approach which involves a U matrix with 1's on the diagonal; it is called *Crout decomposition*.

c) With at least one zero pivot

This version of LU decomposition is sometimes called the **PALU** decomposition. When a zero pivot occurs in the course of Gaussian elimination, then row exchanges (permutations) are necessary to complete the elimination. Indeed, even if the pivot is not exactly zero, a very small value can leads to big roundoff errors in computers. For very big matrices, one can easily lose all accuracy in the solution. Then row permutations allows us to place the largest element (in absolute value) in the pivot.

In that case, the coefficient matrix A can be factorized in such a way that

$$PA = LU,$$

where P is a permutation matrix. We denote by P_{nm} the permutation matrix that interchanges the nth and mth rows or columns of an arbitrary matrix when acted on. P_{nm} is obtained from the identity matrix I by permuting the nth and mth rows or columns of I.

Example: The matrix

$$P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ obtained as } I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \equiv P_{12}$$

permutes the 1st and 2nd rows or columns of a 3-by-3 matrix.

When acted from the left on A, the matrix P_{nm} permutes the nth and mth *rows* of A. Otherwise, P_{nm} permutes the nth and mth *columns* of A (i.e., when it multiplies A from the right). We have

$$P_{12}A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$AP_{12} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Note that permutation matrices are their own inverses.

Solved exercise. Using PALU decomposition, solve the system with coefficient matrix

$$A = \begin{bmatrix} -2 & 2 & -1 \\ 6 & -6 & 7 \\ 3 & -8 & 4 \end{bmatrix}.$$

• Step 1: Since $6 > 2$ (in column 1) we exchange rows 1 and 2 before performing the first elimination.

$$A = \begin{bmatrix} -2 & 2 & -1 \\ 6 & -6 & 7 \\ 3 & -8 & 4 \end{bmatrix} \sim A' = \begin{bmatrix} 6 & -6 & 7 \\ -2 & 2 & -1 \\ 3 & -8 & 4 \end{bmatrix} \Longrightarrow A' = P_{12}A, \text{ with } P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The elimination is carried out with A'.

$$A' = \begin{bmatrix} 6 & -6 & 7 \\ -2 & 2 & -1 \\ 3 & -8 & 4 \end{bmatrix} \sim A'' = \begin{bmatrix} 6 & -6 & 7 \\ 0 & 0 & 4/3 \\ 0 & -5 & 1/2 \end{bmatrix} \Longrightarrow A' = L_1 A'', \text{ with } L_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1/3 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix}.$$

• Step 2: Since $5 > 0$ (in the subsequent part of column 2) we exchange rows 2 and 3 before pursuing the elimination process.

$$A'' = \begin{bmatrix} 6 & -6 & 7 \\ 0 & 0 & 4/3 \\ 0 & -5 & 1/2 \end{bmatrix} \sim A''' = \begin{bmatrix} 6 & -6 & 7 \\ 0 & -5 & 1/2 \\ 0 & 0 & 4/3 \end{bmatrix} \Longrightarrow A''' = P_{23}A'', \text{ with } P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

No further elimination will be carried out because A''' accidentally turns out to be upper triangular $A''' = U$.

Hence

$$\begin{array}{rcl|rcl} A''' = P_{23}A'' \Longleftrightarrow U & = & P_{23}A'' & U & = & P_{23}L_1^{-1}P_{12}A \\ & = & P_{23}L_1^{-1}A' & & = & P_{23}L_1^{-1}\underbrace{P_{23}P_{23}}_{I}P_{12}A \\ & = & P_{23}L_1^{-1}P_{12}A & \Longleftrightarrow \left(P_{23}L_1^{-1}P_{23}\right)^{-1}U & = & P_{23}P_{12}A \end{array}$$

Let $P = P_{23}P_{12}$ and $L = \left(P_{23}L_1^{-1}P_{23}\right)^{-1}$. Then we have

$$PA = LU.$$

L can be obtained by exchanging both the 2nd and 3rd rows and columns of L_1^{-1} and inverting the result:

$$L_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ +1/3 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix} \Longrightarrow L_1^{-1}P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ +1/3 & 0 & 1 \\ -1/2 & 1 & 0 \end{bmatrix} \Longrightarrow P_{23}L_1^{-1}P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ +1/3 & 0 & 1 \end{bmatrix}.$$

The net operation on L_1^{-1} is an interchange of the nondiagonal elements $-1/2$ and $1/3$. Finally

$$L = \begin{bmatrix} 1 & 0 & 0 \\ +1/2 & 1 & 0 \\ -1/3 & 0 & 1 \end{bmatrix}.$$

Obviously, we could have written $L = \left(P_{23}L_1^{-1}P_{23}\right)^{-1} = P_{23}L_1P_{23}$ since P_{23} is its own inverse.

5.2 Cholesky method

Cholesky decomposition (or square root method) is an LU decomposition for symmetric matrices ($A^T = A$). The symmetric matrix is decomposed as

$$A = LL^T.$$

The L matrix entries can be calculated as

$$l_{ki} = \frac{a_{ki} - \sum_{j=1}^{i-1} l_{ij}l_{kj}}{l_{ii}}, \text{ for } i = 1, 2, \cdots, k-1$$

and thus, for diagonal elements

$$l_{kk} = \sqrt{a_{kk} - \sum_{j=1}^{k-1} l_{kj}^2}, \text{ for } k = 1, 2, \cdots, n$$

Once the Cholesky decomposition has been done, the solution can be obtained like in the LU decomposition.

In general, this method may lead to an execution error if diagonal entries of A are negative. However, the Cholesky algorithm find wide application because many symmetric matrices encountered in engineering are, in fact, positive definite. Note that a matrix A is *positive definite* when $\mathbf{x}^T A\mathbf{x}$ is greater than zero for all nonzero vector $\mathbf{x}$.

Example Perform a Cholesky decomposition of the following symmetric system by hand,

$$A = \begin{bmatrix} 4 & 10 & 20 \\ 10 & 41 & 74 \\ 20 & 74 & 145 \end{bmatrix}.$$

The matrix is symmetric, then we have

$$\begin{aligned}
l_{11} &= \sqrt{a_{11}} \\
l_{21} &= \frac{a_{21}}{l_{11}}, \ l_{22} = \sqrt{a_{22} - l_{21}^2} \\
l_{31} &= \frac{a_{31}}{l_{11}}, \ l_{32} = \frac{a_{32} - l_{21}l_{31}}{l_{22}}, \ l_{33} = \sqrt{a_{33} - (l_{31}^2 + l_{32}^2)}.
\end{aligned}$$

From where

$$\begin{aligned}
l_{11} &= 2 \\
l_{21} &= \frac{10}{2} = 5, \ l_{22} = \sqrt{41 - 5^2} = 4 \\
l_{31} &= \frac{20}{2} = 10, \ l_{32} = \frac{74 - (5)(10)}{4} = 6, \ l_{33} = \sqrt{145 - (10^2 + 6^2)} = 3.
\end{aligned}$$

Hence $A = L\,L^T$, with

$$L = \begin{bmatrix} 2 & 0 & 0 \\ 5 & 4 & 0 \\ 10 & 6 & 3 \end{bmatrix} \text{ i.e., } A = \begin{bmatrix} 2 & 0 & 0 \\ 5 & 4 & 0 \\ 10 & 6 & 3 \end{bmatrix} \begin{bmatrix} 2 & 5 & 10 \\ 0 & 4 & 6 \\ 0 & 0 & 3 \end{bmatrix}.$$

Note: Suppose $A = LU$. Taking $A = A^T \implies LU = U^T L^T$, whose solution is $U = L^T$ or $L = U^T$.