Differential Calculus in $\mathbb{R}^n$

Continuity in $\mathbb{R}^n$

Definition Continuity: $U \subseteq \mathbb{R}^n$, $f : U \to \mathbb{R}^m$.

1. $x_0 \in U$. f continuous at x_0 if $\forall \epsilon > 0$, $\exists \delta > 0, \forall x \in U$:
$$\|x - x_0\| < \delta \Rightarrow \|f(x) - f(x_0)\| < \epsilon$$
2. f continuous on U if cont. $\forall x_0 \in U$

Definition Convergence: $(x_k)_{k\in\mathbb{N}}, x_k \in \mathbb{R}^n$.
$$x_k = (x_{k,1}, ..., x_{k,n})$$
$y = (y_1, ..., y_n) \in \mathbb{R}^n$. (x_k) converges to y $(k \to +\infty)$ if $\forall \epsilon > 0, \exists N \geq 1, \forall n \geq N$
$$\|x_k - y\| < \epsilon$$
we usually write $\lim_{n\to\infty} x_n = y$.
Lemma: (x_k) converges to $y \iff$

- $\forall i, 1 \leq i \leq n$: $(x_{k,i})$ converges to y_i.
- $\|x_k - y\|$ converges to 0 $(k \to +\infty)$.

To disprove continuity:
Lemma: $U \subset \mathbb{R}^n$, $f : U \to \mathbb{R}^m$. $x_0 \in U$. f continuous at x_0 $\Leftrightarrow$ $\forall (x_k)_{k\geq 1}$ in U with $\lim_{k\to\infty} x_k = x_0$, $(f(x_k))_{k\geq 1}$ in $\mathbb{R}^m$ converges to $f(x_0)$ $(\lim_{x\to\infty} f(x_k) = f(x_0))$
Lemma: $U \subseteq \mathbb{R}^n, V \subseteq \mathbb{R}^m$, $1 \leq p \in \mathbb{N}$. $f : U \to V, g : V \to \mathbb{R}^p$ continuous $\Rightarrow g \circ f$ continuous.

Proof. Composition with last Lemma. □

- $\lim_{x\to x_0}(f(x) + g(x)) = \lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x)$
- $\lim f(x)g(x) = \lim f(x) \cdot \lim g(x)$

Definition Function Convergence: $U \subseteq \mathbb{R}^n$, $f : U \to \mathbb{R}^m$. $x_0 \in U$, $y \in \mathbb{R}^m$. f has limit y $(x \to x_0, x \neq x_0)$ if $\forall \epsilon > 0, \exists \delta > 0$, $\forall x \in U, x \neq x_0$
$$\|x - x_0\| < \delta \Rightarrow \|f(x) - y\| < \epsilon$$
We write $\lim_{\substack{x\to x_0\\x\neq x_0}} f(x) = y$. $f(x_0)$ has no impact on $\lim_{\substack{x\to x_0\\x\neq x_0}} f(x)$.

Lemma Continuity & Convergence: $U \subseteq \mathbb{R}^n$, $f : U \to \mathbb{R}^m$. $x_0 \in U$. f continuous at $x_0 \Leftrightarrow \lim_{\substack{x\to x_0\\x\neq x_0}} f(x) = f(x_0)$
Lemma Function & Sequence Convergence: $U \subset \mathbb{R}^n$, $f : U \to \mathbb{R}^m$. $x_0 \in U$, $y \in \mathbb{R}^m$. $\lim_{\substack{x\to x_0\\x\neq x_0}} f(x) = y \Leftrightarrow \forall (x_k) \to x_0$ in U, $x_k \neq x_0$, $(f(x_k))$ in $\mathbb{R}^m$ converges to y.

Disprove continuity/convergence with sequence. Prove directly and using comparison if difficult.

Intervals

Definition:

1. $U \subset \mathbb{R}^n$ **bounded** if $\|x\|$, $x \in U$ is bounded in $\mathbb{R}$.
2. $U \subset \mathbb{R}^n$ **closed** if $\forall (x_k)$ in U $(\lim_{k\to\infty} x_k = y \in \mathbb{R}^n)$: $y \in X$
3. $U \subset \mathbb{R}^n$ **compact** if bounded and closed
4. $U \subset \mathbb{R}^n$ **open** if $\forall x = (x_1, ..., x_n) \in U$, $\exists \delta > 0$ such that
$$\{y = (y_1, ..., y_n) \in \mathbb{R}^n \mid |x_i - y_i| < \delta\} \subseteq U$$

Lemma: $U \subset \mathbb{R}^n$ open $\Leftrightarrow$ cplmnt. closed.
$\mathbb{R}$ & $\varnothing$ closed & open
Lemma: $f : \mathbb{R}^n \to \mathbb{R}^m$ continuous. Then:

1. $U \subseteq \mathbb{R}^m$ open $\Rightarrow f^{-1}(U) \subseteq \mathbb{R}^n$ open
2. $U \subseteq \mathbb{R}^m$ closed $\Rightarrow f^{-1}(U) \subseteq \mathbb{R}^n$ closed

Proof. U closed. $(x_i) \to y \in \mathbb{R}^n$ in $f^{-1}(U)$. Continuity of f: $f(y) = f(\lim_{k\to\infty} x_k) = \lim_{k\to\infty} f(x_k) \overset{U \text{ closed}}{\Rightarrow} \lim_{k\to\infty} f(x_k) \in U \Rightarrow f(y) \in U \Rightarrow y \in f^{-1}(U)$.
$(1) \Leftarrow (2)$ □

Considering images preserves compactness:
Theorem: $U \subset \mathbb{R}^n$ non-empty compact set, $f : U \to \mathbb{R}$ continuous $\Rightarrow f$ bounded and achieves maximum&minimum/$\exists x_+, x_- \in U$
$$f(x_+) = \sup_{x\in U} f(x) \qquad f(x_-) = \inf_{x\in U} f(x)$$

Partial Derivatives

Definition Partial Derivatives: $U \subseteq \mathbb{R}^n$ open set. $f : U \to \mathbb{R}^m$. $1 \leq i \leq n$. f has partial derivative on U with respect to i-th variable at $x_0 = (x_{0,1}, ..., x_{0,n}) \in U$ if $g(t) = f(x_{0,1}, ..., x_{0,i-1}, t, x_{0,i+1}, ..., x_{0,n})$ defined on $I = \{t \in \mathbb{R} | (x_{0,1}, ..., x_{0,i-1}, t, x_{0,i+1}, ..., x_{0,n}) \in U\}$ is differentiable at $t = x_{0,i}$ $(f_1, f_2, ..., f_m$ component-wise). Derivative at $x_{0,i}$ $g'(x_{0,i})$ is denoted
$$\frac{\partial f}{\partial x_i}(x_0) = \partial_{x_i} f(x_0) = \partial_i f(x_0)$$
f has partial derivative on U with respect to i-th variable, if differentiable on U with respect to the i-th variable $\forall$ valid $x_0 \in U$.
$\forall x \in U, \exists \partial_{x_i} f \Rightarrow \partial_{x_i} f(x) : U \to \mathbb{R}^m \Rightarrow \exists$:
$$\partial_{x_j}\partial_{x_i} f = \partial_{ji} f = \frac{\partial^2 f}{\partial_{x_j}\partial_{x_i}} = f_{x_j x_i}$$
Lemma ∂ Rules: $U \subseteq \mathbb{R}^n$ open, $f, g : U \to \mathbb{R}^n$, $1 \leq i \leq n$, f, g have i-th ∂

- $\Rightarrow \partial_i(f + g) = \partial_i f + \partial_i g$
- $m = 1 \Rightarrow \partial_i(fg) = \partial_i(f)g + f\partial_i(g)$
- $\forall x \in U, g(x) \neq 0 \Rightarrow \partial_i(\frac{f}{g}) = \frac{\partial_i(f)g - f\partial_i(g)}{g^2}$

Definition Jacobian: $U \subseteq \mathbb{R}^n, f : U \to \mathbb{R}^m$ has partial derivatives on U at $x \in U$:
$$J_f(x) = (\partial_{x_j} f_i(x))_{\substack{1\leq i\leq m\\1\leq j\leq n}}$$
is the **Jacobi matrix** of f at x.
Definition Gradient: $U \subseteq \mathbb{R}^n$ open, $f : U \to \mathbb{R}$, all partial derivatives exist at $x_0 \in U$:
$$\nabla f(x_0) = \begin{pmatrix} \partial_{x_1} f(x_0) \\ \vdots \\ \partial_{x_n} f(x_0) \end{pmatrix} = \text{grad} f(x_0) = J_f(x_0)^T$$
is the gradient of f at x_0.
Definition Divergence: $U \subseteq \mathbb{R}^n$, $f : U \to \mathbb{R}^n$, all ∂ exist at $x_0 \in U$. Divergence of f at x_0
$$Tr(J_f(x_0)) = \sum_{i=1}^{n} \partial_{x_i} f_i(x_0) = \text{div} f(x_0)$$

The Differential

Definition Differential: $U \subseteq \mathbb{R}^n$ open, $f : U \to \mathbb{R}^m$. $u : \mathbb{R}^n \to \mathbb{R}^m$ linear map, $x_0 \in U$. f differentiable at x_0 with differential u if
$$\lim_{\substack{x\to x_0\\x\neq x_0}} \frac{1}{\|x - x_0\|}(f(x) - f(x_0) - u(x - x_0)) = 0$$
Denote $df(x_0) = Df(x_0) = u$.
f differentiable at every $x_0 \in U \Rightarrow f$ differentiable on U.
$df(x_0) = A \Leftrightarrow f$ has "Dreiecksentwicklung"/"Dreigliedentwicklung"
$$f(x) = f(x_0) + A(x - x_0) + \sigma(\|x - x_0\|)$$
f differentiable at $x_0 \Leftrightarrow$ all $f_i : U \to \mathbb{R}$ differentiable at x_0.
Use this to show sth. is (not) differentiable:
Lemma: $U \subseteq \mathbb{R}^n$ open, $f : U \to \mathbb{R}^m$. f has all partial derivatives. partial derivatives of f are continuous on $U \Rightarrow f$ differentiable on U.
$df(x_0)$ (canonical basis of $\mathbb{R}^n$, $\mathbb{R}^m$) is Jacobi matrix of f at x_0.
all ∂ continuous $\Rightarrow f$ cont. differentiable
Lemma: $U \subseteq \mathbb{R}^n$ open, $f : U \to \mathbb{R}^m$ differentiable on U.

- f is continuous on U
- f admits partial derivatives on X with respect to each variable.
- $df(x_0)$ in stand. basis is Jacobi matrix

Lemma: $U \subseteq \mathbb{R}^n$ open, $f, g : U \to \mathbb{R}^m$ differentiable on U.

- $f + g$ differentiable: $d(f + g) = df + dg$
- $m = 1 \Rightarrow fg$ is differentiable
- $m = 1$, $g(x) \neq 0, \forall x \in U \Rightarrow f/g$ differentiable.

Lemma Chain Rule: $U \subseteq \mathbb{R}^n$ open, $V \subseteq \mathbb{R}^m$ open, $f : U \to V$ diffbar, $g : V \to \mathbb{R}^p$ diffbar $\Longrightarrow g \circ f : U \to \mathbb{R}^p$ diffbar on U - $\forall x \in U$:
$$d(g \circ f)(x_0) = dg(f(x_0)) \circ df(x_0)$$
$$J_{g\circ f}(x_0) = J_g(f(x_0))J_f(x_0)$$
Lemma Produce Rule: $f, g : \mathbb{R}^n \to \mathbb{R}$
$$\nabla_{fg}(x_0) = \nabla_f(x_0)g(x_0) + \nabla_g(x_0)f(x_0)$$
$h : \mathbb{R}^n \to \mathbb{R}^2, h(x) = (f(x), g(x))$ and $m : \mathbb{R}^2 \to \mathbb{R}, m(u, v) = u \cdot v$
Definition Tangent Space: $U \subseteq \mathbb{R}^n$ open, $f : U \to \mathbb{R}^m$ differentiable. $x_0 \in U$, $u = df(x_0)$.Graph of affine linear approximat.
$$g(x) = f(x_0) + u(x - x_0)$$
$$\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m | y = f(x_0) + u(x - x_0)\}$$
tangent space at x_0 to the graph of f.
Definition Directional Derivative: $U \subseteq \mathbb{R}^n$ open, $f : U \to \mathbb{R}^m$, $0 \neq v \in \mathbb{R}^n$, $x_0 \in U$. f has directional derivative $w \in \mathbb{R}^m$ in direction v, if $g : \{t \in \mathbb{R} | x_0 + tv \in U\} \to \mathbb{R}^m$
$$g(t) = f(x_0 + tv)$$

has derivative w at $t=0$.

$$D_v f(x_0) = J_g(0) = \begin{pmatrix} \partial_t g_1(0) \\ \vdots \\ \partial_t g_m(0) \end{pmatrix} \in \mathbb{R}^m$$

For partial derivatives: $\partial_{x_i} f(x_0) = D_{e_i} f(x_0)$

Lemma: $U \subseteq \mathbb{R}^n$ open, $f: U \to \mathbb{R}^m$ diffbar. $\forall x \in U, 0 \neq v \in \mathbb{R}^n$: $D_v f(x_0) = df(x_0)(v)$. $D_{v+w} f(x_0) = D_v f(x_0) + D_w f(x_0)$

$h: \mathbb{R}^n \to \mathbb{R}^m$ cont. diffbar. $\Leftrightarrow$ all directional derivatives exist and cont..

Higher Derivatives

Definition: $U \subseteq \mathbb{R}^n$ open, $f: U \to \mathbb{R}^m$.

- f is of class C^1 if f is diffbar and all its ∂s cont. (continuously differentiable). $C^1(U, R^m)$: C^1 functions from U to $\mathbb{R}^m$.

- $k \geq 2$. f is of class C^k if f is diffbar and all $\partial_{x_i} f: U \to \mathbb{R}^m$ are of class C^{k-1}. $C^k(U, \mathbb{R}^m)$: C^k functions from U to $\mathbb{R}^m$.

- $f \in C^k(U, \mathbb{R}^m)$, $\forall k \geq 1 \Rightarrow f$ is of class C^∞. $C^\infty(U, \mathbb{R}^m)$: C^∞ functiosn from U to $\mathbb{R}^m$.

- $C^0 \supseteq C^k \supseteq C^\infty$.

C^k closed under +/·/concatentionation → vector space

Definition Multi-dim. Polynomials: Monomial: $\lambda x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}, \lambda \in \mathbb{R}, a_i \in \{0,1,2,\ldots\}$. Degree of monomial: $a_1 + a_2 + \ldots + a_m$.

Polynomial: Sum of Monomials

Definition Hessian: $U \subseteq \mathbb{R}^n$ open, $f: U \to \mathbb{R}$ is C^2. $x \in U$: Hessian matrix of f at x:

$$Hess_f(x_0) = H_f(x_0) = (\partial_{x_i}\partial_{x_j} f(x_0))_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}}$$

symmetric and square.

Lemma: $k \geq 2$. $U \subseteq \mathbb{R}^n$ open, $f: U \to \mathbb{R}^m$ is C^k. ∂s order k independent of the order in which the partial derivatives are taken

$$\partial_{x,y} f = \partial_{y,x} f$$

Landau Symbol - little o notation

Definition: $U \subseteq \mathbb{R}^n, g: U \to \mathbb{R}, x_0 \in U$ $\mathrm{o}(g)$: set of $f: U \to \mathbb{R}$ with

$$\lim_{\substack{x \to x_0 \\ x \neq x_0}} \left| \frac{f(x)}{g(x)} \right| = 0$$

$f = o(g) :\Leftrightarrow f \in \mathrm{o}(g)$.

$\mathrm{o}(f) = \mathrm{o}(g) :\Leftrightarrow \mathrm{o}(f) \subseteq \mathrm{o}(g)$.

- $\mathrm{o}(x^a) + \mathrm{o}(x^b) = \mathrm{o}(x^{\min\{a,b\}})$
- $\mathrm{o}(x^a) \cdot \mathrm{o}(x^b) = \mathrm{o}(x^{a+b})$
- $x^a \cdot \mathrm{o}(x^b) = \mathrm{o}(x^{a+b})$
- $P = \mathrm{o}(\|x\|^k)$ if $\deg P > k$
- $\mathrm{o}(P) = \mathrm{o}(\|x\|^k)$ if $\deg P \geq k$
- $P \cdot \mathrm{o}(\|x\|^k) = \mathrm{o}(\|x\|^{k + \deg P})$

Helpful to combine Taylor Polynomials.

Taylor Polynomials

$$f(x) = \sum_{j=0}^{k} \frac{1}{j!} f^{(j)}(x_0)(x-x_0)^i + \sigma(|x-x_0|^k)$$

Definition Taylorpolynomials: $1 \leq k \in \mathbb{N}$. $f: U \to \mathbb{R}$ is C^k. k-th Taylor polynomial of f at $x_0 \in U$ is:

$$T_k(f) = \sum_{\substack{m_1,\ldots,m_n \geq 0 \\ m_1+\ldots+m_n \leq k}} \frac{1}{m_1! m_2! \ldots m_n!}$$

$$\partial_1^{m_1} \partial_2^{m_2} \ldots \partial_n^{m_n} f(x_0) \cdot y_1^{m_1} \ldots y_n^{m_n}$$

(polynomial in n variables of degree $\leq k$)

Abbreviated notation (m representative for $m_1, \ldots, m_n$ and its respective enumerations)

$$T_k f(x) = \sum_{|m| \leq k} \frac{1}{m!} \partial_x^m f(x_0)(x-x_0)^m$$

$T_k f(x)$ only polynomial with $\deg P \leq k$ and

$$\partial_1^{m_1} \ldots \partial_2^{m_2} P(x_0) = \partial_1^{m_1} \ldots \partial_2^{m_2} f(x_0)$$

for all $m_1, \ldots, m_n$ mit $m_1 + \ldots + m_n \leq k$.

Theorem Taylor Approximation: $1 \leq k \in \mathbb{N}$. $U \subset \mathbb{R}^n$ open, $f: U \to \mathbb{R}$ is C^k. $x_0 \in U$:

$$f(x) = T_k f(x_0)(x-x_0) + E_k f(x_0)$$

$$f(x) = T_k f(x)(x-x_0) + \mathrm{o}(\|y\|^k)$$

$T_2 f(x_0)(x-x_0) = f(x_0) + \langle \nabla_f(x_0), y \rangle + \frac{1}{2} y^\top H_f(x_0) y$

Critical Points

Definition Extrema: $U \subseteq \mathbb{R}^n, f: U \to \mathbb{R}$. $x_0 \in U$ is called...

- local minimum if some $\epsilon > 0$ exists so that $\|x - x_0\| < \epsilon, x \in U \Rightarrow f(x_0) \leq f(x)$
- local maximum if some $\epsilon > 0$ exists so that $\|x - x_0\| < \epsilon, x \in U \Rightarrow f(x_0) \geq f(x)$
- local extremum if x_0 is a local minimum or a local maximum.
- global minimum if for all $x \in U$: $f(x_0) \leq f(x)$
- global maximum if for all $x \in U$: $f(x_0) \geq g(x)$
- global extremum if x_0 is a global minimum or a global maximum

Definition Critical Point: $U \subseteq \mathbb{R}^n$ open, $f: U \to \mathbb{R}$ diffbar. $x_0 \in U$ with $\nabla f(x_0) = 0$ is a critical point of f.

Definition Saddle Point: A saddle point is a critical point, which is not a an extremum.

Lemma: $U \subseteq \mathbb{R}^n$ open, $f: U \to \mathbb{R}$ diffbar. $x_0 \in U$ is local extremum $\Rightarrow x_0$ is critical point.

Understand through Taylor Polynomial.

In 1D: 2nd derivative may imply local min/max. In mult. dim.: $y^\top H_f(x_0) y$ as 2nd.

Definition: $m \times n$ matrix A is called

- positiv definite $\Leftrightarrow \forall y \neq 0$: $y^\top A y > 0$
- negative definite $\Leftrightarrow \forall y \neq 0$: $y^\top A y < 0$
- indefinite $\Leftrightarrow \exists y, z, y^\top A y > 0$ & $z^\top A z < 0$

Theorem: $U \subseteq \mathbb{R}^n$ open, $f: U \to \mathbb{R}$ is C^2. x_0 critical point. Then:

- $H_f(x_0)$ positive definite $\Rightarrow x_0$ is a local minimum
- $H_f(x_0)$ negative definite $\Rightarrow x_0$ is a local maximum
- $H_f(x_0)$ indefinite $\Rightarrow x_0$ not a local extremum, but a saddle point

H_f is symmetric $\Rightarrow H_f$ is diagonalizable.

- H_f positive definite $\Leftrightarrow$ all EV are positive
- H_f negative definite $\Leftrightarrow$ all EV are negative
- H_f indefinite $\Leftrightarrow$ at least one positive and at least one negative EV

Definition: $U \subseteq \mathbb{R}^n$ open, $f: U \to \mathbb{R}$ is C^2. Critical point x_0 of f is called

- degenerate if $\det H_f(x_0) = 0$
- non-degenerate if $\det H_f(x_0) \neq 0$

$H_f(x_0) \neq 0$/non-degenerate $\Rightarrow$ p.d./n.d./i.d.

For critical points, even degenerate ones:

- H_f has positive EV $\Rightarrow$ not a local maximum
- H_f has negative EV $\Rightarrow$ not a local minimum

Theorem Sylvesters Criteria: Symmetric $A \in \mathbb{R}^{n,n}$ p.s.$\Leftrightarrow \forall i, 1 \leq i \leq n$: $\det A_{:i,:i} > 0$

Lagrange Multipliers

Goal: Find extrema of function on root root set of other function.

Idea 1: Parameterize root set→new function

Lemma: $U \subseteq \mathbb{R}^n$ open, $f, g: U \to \mathbb{R}$ are C^1. $x_0 \in U$ local extremum of f restricted to $Y = \{x \in U | g(x) = 0\}$/local extremum of $f|_{g^{-1}(0)} \Rightarrow$ either

- $\nabla g(x_0) = 0$, or

- $\exists \lambda \in \mathbb{R}$ such that $\begin{cases} \nabla f(x_0) = \lambda \nabla g(x_0) \\ g(x_0) = 0 \end{cases}$

The Inverse and Implicit Function Theorems

Definition Change of Variable: $U \subseteq \mathbb{R}^n$ open, $f: U \to \mathbb{R}^n$ diffbar vector field. $x_0 \in U$. f is a change of variable around/"lokal invertierbar bei" x_0 if:

$\exists$ open set B, $x_0 \in B$, $f(B) \subseteq \mathbb{R}^n$ open, and $\exists$ diffbar $g: f(B) \to B$ with $f \circ g = id_{f(B)}, g \circ f = id_B$.

Theorem Inverse Function Theorem: $U \subseteq \mathbb{R}^n$ open, $f: U \to \mathbb{R}^n$ diffbar. If $\exists x_0 \in U$ with $\det(J_f(x_0)) \neq 0 \Rightarrow f$ is a change of variable around/"lokal invertierbar bei" x_0.

$$J_g(f(x_0)) = J_f(x_0)^{-1}$$

g local inverse. f is $C^k \Rightarrow g$ is C^k too.

local c.o.v. everywhere $\not\Rightarrow$ globally invertible!

Example: $U = \{(r, \Theta) \in \mathbb{R}^2 | r > 0\}$ open, $f: U \to \mathbb{R}^2$, $f(r, \Theta) = (r\cos\Theta, r\sin\Theta)$. Locally invertible with $g(x,y) \mapsto \left[\sqrt{x^2+y^2}, \arctan(\frac{y}{x})\right]$. Not global

Integration in $\mathbb{R}^n$

Definition: $I = [a, b] \subset \mathbb{R}$ closed, bounded. $f(t) = (f_1(t), ..., f_n(t))$ continuous $I \to \mathbb{R}^n$

$$\int_a^b f(t)dt = \left(\int_a^b f_1(t)dt, ..., \int_a^b f_n(t)dt\right) \in \mathbb{R}^n$$

Line Integrals

$\mathbb{R}^n \to \mathbb{R}^n$ functions. Line integral: integrating functions while walking along path

Definition Parameterized Curve: in $\mathbb{R}^n$ is continuous map $\gamma : [a, b] \to \mathbb{R}^n$: is piecewise C^1 / $\exists k \geq 1$ and partition

$$a = t_0 < t_1 < \cdots < t_{k-1} < t_k = b$$

f on $]t_{j-1}, t_j[$ is $C^1$for $1 \leq j \leq k$.

Definition Line Integral: $\gamma : [a, b] \to \mathbb{R}^n$ parameterized curve. $U \subseteq \mathbb{R}^n$ subset containing image of γ, $f : U \to \mathbb{R}^n$ cont. Line integral of f along γ:

$$\int_\gamma f(s) \cdot ds = \int_a^b f(\gamma(t)) \cdot \gamma'(t)dt \in \mathbb{R}$$

- $\int_a^b f + g\ dt = \int_a^b fdt + \int_a^b gdt$
- $\int_a^b fdt + \int_b^c fdt = \int_a^c fdt$
- $\int_b^a fdt = -\int_a^b fdt$

Definition Oriented Reparameterization: $\gamma : [a, b] \to \mathbb{R}^n$ parameterized curve. Oriented reparameterization ("orientierte Umparameterisierung") of γ: parameterized curve $\sigma : [c, d] \to \mathbb{R}^n$, $\sigma = \gamma \circ \phi$, $\phi : [c, d] \to [a, b]$ cont. map, diffbar on $]c, d[$, strictly increasing, $\phi(c) = a$, $\phi(d) = b$. ϕ is bijective.

Image of o.r.s identical. γ also o.r. of σ.

Lemma: γ parameterized curve in $\mathbb{R}^n$, σ oriented reparameterization. U set containing image of γ/σ, $f : U \to \mathbb{R}^n$ cont. function:

$$\int_\gamma f(s) \cdot ds = \int_\sigma f(s) \cdot ds$$

Proof.

$$\int_\gamma f(s) \cdot ds = \int_a^b f(\gamma(t)) \cdot \gamma'(t)dt$$

substitution with $t = \phi(u)$

$$= \int_{\phi^{-1}(a)}^{\phi^{-1}(b)} f(\gamma(\phi(u))) \cdot \gamma'(\phi(u))\phi'(u)du$$

$$= \int_c^d f(\sigma(u)) \cdot \sigma'(u)dt = \int_\sigma f(s) \cdot ds$$

□

γ, σ param. curve - same but opp. directions

$$\int_\sigma f(s) \cdot d\vec{s} = -\int_\gamma f(s) \cdot d\vec{s}$$

Definition Conservative Vector Field: $U \subseteq \mathbb{R}^n$, $f : U \to \mathbb{R}^n$ cont.: $\forall x_1, x_2 \in U$, γ param. curve from x_1 to x_2 the line integral identical $\Rightarrow$ f is conservative.

Lemma: C^1 $g : U \to \mathbb{R}$, $f = \nabla_g \Rightarrow f$ cons.

Proof.

$$\int_\gamma f(s) \cdot ds = \int_a^b \nabla_g(\gamma(t)) \cdot \gamma'(t)dt$$

$$= \int_a^b dg(\gamma(t))(\gamma'(t))dt = \int_a^b \frac{d}{dt} g(\gamma(t))dt$$

$$= [g(\gamma(t))]_a^b = g(\gamma(b)) - g(\gamma(a))$$

□

Definition Closed Curve: $\gamma(a) = \gamma(b) \Rightarrow \gamma$ closed curve. Write $\oint_\gamma$ for $\int_\gamma$.

Lemma: f cons. $\Leftrightarrow \oint_\gamma f(s) \cdot d\vec{s} = 0$.

Proof. $\Longrightarrow$: trivial

$\Longleftarrow$: $\gamma, \tilde{\gamma} : [a, b] \to \mathbb{R}^n$

$\gamma(a) = \tilde{\gamma}(a), \gamma(b) = \tilde{\gamma}(b)$

$$\sigma(t) = \begin{cases} \gamma(a + (b - a)t), & 0 \leq t \leq 1 \\ \tilde{\gamma}(a + (b - a)(2 - t)), & 1 < t \leq 2 \end{cases}$$

$$\Rightarrow \oint_\sigma f(s)d\vec{s} = \int_\gamma f(s)d\vec{s} - \int_{\tilde{\gamma}} f(s)d\vec{s} = 0$$

□

Cons. vector fields are combinable.

Definition path-connected: $U \subseteq \mathbb{R}^n$ is "wegzusammenhängend"/path-connected $\Leftrightarrow$ $\forall x, y \in U, \exists$ curve γ from x to y, $Image(\gamma) \subseteq U$.

Theorem: $U \subseteq \mathbb{R}^n$ open, f cons. v.f. $\Rightarrow \exists C^1$ function g on U, $f = \nabla_g$.

If U path-cnnct. $\Rightarrow$ g unique up to addition of a constant. Unique g is called potential of f.

Construct g: Choose arbitrary $x_0 \in U$:

$$g(x) := \int_{\gamma_x} f(s) \cdot ds$$

Alternative: Integrate coordinates of f, maintain $h(y, z)$ or similar for unknown variables

Lemma Necessary for conservative: $U \subseteq \mathbb{R}^n$ open, $f : U \to \mathbb{R}^n$ is C^1. Write $f(x) = (f_1(x), \ldots, f_n(x))$. f conservative $\Rightarrow$

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}, \quad \forall 1 \leq i \neq j \leq n$$

Definition star-shaped: $U \subseteq \mathbb{R}^n$ is start shaped/"sternförmig":= $\exists x_0 \in U, \forall x \in U$: line segment joining x_0 to x is contained in U. Then: U is star-shaped around x_0.

Sufficient but not necessary:

Theorem: $U \subseteq \mathbb{R}^n$ s.-s., open. f is C^1 v.f. $\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}, \forall 1 \leq i \neq j \leq n \Rightarrow f$ is cons.

Definition curl: $U \subseteq \mathbb{R}^3$ open, $f : U \to \mathbb{R}^3$ is C^1. Curl (continuous):

$$curl(f) = \begin{pmatrix} \partial_y f_3 - \partial_z f_2 \\ \partial_z f_1 - \partial_x f_3 \\ \partial_x f_2 - \partial_y f_1 \end{pmatrix}$$

$$curl\ f = 0 \Longleftrightarrow J_f \text{ symmetric}$$

The Riemann Integral in $\mathbb{R}^n$

$U \subseteq \mathbb{R}^n$ compact, $f : U \to \mathbb{R}$ continuous.

Properties

Definition Compatibility: $n = 1, U = [a, b], a \leq b$: $\int_{[a,b]} f(x)dx = \int_a^b f(x)dx$

Definition Linearity: f, g cont., $a, b \in \mathbb{R}$

$$\int_U (af(x) + bg(x))dx = a \int_U f(x)dx + b \int_U g(x)dx$$

Definition Positivity: $f \leq g$

$$\int_U f(x)dx \leq \int_U g(x)dx$$

$f \geq 0 : \int_U f(x)dx \geq 0$

$V \subseteq U$ compact, $f \geq 0$

$$\int_V f(x)dx \leq \int_U f(x)dx$$

Definition Follows from Positivity:

$$\left|\int_U f(x)dx\right| \leq \int_U |f(x)|dx$$

$$\left|\int_U (f(x) + g(x))dx\right| \leq \int_U |f(x)|dx + \int_U |g(x)|dx$$

Definition Volume: $f = 1$. volume of U := $\int_U f(x)dx$ - generally: volume of

$$\{(x, y) \in U \times \mathbb{R} | 0 \leq y \leq f(x)\} \subseteq \mathbb{R}^{n+1}$$

volume of U: $vol(U) = Vol(U) = vol_n(U)$

Theorem Fubinis Theorem: $n_1, n_2 \geq 1$, $n = n_1 + n_2$. $x_1 \in \mathbb{R}^{n_1}$: compact V, U_1:

$$V(x_1) = \{x_2 \in \mathbb{R}^{n_2} | (x_1, x_2) \in U\} \subseteq \mathbb{R}^{n_2}$$
$$U_1 = \{x_1 \in \mathbb{R}^{n_1} | V(x_1) \neq \varnothing\} \subseteq \mathbb{R}^{n_1}$$

$$\Longrightarrow \int_U f(x_1, x_2)dx = \int_{U_1} g(x_1)dx$$

$$= \int_{U_1} \left(\int_{V(x_1)} f(x_1, x_2)dx_2\right) dx_1$$

Definition Domain Additivity: $U_1, U_2 \subseteq \mathbb{R}^n$ compact, $f : U_1 \cup U_2 \to \mathbb{R}$ continuous

$$\int_{U_1 \cup U_2} f(x)dx = \int_{U_1} f(x)dx + \int_{U_2} f(x)dx - \int_{U_1 \cap U_2} f(x)dx$$

More

Definition parameterized m-set: $1 \leq m \leq n$. parameterized m-set in $\mathbb{R}^n$ is continuous map

$$f : [a_1, b_1] \times \cdots \times [a_m, b_m] \to \mathbb{R}^n$$
$$C^1 \text{ on }]a_1, b_1[\times \cdots \times]a_m, b_m[$$

Definition Neglibibility: $B \subseteq \mathbb{R}^n$ negligible/'vernachlässigbar' if: $\exists\ k \geq 0$ and parameterized m_i-sets $f_i : U_i \to \mathbb{R}^n$, with $1 \leq i \leq k$ and $m_i < n$ such that

$$U \subseteq f_1(U_1) \cup \dots \cup f_k(U_k)$$

Lemma: $U \subseteq \mathbb{R}^n$ compact,negligible.f cont.

$$\int_U f(x)dx = 0$$

Improper Integrals

Consider only special improper integral in xD.
Definition: $I \subseteq \mathbb{R}$ compact interval, $a \in \mathbb{R}$, $f : [a, \infty) \times I \to \mathbb{R}$ continuous.

$$\int_{[a,\infty)\times I} f(x,y)dxdy = \lim_{b\to\infty} \int_{[a,b]\times I} f(x,y)dxdy$$

(if exists)
Fubini $\Rightarrow$ [$(*)$ only if inner integral $\exists$&cont.]

$$\int_{[a,\infty)\times I} f(x,y)dxdy = \int_a^\infty \int_I f(x,y)dydx \overset{(*)}{=} \int_I \int_a^\infty f(x,y)dxdy$$

Definition: If limes on right side $\exists$:

$$\int_{\mathbb{R}^2} f(x,y)dxdy := \lim_{R\to\infty} \int_{[-R,R]^2} f(x,y)dxdy$$

$$= \lim_{R\to\infty} \int_{B_R(0)} f(x,y)dxdy = \int_{-\infty}^\infty \int_{-\infty}^\infty f(x,y)dxdy = \int_{-\infty}^\infty \int_{-\infty}^\infty f(x,y)dydx$$

Lemma Comparison: $|f| \leq g$, $I \subseteq \mathbb{R}$ bounded interval, $J = [a, \infty)$, $a \in \mathbb{R}$

if $\exists$

$$\int_{J\times I} g(x,y)dxdy \text{ or } \int_{\mathbb{R}^2} g(x,y)dxdy$$

$\Rightarrow \exists$

$$\int_{J\times I} f(x,y)dxdy \text{ or } \int_{\mathbb{R}^2} f(x,y)dxdy$$

Change of Variable

f cont., g is C^1 on (a,b), g cont. on $[a,b]$

$$1D: \quad \int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(y)dy$$

Theorem Change of Variable Formula:

$$\int_{\overline{U}} f(\varphi(x)) \cdot |\det J_\varphi(x)|dx = \int_{\overline{V}} f(y)dy$$

holds if:

- $\overline{U}$ is compact and $\overline{U} = U \cup B$, U open, B negligible (understand B as the edge and U as the 'content' of $\overline{U}$)
- $\overline{V}$ is compact and $\overline{V} = V \cup C$, V open, C negligible (understand C as the edge and V as the 'content' of $\overline{V}$)
- $\varphi : \overline{U} \to \overline{V}$ is continuous and C^1 on U
- $\varphi(U) = V$, $\varphi : U \to V$ is bijective (must not be bijective on the border B)
- Some continuous function $\overline{U} \to \mathbb{R}$ exists, which equals $|\det J_\varphi|$ if constrained to U.

$|\det J_\varphi(x)|$ represents intgral change under φ.
Polar Coordinates

$$\varphi : [0,R] \times [-\pi, \pi] \to B_R(0), \quad \varphi(r, \Theta) = (r\cos\Theta, r\sin\Theta)$$

$$J_\varphi(r,\Theta) = \begin{pmatrix} \cos\Theta & -r\sin\Theta \\ \sin\Theta & r\cos\Theta \end{pmatrix}$$

$$\Rightarrow \det|J_\varphi(r,\Theta)| = |r\cos^2\Theta + r\sin^2\Theta| = r$$

$$\Rightarrow \text{"}dxdy = rdrd\Theta\text{"}$$

3D Polar Coordinates

$$\xi : [0,R] \times [0, 2\pi] \times [0,\pi]$$

$$\xi(r,\Theta,\varphi) = \begin{pmatrix} r\sin\varphi\cos\Theta \\ r\sin\varphi\sin\Theta \\ r\cos\varphi \end{pmatrix}$$

- z-axis to the top
(Θ rot. around z, $= 0 \leftrightarrow x$-axis, clock-wise)
(φ deviation from z)
- x-axis to the right
- y-axis to the front

$$J_\xi(r,\Theta,\varphi) = \begin{pmatrix} \sin\varphi\cos\Theta & -r\sin\varphi\sin\Theta & r\cos\varphi\cos\Theta \\ \sin\varphi\sin\Theta & r\sin\varphi\cos\Theta & r\cos\varphi\sin\Theta \\ \cos\varphi & 0 & -r\sin\varphi \end{pmatrix}$$

$$\Rightarrow |\det J_\xi(r,\Theta,\varphi)| = |r^2(-\sin^3\varphi\cos^2\Theta - \sin\varphi\cos^2\varphi\sin^2\Theta - \sin\varphi\cos^2\varphi\cos^2\Theta - \sin^3\varphi\sin^2\Theta)$$
$$= |r^2(-\sin^3\varphi - \sin\varphi\cos^2\varphi)| = |-r^2\sin\varphi|$$

$$\Rightarrow \text{"}dxdydz = r^2\sin\varphi drd\Theta d\varphi\text{"}$$

Geometric Applications

Definition center of Gravity: $U \subseteq \mathbb{R}^n$ compact with positive volume. Center of mass/gravity (barycenter) of U is $\overline{x} \in \mathbb{R}^n$, $\overline{x} = (\overline{x}_1, \dots, \overline{x}_n)$ with

$$\overline{x}_i = \frac{1}{vol(U)} \int_U x_i dx$$

Theorem Surface Area: let $f : [a,b] \times [c,d] \to \mathbb{R}$ be C^1 on $(a,b) \times (c,d) \to \mathbb{R}$. Let $\Gamma = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in [a,b] \times [c,d], z = f(x,y)\} \subset \mathbb{R}^3$ be the graph of f. Intuitively, this is a surface, and it should have an area. This is in fact given by

$$\int_a^b \int_c^d \sqrt{1 + (\partial_x f(x,y))^2 + (\partial_y f(x,y))^2} dxdy$$

Such a result also holds for the graphs of functions defined on other sets, such as discs, provided they are C^1 in the "interior" of the domain.
Theorem Graph Length:
For the length of the graph of a function $f : [a,b] \to \mathbb{R}$ we have

$$\int_a^b \sqrt{1 + f'(x)^2} dx$$

For length as motion in space of $\phi : \mathbb{R} \to \mathbb{R}^{\text{typicaly} > 1} : \int_a^b |\phi'(t)|dt$

Important Integrals

1.

$I := \int_{-\infty}^\infty e^{-x^2}dx$, $J := \int_{\mathbb{R}^2} e^{-x^2-y^2}dxdy$

$$J = \lim_{R\to\infty} \int_{B_R(0)} e^{-x^2-y^2}dxdy = \lim_{R\to\infty} \int_0^R \int_{-\pi}^\pi e^{-r^2} r d\Theta dr = \lim_{R\to\infty} 2\pi \int_0^R re^{-r^2}dr$$
$$= \lim_{R\to\infty} 2\pi \left[-\frac{1}{2}e^{-r^2}\right]_0^R = \lim_{R\to\infty} 2\pi \left(\frac{1}{2} - \frac{1}{2}e^{-R^2}\right) = \pi$$

$$J = \lim_{R\to\infty} \int_{-R}^R \int_{-R}^R e^{-x^2-y^2}dxdy = \lim_{R\to\infty} \int_{-R}^R e^{-y^2}\left[\int_{-R}^R e^{-x^2}dx\right]dy$$
$$= \lim_{R\to\infty} \left(\int_{-R}^R e^{-x^2}dx\right) \cdot \left(\int_{-R}^R e^{-y^2}dy\right) = \lim_{R\to\infty} \left(\int_{-R}^R e^{-x^2}dx\right)^2 = \left(\lim_{R\to\infty} \int_{-R}^R e^{-x^2}dx\right)^2 = I^2$$

Ordinary Differential Equations

Funktion f unbekannte - $f, f', ...$ in Formel

ODE: f hat einen Parameter - PDE: mehrere

Definition gewöhnliche Differentialgleichungen: $G(y, y', ..., y^{(k)}, x) = 0, k \geq 1$. y Funktion in einer Variablen x. k: Ordnung.

Lösung: $f : I \to \mathbb{R}$ mit $G(f(x), f'(x), ..., f^{(k)}(x), x) = 0, \forall x \in I$.

$y' = g(x) \Rightarrow y = G(x) = \int g(x)dx$.

Definition Anfangswertproblem: Ordnung k und $y(x_0) = y_0, y'(x_0) = y_1, ..., y^{(k-1)}(x_0) = y_{k-1}$ bekannt.

Even ODE defined on $\mathbb{R}$, may only solvable on subset.

Separation of Variable ODE $y' = \frac{1}{a(y)} b(x)$, a, b continuous, $a(y) \neq 0$:

$$y' = \frac{1}{a(y)} b(x) \Leftrightarrow a(y) \cdot y' = b(x)$$

$$\Leftrightarrow \int a(y) \cdot y'(x)dx = \int b(x)dx + c, c \in \mathbb{R}$$

$$\Leftrightarrow A(y) = B(x) + c$$

$$\Rightarrow y = A^{-1}(B(x) + c)$$

Theorem Picard's Existence Theorem (Existenz-&Eindeutigkeitssatz): $(x_0, y_0 \in \mathbb{R})$. $F : \mathbb{R}^2 \to \mathbb{R}$ continuously differentiable near (x_0, y_0). Let . ODE $y' = F(x, y)$ has unique solution f defined on a "largest" open interval I containing x_0 such that $f(x_0) = y_0$. I.e., an initial value problem $y' = F(y, x)$ with $y(x_0) = y_0$ with F being continuously differentiable has exactly one maximum solution.

'Maximum': All other solutions $g : J \to \mathbb{R}$: $J \subseteq I, f|_J = g$.

Reduce ODE Order from ≥ 2 $F(x) = \begin{pmatrix} f_0(x) \\ f_1(x) \\ \dots \\ f_{k-1}(x) \end{pmatrix}$, $f_i'(x) = f_{i+1}(x)$, $f_0 = y$.

$$\begin{pmatrix} f_0'(x) \\ f_1'(x) \\ \dots \\ f_{k-1}'(x) \end{pmatrix} = F'(x)$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ & & \vdots & & & \\ & & \text{ODE} & & & \end{pmatrix} \begin{pmatrix} f_0(x) \\ f_1(x) \\ \vdots \\ f_{k-1}(x) \end{pmatrix}$$

Linear Differential Equations

Definition Linear Differential Equations: $I \subset \mathbb{R}$ an open interval, $k \geq 1$ integer. An **homogenous linear ordinary differential equation** of order k on I:

$$y^{(k)} + a_{k-1}y^{(k-1)} + ... + a_1 y' + a_0 y = 0$$

$a_0, ..., a_{k-1}$ complex-valued functions on I.

$$y^{(k)} + a_{k-1}y^{(k-1)} + ... + a_1 y' + a_0 y = b$$

$b : I \to \mathbb{C}$, is an inhomogenous linear ODE, associated homogeneous equation $b = 0$.

Solution is k-times differentiable function $f : I \to \mathbb{C}$ with $f^{(k)}(x) + a_{k-1}f^{(k-1)}(x) + ... + a_0(x)f'(x) = b(x)$, $\forall x \in I$. $(f'(x) = (\text{Re} f(x))' + i(\text{Im} f(x))')$.

LEADING COEFFICIENT 1!

Theorem: $I \subset \mathbb{R}$ open interval, $k \geq 1$

$$y^{(k)} + a_{k-1}y^{(k-1)} + ... + a_1 y' + a_0 y = 0$$

be a linear ODE, continuous coefficients.

1. Set S of k-times differentiable solutions $f : I \to \mathbb{C}$ is complex vector space, subspace of complex-valued functions on I.

   a_i real-valued: Set S real-valued solutions is real vector space, subspace of space of real-valued functions on I.
2. $\dim S = k$, and for any $x_0 \in I$, any $(y_0, ..., y_{k-1}) \in \mathbb{C}^k$: unique $f \in S$ with

   $$f(x_0) = y_0, f'(x_0) = y_1, ..., f^{(k-1)}(x_0) = y_{k-1}$$

   a_i real-valued: $\dim S = k$, and for any $x_0 \in I$, any $(y_0, ..., y_{k-1}) \in \mathbb{R}^k$: unique real-valued f:

   $$f(x_0) = y_0, f'(x_0) = y_1, ..., f^{(k-1)}(x_0) = y_{k-1}$$
3. b continuous on I. $\exists$ solution f_0 for

   $$y^{(k)} + a_{k-1}y^{(k-1)} + ... + a_1 y' + a_0 y = b$$

   solutions $S_b = \{f + f_0 | f \in S\}$, f_0 solut.
4. $x_0 \in I$, $(y_0, ., y_{k-1}) \in \mathbb{C}^k$: unique $f \in S_b$

   $$f(x_0) = y_0, f'(x_0) = y_1, ..., f^{(k-1)}(x_0) = y_{k-1}$$

b, a_i real-valued $\Rightarrow \exists$ real-valued solution

If f_1, f_2 solutions of homogenous linear ODE $\Rightarrow f = z_1 f_1 + z_2 f_2$, $z_1, z_2 \in \mathbb{C}$ solutions too.

Definition Superposition (Superpositionsprinzip): f_1, f_2 solve inhom. linear ODE (right-hand sides b_1, b_2) $\Rightarrow f_1 + f_2$ solves the inhom. linear ODE (right-hand side $b_1 + b_2$)

Solution Strategy Initial Value Problem

1. Find basis $f_1, ... f_n$ of the hom. ODE.
2. Find solution f_0 of the inhom. ODE ("Partikulärlösung"). General solution: $f_0 + \sum_{j=1}^{n} \lambda_j f_j, \lambda_j \in \mathbb{C}$
3. Solve LSE of general solution with initial values for $\lambda_1, ..., \lambda_k$ (unique)

Linear ODEs of Order 1

$y' + ay = b$ - a, b continuous on I.

Step 1 of Solution Strategy.

Theorem Homog. Sol: Solution has form

$$f(x) = z \exp(-A(x)) = ze^{-A(x)}$$

A primitive of a, $z \in \mathbb{C}$. $f(x_0) = y_0$: unique.

$$f(x) = y_0 \exp(A(x_0) - A(x)) = y_0 e^{A(x_0) - A(x)}$$

Proof. $\exists x_0, f(x_0) = 0 \Rightarrow f = 0$ is solution (only as unique by Picard)

Else: separation of variable $\square$

Step 2 of Solution Strategy. Guess. Otherwise: "Variation of the constant" - make z a variable: $f(x) = z(x)e^{-A(x)}$

$$y' + ay = b \Rightarrow f'(x) + a(x)f(x) = b(x)$$

$$\Leftrightarrow z'(x)e^{-A(x)} = b(x) \Leftrightarrow z'(x) = e^{A(x)}b(x)$$

Step 3 of Solution Strategy. Use initial values

Linear ODEs with Constant Coefficients

Same solution strategy

$$y^{(k)} + a_{k-1}y^{(k-1)} + ... + a_0 y = b$$
$$a_0, ..., a_{k-1} \in \mathbb{C}, b : I \to \mathbb{C}$$

Step 1 of Solution Strategy. Characteristic polynomial $P(t) = t^k + a_{k-1}t^{k-1} + ... + a_0$. α root $\Rightarrow e^{\alpha x}$ solution

Theorem: P no roots with multiplicity $\Rightarrow$ $\{e^{\alpha x} | \alpha \in \mathbb{C}, P(\alpha) = 0\}$ is basis of solution space

Proof. k unique $\alpha \Rightarrow k$ lin. indep. func. $\square$

Theorem: P roots $\alpha_1, ..., \alpha_l$, multiplicities $v_1, ..., v_l \geq 1 \Rightarrow$ basis of solution space:

$$\{x^j e^{\alpha_i x} | i \in \{1, ..., l\}, j \in \{0, ..., v_i - 1\}$$

Proof. $(xe^{\alpha x})^{(i)} = \alpha^i x e^{\alpha i} + i\alpha^{i-1}e^{\alpha x}$

in ODE $\to$ sums with factor $P^{(k)}(\alpha)$, are $= 0$ from multiplicity of α $\square$

Can make complex basis real if $a_i \in \mathbb{R}$. $\alpha, \overline{\alpha}$ roots. Replace $e^{\alpha x}, e^{\overline{\alpha}x}$ with

$$\underbrace{\begin{array}{c} e^{\alpha x} = e^{\beta x}e^{\gamma i x} = e^{\beta x}(\cos(\gamma x) + i \sin(\gamma x)) \\ e^{\overline{\alpha} x} = e^{\beta x}(\cos(\gamma x) - i \sin(\gamma x)) \end{array}}$$

$$\Rightarrow \text{basis:} \{e^{\beta x}\cos(\gamma x), e^{\beta x}\sin(\gamma x)\}$$

Step 2 of Solution Strategy. Guess. Or use those 'tricks':

- Superposition Principle

  f_0/g_0 sol. with right-hand side $b/c \Rightarrow \lambda f_0 + \mu g_0$ sol. with r.-h. side $\lambda b + \mu c$
- Method of Underdetermined Coefficients. Idea: f similar to b.

  $b(x) = x^d e^{\beta x}$. Solution: $Q(x)e^{\beta x}$, Q polynomial, $\deg Q \leq d + v$ (v multiplicity of β as root of P)

  $b(x) = x^d \cos(\beta x)$ or $b(x) = x^d \sin(\beta x)$. Solution: $Q_1(x)\cos(\beta x) + Q_2 \sin(\beta x)$, Q_1, Q_2 polynomials, $\deg Q_1, \deg Q_2 \leq d + v$ (v multiplicity of βi as root of P)

1. $ODE = Q(x)$...(approx.), 2. solve factors of approx. by comparing coefficients

Variation of Constants (complex b, non-constant coefficients)

$$y^{(k)} + a_{k-1}y^{(k-1)} + ... + a_1 y' + a_0 y = b$$

$(f_1, ..., f_k)$ basis of hom. sys.

$$f(x) = z_1(x)f_1(x) + ... + z_k(x)f_k(x)$$

$$\left\{ \begin{array}{c} z_1'(x)f_1(x) + ... + z_k' f_k(x) = 0 \\ z_1'(x)f_1'(x) + ... + z_k' f_k'(x) = 0 \\ \vdots \\ z_1'(x)f_1^{(k-2)}(x) + ... + z_k' f_k^{(k-2)}(x) = 0 \end{array} \right\}$$

$$\Rightarrow k \text{ equations for } z_i, i \in [k]$$

Insert f into ODE, solve system after simplification as f_i are sol. of hom. sys.

Example: $y'' + a_1 y' + a_0 y = b$

$$\left\{ \begin{array}{c} f_0(x) = z_1(x)f_1(x) + z_2(x)f_2(x) \\ z_1' f_1(x) + z_2' f_2(x) = 0 \end{array} \right\}$$

$$\xRightarrow{\text{derive for use in ODE}}$$

$$\left\{ \begin{array}{c} f_0' = z_1 f_1' + z_2 f_2' \\ f_0'' = z_1' f_1' + z_1 f_1'' + z_2' f_2' + z_2 f_2'' \end{array} \right\}$$

$$\xRightarrow{\text{insert into ODE}}$$

$$z_1' f_1' + z_1 f_1'' + z_2' f_2' + z_2 f_2'' + a_1 z_1 f_1' + a_1 z_2 f_2' + a_0 z_1 f_1 + a_0 z_2 f_2 = b$$

$$\xRightarrow{f_1 \text{ and } f_2 \text{ solutions of ODE can factor out } z_1 \text{ and } z_2 \text{ to get:}}$$

$$z_1' f_1' + z_2' f_2' = b$$

$$\Longrightarrow$$

$$\left\{ \begin{array}{c} z_1' f_1 + z_2' f_2 = 0 \\ z_1' f_1' + z_2' f_2' = b \end{array} \right\}$$

$$\Leftrightarrow \begin{pmatrix} f_1 & f_2 \\ f_1' & f_2' \end{pmatrix} \begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = A^{-1} \begin{pmatrix} 0 \\ b \end{pmatrix}$$

wichtige ODEs

- $y' = \frac{1}{\cos^2 x} \Rightarrow y = \tan(x) + c$
- $y'' = y \Rightarrow y = c_1 e^x + c_2 e^{-x}$
- Harmonic Oscillator: $y(x)$ vertical deviation, $my'' = -ky - by'$

(Hopefully) Helpful stuff

trigonometrische Funktionen

$\sin z := \sum_{m=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + ...$

$\cos z := \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + ...$

konvergiert absolut $\forall z \in \mathbb{C}$ (Quotientenkriterium), $\rho = \infty$

Theorem 3.8.1: $\sin, \cos \in \mathbb{R}^{\mathbb{R}}$ sind stetig

Theorem 3.8.2:

1. $\exp(iz) = \cos z + i \sin z, \forall z \in \mathbb{C}$
2. $\cos z = \cos(-z) \& \sin(-z) = -\sin z, \forall z \in \mathbb{C}$
3. $\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \cos z = \frac{e^{iz} + e^{-iz}}{2}$
4. $\sin(z + w) = \sin z \cos w + \cos z \sin w$
   $\cos(z + w) = \cos z \cos w - \sin z \sin w$
5. $\cos^2 z + \sin^2 z = 1, \forall z \in \mathbb{C}$

Corollary 3.8.3: $\sin(2z) = 2 \sin z \cos z$ & $\cos(2z) = \cos^2 z - \sin^2 z$

Corollary: $\cos^3 x = \frac{3}{4}\cos x + \frac{1}{4}\cos(3x)$

$\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin(3x)$

Useful:

$\sin^2 x = \frac{1}{2}(1 - \cos(2x))$

$\cos^2 x = \frac{1}{2}(1 + \cos(2x))$

wichtige Konvergenzen

Grenzwerte

- $\frac{1}{n} \to 0$ (direkt von Definition)
- $\frac{1}{n^2} \to 0$ (Vergleichsprinzip)
- $\frac{1}{2^n} \to 0$ (Vergleichsprinzip)
- $\frac{1}{n!} \to 0$ (Vergleichsprinzip)
- $\frac{\cos(...)}{n} \to 0$ ($|\cos(...)| \leq 1$)
- $\frac{n+1}{n} = 1 + \frac{1}{n} \to 1$
- $\lim n^a q^n = 0, 0 \leq q < 1, a \in \mathbb{Z}$ (fallend + Methode "Grenzwert monotoner Folgen")
- $a_1 = c, a_{n+1} = \frac{1}{2}(a_n + \frac{c}{a_n})$ konvergiert $\sqrt{c}$ (fallend + Methode "Grenzwert monotoner Folgen")
- $\lim_{n\to\infty} n^a q^n = 0$ ($a \in \mathbb{Z}, 0 \leq q < 1$) monoton fallend für groß genug n, nach unten beschränkt, Weierstrass
- $\lim_{n\to\infty} \sqrt[n]{n} = 1$
  $n \geq 1$ von $(b^n - a^n) = (b - a)(b^{n-1} + b^{n-2}a + ...)$ & $\lim \frac{n}{(1+\epsilon)^n} = 0$
- $a_1 = c > 1, a_{n+1} = \frac{1}{2}\left(a_n + \frac{c}{a_n}\right)$: $\lim a_n = \sqrt{c}$
  Da monoton fallend, >0 & Weierstrass
- $\lim_{x\to\infty} \sqrt{x^2 + 5} - x = \lim \frac{(\sqrt{x^2+5}-x)(\sqrt{x^2+5}+x)}{\sqrt{x^2+5}+x} = \lim \frac{x^2+5-x^2}{\sqrt{x^2+5}+x} = \lim \frac{5}{\sqrt{x^2+5}+x} = 0$

Reihen

- $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n^2}$ (konvergiert nach Cauchy Corollary), $= \frac{\pi^2}{6}$
  alternativ Vergleich zu $\sum \frac{1}{(k-1)k}$
- $a_n = \sum_{i=1}^{n} \frac{1}{n^3}$ konvergiert, Grenzwert ?
- $\sum_{i=1}^{n} \frac{1}{i} \to \infty$
- $\sum \frac{1}{n} = \infty$ (increasing + Cauchy (not bound))
- $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$
- $\sum q^n = \frac{1}{1-q}, |q| < 1, q \in \mathbb{C}$ ($a_n = \frac{1-q^{n+1}}{1-q}$ Induktion, $|a_n - \frac{1}{1-q}| \to 0$)
- $\sum \frac{(-1)^k}{k^2}$ konvergiert absolut
- $\sum (-1)^{k+1} \frac{1}{k}$ konvergiert, aber nicht absolut
- $\sum \frac{n!}{2^n}$ divergiert (Quotientenkriterium)
- $\sum q^n$ konvergiert $|q| < 1$, divergiert ≥ 1 (Quotientenkriterium, besonders für $= 1$)
- $\sum \frac{z^k}{k!}$ konvergiert

- $\sum a_n$ konvergiert, $a_n = x_0 + ... + x_n$, $x_n = a_n - a_{n-1} \to 0$
- $\sum x_n = \sum (x_{2n+1} + x_{4n+2} + x_{4n+4})$ falls $\sum x_n$ absolut konvergiert
- $\sum x_n$ konvergiert (nicht absolut). $\forall m \in \mathbb{R}$, Bijektion $j : \mathbb{N} \to \mathbb{N}$ existiert $\sum \frac{(-1)^{j(k)-1}}{j(k)} = m$

Doppelte Summation Vertauschung von Index
- funktioniert bei $a_{m,n} = (\frac{1}{2} + \frac{1}{m})^k$
- funktioniert nicht bei $a_{m,n} = \begin{cases} 1, & m = n \\ -1, & m + 1 = n \\ 0, & \text{else} \end{cases}$

Cauchy Produkt

$a_n = \frac{(-1)^n}{\sqrt{n+1}}$. $|\sum_{j=0}^{n} a_{n-j} a_j| = \sum_{j=0}^{n} \frac{1}{\sqrt{(n-j+1)(j+1)}} \geq \sum_{j=0}^{n} \frac{1}{\sqrt{(n+1)^2}} = 1$

Polynome

$p(x) = a_d x^d + a_{d-1} x^{d-1} + ... + a_0, a_d \neq 0$
$q(x) = b_c x^c + ... + b_0, b_c \neq 0$
Betrachtung von $\lim \frac{p(n)}{q(n)}$.

- $d > c$:
  $\frac{a_d}{b_c} > 0 \Rightarrow \lim \frac{p(n)}{q(n)} = +\infty$
  $\frac{a_d}{b_c} < 0 \Rightarrow \lim \frac{p(n)}{q(n)} = -\infty$
- $d = c$
  $\lim \frac{p(n)}{q(n)} = \frac{a_d}{b_c}$
- $d < c$
  $\lim \frac{p(n)}{q(n)} = 0$

Euler

$e = \lim_{n\to\infty} (1 + \frac{1}{n})^n$ (konvergiert da fallend mit 2.2.7)

Exponentialfunktion

Betrachte $\sum \frac{z^n}{n!}$. Konvergiert $\forall z$ (Qutoentenkriterium). $\exp(z) := \sum \frac{z^n}{n!}$

Mit 2.7.26: $\exp(z + w) = \exp(w)\exp(z)$
$\exp(z) \neq 0$ as inverse $\neq 0$ with $\exp(z - z) = 1$.

$\exp(1) := e \approx 2.718281828... \Rightarrow \exp(z) = \exp(1 + ... + 1) = \exp(1)^z = e^z$

Stetigkeit Let $x_0 \in \mathbb{R}$. $\exp(x) - \exp(x_0) = \exp(x_0)(\exp(x - x_0) - 1)$ and $exp(x - x_0) - 1 = \sum_{n=1}^{\infty} \frac{(x-x_0)^n}{n!}$. With the triangle inequality and $\frac{1}{n!} \leq 1$, we get $|\exp(x - x_0) - 1| \leq \sum_{n=1}^{\infty} \frac{|x-x_0|^n}{n!} \leq \sum_{n=1}^{\infty} |x - x_0|^n = \frac{1}{1-|x-x_0|} - 1 = \frac{|x-x_0|}{1-|x-x_0|}$. Then, for $\epsilon > 0$, let $\delta = \min(\frac{1}{4}, \frac{\epsilon}{4\exp(x_0)})$. With $|x - x_0| < \delta$, we get $|\exp(x) - \exp(x_0)| < 2\exp(x_0)\frac{\epsilon}{4\exp(x_0)} = \frac{\epsilon}{2} < \epsilon$

Grenzwerte von Funktionen

$\lim_{\substack{x\to 0 \\ x<0}} \frac{\sin(x)}{x} = \lim(\frac{1}{x}(x - \frac{x^3}{6} + ...)) = \lim(1 - \frac{x^2}{6} + ...) = 1$

- $\lim \frac{x^2+1}{x^2-1} = 1$
- $\lim e^x = \infty$
- $\lim_{x\to-\infty} e^x = 0$
- $\lim \frac{e^x}{x^a} = \infty$
- $\lim x^a e^{-x} = 0$

Trigonometry

Corollary 4.2.8: $[0, 2\pi[\to \mathbb{R}^2, t \mapsto (\cos t, \sin t)$ is Bijektion von $[0, 2\pi[$ nach $K = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$.

Proof of existence: Since $x^2 + y^2 = 1$ we get $0 \leq x^2 \leq 1$ and $-1 \leq x \leq 1$. This means that there is a unique $u \in [0,\pi]$ such that $\cos(u) = x$. Then, from $1 = x^2 + y^2 = \cos^2(u) + \sin^2(u) = x^2 + \sin^2(u)$ we get $y^2 = \sin^2(u)$ wo $y = \pm\sin(u)$. Case 1: If $y \geq 0$, then $y = \sin(u)$ since $0 \leq u \leq \pi$. We take $t = u$. Case 2: If $y < 0$, then $y = -\sin(u) = \sin(2\pi - u)$. But then also $x = \cos(u) = \cos(2\pi - u)$ So we can take $t = 2\pi - u \in [\pi, 2\pi]$.

- $\lim_{\substack{x\to 0\\ x>0}} \frac{1-\cos(x)}{\sin(x)} = \lim_{\substack{x\to 0\\ x>0}} \frac{\sin(x)}{\cos(x)} = \frac{0}{1} = 0$

Corollary 4.2.18, Example!: $\sqrt[n]{x_1 \cdots x_n} \leq \frac{x_1 + ... + x_n}{n}$

We consider $f(x) = -\ln x$ with $f'(x) = -\frac{1}{x}$ and $f''(x) = \frac{1}{x^2}, x \in]0,\infty[$. Hence, f is convex. From 4.2.14 with $I =]0,\infty[$ and $\lambda_1 = ... = \lambda_n = \frac{1}{n}$ we get $-\ln(\frac{1}{n}\sum_{i=1}^n x_i) \leq \sum_{i=1}^n -\frac{1}{n}\ln x_i = -\frac{1}{n}\ln(x_1 \cdots x_n)$ Now we use that exp is increasing:

$$\exp(\frac{\log x_1}{n} + ... + \frac{\log x_n}{n}) \leq \frac{x_1 + ... + x_n}{n}$$
$$\Leftrightarrow \exp(\frac{\log x_1}{n}) \cdots \exp(\frac{\log x_n}{n}) \leq \frac{x_1 + ... + x_n}{n}$$
$$\Leftrightarrow \sqrt[n]{x_1 \cdots x_n} \leq \frac{x_1 + ... + x_n}{n}$$

Zeta-Funktion

Für $s > 1$ konvergiert $\zeta(s) = \sum \frac{1}{n^s}$.
$S_N = 1 + \frac{1}{2^s} + \frac{1}{3^s} + ... + \frac{1}{N^s}$. Mit $k \geq 1, N \leq 2^k$, dann $S_N \leq S_{2^k}$.

$$S_{2^k} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + ...$$
$$\leq 1 + \frac{1}{2^s} + \frac{1}{2^{s-1}} + \frac{1}{4^{s-1}} + ...$$
$$= 1 + \frac{1}{2^s} + \frac{1}{2^{s-1}} + \frac{1}{(2^{s-1})^2} + \frac{1}{(2^{s-1})^4} + ...$$

Folgt mit Vergleichssatz und geometrischer Reihe.

Gamma Funktion

$\int_0^b x^n e^{-x} dx = -b^n e^{-b} + n\int_0^b x^{n-1}e^{-x}dx$. Mit $\lim_{b\to+\infty} b^n e^{-b} = 0$: $\int_0^\infty x^n e^{-x}dx = n\int_0^\infty x^{n-1}e^{-x}dx$. Es folgt: $\int_0^\infty x^n e^{-x}dx = n(n-1)...1\int_0^\infty e^{-x}dx = n!$.

Konvergenzen

Example

$\sum_{n=2}^\infty \frac{1}{n(\ln n)^\beta}$ konvergiert gdw. $\int_2^\infty \frac{1}{n(\ln n)^\beta}dx$ konvergiert. $b > 2, x = e^u, u \in [\ln 2, \ln b]$.
$\int_2^b \frac{1}{x(\ln x)^\beta}dx = \int_{\ln 2}^{\ln b} \frac{1}{e^u u^\beta} e^u du = \int_{\ln 2}^{\ln b} \frac{1}{u^\beta}du$.

Ableitungen

- $\exp' = \exp$
- $\ln'(x) = \frac{1}{x}$ (4.1.12)
- $(\log)^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{x^n}$
- $\sin' = \cos, \cos' = -\sin$
- $(x^n)' = nx^{n-1}$
- $\tan' x = \frac{1}{\cos^2 x} = 1 + \tan^2 x$ (4.1.9(3))
- $\cot' x = -\frac{1}{\sin^2 x}$

Notice: f even ($f(-x) = f(x)) \Rightarrow f'$ uneven ($f(-x) = -f(x)$) & f uneven $\Rightarrow f'$ even

x^2

$f: \mathbb{R} \to \mathbb{R}, f(x) = x^2$ Then $f'(x_0) = 2x_0, \forall x_0 \in \mathbb{R}$ Follows from $f(x) - f(x_0) = x^2 - x_0^2 = (x - x_0)(x + x_0)$ For $x \neq x_0$ then $\lim \frac{f(x)-f(x_0)}{x-x_0} = \lim x + x_0 = 2x_0$

$ax + b$

$f(x) = ax+b$ f is differentiable with $f'(x) = a$ as $\frac{f(y)-f(x)}{y-x} = \frac{a(y-x)}{y-x} = a$.

arcsin

$\sin' = \cos, \cos x > 0 (\forall x \in]-\frac{\pi}{2}, \frac{\pi}{2}[) \overset{5.2.5(4)}{\Longrightarrow}$ sin strikt monoton wachsend auf $]-\frac{\pi}{2}, \frac{\pi}{2}[$ / $\sin: [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1,1]$ Bijektion. $\arcsin: [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ Umkehrfunktion. Differenzierbar $]-1,1[$. $\arcsin' y = \frac{1}{\sin' x} = \frac{1}{\cos x} = \frac{1}{\sqrt{1-y^2}}$.
With 4.1.12 we know that arcsin is differentiable on $]-1,1[$ and with $y = \sin x$ get $\arcsin'(y) = \frac{1}{\sin'(x)} = \frac{1}{\cos(x)}$ We can use $\sin^2(x) + \cos^2(x) = 1 \Rightarrow y^2 = \sin^2(x) = 1 - \cos^2(x)$. With $\cos(x) > 0$ (which holds for $[-\frac{\pi}{2}, \frac{\pi}{2}]$) we get: $\cos(x) = \sqrt{1-y^2}$. So, $\forall y \in]-1,1[$ we get $\arcsin'(y) = \frac{1}{\sqrt{1-y^2}}$

arccos

$\arccos: [-1,1] \to [0,\pi]$. $\arccos' y = \frac{-1}{\sqrt{1-y^2}}$.

tan

$\tan' x = \frac{1}{\cos^2 x}$

arctan

$\tan' x = \frac{1}{\cos^2 x} \Rightarrow \arctan:]-\infty,\infty[\to]-\frac{\pi}{2}, \frac{\pi}{2}[$. $\arctan' y = \cos^2 x = \frac{1}{1+y^2}$.

arccot

$\cot x = \frac{\cos x}{\sin x}, \cot' x = -\frac{1}{\sin^2 x}$
$\text{arccot}:]-\infty,\infty[\to]0,\pi[, \text{arccot}' y = -\frac{1}{1+y^2}$

Hyperbel & Areafunktionen

$\cosh x = \frac{e^x+e^{-x}}{2}, \sinh x = \frac{e^x-e^{-x}}{2}, \tanh x = \frac{\sinh x}{\cosh x}, \cosh' x = \sinh x, \sinh' x = \cosh x, \text{arcosh}' y = \frac{1}{\sqrt{y^2-1}}, \text{arsinh}' y = \frac{1}{\sqrt{1+y^2}}, \tanh' x = \frac{1}{\cosh^2 x} > 0, \text{artanh}' y = \frac{1}{1-y^2}$

Stuff

4.3.4: $\exp, \sin, \cos \sinh, \cosh, \tanh, \ldots$ sind glatt auf ganz $\mathbb{R}$
Polynome sind glatt auf ganz $\mathbb{R}$
ln ist glatt

Integrals

- $\int e^x dx = e^x$
- $\int \cos(2x)dx = \frac{1}{2}\sin(2x)$
- $\int e^x dx = e^x + C$
- $\int \frac{1}{x}dx = \log x + C, x > 0$
- $\int x^s dx = \begin{cases} \frac{x^{s+1}}{s+1} + C, & s \neq -1 \\ \ln x + C, & x > 0 \end{cases}$
- $\int \sin x dx = -\cos x + C$
- $\int \sinh x dx = \cosh x + C$
- $\int \cos x dx = \sin x + C$
- $\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x + C$
- $\int \cosh x dx = \sinh x + C$
- $\int \frac{1}{\sqrt{1+x^2}}dx = \text{arsinh} x + C$
- $\int \frac{1}{1+x^2}dx = \arctan x + C$
- $\int e^x dx = e^x + C$
- $\int \frac{1}{\sqrt{x^2-1}}dx = \text{arcosh} x + C$
- $\int \cos(ax) = \frac{1}{a}\sin(ax)$

$f(x) = x$

Let $f(x) = x$ on $[a,b]$. Let $P_n = \{a + i \cdot h | 0 \leq i \leq n\}$ with $h = \frac{b-a}{n}$. Then

$$s(f, P_n) = \sum_{i=1}^n x_{i-1}(x_i - x_{i-1})$$
$$= \frac{b-a}{n}\sum_{i=1}^n (a + (i-1)h)$$
$$= \frac{b-a}{n}\left(na + h\frac{n(n-1)}{2}\right)$$
$$= (b-a)a + \frac{(b-a)^2}{2}(\frac{n-1}{n})$$

And $S(f, P_n) = ... = (b-a)a + \frac{(b-a)^2}{2}(\frac{n-1}{n})$ So, $\lim_{n\to\infty} S(f,P_n) = \frac{b^2-a^2}{2} = \lim_{n\to\infty} s(f,P_n)$ Hence, f is integrable with $\int_a^b f(x)dx = \frac{b^2-a^2}{2}$.

(ir)rational

- $f(x) = \begin{cases} 1, x \text{ rational} \\ 0, x \text{ irrational} \end{cases}$ nur Lebesgue differenzierbar.
- Let $f: [0,1] \to \mathbb{R}$ with $f(x) = \begin{cases} 0, x \text{ irrational or } x=0 \\ \frac{1}{q}, x=\frac{p}{q}, p,q \text{ natural numbers, relatively prime} \end{cases}$ One can show $\int_0^1 f(x)dx = 0$.

$\frac{\sin x}{\cos x}$

$\int_a^b \frac{\sin t}{\cos t}dt$ for $-\frac{\pi}{2} < a < b < \frac{\pi}{2}$. We can use substitution: $\int_a^b \frac{\sin t}{\cos t}dt = -\int_a^b \frac{\cos' t}{\cos t}dt = -\int_a^b f(\cos t)\cos' t\, dt$ with $f(y) = \frac{1}{y}$. Using substitution (5.4.6) we then get $\int_a^b \frac{\sin t}{\cos t}dt = -\int_{\cos(a)}^{\cos(b)} \frac{1}{x}dx = -\log(\cos(b)) + \log(\cos(a))$
It follows that an antiderivative of $\tan(t)$ is $-\log(\cos(t))$ for $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

example

$b > 1$.

$$\int_0^\infty \frac{1}{1+x^\alpha}dx = \int_0^1 \frac{1}{1+x^\alpha}dx + \int_1^b \frac{1}{1+x^\alpha}d$$

$\frac{1}{2x^\alpha} \leq \frac{1}{1+x^\alpha} \leq \frac{1}{x^\alpha}, \forall x \geq 1$.

example

$\int \sin^2 x dx = \int \frac{1}{2}(1 - \cos(2x))dx$

example

$\int \frac{1}{\sqrt{e^x - e^2}}$ with $u = e^x - e^2$: $\int \frac{1}{\sqrt{u}(u+e^2)}du$ with $v = \frac{\sqrt{u}}{e}$: $\int \frac{2e^2 v}{ve(v^2e^2+e^2)}dv = \int \frac{2}{v^2e+e}dv = \frac{2}{e}\arctan(v)$

example

$\int \frac{1}{1+\cos(x)}dx = \int \frac{1}{1+\cos(x)}\frac{1-\cos(x)}{1-\cos(x)}dx = \int \frac{1}{\sin^2(x)}dx - \int \frac{\cos(x)}{\sin^2(x)}dx = -\cot(x) + \frac{1}{\sin(x)}$

example

$\int \frac{1}{\cos^4(x)}dx = \int \frac{1}{\cos^2(x)}\frac{1}{\cos^2(x)}dx$ with $u = \tan(x)$ we get $\int u^2 + 1 du = \frac{1}{3}\tan^3(x) + \tan(x)$

example

$\int_{-e}^e \sin(-x^3)dx = \int_{-e}^0 ...dx + \int_0^e ...dx = 0$

example

$\int \tan^4(x)dx = \int \tan^2(x)\tan^2(x)dx = \int \tan^2(x)\frac{1}{\cos^2(x)} - \tan^2(x)dx = \int \tan^2(x)\frac{1}{\cos^2(x)}dx - \int \frac{1}{\cos^2(x)}dx + \int 1dx$ with

$u = \tan x$ we get $\int u^2 du - \int 1 du + \int 1 dx = \dots$

example

$\int (1+2ax^2)e^{ax^2}dx$
$= \int e^{ax^2}dx + 2a\int x^2 e^{ax^2}dx$
$= xe^{ax^2} - 2a\int x^2 e^{ax^2}dx + 2a\int x^2e^{ax^2}dx$
$= xe^{ax^2}$

Area of Half Circle

Application 5.4.7 - lecture approach 1

Let $r > 0$ and $f(x) = \sqrt{r^2 - x^2}$ be defined on $[-r, r]$. Graphically, this corresponds to the half-circle above the x-axis with radius r. We want to compute a portion of the area of that half circle, i.e., $\int_a^b \sqrt{r^2-x^2}dx$ with $-r \le a < b \le r$. We start by using a trick. That is, to use partial integration, we multiply by function to be integrated by 1, which we consider our g'. Then we have our function as f and $g(x) = x$. We get

$$\int_a^b \sqrt{r^2-x^2}\,dx = \int_a^b 1\cdot\sqrt{r^2-x^2}\,dx = \Big[x\sqrt{r^2-x^2}\Big]_a^b - \int_a^b x\frac{-2x}{2\sqrt{r^2-x^2}}\,dx$$

$$= \Big[x\sqrt{r^2-x^2}\Big]_a^b + \int_a^b \frac{x^2}{\sqrt{r^2-x^2}}\,dx$$

Now, we employ a second trick: $x^2 = x^2 - r + r$ using that, we get

$$= \Big[x\sqrt{r^2-x^2}\Big]_a^b - \int_a^b \sqrt{r^2-x^2}\,dx + \int_a^b \frac{r}{\sqrt{r^2-x^2}}\,dx$$

Now, we consider the special case $r = 1$ and with $\frac{1}{\sqrt{1-x^2}} = \arcsin'(x)$ get

$$= \Big[x\sqrt{1-x^2}\Big]_a^b - \int_a^b \sqrt{1-x^2}\,dx + \int_a^b \frac{1}{\sqrt{1-x^2}}\,dx$$

$$\Rightarrow 2\int_a^b \sqrt{1-x^2}\,dx = \Big[x\sqrt{1-x^2}\Big]_a^b + \Big[\arcsin(x)\Big]_a^b$$

Thus, one antiderivative of $\sqrt{1-x^2}$ is $S(x) = \frac{1}{2}(x\sqrt{1-x^2} + \arcsin(x))$.

Application 5.4.7 - generalized lecture approach 2

Let $r > 0$ and $f(x) = \sqrt{r^2-x^2}$ be defined on $[-r, r]$. Graphically, this corresponds to the half-circle above the x-axis with radius r. We want to compute the area of that half circle, i.e., $\int_a^b \sqrt{r^2-x^2}dx$ with $-r \le a < b \le r$. Now let $\phi : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-r, r], t \mapsto r\cdot\sin t$ We then also have for the inverse of ϕ^{-1} : $t \mapsto \arcsin(\frac{t}{r})$. We get with by using $t \in [-\frac{\pi}{2}, \frac{\pi}{2}], \cos t \ge 0$ in the last step

$$\int_a^b f(x)\,dx = \int_{\phi(\phi^{-1}(a))}^{\phi(\phi^{-1}(b))} f(x)\,dx$$

$$= \int_{\phi^{-1}(a)}^{\phi^{-1}(b)} f(\phi(t))\phi'(t)\,dt$$

$$= \int_{\phi^{-1}(a)}^{\phi^{-1}(b)} \sqrt{r^2 - r^2\sin^2 t}\cdot r\cdot\cos t\,dt$$

$$= r^2\int_{\phi^{-1}(a)}^{\phi^{-1}(b)} \cos^2 t\,dt$$

To compute $\int_\alpha^\beta \cos^n\ dt$ one can use $\cos t = \frac{e^{it}+e^{-it}}{2}$. For $\cos^2 t$ we get $\cos^2 t = (\frac{e^{it}-e^{-it}}{2})^2 = \frac{e^{2it}+2+e^{-2it}}{4} = \frac{1}{2}\cos(2t) + \frac{1}{2}$ for the integral we then get

$$\int_\alpha^\beta \cos^2(t)\,dt = \frac{1}{2}\int_\alpha^\beta \cos(2t)\,dt + \frac{1}{2}(\beta-\alpha)$$

$$= \frac{1}{4}\int_{2\alpha}^{2\beta}\cos(y)\,dy + \frac{\beta-\alpha}{2}$$

$$= \frac{1}{4}\Big[\sin y\Big]_{2\alpha}^{2\beta} + \frac{\beta-\alpha}{2}$$

$$= \frac{1}{4}(\sin(2\beta) - \sin(2\alpha)) + \frac{\beta-\alpha}{2}$$

Now, one would only have to substitute α / β with $\phi^{-1}(a)/\phi^{-1}(b)$.

Application 5.4.7

[...] (Some general stuff repeating the meaning of integral as area.) Let $r > 0$ and $f(x) = \sqrt{r^2-x^2}$ be defined on $[-r, r]$. Graphically, this corresponds to the half-circle above the x-axis with radius r. We want to compute the area of that half circle, i.e., $\int_{-r}^r \sqrt{r^2-x^2}dx$ Now, let $\phi : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-r, r], t \mapsto r\cdot\sin t$ We can use 5.4.6 to get

$$\int_{-r}^{r} f(x)dx = \int_{\phi(-\frac{\pi}{2})}^{\phi(\frac{\pi}{2})} f(x)dx$$

$$= \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} f(\phi(t))\phi'(t)dt$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{r^2 - r^2\sin^2 t}r\cos t\ dt$$

$$= r^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\cos^2 t}\cos t\ dt$$

As $\cos t \ge 0, t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, we get $\sqrt{\cos^2 t}\cos t$ and $\int_{-r}^r \sqrt{r^2-x^2}dx = r^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 t\ \ dt$ So, now we want to compute $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 t\ \ dt$ for which we use partial integration with $f(t) = \cos t$ and $g'(t) = \cos t$. Then, $g(t) = \sin(t)$. We use 5.4.5 to compute this integral

$$(\cos\frac{\pi}{2}\sin\frac{\pi}{2} - \cos(-\frac{\pi}{2})\sin(-\frac{\pi}{2})) - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(-\sin t)\sin t\,dt$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^2 t\ dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1-\cos^2 t)\ dt$$

So we see that $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 t\ \ dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1\ \ dt - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 t\ \ dt$ From that follows $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 t\ \ dt = \frac{\pi}{2}$ Hence: $\int_r^r\sqrt{r^2-x^2}dx = \frac{\pi}{2}r^2$

Integration of Converging Series

Consider $f(x) = \frac{1}{1-x}, |x| < 1$ which equals $= \sum_{n=0}^\infty x^n$ with convergence radius $\rho = 1$. For valid x ($|x| < 1$): $\int_0^x f(t)dt = \sum_{n=0}^\infty \frac{x^{n+1}}{n+1} = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots$ In the other direction: $\int_0^x \frac{1}{1-t}dt = \Big[-\log(1-t)\Big]_0^x = -\log(1-x) - (-\log(1)) = -\log(1-x) = \log(\frac{1}{1-x})$ So: $\log(\frac{1}{1-x}) = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots$

Approximation von Summen

Beispiel 1

$f(x) = x^a, a \ge 0$ With $\int_1^n f(x)dx = \frac{x^{a+1}-1}{a+1}$ we can approximate a sum $\frac{n^{a+1}-1}{a+1} \le 1^a + \dots + n^a \le \frac{(n+1)^{a+1}-1}{a+1}$ From that we get $1 - \frac{1}{n^{a+1}} \le \frac{1^a+\dots+n^a}{\frac{n^{a+1}}{a+1}} \le \frac{(n+1)^{a+1}(a+1)}{(a+1)n^{a+1}} - \frac{(a+1)}{(a+1)n^{a+1}}$ As we have $\to_{n\to\infty} 1$ for the lower and upper bound, we get $\lim_{n\to\infty}\frac{1^a+\dots+n^a}{\frac{n^{a+1}}{a+1}} = 1$ So, with $a = e$, we for instance get $1^e + \dots + n^e \approx \frac{n^{e+1}}{e+1}$

Beispiel 2

$f(x) = \log x$, $S_n = \log(n!)$. We have:

$$\int_1^n \log t\ dt = \Big[x\log x - x\Big]_1^n$$

$$= (n\log n - n) - (-1) = n\log n - n + 1$$

$$\Rightarrow n\log n - n + 1 \le \log n!$$

$$\le (n+1)\log(n+1) - (n+1) + 1$$

Because $\frac{n\log n - n + 1}{n\log n - n} \to_{n\to\infty} 1$ and $\frac{(n+1)\log(n+1)-(n+1)+1}{n\log n - n} \to_{n\to\infty} 1$ we get $\lim_{n\to\infty}\frac{\log n!}{n\log n - n} = 1$. So, we get $n! \approx \exp(n\log n - n) = (\frac{n}{e})^n$.

Beispiel 3

$$\sum_{n=2}^\infty \frac{1}{n^\alpha} \le \int_1^\infty \frac{1}{x^\alpha}dx \le \sum_{n=1}^\infty \frac{1}{n^\alpha}$$

$$\frac{1}{\alpha-1} \le \sum_{n=1}^\infty \frac{1}{n^\alpha} \le \frac{\alpha}{\alpha-1}$$

uneigentliche Integrale

1. $f(x) = e^{-cx}, c > 0$ is integrable on $[a, +\infty[$.
2. $f(x) = \frac{1}{x^a}$ is integrable on $[1, \infty[$ if $a > 1$. We have $\int_1^\infty \frac{1}{x^a}dx = \frac{1}{a-1}$.
3. $f(x) = \frac{1}{x^a}$ is integrable on $]0, 1]$ if $a < 1$. For example: $\int_x^1 \frac{1}{\sqrt{t}}dt = [2\sqrt{t}]_x^1 = 2 - 2\sqrt{x} \to_{x\to 0} 2$

McLaurin

We want to check whether $\sum_{n=2}^\infty \frac{1}{n(\log n)^a}$ exists or not. So we consider $\int_2^\infty \frac{1}{t(\log t)^a}dt = \int_{\log 2}^{\log x}\frac{1}{y^a}dy$ with $y = \log t$. From an example above we know that $\int_{\log 2}^\infty \frac{1}{y^a}dy$ exists if and only if $a > 1$.

Differenzierbare Funktionen

Ableitung: Definition +

Definition 4.1.1: f differenzierbar in x_0 falls $\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$ existiert. Grenzwert: $f'(x_0)$.
Äquivalent/Alt.: $f'(x_0) = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$
Tangente in x_0 : $f(x) = f'(x)(x-x_0)+f(x_0)$.
Theorem 4.1.3, Weierstrass: $f \in \mathbb{R}^D, x_0 \in D$ Häufungspunkt von D. Äquivalent:

1. f differenzierbar in x_0
2. $\exists c \in \mathbb{R}, r \in \mathbb{R}^D$
   - $f(x) = f(x_0)+c(x-x_0)+r(x)(x-x_0)$
   - $r(x_0) = 0$ und r ist stetig in x_0
   Dann $c = f'(x_0)$ eindeutig bestimmt.

Theorem 4.1.4: $f \in \mathbb{R}^D$ differenzierbar in $x_0 \Leftrightarrow \exists \phi \in \mathbb{R}^D$ stetig in $x_0, f(x) = f(x_0)+\phi(x)(x-x_0)(\forall x \in D)$. Dann $\phi(x_0) = f'(x_0)$.

Corollary 4.1.5: f differenzierbar in $x_0 \Rightarrow f$ stetig in x_0

Definition 4.1.7: $f \in \mathbb{R}^D$ differenzierbar in $D \Leftrightarrow f$ differenzierbar $\forall x_0 \in D$ Häufungspunkt
Theorem 4.1.9: $D \subset \mathbb{R}, x_0 \in D$ Häufungspunkt, $f, g \in \mathbb{R}^D$ differenzierbar in x_0:

- $f+g$ differenzierbar in x_0: $(f+g)'(x_0) = f'(x_0)+g'(x_0)$
- $f\cdot g$ differenzierbar in x_0: $(f\cdot g)'(x_0) = f'(x_0)g(x_0)+f(x_0)g'(x_0)$
- $f(x_0) \neq 0$: $\frac{f}{g}$ differenzierbar in x_0: $(\frac{f}{g})'(x_0) = \frac{f'(x_0)g(x_0)-f(x_0)g'(x_0)}{g(x_0)^2}$

Theorem 4.1.11: $D, E \subset \mathbb{R}, x_0 \in D$ Häufungspunkt von D. $f \in E^D$ differenzierbar in x_0, $y_0 := f(x_0)$ Häufungspunkt von E, $g \in \mathbb{R}^E$ differenzierbar in y_0. Dann: $g \circ f \in \mathbb{R}^D$ differenzierbar in x_0 mit $(g\circ f)'(x_0) = g'(f(x_0))f'(x_0)$.

Corollary 4.1.12: $f \in E^D$ Bijektion differenzierbar in x_0, $x_0 \in D$ Häufungspunkt, $f'(x_0) \neq 0$, f^{-1} stetig in $y_0 = f(x_0)$. Dann: y_0 Häufungspunkt von E, f^{-1} differenzierbar in $y_0 : (f^{-1})'(y_0) = \frac{1}{f'(x_0)}$.

erste Ableitung

Definition 4.2.1: $f \in \mathbb{R}^D, x_0 \in D, D \subset \mathbb{R}$

1. f lokales Maximum in $x_0 \Leftrightarrow \exists\delta > 0 : f(x) \leq f(x_0)(\forall x \in]x_0-\delta, x_0+\delta[\cap D)$
2. f lokales Minimum in $x_0 \Leftrightarrow \exists\delta > 0 : f(x) \geq f(x_0)(\forall x \in]x_0-\delta, x_0+\delta[\cap D)$
3. f lokales Extremum in $x_0 \Leftrightarrow$ lokales Minimum oder Maximum in x_0

Theorem 4.2.2: $f :]a,b[\to \mathbb{R}, x_0 \in]a,b[, f$ differenzierbar in x_0.

1. $f'(x_0) > 0$: $\exists\delta > 0$
   - $f(x) > f(x_0), \forall x \in]x_0, x_0+\delta[$
   - $f(x) < f(x_0), \forall x \in]x_0-\delta, x_0[$
2. $f'(x_0) < 0$: $\exists\delta > 0$
   - $f(x) < f(x_0), \forall x \in]x_0, x_0+\delta[$
   - $f(x) > f(x_0), \forall x \in]x_0-\delta, x_0[$
3. f lokales Extremum in $x_0 \Rightarrow f'(x_0) = 0$

Theorem 4.2.3, Rolle 1690: $f : [a,b] \to \mathbb{R}$ stetig, differenzierbar in $]a,b[$. $f(a) = f(b) \Rightarrow \exists\zeta \in]a,b[, f'(\zeta) = 0$
Theorem 4.2.4, Lagrange 1797: $f : [a,b] \to \mathbb{R}$ stetig, differenzierbar in $]a,b[$. $\Rightarrow \exists\zeta \in]a,b[$: $f(b)-f(a) = f'(\zeta)(b-a)$
Corollary 4.2.5: $f, g : [a,b] \to \mathbb{R}$ stetig und differenzierbar in $]a,b[$

1. $f'(\zeta) = 0, \forall\zeta \in]a,b[\Rightarrow f$ konstant
2. $f'(\zeta) = g'(\zeta), \forall\zeta \in]a,b[\Rightarrow \exists c \in \mathbb{R}, f(x) = g(x)+c, \forall x \in [a,b]$
3. $f'(\zeta) \geq 0, \forall\zeta \in]a,b[\Rightarrow f$ monoton wachsend auf $[a,b]$
4. $f'(\zeta) > 0, \forall\zeta \in]a,b[\Rightarrow f$ strikt monoton wachsend auf $[a,b]$
5. $f'(\zeta) \leq 0, \forall\zeta \in]a,b[\Rightarrow f$ monoton fallend auf $[a,b]$
6. $f'(\zeta) < 0, \forall\zeta \in]a,b[\Rightarrow f$ strikt monoton fallend auf $[a,b]$
7. $\exists M \geq 0, |f'(\zeta)| \leq M, \forall\zeta \in]a,b[\Rightarrow \forall x_1, x_2 \in [a,b], |f(x_1)-f(x_2)| \leq M|x_1-x_2|$.

Theorem 4.2.9, Cauchy: $f, g : [a,b] \to \mathbb{R}$ stetig, differenzierbar in $]a,b[\Rightarrow \exists\zeta \in]a,b[, g'(\zeta)(f(b)-f(a)) = f'(\zeta)(g(b)-g(a))$. Falls $g'(x) \neq 0, \forall x \in]a,b[$: $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\zeta)}{g'(\zeta)}$

Theorem 4.2.10, l'Hospital: $f, g :]a,b[\to \mathbb{R}$ differenzierbar, $g'(x) \neq 0, \forall x \in]a,b[$. Wenn $\lim_{x\to b^-} f(x) = 0, \lim_{x\to b^-} g(x) = 0$ und $\lim_{x\to b^-}\frac{f'(x)}{g'(x)} =: \lambda$ existiert $\Rightarrow \lim_{x\to b^-}\frac{f(x)}{g(x)} = \lim_{x\to b^-}\frac{f'(x)}{g'(x)}$. Also with $b = +\infty, \lambda = +\infty, x \to a^+$.

Theorem 4.2.16: $f \in \mathbb{R}^{]a,b[}$ differenzierbar in $]a,b[$. (streng) konvex $\Leftrightarrow f'$ (streng) monoton wachsend
Corollary 4.2.17: $f \in \mathbb{R}^{]a,b[}$ zweimal differenzierbar in $]a,b[$. f (streng) konvex $\Leftarrow f'' \geq 0 (f'' > 0)$ auf $]a,[$.

Theorem 4.3.3: $n \geq 1, f, g \in \mathbb{R}^D$ n-mal differenzierbar in D

1. $f+g$ n-mal differenzierbar: $(f+g)^{(n)} = f^{(n)}+g^{(n)}$
2. $f\cdot g$ n-mal differenzierbar: $(f\cdot g)^{(n)} = \sum_{k=0}^{n}\binom{n}{k}f^{(k)}g^{(n-k)}$

Corollary 4.4.6, Taylor Approximation: $f : [c,d] \to \mathbb{R}$ stetig, $(n+1)$-mal differenzierbar in $]c,d[$. $\forall a \in]c,d[, \forall x \in [c,d], \exists\zeta$ zwischen x, a :
$f(x) = \sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k+\frac{f^{(n+1)}(\zeta)}{(n+1)!}(x-a)^{n+1}$

Means: Taylor polynomial good approximation for f near x_0
Corollary 4.4.7: $n \geq 0, a < x_0 < b, f \in \mathbb{R}^{[a,b]}$ $(n+1)$-mal differenzierbar in $]a,b[$. Wenn $f^{(1)} = f^{(2)} = ... = f^{(n)} = 0$

1. n gerade, x_0 lokale Extremstelle $\Rightarrow f^{(n+1)}(x_0) = 0$
2. n ungerade, $f^{(n+1)}(x_0) > 0 \Rightarrow x_0$ strikte lokale Minimalstelle
3. n ungerade, $f^{(n+1)}(x_0) < 0 \Rightarrow x_0$ strikte lokale Maximalstelle

Corollary 4.4.8: $f : [a,b] \to \mathbb{R}$ stetig, zweimal differenzierbar in $]a,b[$. $a < x_0 < b$. Falls $f'(x_0) = 0$

1. $f^{(2)}(x_0) > 0 \Rightarrow x_0$ strikte lokale Minimalstelle
2. $f^{(2)}(x_0) < 0 \Rightarrow x_0$ strikte lokale Maximalstelle

Riemann Integral

$a < b, I = [a,b]$

Itegrierbare Funktionen

Theorem 5.2.1: $f, g \in \mathbb{R}^{[a,b]}$ beschränkt, integrierbar, $\lambda \in \mathbb{R} \Rightarrow f+g, \lambda\cdot f, f\cdot g, |f|, \max(f,g), \min(f,g), \frac{f}{g}(g(x) \neq 0, \forall x \in [a,b])$.
Corollary 5.2.2: $\psi : [c,d] \to \mathbb{R}$ beschränkt $\Rightarrow \sup_{x,y\in[a,b]}|\psi(x)-\psi(y)| = \sup_{x\in[c,d]}\psi(x) - \inf_{x\in[c,d]}\psi(x)$

Corollary 5.2.3: P, Q Polynome, $[a,b]$ ohne Q Nullstelle. Dann $[a,b] \to \mathbb{R}, x \mapsto \frac{P(x)}{Q(x)}$ integrierbar.

Definition 5.2.4: $f \in \mathbb{R}^D$ gleichmässig stetig in $D \subset \mathbb{R} \Leftrightarrow \forall\epsilon > 0, \exists\delta > 0, \forall x, y \in D : (|x-y| < \delta \Rightarrow |f(x)-f(y)| < \epsilon)$
Theorem 5.2.6, Heine: $f : [a,b] \to \mathbb{R}$ stetig in $[a,b]$. f gleichmässig stetig in $[a,b]$.

Theorem 5.2.7: $f \in \mathbb{R}^{[a,b]}$ stetig $\Rightarrow f$ integrierbar
Theorem 5.2.8: $f : [a,b] \to \mathbb{R}$ monoton $\Rightarrow f$ ist integrierbar

Corollary 5.2.9: $a < b < c, f \in \mathbb{R}^{[a,c]}$ beschränkt, $f|_{[a,b]}$ und $f|_{[b,c]}$ integrierbar $\Rightarrow f$ integrierbar mit $\int_a^c f(x)dx = \int_a^b f(dx)dx + \int_b^c f(x)dx$.
Definition: $\int_a^a f(x)dx = 0$ und wenn $a < b$: $\int_b^a (fx)dx := -\int_a^b f(x)dx$.

Theorem 5.2.10: $I \subsetneq \mathbb{R}$ kompaktes Intervall $[a,b]$, $f_1, f_2 \in \mathbb{R}^I$ beschränkt integrierbar, $\lambda_1, \lambda_2 \in \mathbb{R}$: $\int_a^b (\lambda_1 f_1(x) + \lambda_2 f_2(x))dx = \lambda_1 \int_a^b f_1(x)dx + \lambda_2 \int_a^b f_2(x)dx$

Ungleichungen und der Mittelwertsatz

Theorem 5.3.1: $f, g : [a,b] \to \mathbb{R}$ beschränkt integrierbar, $f(x) \leq g(x), \forall x \in [a,b] \Rightarrow \int_a^b f(x)dx \leq \int_a^b g(x)dx$

Corollary 5.3.2: $f : [a,b] \to \mathbb{R}$ beschränkt integrierbar $\Rightarrow \left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)dx|dx$

Theorem 5.3.3, Cauchy-Schwarz-Ungleichung: $f, g : [a,b] \to \mathbb{R}$ beschränkt integrierbar $\Rightarrow \left|\int_a^b f(x)g(x)dx\right| \leq \sqrt{\int_a^b f^2(x)dx}\sqrt{\int_a^b g^2(x)dx}$

Theorem 5.3.4, Mittelwertsatz: $f \in \mathbb{R}^{[a,b]}$ stetig $\Rightarrow \exists \zeta \in [a,b], \int_a^b f(x)dx = f(\zeta)(b-a)$

Theorem 5.3.6: $f, g \in \mathbb{R}^{[a,b]}$, f stetig, g beschränkt integrierbar, $g(x) \geq 0, \forall x \in [a,b] \Rightarrow \exists \zeta \in [a,b]$, $\int_a^b f(x)g(x)dx = f(\zeta)\int_a^b g(x)dx$

Fundamentalsatz der Differentialrechnung

Theorem 5.4.1: $a < b, f : [a,b] \to \mathbb{R}$ stetig. $F(x) = \int_a^x f(t)dt, a \leq x \leq b$, stetig differenzierbar in $[a,b]$ und $F'(x) = f(x), \forall x \in [a,b]$

Definition 5.4.2: $a < b, f \in \mathbb{R}^{[a,b]}$ stetig. $f : [a,b] \to \mathbb{R}$ ist Stammfunktion von $f \Leftrightarrow F$ (stetig) differenzierbar in $[a,b], F' = f$ in $[a,b]$.

Theorem 5.4.3, Fundamentalsatz der Differenzialrechnung: $f : [a,b] \to \mathbb{R}$ stetig $\Rightarrow \exists F$ von f (eindeutig bis auf additive Konstante): $\int_a^b f(x)dx = F(b) - F(a)$

Notation: $[f(x)]_a^b := f(b) - f(a)$

Theorem 5.4.5, Partielle Integration: $a < b \in \mathbb{R}$, $f, g \in \mathbb{R}^{[a,b]}$ stetig differenzierbar $\Rightarrow \int_a^b f(x)g'(x)dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)dx$.

Theorem 5.4.6, Substitution: $a < b$, $\phi \in \mathbb{R}^{[a,b]}$ stetig differenzierbar, $\phi([a,b]) \subset I \subset \mathbb{R}, I$ interval, $f : \mathbb{R}^I$ stetig $\Rightarrow \int_{\phi(a)}^{\phi(b)} f(x)dx = \int_a^b f(\phi(t))\phi'(t)dt$

Corollary 5.4.8: $I \subset \mathbb{R}, f \in \mathbb{R}^I, a, b, c \in \mathbb{R}$

- $[a+c, b+c] \subset I, \int_{a+c}^{b+c} f(x)dx = \int_a^b f(t+c)dt$
- $c \neq 0, [ac, bc] \subset I, \int_a^b f(ct)dt = \frac{1}{c}\int_{ac}^{bc} f(x)dx$

Integration konvergenter Reihen

Theorem 5.5.1: $f_n : [a,b] \to \mathbb{R}$ Folge beschränkter, integrierbarer Funktionen, gleichmässig konvergent zu $f : [a,b] \to \mathbb{R} \Rightarrow f$ beschränkt integrierbar $\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$.

Corollary 5.5.2: $f_n : [a,b] \to \mathbb{R}$ Folge beschränkter integrierbarer Funktionen so dass $\sum_{n=0}^\infty f_n$ gleichmässig konvergiert auf $[a,b] \Rightarrow \sum_{n=0}^\infty \int_a^b f_n(x)dx = \int_a^b (\sum_{n=0}^\infty f_n(x))dx$

Corollary 5.5.3: $f(x) = \sum_{n=0}^\infty c_k x^k$ Potenzreihe mit $\rho > 0 \Rightarrow \forall 0 \leq r < \rho, f$ integrierbar auf $[-r, r], \forall x \in]-\rho, \rho[$: $\int_0^x f(t)dt = \sum_{n=0}^\infty \frac{c_n}{n+1}x^{n+1}$

uneigentliche Integrale

Definition 5.8.1: $f : [a, \infty[\to \mathbb{R}$ beschränkt und integrierbar auf $[a,b], \forall b > a$. Wenn $\lim_{b\to\infty} \int_a^b f(x)dx$ existiert, bezeichnen $\int_a^\infty f(x)dx$. "f integrierbar auf $[0,\infty[$".

Lemma 5.8.3: $f : [a, \infty[\to \mathbb{R}$ beschränkt, integrierbar $[a,b], \forall b > a$

1. $|f(x)| \leq g(x) (\forall x \geq a), g(x)$ integrierbar auf $[a, \infty[\Rightarrow f$ integrierbar auf $[a, \infty[$
2. $0 \leq g(x) \leq f(x), \int_a^\infty g(x)dx$ divergent $\Rightarrow \int_a^\infty f(x)dx$ divergent

Theorem 5.8.5, McLaurin: $f : [1, \infty[\to [0, \infty[$ monoton fallend. $\sum_{n=1}^\infty f(n)$ konvergiert $\Leftrightarrow \int_1^\infty f(x)dx$ konvergiert

Definition 5.8.8: $f :]a,b] \to \mathbb{R}$ (auf $[a+\epsilon, b], \epsilon > 0$, beschränkt und integrierbar) integrierbar $\Leftrightarrow \lim_{\epsilon\to 0^+} \int_{a+\epsilon}^b f(x)dx$ existiert. Grenzwert: $\int_a^b f(x)dx$

Gamma Funktion in der Vorlesung nicht betrachtet.

unbestimmte Integrale

$f \in \mathbb{R}^I, I \subset \mathbb{R}$. f stetig $\Rightarrow \int f(x)dx = F(x) + C$ für Stammunktion F.

Theorem Partielle Integration: $\int f \cdot g' = f \cdot g - \int f' \cdot g$

Theorem Substitution: $\int f(\phi(u))\phi'(u)du = F \circ \phi(u)$

Stammfunktionen rationaler Funktionen

$R(x) = \frac{P(x)}{Q(x)}$: $\int R(x)dx$ lässt sich als elementare Funktion darstellen.

1. Reduktion auf $\deg(P) < \deg(Q)$. Verwende: Euklidischer Algorithmus.
2. Zerlegung in Summe von Brüchen bestimmter Formen
   (a) Einfache Polynome sind bereits bekannt.
   (b) $\int \frac{a}{bx+c}dx = \frac{a}{b}\int \frac{dy}{y}$ (with $y = bx+c, dy = bdx) = \frac{a}{b}(\log(y) + C) = \frac{a}{b}(\log(bx+c) + C)$
   (c) must have $d^2 - 4ec < 0, c \neq 0$
   $\boxed{\int \frac{ax+b}{cx^2+dx+e}dx}$ (with $y = x + \frac{d}{2c}$ / $x = y - \frac{d}{2c}$, $dx = dy, \alpha = \frac{e}{c} - \frac{d^2}{4c^2}$)
   $\boxed{= \int \frac{a(y - \frac{d}{2c}) + b}{c(y^2) + \alpha}dy}$ (with $w = \frac{y}{\sqrt{\alpha}}, dw = \frac{1}{\sqrt{\alpha}}dy, y = w\sqrt{\alpha})$
   $\boxed{= \sqrt{\alpha}\int \frac{a(w\sqrt{\alpha} - \frac{d}{2c}) + b}{c\alpha(w^2+1)}dw}$
   Mit $\int \frac{w}{w^2+1} = \frac{1}{2}\int \frac{2w}{w^2+1}dw = \frac{1}{2}\log(w^2+1)$ & $\int \frac{1}{w^2+1}dw = \arctan(w)$ gelöst werden.
3. Integration der Partialbrüche

Example for (b)

$\int \frac{x^3}{x^2-1}$. First, $\frac{x^3}{x^2-1} = \frac{x\cdot(x^2-1)+x}{x^2-1} = x + \frac{x}{x^2-1}$. Then, $\frac{x}{x^2-1} = \frac{1}{2}(\frac{1}{x-1} + \frac{1}{x+1}) \Rightarrow \frac{x^3}{x^2-1} = x + \frac{\frac{1}{2}}{x-1} + \frac{1}{2}\frac{1}{x+1}$. Now, we can aply 2 and get: $\int \frac{x^3}{x^2-1}dx = \frac{x^2}{2} + \frac{1}{2}\log(x-1) + \frac{1}{2}\log(x+1) + C$

Example for (c)

We have $\int \frac{dx}{x^2+x+1}$ This satisfies all requirements regarding the polynomial coefficients. Notice that $x^2+x+1 = (x+\frac{1}{2})^2 + 1 - \frac{1}{4} = (x+\frac{1}{2})^2 + \frac{3}{4}$ We substitute $y = x + \frac{1}{2}$ and get $\int \frac{dy}{y^2+\frac{3}{4}}$ Then, we substitute $y = \frac{\sqrt{3}}{2}w, dy = \frac{\sqrt{3}}{2}dw$ and get $\frac{\sqrt{3}}{2}\int \frac{1}{\frac{3}{4}(w^2+1)}dw$ So we get this and resubstitute: $\int \frac{dx}{x^2+x+1} = \frac{\sqrt{3}}{2}\frac{4}{3}\arctan(w) = \frac{2}{\sqrt{3}}\arctan(\frac{2y}{\sqrt{3}}) = \frac{2}{\sqrt{3}}\arctan(\frac{2}{\sqrt{3}}(x+\frac{1}{2}))$

(c) w/o condition example Concept, not entirely correct.

$$\int \frac{ax+b}{cx^2+dx+e}dx$$
$$= \int \frac{x+1}{x^2+4x+2}dx$$
with $y = x+2, dy = dx$
$$= \int \frac{y-1}{y^2-2}dy$$
$$= -\frac{1}{2}\int \frac{y-1}{1-\frac{y^2}{2}}dy$$
with $\omega = \frac{y}{\sqrt{2}}$
$$= -\frac{1}{\sqrt{2}}\int \frac{\omega\sqrt{2}-1}{1-\omega^2}d\omega$$
$$= -\int \frac{\omega}{1-\omega^2}d\omega + \frac{1}{\sqrt{2}}\int \frac{1}{1-\omega^2}d\omega$$
$$= \frac{1}{2}\int \frac{-2\omega}{1-\omega^2}d\omega + \frac{1}{\sqrt{2}}\tanh^{-1}(\omega)$$
$$= \frac{1}{2}\ln(1-\omega^2) + \frac{1}{\sqrt{2}}\tanh^{-1}(\omega)$$
$$= \frac{1}{2}(\ln(1-\frac{y^2}{2}) + \frac{1}{\sqrt{2}}\tanh^{-1}(\frac{y}{\sqrt{2}})$$
$$= \frac{\ln(1-\frac{(x+2)^2}{2})}{2} + \frac{1}{\sqrt{2}}\tanh^{-1}(\frac{x+2}{\sqrt{2}})$$