Inference in regression

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Recall our model and fitted values

- Consider the model

$$Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$$

- $\epsilon \sim N(0, \sigma^2)$.
- We assume that the true model is known.
- We assume that you've seen confidence intervals and hypothesis tests before.
- $\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}$
- $\hat{\beta}_1 = Cor(Y, X) \frac{Sd(Y)}{Sd(X)}$.

Review

- Statistics like $\frac{\hat\theta - \theta}{\hat\sigma_{\hat\theta}}$ often have the following properties.
    1. Is normally distributed and has a finite sample Student's T distribution if the estimated variance is replaced with a sample estimate (under normality assumptions).
    2. Can be used to test $H_0 : \theta = \theta_0$ versus $H_a : \theta >, <, \neq \theta_0$.
    3. Can be used to create a confidence interval for θ via $\hat\theta \pm Q_{1-\alpha/2} \hat\sigma_{\hat\theta}$ where $Q_{1-\alpha/2}$ is the relevant quantile from either a normal or T distribution.
- In the case of regression with iid sampling assumptions and normal errors, our inferences will follow very similarily to what you saw in your inference class.
- We won't cover asymptotics for regression analysis, but suffice it to say that under assumptions on the ways in which the X values are collected, the iid sampling model, and mean model, the normal results hold to create intervals and confidence intervals

Standard errors (conditioned on X)

$$
\begin{aligned}
\mathrm{Var}(\hat{\beta}_1) &= \mathrm{Var}\left(\frac{\sum_{i=1}^{n}(Y_i - \bar{Y})(X_i - \bar{X})}{\sum_{i=1}^{n}(X_i - \bar{X})^2}\right) \\
&= \frac{\mathrm{Var}\left(\sum_{i=1}^{n} Y_i(X_i - \bar{X})\right)}{\left(\sum_{i=1}^{n}(X_i - \bar{X})^2\right)^2} \\
&= \frac{\sum_{i=1}^{n} \sigma^2 (X_i - \bar{X})^2}{\left(\sum_{i=1}^{n}(X_i - \bar{X})^2\right)^2} \\
&= \frac{\sigma^2}{\sum_{i=1}^{n}(X_i - \bar{X})^2}
\end{aligned}
$$

Results

- $\sigma^2_{\hat\beta_1} = Var(\hat\beta_1) = \sigma^2 / \sum_{i=1}^n (X_i - \bar X)^2$
- $\sigma^2_{\hat\beta_0} = Var(\hat\beta_0) = \left(\frac{1}{n} + \frac{\bar X^2}{\sum_{i=1}^n (X_i - \bar X)^2}\right)\sigma^2$
- In practice, σ is replaced by its estimate.
- It's probably not surprising that under iid Gaussian errors

$$\frac{\hat\beta_j - \beta_j}{\hat\sigma_{\hat\beta_j}}$$

follows a t distribution with $n-2$ degrees of freedom and a normal distribution for large n.

- This can be used to create confidence intervals and perform hypothesis tests.

Example diamond data set

```r
library(UsingR); data(diamond)
y <- diamond$price; x <- diamond$carat; n <- length(y)
beta1 <- cor(y, x) * sd(y) / sd(x)
beta0 <- mean(y) - beta1 * mean(x)
e <- y - beta0 - beta1 * x
sigma <- sqrt(sum(e^2) / (n-2))
ssx <- sum((x - mean(x))^2)
seBeta0 <- (1 / n + mean(x) ^ 2 / ssx) ^ .5 * sigma
seBeta1 <- sigma / sqrt(ssx)
tBeta0 <- beta0 / seBeta0; tBeta1 <- beta1 / seBeta1
pBeta0 <- 2 * pt(abs(tBeta0), df = n - 2, lower.tail = FALSE)
pBeta1 <- 2 * pt(abs(tBeta1), df = n - 2, lower.tail = FALSE)
coefTable <- rbind(c(beta0, seBeta0, tBeta0, pBeta0), c(beta1, seBeta1, tBeta1, pBeta1))
colnames(coefTable) <- c("Estimate", "Std. Error", "t value", "P(>|t|)")
rownames(coefTable) <- c("(Intercept)", "x")
```

Example continued

```
coefTable
```

```
             Estimate Std. Error t value   P(>|t|)
(Intercept)    -259.6      17.32  -14.99 2.523e-19
x              3721.0      81.79   45.50 6.751e-40
```

```
fit <- lm(y ~ x);
summary(fit)$coefficients
```

```
             Estimate Std. Error t value  Pr(>|t|)
(Intercept)    -259.6      17.32  -14.99 2.523e-19
x              3721.0      81.79   45.50 6.751e-40
```

Getting a confidence interval

```
sumCoef <- summary(fit)$coefficients
sumCoef[1,1] + c(-1, 1) * qt(.975, df = fit$df) * sumCoef[1, 2]
```

```
[1] -294.5 -224.8
```

```
sumCoef[2,1] + c(-1, 1) * qt(.975, df = fit$df) * sumCoef[2, 2]
```

```
[1] 3556 3886
```

With 95% confidence, we estimate that a 0.1 carat increase in diamond size results in a 355.6 to 388.6 increase in price in (Singapore) dollars.

Prediction of outcomes

- Consider predicting Y at a value of X
  - Predicting the price of a diamond given the carat
  - Predicting the height of a child given the height of the parents
- The obvious estimate for prediction at point x_0 is

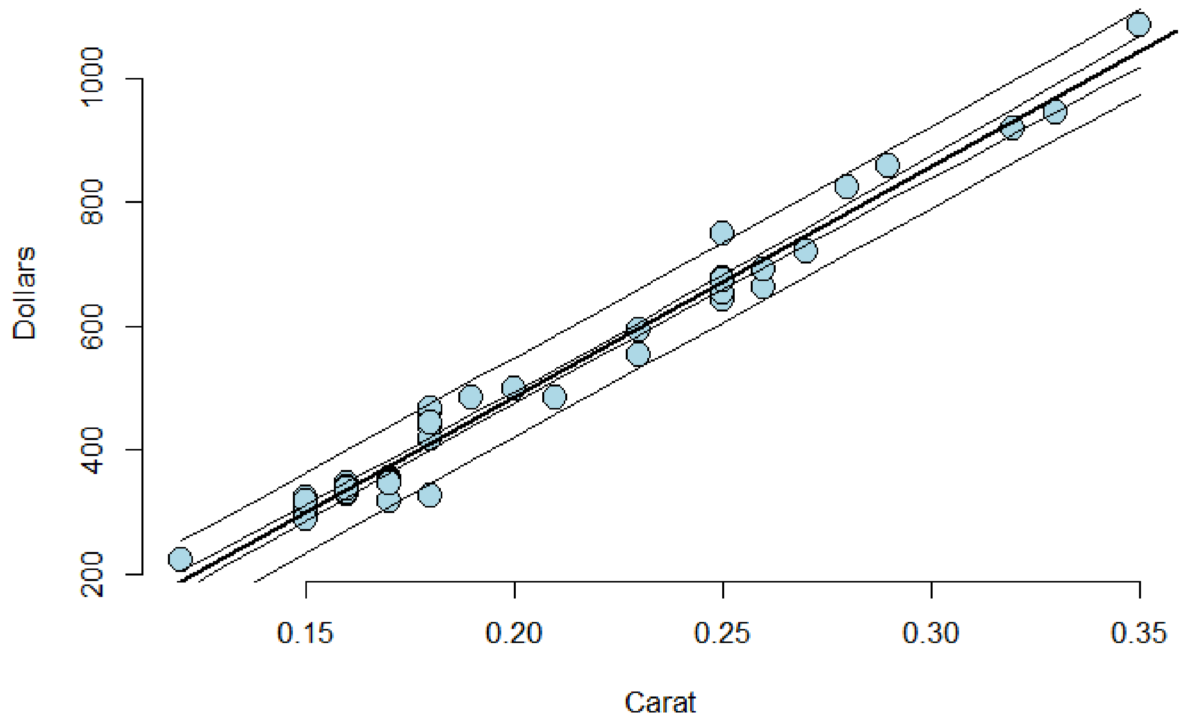
$$\hat{\beta}_0 + \hat{\beta}_1 x_0$$

- A standard error is needed to create a prediction interval.
- There's a distinction between intervals for the regression line at point x_0 and the prediction of what a y would be at point x_0.
- Line at x_0 se, $\hat{\sigma}\sqrt{\frac{1}{n} + \frac{(x_0 - \bar{X})^2}{\sum_{i=1}^n (X_i - \bar{X})^2}}$
- Prediction interval se at x_0, $\hat{\sigma}\sqrt{1 + \frac{1}{n} + \frac{(x_0 - \bar{X})^2}{\sum_{i=1}^n (X_i - \bar{X})^2}}$

Plotting the prediction intervals

```
plot(x, y, frame=FALSE,xlab="Carat",ylab="Dollars",pch=21,col="black", bg="lightblue", cex=2)
abline(fit, lwd = 2)
xVals <- seq(min(x), max(x), by = .01)
yVals <- beta0 + beta1 * xVals
se1 <- sigma * sqrt(1 / n + (xVals - mean(x))^2/ssx)
se2 <- sigma * sqrt(1 + 1 / n + (xVals - mean(x))^2/ssx)
lines(xVals, yVals + 2 * se1)
lines(xVals, yVals - 2 * se1)
lines(xVals, yVals + 2 * se2)
lines(xVals, yVals - 2 * se2)
```

Plotting the prediction intervals



Discussion

- Both intervals have varying widths.
  - Least width at the mean of the Xs.
- We are quite confident in the regression line, so that interval is very narrow.
  - If we knew β_0 and β_1 this interval would have zero width.
- The prediction interval must incorporate the variabilibity in the data around the line.
  - Even if we knew β_0 and β_1 this interval would still have width.

In R

```r
newdata <- data.frame(x = xVals)
p1 <- predict(fit, newdata, interval = ("confidence"))
p2 <- predict(fit, newdata, interval = ("prediction"))
plot(x, y, frame=FALSE,xlab="Carat",ylab="Dollars",pch=21,col="black", bg="lightblue", cex=2)
abline(fit, lwd = 2)
lines(xVals, p1[,2]); lines(xVals, p1[,3])
lines(xVals, p2[,2]); lines(xVals, p2[,3])
```

In R