Surface Loading effects on the LHC tunnel

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1 Disclaimer

The stuff below still needs to be checked thoroughly and may be error-prone. It contains some thoughts and a theoretical framework are gathered here to quantify the circumference changes due to surface loading induced deformations of the bedrock.

2 Horizontal deformations due to surface loading

3 LHC circumference changes due to horizontal surface deformations

Assumptions:

- ring is positioned horizontally (it is only approximately horizontal) is)
- Horizontal deformations in and around the ring can be linearized wrt to the center point.

The circumference of the LHC ring, ΔL, may be computed by a path integral over the ring which is deformed from a perfect circle by the deformation vector $\mathbf{h}(x,y)$:

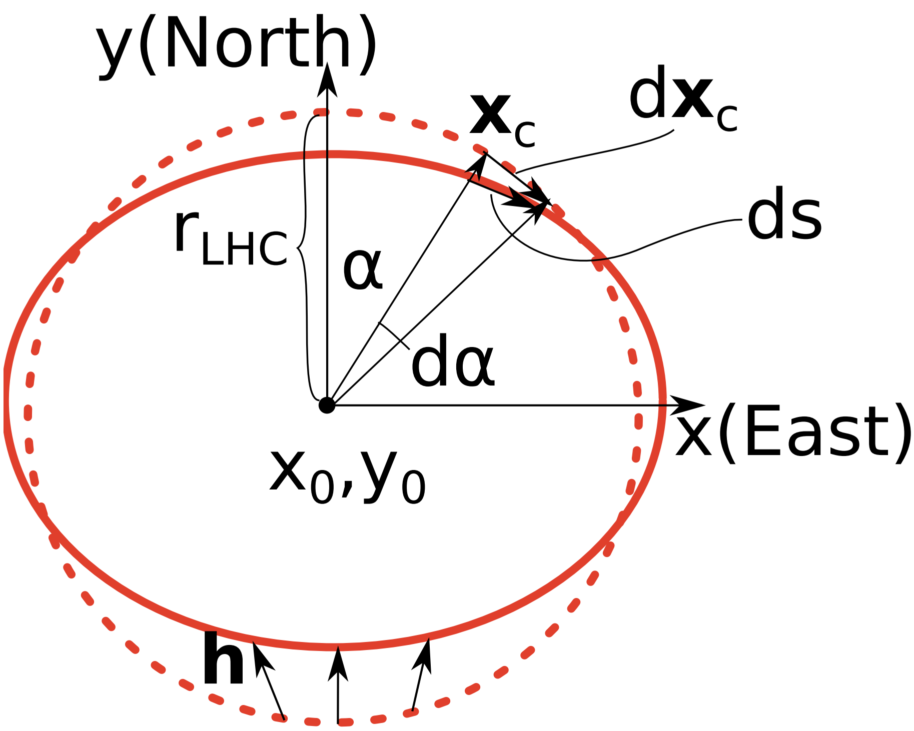
$$L = \oint_{deformedring} ds(x,y) \tag{1}$$

We parameterize the path in terms of the angle α. The infinitesimal path length $ds(x,y)$, can then be linked to $d\alpha$ by:

$$\begin{aligned} ds(x,y) = |\mathbf{x}_c(\alpha+d\alpha) + \mathbf{h}(\alpha+d\alpha) - \mathbf{x}_c(\alpha) + \mathbf{h}(\alpha)| \\ = |d\mathbf{x}_c(\alpha) + d\mathbf{h}(\alpha)| \end{aligned} \tag{2}$$

In terms of vector products ds can be written as:

$$\begin{aligned} ds &= \sqrt{[d\mathbf{x}_c + d\mathbf{h}] \cdot [d\mathbf{x}_c + d\mathbf{h}]} \\ &= \sqrt{d\mathbf{x}_c \cdot d\mathbf{x}_c + 2 d\mathbf{x}_c \cdot d\mathbf{h} + d\mathbf{h} \cdot d\mathbf{h}} \end{aligned} \tag{3}$$



We now approximate the horizontal deformation $\mathbf{h}$ as a first order taylor series around the coordinate origin, $\mathbf{x}_0$:

$$\mathbf{h}(\mathbf{x}_c) \approx \mathbf{h}\Big|_{\mathbf{x}_0} + \mathbf{J}(\mathbf{h})\Big|_{\mathbf{x}_0} [\mathbf{x}_c - \mathbf{x}_0] \tag{4}$$

Where the Jacobian $\mathbf{J}$ is composed of:

$$\mathbf{J}(\mathbf{h}) = \begin{bmatrix} \frac{\partial h_x}{\partial x} & \frac{\partial h_x}{\partial y} \\ \frac{\partial h_y}{\partial x} & \frac{\partial h_y}{\partial y} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \tag{5}$$

Building the difference vector between the locations associated with $\alpha + d\alpha$ and α thus yield:

$$d\mathbf{h}(\alpha) = \mathbf{J}(\mathbf{h})\Big|_{\mathbf{x}_0} d\mathbf{x}_c(\alpha) \tag{6}$$

Substituting this in Eq. 3 yields:

$$ds = \sqrt{d\mathbf{x}_c \cdot d\mathbf{x}_c + 2d\mathbf{x}_c \cdot (\mathbf{J}(\mathbf{h})\Big|_{\mathbf{x}_0} d\mathbf{x}_c) + (\mathbf{J}(\mathbf{h})\Big|_{\mathbf{x}_0} d\mathbf{x}_c) \cdot (\mathbf{J}(\mathbf{h})\Big|_{\mathbf{x}_0})d\mathbf{x}_c)} \tag{7}$$

Since the expected horizontal deformations (order of mm) vary only slightly over several 10's of km, the last term may be safely discarded as it is scaled by the squares of these small horizontal gradients In terms of α, $d\mathbf{x}_c$ is oriented tangential to the circle and can be written as:

$$d\mathbf{x}_c = r_{LHC} \begin{bmatrix} \cos\alpha \\ -\sin\alpha \end{bmatrix} d\alpha \tag{8}$$

Subtituting the above in the integral of Eq. 1 and evaluating the vector products yields:

$$L = r_{LHC} \oint_0^{2\pi} \sqrt{1 + J_{11}\cos^2\alpha + J_{22}\sin^2\alpha - (J_{12} + J_{21})\sin\alpha\cos\alpha d\alpha} \tag{9}$$

The above integral has similarities to the complete Elliptic integral of the second kind, although an additional mixed term involving $\sin\alpha\cos\alpha$ is present, and the bounds run to 2π. In any case no analytical solution exists for this integral, so one may consider computing this numerically. However another approach would be to linearize the state of the deformed ring against the undeformed state (i.e. a perfect circle). This is generally justified as the deformations are considered small relative to the ring geometry. Note that in the the undeformed state, $\mathbf{J}(\mathbf{h}) = \mathbf{0}$, so we can linearize around the situation where all $J_{ij} = 0$. The circumference can then be approximated as:

$$L \approx L\Big|_{circle} + \sum_{i=1}^{2}\sum_{j=1}^{2} \frac{\partial L}{\partial J_{ij}}\Big|_{circle} J_{ij} = 2\pi r_{LHC} + \Delta L \tag{10}$$

Using Leibniz integral rule we can bring the differentation against J_{ij} within the integral. So one can write for the change of circumference, ΔL:

$$\Delta L = r_{LHC}\left[J_{11}\int_0^{2\pi}\cos^2 d\alpha + J_{22}\int_0^{2\pi}\sin^2 d\alpha - (J_{12} + J_{21}) r_{LHC}\int_0^{2\pi}\cos\alpha\sin\alpha d\alpha\right] \tag{11}$$

These trigonometric integrals can be evaluated analytically where the last terms yields zero over the domain $0\ldots 2\pi$ such that:

$$\Delta L = \frac{\pi r_{LHC}}{2}(J_{11} + J_{22}) = \frac{\pi r_{LHC}}{2}\left(\frac{\partial h_x}{\partial x} + \frac{\partial h_y}{\partial y}\right)\Big|_{\mathbf{x}_0} \tag{12}$$

Circumference changes caused by horizontal Earth deformations are thus linearly proportional to the radius of the ring. from a signal to noise perspective this means that large rings have clearly an advantage when loading signals are to be recovered. To compute circumference changes according to Eq. 12, we will (only) need the compute the 2 gradients $\frac{\partial h_x}{\partial x}, \frac{\partial h_y}{\partial y}$ at the ring's center. For surface loading phenomena this will be the topic of the next sections.

3.1 Hydrology and atmospheric induced LHC circumference changes from GRACE data

GRACE results are commony provided in terms of spherical harmonic Stokes coefficients which describe the Earth's potential field. Most variability of the gravity field is caused by mass redistribution close to the Earth's surface (i.e. a thin shell). Using a thin shell assumption together with an elastic Earth model, geoid changes, horizonta/vertical deformations can be computed from those Stokes coefficients.

A surface load (in equivalent water height),$T(\theta,\lambda)$, at longitude λ and co-latitude θ,can be be described in terms of a spherical harmonic expansion:

$$T(\theta,\lambda) = a\sum_{n=0}^{\infty}\sum_{m=-n}^{n} T_{nm}\bar{Y}(\theta,\lambda) \tag{13}$$

Where a is the mean Earth radius, and the spherical harmonic functions are normalized 4π -'geodesy-style'. Which means that:

$$\bar{Y}_{nm} = \begin{cases} N_{nm}P_{nm}(\cos\theta)\cos m\lambda, \; m \geq 0 \\ N_{n|m|}P_{n|m|}(\cos\theta)\sin|m|\lambda, \; m < 0 \end{cases} \tag{14}$$

Where P_{nm} are the associated Legendre functions, without a Condon-Shortly Phase applied. The normalization factor is defined as:

$$N_{nm} = \sqrt{(2-\delta_{0m})(2n+1)\frac{(n-m)!}{(n+m)!}} \tag{15}$$

In practice, the combined product $N_{n|m|}P_{n|m|}$ may be directly computed by recursion, as P_{nm} will be numerically unstable for higher degrees n.

In that form the following orrhogonality property holds:

$$\oint \bar{Y}_{nm}(\omega)\bar{Y}_{n'm'}(\omega)d\omega = 4\pi\delta_{nn'}\delta_{mm'} \tag{16}$$

Horizontal deformation can be comuted from the surface loading coefficients T_{nm} by (see e.g. Eq. 4.3 Dissertation Rietbroek 2014):

$$\mathbf{h}(\theta,\lambda) = \frac{3a\rho_w}{\rho_e}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\frac{l'_n}{2n+1}\begin{bmatrix} \frac{\partial\bar{Y}_{nm}}{\sin\theta\partial d\lambda} \\ -\frac{\partial\bar{Y}_{nm}}{\partial\theta}\end{bmatrix} T_{nm} \tag{17}$$

The Spherically symmetric Non-rotating Elastic Isotropic (SNREI) Earth model is contained within the so-called load Love numbers, l'_n,which are only dependent on the degree, and describe the horizontal deformation response due to a surface load. For the circumference changes we need the derivative of those contributions. Noting that:

$$\begin{aligned} \partial x &= a\sin\theta\partial\lambda && (18) \\ \partial y &= -a\partial\theta && (19) \\ & && (20) \end{aligned}$$

and simplifying the notation by introducing the surface gradient operator:

$$\nabla_\Omega = \begin{bmatrix} \frac{\partial}{\sin\theta\partial\lambda} \\ \frac{\partial}{\partial\theta} \end{bmatrix} \tag{21}$$

yields a cicrumference change of:

$$\Delta L = \frac{\pi r_{LHC}}{2}\frac{3\rho_w}{\rho_e}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\frac{l'_n}{2n+1}\left[\nabla_\Omega\cdot(\nabla_\Omega\bar{Y}_{nm})\right]T_{nm} \tag{22}$$

The terms with the surface gradient operator can furthermore be simplified to ease its numerical computation. Realizing that the surface spherical harmonics obey the following differenttial equation:

$$r^2 \nabla^2 \bar{Y}_{nm} = -n(n+1)\bar{Y}_{nm} \tag{23}$$

which in spherical coordinates is:

$$\frac{\cos\theta}{\sin\theta}\frac{\partial \bar{Y}_{nm}}{\partial\theta} + \frac{\partial^2 \bar{Y}_{nm}}{\partial\theta^2} + \frac{\partial^2 \bar{Y}_{nm}}{\sin^2\theta\partial\lambda^2} = -n(n+1)\bar{Y}_{nm} \tag{24}$$

So:

$$\Delta L = \frac{\pi r_{LHC}}{2}\frac{3\rho_w}{\rho_e}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\frac{-l'_n}{2n+1}\left[\frac{\cos\theta}{\sin\theta}\frac{\partial \bar{Y}_{nm}}{\partial\theta} + n(n+1)\bar{Y}_{nm}\right]T_{nm} \tag{25}$$

The spherical harmonic and its derivative can be computed numerically for the center position of the ring.

Using thin shell assumptions, the fully normalized Stokes coefficients (relative to a static field) provided by GRACE, δC_{nm}, can be linked to the Surface laoding coefficients by:

$$\delta C_{nm} = \frac{1+k'_n}{2n+1}\frac{3\rho_w}{\rho_e}T_{nm},\ n > 0 \tag{26}$$

The load Love number k'_n now describes the change in potential of the Earth upon a forcing by a surface load. So equation 25 can also be written in terms of Stokes coefficient (anomalies)

$$\Delta L = \frac{\pi r_{LHC}}{2}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\frac{-l'_n}{1+k'_n}\left[\frac{\cos\theta}{\sin\theta}\frac{\partial \bar{Y}_{nm}}{\partial\theta} + n(n+1)\bar{Y}_{nm}\right]\delta C_{nm} \tag{27}$$

TODONOTE: Consistent treatment of reference systems (i.e. degree 1 contributions)

3.2 LHC circumference changes due to water level changes in Lake Geneva

A Deformation relative to a deforming surface point

The deformation in an 'isomorphic' reference frame induced by a surface load on an elastic earth can be described in the local East, North, Up axis system as:

$$\mathbf{d}(\theta,\lambda) = \frac{3a\rho_w}{\rho_e}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\frac{1}{2n+1}\begin{bmatrix} l'_n \frac{\partial \bar{Y}_{nm}}{\sin\theta\partial d\lambda} \\ -l'_n \frac{\partial \bar{Y}_{nm}}{\partial\theta} \\ h'_n \bar{Y}_{nm}\end{bmatrix} T_{nm} \tag{28}$$

The choice of the degree 1 load Love numbers (i.e. h'_1, l'_1) determines which isomorphic reference is used to describe the deformation.

Now if we want to shift the reference frame origin, to follow that of a surface point λ_0, θ_0, we need to express this offset first in terms of cartesian coordinates apply the shift and then rotate it back in the local ENU system. On a spherical Earth the rotation matrix is described as follows:

$$\mathbf{R}_{xyz \to ENU}(\theta, \lambda) = \begin{bmatrix} -\sin\lambda & \cos\lambda & 0 \\ -\cos\lambda\cos\theta & -\sin\lambda\cos\theta & \sin\theta \\ \cos\lambda\sin\theta & \sin\lambda\sin\theta & \cos\theta \end{bmatrix} \quad (29)$$

Furthermore, one should note that the above rotation matrix can also be written in terms of degree 1 spherical harmonics and its derivatives:

$$\mathbf{R}_{xyz \to ENU}(\theta, \lambda) = \frac{1}{\sqrt{3}} \begin{bmatrix} \frac{\partial \bar{Y}_{11}}{\sin\theta \partial d\lambda} & \frac{\partial \bar{Y}_{1-1}}{\sin\theta \partial d\lambda} & 0 \\ -\frac{\partial \bar{Y}_{11}}{\partial \theta} & -\frac{\partial \bar{Y}_{1-1}}{\partial \theta} & -\frac{\partial \bar{Y}_{10}}{\partial \theta} \\ \bar{Y}_{11} & \bar{Y}_{1-1} & \bar{Y}_{10} \end{bmatrix} \quad (30)$$

We can shift the deformation $\mathbf{d}$ to a frame which moves with the deformation at the reference surface point, $\mathbf{d}_0 = \mathbf{d}(\lambda_0, \theta_0)$

$$\tilde{\mathbf{d}} = \mathbf{d} - \mathbf{R}(\theta, \lambda)\mathbf{R}^T(\theta_0, \lambda_0)\mathbf{d}_0 \quad (31)$$

It can be shown that this expression for this relative deformation is independent of the chosen (isomorphic) reference frame (and thus the associated reference frame origin of the degree 1 Love numbers are irrelavant). To see this note that the degree 1 contributions to the relative deformation above can be written in terms of the rotation matrix int he cartesian frame:

$$\tilde{\mathbf{d}}(\theta, \lambda, n = 1) = \frac{\sqrt{3} a \rho_w}{\rho_e} \left[\mathbf{\Lambda R}(\theta, \lambda) - \mathbf{R}(\theta, \lambda)\mathbf{R}^T(\theta_0, \lambda_0)\mathbf{\Lambda R}(\theta_0, \lambda_0) \right] \begin{bmatrix} T_{11} \\ T_{1-1} \\ T_{10} \end{bmatrix} \quad (32)$$

where the load Love numbers are contained within the diagonal matrix:

$$\mathbf{\Lambda} = \begin{bmatrix} l'_1 & 0 & 0 \\ 0 & l'_1 & 0 \\ 0 & 0 & h'_1 \end{bmatrix} \quad (33)$$

A shifted reference frame would be accomplished by adding a unit diagonal matrix scaled by an isomorphic frame parameter α:

$$\mathbf{\Lambda}^* = \mathbf{\Lambda} + \alpha \boldsymbol{I} \quad (34)$$

The frame independence can be tested by substituting $\mathbf{\Lambda}^*$ which will result in the same expression as using $\mathbf{\Lambda}$.