Proof of Greedy-EFT Optimality

Question

You are given a set of intervals $\mathcal{I} = \{I_1, I_2, \ldots, I_n\}$, where each interval $I_i = [s_i, f_i)$ has a start time s_i and finish time f_i.
The goal of the **Interval Scheduling Problem** is to select the largest possible subset of mutually non-overlapping intervals.
The proposed greedy algorithm, known as the **Earliest Finish Time (EFT)** algorithm, proceeds as follows:
Prove that the EFT greedy algorithm always returns an optimal solution. Use a proof by contradiction, induction, or direct argument — but **do not** use an exchange argument. Your proof should be rigorous and clearly structured.

Preliminaries

Assume the input intervals (tasks) are sorted by non-decreasing finish time

$$f_1 \;\le\; f_2 \;\le\; \cdots \;\le\; f_n.$$

Let $G = (g_1, g_2, \ldots, g_k)$ be the set of intervals selected by the *Greedy–Earliest-Finish-Time (EFT)* algorithm, where each g_i is the earliest-finishing interval compatible with all previously chosen ones. Hence

$$f_{g_1} \;\le\; f_{g_2} \;\le\; \cdots \;\le\; f_{g_k}.$$

Our goal is to show that G is an *optimal* schedule—i.e. its size k is the maximum possible. :contentReference[oaicite:0]index=0

Proof by Contradiction

Suppose, for contradiction, that G is *not* optimal. Let

$$O = (o_1, o_2, \ldots, o_m), \qquad m > k,$$

be an optimal solution that agrees with G for as many initial intervals as possible. Without loss of generality, order O by finish time:

$$f_{o_1} \;\le\; f_{o_2} \;\le\; \cdots \;\le\; f_{o_m}.$$

Define

$$r \;=\; \min\{\, i \mid g_i \neq o_i \,\}.$$

Thus $g_1 = o_1, \ldots, g_{r-1} = o_{r-1}$ but $g_r \neq o_r$.

Constructing a better optimal schedule. Because Greedy–EFT chose g_r instead of o_r, we must have $f_{g_r} \le f_{o_r}$. Form a new schedule

$$O' = (o_1, \dots, o_{r-1}, g_r, o_{r+1}, \dots, o_m).$$

- *Feasibility:* By construction, g_r is compatible with $g_{r-1} = o_{r-1}$. Because $f_{g_r} \le f_{o_r}$ and the suffix $o_{r+1}, \dots, o_m$ was compatible with o_r, it remains compatible after the replacement. Hence O' is feasible.
- *Cardinality:* $|O'| = |O| = m$.
- *Agreement with G:* O' matches G for the first r positions, one more than O did, contradicting the way we chose O.

This contradiction shows our assumption was false.

Conclusion

The Greedy–EFT algorithm produces a schedule G of maximum possible size; therefore, it is optimal.