Graph Theory Recap

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1 Lectures

1. Intro

2. Basic notions, statement and proof of Schurs thm ($\exists$ mono $a, b, c \in [n]$ st $a + b = c$, ramsey type proof). Reformulate as k-colour K_n to assist in induction.

3. Prove FINITE RAMSEY in k colours, find mono triangle (induct, set $n = k(m-1)+2$, pick vertex, has at least m edges of some colour, two cases, done). Generalise to finite ramsey, define $R(s,t)$, similar strat as with triangle to get UB $R(s,t) \leq R(s-1,t) + R(s,t-1)$ (induct on $s+t$, simple if one is 2). Get $R(s,t) \leq 2^{s+t}$. Note use this to prove $R(s,t) \leq \binom{s+t-2}{s-1}$ on sheet 1.

4. Prove multicoloured finite ramsey, induct on number of colours, colour red and not red. (Note $R_{k+1}(s) \leq R(s, R_k(s))$ used on sheet 1). INFINITE RAMSEY stated and proved, use two pass proof.

5. Prove Bolzano Weierstrass from infinite ramsey, back to basic notions. Define: order, nbhd, degree, k-regular, handshake lemma, paths, connected, tree is a connected forest, induced subgraph, complement.

6. Define O notation (capitals mean ineqs, lower case means limit, $l$ose, ωin, ΘIE fighter). EXTREMAL GRAPH THEORY. Def $\text{ex}(n, H)$ to be the most edges you can add to n verticies without accidentally creating an H (ie $\text{ex}(n, H) = \max\{e(G) : |G| = n, H \not\subseteq G\}$). Def bipartite, state bipartite iff no odd cycles. Def complete bipartite $K_{s,t}$. MANTELS thm: $n \geq 3$, $\text{e}(G) \geq \lfloor \frac{n^2}{4} \rfloor$, $K_3 \not\subset G$ then $G \cong K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$. Proof: remove edges then min deg vertex then ind hyp then add them back.

7. Def r-partite, def Turan graph $T_r(n)$ to be complete r-partite graph on n verticies with classes as equal as possible, write $t_r(n) = \text{e}(T_r(n))$. List elementary observations: $K_{r+1} \not\subset T_r(n)$ and any edge added gives K_{r+1}, has most edges of any r-partite graph on n verts, if $r|n$ then regular, if not vertex degrees differ by at most 1. Deleting a vertex of min degree removes a vert from a big class so makes $T_r(n-1)$ so $t_r(n-1) = t_r(n) - \delta(T_r(n))$. Given $T_r(n-1)$, want to add a vertex and as many edges wihtout making K_{r+1}. End up adding at most $t_r(n) - t_r(n-1)$ edges and making $T_r(n)$. State TURANS THM (straight generalisation of mantel) $n \geq r+1$, $\text{e}(G) \geq t_r(n)$, $K_{r+1} \not\subset G$ then $G \cong T_r(n)$. Induct using these observations and delete edges then min degree then add back. Cor: $\text{ex}(n, K_{r+1}) \sim (1 - \frac{1}{r})\binom{n}{2}$.

8. $\text{ex}(n, C_4)$? Note C_4 is made from two P_2s, count P_2s by middle vertex, notice that if no C_4 then each vertex is at the end of at most one P_2, use to bound then Jensen then bound by larger root of quadratic, so $\text{ex}(n, C_4) \leq \frac{n}{4}(1 + \sqrt{4n-3}) = O(n^{3/2})$. But this generalises! Same idea to show $\text{ex}(n, K_{t,t}) = O(n^{2-\frac{1}{t}})$. Problem of Zarankiewicz, find $Z(n,t)$, largest number of edges in bipartite graph, n verts in each class and no $K_{t,t}$. Thm: $Z(n,t) = O(n^{2-\frac{1}{t}})$, same idea, count t-fans. Def: $\text{ex}(H) = \lim_{n \to \infty} \frac{\text{ex}(n,H)}{\binom{n}{2}}$. Show this exists by showing decreasing bounded below (better explanation??).

9. Find $\text{ex}(H)$ in general? First def complete r-partite graph $K_r(t) = T_r(rt)$. ERDOS STONE THEOREM. Let $r, t \geq 1$ be integers, $\varepsilon > 0$. Then $\exists n_0$ st $\forall n \geq n_0$, $|G| = n, \text{e}(G) \geq (1 - \frac{1}{r} + \varepsilon)\binom{n}{2} \Rightarrow K_{r+1}(t) \subset G$. Def chromatic number $\chi(H)$ to be least r st H is r-partite. Cor: $\text{e}(H) > 0 \Rightarrow 1 - \frac{1}{\chi(H)-1}$. ($\chi(H) = r+1 \Rightarrow H \subset K_{r+1}(t)$). So $\text{ex}(n, H) \sim (1 - \frac{1}{\chi(H)-1})\binom{n}{2}$. Def density $D(G) = \text{e}(G)/\binom{|G|}{2}$, upper density $\text{ud}(G)$ to be lim sup of this over all subgraphs of given size. Cor of ES: $\text{ud}(G) \in \{0,1\} \cup \{1 - \frac{1}{r} : r \in \mathbb{N}\}$.

10. Non examble proof of ES, state in min degree form: $\delta(G) \geq (1 - \frac{1}{r} + \varepsilon)n \Rightarrow K_{r+1}(t) \subset G$. HAMILTONIAN GRAPHS. A graph of order n is hamiltonian if $\exists C_n \subset G$. DIRAC'S THM. Let G be a connected graph order $n \geq k \geq 3$ st $\delta(G) \geq k/2$. Then G contains a path of length k and a cycle of at least $(k+2)/2$(is this correct??).

11. Proof of Dirac: show cond gives diam(G)≤ 2 (via $x \nsim y \Rightarrow |\Gamma(x) \cap \Gamma(y)| \geq 2 > 0$), take maxml path, ends can only be connected to things inside path, either get big cycle or relable for contradiction. Def Euler circuit as path that travels each edge exactly once, show Eulerian iff all degrees even by induction by deleting maxml circuit, remaining edges (if any) have euler circuit by ind hyp, add back for larger circuit, #. GRAPH COLOURING. Def k-colouring.

12. Planar Graphs. Formally def drawing, show K_5, $K_{3,3}$ not planar, def subdivision. KURATOWSKI'S THM: G planar iff contains no subdivision of K_5 or $K_{3,3}$ (proof omitted). Def leaf, show trees of order ≥ 2 have some leaf (look at ends of maxml path). Let T be a tree order n, then $\mathrm{e}(T) = n - 1$ (pick off leaf (min deg), ind hyp, add back). Trees are planar (pick off leaf, ind hyp, draw on leaf). Def faces, note same number of faces in each drawing (include infinite face).

13. G planar connected, $V(G) = n$, $E(G) = m$, G has l faces. Then $n - m + l = 2$ (induct on m, if $l = 1$ then G is a tree, else see how cycle changes faces). Cor: G planar order ≥ 3 then $\mathrm{e}(G) \leq 3n - 6$ (use prev result). SIX COLOUR THEOREM, induct using prev bound to get min deg ≤ 5 delete, colour (ind hyp), add back. 5CT: pick min deg as before, either have colour free and win, else colour nbrs 1,..,5 then def ij-path and ij-cpt (not really sure whats going on). State 4CT.

14. Bounds on $\chi(G)$. Lower bounds: def clique $\omega(G) \leq \chi(G)$ (sometimes bad bound cf prob. method), def indep number $\alpha(G)$ size of maxml indep set, so $\chi(G) \geq |G|/\alpha(G)$ (sometimes bad cf $G = K_k \cup E_k$). Upper bounds: Greedy gives $\chi(G) \leq \Delta(G) + 1$ (again bad cf $K_{1,t}$). This sharp if G complete or odd cycle, and by BROOK'S THM, this is iff, ie G connected not complete or odd cycle, then $\chi(G) \leq \Delta(G)$ (prf: induct on $|G|$, rule out cases where $\Delta < 3$ so that $\Delta \geq 3$. pick arb $v \in G$, let $H = G - v$, either $\Delta(H) < \Delta$, done by greedy, or theyre equal and look at H again, H not odd cycle as $\Delta(H) = \Delta = 3$, H connected, and H not regular since if $u \sim v$ in G then $d(u) \leq \Delta$ so $d_H(v) \leq \Delta - 1$. By ind hyp $\chi(H) \leq \Delta$, repeat for each path cpt of $G - v$. Now seek to colour v whilst not raising colouring).

15. Aim to generalise planar to compact surfaces. State classification from alg top into T_g, S_g orientable and non orientable of genus g, with Euler char $2 - 2g, 2 - g$ resp. State Euler-Poincare formula: $n - m + l \geq E$. HEAWOOD'S THM: Let S be a closed boundaryless surface with Euler char $E \leq 1$. Then $\chi(S) \leq \lfloor \frac{7+\sqrt{49-24E}}{2} \rfloor$ (prf:)

16. Def edge colouring, edge chromatic number $\chi'(G)$. Easy bounds $\Delta(G) \leq \chi'(G) \overset{\text{greedy}}{\leq} 2\Delta(G) - 1$. Better than could hope for: VIZING' S THM: $\chi'(G) \leq \Delta(G) + 1$ (induct on edges, base case trivial, start by picking some use ij cpt ideas). CONNECTIVITY. Def a matching $X \to Y$ as a subset of edges of bipartition st looks like injection $X \hookrightarrow Y$. HALL's MARRIAGE THM: $\exists$ matching $X \to Y$ iff $\forall A \subset X, |\Gamma(A)| \geq |A|$ ($\Rightarrow$ obv $\Leftarrow$ ind on $|X|$, strict ineq easy, equality case write as two disjoint graphs, use incln excln).

17. Def indep set of edges, note a matching is just a set of $|X|$ indep edges . Cor: (Defect hall)$d \geq 1$, then G contains $|X| - d$ indep edges iff $\forall A \subset X, |\Gamma(X)| \geq |A| - d$ (introduce imaginary perfect men suitable for all women, find matching by Hall, delete perfect men). Cor: polyandrous hall. Def k-connected if deleting $< k$ verticies never disconnectes the graph. Def cutvertex, say paths from a to b are indep if only meet at a and b. Note if each distinct pair $a, b \in G$ have k indep paths then G is k-connected. Converse? Not trivially true (can't argue 'delete one vertex from each $k - 1$ paths to disconnect'). Def AB-path as has first vertex in A, last in B and never in $A \cup B$ otherwise. Say $W \subset V(G)$ is AB separator if $G - W$ has no AB-path (note A and B are AB separators). Big thm: let $k = \min\{|W| : W \text{ is an } AB\text{-separator}\}$. Then G contains k vertex disjoint AB-paths (NB vertx disjoint paths don't even meet in A or B). Cor: MENGER'S THM. Let G be incomplete k-connected graph, $a, b \in V(G)$. Then G has k indep ab-paths (use prev thm, $A = \Gamma(a), B = \Gamma(b)$). Def connectivity $\kappa(G) = \max(\{k \geq 1 : G \text{ is } k\text{-connected}\} \cup \{0\})$ if incomplete, in light of menger def $\kappa(K_n) = n - 1$.

18. Prove big thm from last time: every AB-separator has order $\geq k \Rightarrow \exists k$ vertex-disjoint AB-paths. Induct on $e(G)$, READ PROOF. Cor: Hall's marriage thm. Edge connectivity: def l-edge connected, edge-connectivity $\lambda(G) = \max(\{k \geq 1 : G \text{ is } l\text{-edgeconnected}\} \cup \{0\})$. Cor: Edge Menger G l-edgeconnected then G has l edge disjoint ab-paths in G. PROB METHOD. Thm: $R(s) = \Omega(\sqrt{2}^s)$ (prf: randomly colour, markov on number of mono K_s).

19. (Correction to menger proof - what fix?). PROBSTIC METHOD STRAT: can either do random thing, show $\mathbb{P}$(thing we want) > 0 OR show $\mathbb{P}$(thing we want almost happens) > 0, then modify thing to win. Return to lower bounds for Zarankiewicz; $Z(n,t) = \Omega(n^{2-\frac{2}{t+1}})$ (prf: randomly add edges to $K_{n,n}$, let $A = \mathrm{e}(G), B = \#K_{t,t}$, choose p st $\mathbb{E}(A - B)$ bigg, pick example, remove edges from each $K_{t,t}$). Thm: $\forall g \geq 3, k \geq 2, \exists G$ with no cycles

length $\leq g$ and $\chi(G) \geq k$ (prf: fix n, p, $X = \#$ cycles length $\leq g$, use $\chi(G) \geq \frac{|G|}{\alpha(G)}$, aim to find n, p st $\mathbb{P}(X > n/2) < \frac{1}{2}, \mathbb{P}(\alpha(G) \geq \frac{n}{2k}) < \frac{1}{2}$ then modify).

20. Properties of Expectation: Linearity, Markov: X non neg r.v. with fin mean, $\lambda > 0$ then $\mathbb{P}(X \geq \lambda) \leq \mathbb{E}(X)/\lambda$, chebyshev ineq, fin var σ^2, then $\mathbb{P}(|X-\mu| \geq \varepsilon) \leq \sigma^2/\varepsilon^2$ (useful: $\mathbb{P}(X = 0) \leq \mathbb{P}(|X-\mu| \geq \mu) \leq \sigma^2/\mu^2$). Structure of random graphs. See sharp threshold in $\mathbb{P}(\text{phenomena} \in G)$. Asymptotics on $\mathbb{P}(G \in \mathcal{G}(n,p)$ has an edge). Def *almost every* $G \in \mathcal{G}(n,p)$ *has property* Q as $\mathbb{P}(G \in \mathcal{G}(n,p)$ has $Q) \to 1$ as $n \to \infty$. Show for A_i events and X number of A_i that occur, have $\mathbb{E}(X) = \sum_{i=1}^{n} \mathbb{P}(A_i)$, $\mathrm{Var}(X) = \sum_{i,j} (\mathbb{P}(A_i \cap A_j) - \mathbb{P}(A_i)\mathbb{P}(A_j))$, noting A_i, A_j indep $\Rightarrow \mathbb{P}(A_i \cap A_j) - \mathbb{P}(A_i)\mathbb{P}(A_j) = 0$.

21. Prop: $p = 1/n$ sharp threshold for finding triangles, use prev formulae to help calculate. Typical clique number $\omega(G)$ for $G \in \mathcal{G}(n,p)$. Thm: $\exists d : \mathbb{N} \to \mathbb{N}$ st almost every $G \in \mathcal{G}(n,p)$ has $\omega(G) \in \{d-1, d, d+1\}$. Only prove for $p = \frac{1}{2}$, messiest bounding in the course (nice explanation??). Get $\omega(G) \sim \frac{2\log n}{\log \frac{1}{p}}$.

22. Cor: A.E. $G \in \mathcal{G}(n,p)$ has $\chi(G) \geq (1+o(1))\frac{n \log \frac{1}{1-p}}{2 \log n}$. ALGEBRAIC METHODS. Def distance, diameter of graphs, def Moore graph $|G| = k^2 + 1, \Delta(G) = k, \mathrm{diam}(G) = 2$, show C_5 and petersen graph are first two moore graphs.

23. Def chromatic poly $f_G(k)$ as number of k-colourings of a graph G. Examples: $f_{K_n}(k) = k(k-1)..(k-n+1)$, $f_{\text{tree}}(k) = k(k-1)^{|G|-1}$. Def contraction over edge G/e in topological sense. CUT FUSE RELATION:$f_G(k) = f_{G-e}(k) - f_{G/e}(k)$ (see what happens if ends of e are same or diff colour). Show $f_G(k)$ is a poly for any graph G, ind on #edges, use cut fuse. Cor: $|G| = n, \mathrm{e}(G) = m$, then $f_G(k) = X^n - mX^{n-1} + ..$ (improve to show coeffs alternate in sign and coeff of $n-2$ is $\binom{n}{2} - \#\{\text{triangles in } G\}$). EIGENVALUE METHODS. Def adj matrix, note $\left(A^2\right)_{ij} = |\Gamma(i) \cap \Gamma(j)|$, so $\left(A^2\right)_{ii} = d(i)$ so $\mathrm{Tr}(A^2) = 2\,\mathrm{e}(G)$, generally $(A^\ell)_{ij} = \#\{\text{walks length } \ell \text{ from } i \text{ to } j\}$. Note relabling verticies $\leftrightarrow$ CoB, so evals are indep, and since real symm, evals are real. Examples of spotting evals, def operator norm.

24. Thm: λ eval of $G \Rightarrow |\lambda| \leq \Delta(G)$. If connected then Δ eval iff G is Δ-regular, in which case Δ has multiplicity 1, evec $(1,..,1)$ (use triangle ineq). Def (k,a,b)-strongly-regular as k-regular, $x \sim y \Rightarrow |\Gamma(x) \cap \Gamma(y)| = a$, $x \nsim y \Rightarrow |\Gamma(x) \cap \Gamma(y)| = b$. Note sr graphs are connected with diam≤ 2, note Moore graphs are $(k,0,1)$-sr. Note for G (k,a,b)-sr, then $A^2 = kI + aA + b(J - I - A)$ for J all 1s matrix, apply to evec x not $(1,..,1)$ noting these orthogonal so $Jx = 0$, find evals, then count multiplicities to get RATIONALITY CONDITION: $|G| = n$ (k,a,b)-sr $\Rightarrow \frac{1}{2}\left\{(n-1) \pm \frac{(a-b)(n-1)+2k}{\sqrt{(b-a)^2 - 4(b-k)}}\right\} \in \mathbb{Z}$, use to find Moore graphs (noting $n = k^2 + 1$) to get $k \in \{1, 2, 3, 7, 57\}$.