# Riemann Surfaces lecture recap

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## 1 Lectures

1. Recap of CA, def analytic fn on domain (open path connected set) $\mathcal{U} \subset \mathbb{C}$ (as either everywhere diffble in nbhd or has taylor series everywhere), intro to local param, principle of isolated zeros, identity principle, def removable sing, pole, essential sing, state Casorati-Weierstrass ($a$ essential sing $\Rightarrow$ image of $f$ on any punctured nbd of $a$ is dense in $\mathbb{C}$), recall cts inverse of analytic fn is analytic

2. Def complex log, then fn element/fn germ of direct analytic cntn, then analytic cntn along a path, show this is equiv rel, def complete analytic fn as an equiv class under anltyc cntn along paths. GOAL: construct a surface on which log is analytic by gluing.

3. Def a covering map as local homeo, regular covering map is alg top defn (preimage is disj union of homeo discs). Construct gluing space for log and $z^n$. Power series and continuation: let $T$ be boundary of disc of RoC, say $z \in T$ is *regular* if have analytic fn on some open nbd of $z$, otherwise $z$ is *singular*, nb this is indep of convergence of series at $z$. Propn: if RoC=1, then $\exists$ some singular pt on $T$ (use maxmlty of RoC), say $T$ is natural boundary if every pt on $T$ is singular.

4. DEF OF A RIEMANN SURFACE. RS is a connected Haus top space with an atlas (trans fns between charts are analytic (and in fact conformal equivs), also see that $R$ is path connected (as loc pc and connected). Say atlases are equiv if their refinement (union) is an atlas (check trans maps between charts from diff atlases is analytic), this is equiv rel. An equiv class of this is a *conf structure*. Note connected subset of RS is an RS by restriction of charts. Def analytic map between RSes (analytic on all charts). If have analytic inverse, then $f : R \to S$ is *conf equiv*, eg $(\mathbb{C}, z)$, $(\mathbb{C}, \bar{z})$ conf equiv. Lemma: comp of analytic fns is analytic. Prf: draw 6 blobs, patch down to $\mathbb{C}$, analycity is local, use comp of holo fns in $\mathbb{C}$ is holo.

5. Cor: equivalence of atlases is an equiv rel (prf:comp of id is analytic). Prop: covering map $\pi : \tilde{R} \to R$ to RS $R$ defines unique conf structure on $\tilde{R}$ on which $\pi$ is analytic (prf: charts are proj to $R$ then take charts in image of nbd, use comp of anlytc fns analytc, show atlases are equiv). Shows gluing is conf structure (as defnd by quotient top). Def Riemann sphere, charts $z$ and $1/z$. Inverse function thm: given $g$ analaytic on open $V \subset \mathbb{C}$ and $a \in V$ with $g'(a) \neq 0$, then $\exists$ nbd $N$ of $a$ with $g|_N : N \to g(N)$ conf equiv (prf: wlog $g(a) = 0$, isolated zeros to find disc only 0 on $a$, argument principle to find open nbd, take open disc in this, have bijection from disc to preimage, inv cts by open mapping, cts invereses are conf equivs). Discussion about choosing chart in which $g : z \mapsto z^r$ (take chart, taylor expand $g(z) = (z-a)^r h(z)$, def $f(z) = (z-a)/h(z)^{\frac{1}{r}}$ which gives conf equiv).

6. COMPLEX TORI. Given $\{\tau_1, \tau_2\}$ $\mathbb{R}$-basis of $\mathbb{C}$, $\Lambda = \bigoplus \mathbb{Z}\tau_i$, $T = \mathbb{C}/\Lambda$:

   (a) T is top space (take quotient top $\pi : \mathbb{C} \to T$)

   (b) $\pi$ open ($\pi^{-1}(\pi(V)) = \bigcup_{\omega \in \lambda}(V + \omega)$)

   (c) $\Lambda$ discrete (else $m_k\tau_1 + n_k\tau_2 \to 0 \Rightarrow -\frac{\tau_2}{\tau_1} = \lim \frac{m_k}{n_k} \in \mathbb{R}$, #)

   (d) $\pi$ reg covering map (covering map: $\Lambda$ discrete $\Rightarrow \exists D_z$ st $\pi|_{D_z} : D_z \to \pi(D_z)$ inj so {cts,open,inj,surj} so homeo. Reg: $D_z$ small $\Rightarrow \pi^{-1}(\pi(D_z)) = \coprod_{\omega \in \Lambda}(D_z + \omega)$)

   (e) T admits conf structure (use one from $\mathbb{C}$, show chart transitions are translations by elts of $\Lambda$ so analytic)

   (f) T compact ($T = \pi$(fund parallelogram)).

   OPEN MAPPING FOR RSes (take charts and use complex open mapping). Cor: non-constant analytic $f : R \to S$ with $R$ compact RS then $f$ surj and $S$ compact (image open by open mapping, compact set in hausdorff space so closed, $S$ connected with $f(R) \neq \emptyset$ so surj). Def of harmonic function and a fn $f : R \to \mathbb{R}$ where real part of $f$ on each chart is harmonic.

7. Show non constant harmonic fns are open, so harmonic fns on compact RSes are constant (take charts). Discussion on gluing $\sqrt{z^3-z}$ (READ PROPERLY- DONT JUST THINK SOMETHING SOMETHING ALG GEOM- DIFFERENT CONCEPT). Sketch of how glued structure is torus minus 4 pts. Defn lift of path, show lifts of paths are unique up to start point. Note interaction between path lifting and weaker notion of covering map, note if bad choise of covering space then lifts do not always exist (cf exp). Prop: existence of path lifting for regular covering maps (same as alg top proof, look at sup of pts where lift exists, take open nbhd, local homeo, extend more).

8. Def homotopic as paths, def simply connected, MONODROMY thm (Homotopy lifting for paths), note conds automatically hold for regular covers, proof non examble. SPACE OF GERMS CONSTRUCTION. Def equivalents of fn elements $(f, D) \equiv_z (g, E)$ if $z \in D \cap E$ and $f = g$ on nbd of $z$. Def germ of $f$ at $z$ as such an equiv class, written $[f]_z$, noting $[f_1]_{z_1} = [f_2]_{z_2}$ if $z_1 = z_2$ and locally $f_1 = f_2$ at $z_1$. Def space of germs of $G$ as $\mathcal{G} = \{[f]_z \big| z \in G \text{ and } (f, D) \text{ is a fn element with } z \in D\}$, equip with conf structure:

   (a) Topology on $\mathcal{G}$: given $(f, D)$ of $\mathcal{G}$, write $[f]_D = \{[f]_z : z \in D\}$, def open sets of $\mathcal{G}$ to be unions of sets of the form $[f]_D$. Checks: $D = \emptyset \Rightarrow \emptyset \subset \mathcal{G}$ open, $\mathcal{G}$ open as each $[f]_z \in [f]_D$ some $D$. If $[f]_z \in [g]_D \cap [h]_E$, then $z \in D \cap E$ and $\exists$ nbhd $N$ on which $f = g = h$, so $[f]_N \subset [g]_D \cap [h]_E$, so intersections open.
   (b) $\mathcal{G}$ is Hausdorff: suppose $[f]_z \neq [g]_w$, take representatives, either $z \neq w$ then take disj open sets to separate them and done, or $z = w$ then can use identity principle to show $f = g$ on nbhd of $z = w$, #.
   (c) Connected cpts of $\mathcal{G}$ cover $G$ (nb not regular): Def $\pi : \mathcal{G} \to G, [f]_z \mapsto z$. $\pi$ cts since $\pi^{-1}(V) = \bigcup_{D \subseteq V}\{[f]_D : (f, d) \text{ is a fn element of } V\}$ open by defn of top. $\pi$ locally bijection, with inverse $\pi^{-1}(z) = [f]_z$. $\pi$ open since $\mathcal{U} = \bigcup_{D_\alpha \subseteq D}[f]_{D_\alpha} \Rightarrow \pi(\mathcal{U}) = \bigcup_{D_\alpha \subseteq D} D_\alpha$.
   (d) Atlas on $\mathcal{G}$: Def chart $([f]_D, \pi)$, makes $\pi$ analytic, so connected cpts of $\mathcal{G}$ are RSes.
   (e) Evaluation: $\exists$ a map $\mathcal{E} : \mathcal{G} \to \mathbb{C}, [f]_z \mapsto f(z)$, analytic on connected cpts as in chart $([f]_D, \pi)$, have $(\mathcal{E} \circ \pi^{-1})(z) = f(z)$ analytic.

   Thm: {fn element $(g, E)$ is an analytic continuation of $(f, D)$ along $\gamma : [0, 1] \to G$} $\iff$ {$\exists$ a lift $\tilde{\gamma}$ of $\gamma$ to $\mathcal{G}$ st $\tilde{\gamma}(0) = [f]_{\gamma(0)}$ and $\tilde{\gamma}(1) = [g]_{\gamma(1)}$}. Beginning of proof.

9. Continue (big) proof of thm. Cor: if $\mathcal{F}$ is a complete analytic fn on domain $G$, then $\mathcal{G}_\mathcal{F} = \bigcup_{(f,D) \subset \mathcal{F}}[f]_D \subset \mathcal{G}$ is a connected cpt of $\mathcal{G}$ (prf: $\mathcal{G}$ loc homeo to subdomains of $G$, so every pt has path conn nbhd, so done by thm). Def $\mathcal{G}_\mathcal{F}$ as RS of complete analytic fn $\mathcal{F}$. Example: show for $\mathcal{F} = \sqrt{z^3 - z}$, have $\mathcal{G}_\mathcal{F}$ conf equiv to alg curve construction, by $g : \mathcal{G}_\mathcal{F} \to R'$ by $[f]_z \mapsto (\pi([f]_z), \mathcal{E}([f]_z))$. Show $(g, E), (h, E)$ analytic continutations of $(f, D)$ along same path $\gamma$ in $G$, then $g = h$ on E (use thm). CLASSICAL MONODROMY: suppose $(f, D)$ can be analytically continued along all paths in $G$ starting in $D$. Then IF $(g, E), (h, E)$ are analytic continuations of $(f, D)$ along homotopic paths, then $g = h$ on $E$.

10. Cor of classical monodromy: $G \subset \mathbb{C}$ simp connected, $(f, D)$ is a fn element which can be analytically continued along any path in $G$, then $\exists$ single valued analytic continuation of $f$ to $\mathbb{C}$ (prev cor says gives well defined). Def of MULTIPLICITY $m_f(z)$ of map $f : \mathcal{U} \to \mathbb{C}$ (first non zero power in taylor expansion), show its multiplicative. Def multiplicity of map $f : R \to S$ as mult on charts (nb we checked earlier $f$ local powering map). Def ramification pt ($z$ st $m_f(z) > 1$), branch pt ($f(z)$ st $m_f(z) > 1$), with $m_f(z)$ ramification index. Show for $f$ poly deg $d$, have $m_f(\infty) = d$, show $m_f(z) > 1$ for isolated $z$ (else $f'$ #s isolated zeros). VALENCY THM. $f : R \to S$ non const analytic map of compact RSes, then $n : S \to \mathbb{N}, w \mapsto \sum_{z \in f^{-1}(w)} m_f(w)$ is constant. Proof: check locally constant, note at each pt we are either evenly covered or have some powering maps.

11. Def degree or valency of such an $f : R \to S$ as number of preimages (with multiplicity) of any point. Cor: fund thm of algebra ($f^{-1}(0) = n$). Cor: $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ non const antc is rational (prf: wlog $f(\infty) \in \mathbb{C}$, look at $f^{-1}(\infty)$, taylor expand at each of these pts, subtract negative powers in each taylor expand off of $f$, this bounded ($f(\infty) \in \mathbb{C}$) so constant so $f$ rational), cor: mobius maps are precisely the analytic isoms of $\mathbb{C}_\infty$. GOAL: Riemann-Hurwitz. Def topological triangle, triangulation, euler char $\chi(R)$. Import (i) $\chi(R)$ indep of homeo (ii) every compact RS admits triangulation (iii) every compact RS is orientable. RIEMANN HURWITZ: $f : R \to S$ analytic non const map of RSes of deg $n$, set $e_p = m_f(p)$. Then $\chi(R) = n\chi(S) - \sum_{p \in R}(e_p - 1)$. Start by refining so branch pts $\{Q_1, .., Q_r\}$ verts of triangulation, then (as in valency thm) find finite open cover $\mathcal{U}_1, .., \mathcal{U}_r, \mathcal{U}_{r+1}, .., \mathcal{U}_s$ of $S$ st for $j = r + 1, .., s$, have $f|_V : V \to \mathcal{U}_j$ homeo for any cpt $V$ of $f^{-1}(\mathcal{U}_j)$, and for $j = 1, .., r$, have $f|_V : V \to \mathcal{U}_j$ is $z \mapsto z^{e_p}$ where $p$ is unique preimage of $Q_i$. Refine so each triangle contained in some $\mathcal{U}_j$.

12. Finish proof: each triangle lifts to $n$ triangles in $R$, if no vertex in $Q_i$ then triangles disjoint, else we get same number of triangles but they share $e_p$ verticies, count V,E,F for result. Let $R = w^2 = z^3 - z \subseteq \mathbb{C}^2$. Assume:

$R$ can be compactified to $\bar{R}$ by adding finitely many pts, and in such a way that $\pi_z : R \to \mathbb{C}$ extends to $\bar{\pi}_z : \bar{R} \to \mathbb{C}_\infty$, which has unique branch pt at $\infty$, then use RH to show $\bar{R}$ is a torus. PERIODIC FUNCTIONS. Def meromorphic fn, say $w \in \mathbb{C}$ is a period of $R$ if $f(z+w) = f(z) \forall z \in \mathbb{C}$, such a set $\Omega$ is an additive subgrp of $\mathbb{C}$ with isolated pts. Def $f$ simply periodic if $\Omega = \mathbb{Z}\omega_1$, doubly periodic or elliptic if $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. Show if $f$ meromorphic on $\mathbb{C}$ simply periodic then $\exists!$ meromorphic fn $\tilde{f}$ on $\mathbb{C}^*$ st $f(z) = \tilde{f}(e^{2\pi i z})$.

13. Def period parallelogram $P_z = \{z + t_1\omega_1 + t_2\omega_2 : t_1, t_2 \in [0,1)\}$. Note if $f : \mathbb{C} \to \mathbb{C}_\infty$ then $\exists!$ analytic $\tilde{f}$ st $f = \tilde{f} \circ \pi : \mathbb{C} \to \mathbb{C}/\Lambda \to \mathbb{C}_\infty$. Cor: elliptic fns have well defined degree on period parallelograms. Cor: elliptic fns have deg$\geq 2$ (prf 1: else conf isom torus to sphere. prf 2: residue thm around $P_z$). WEIRSTRASS $\wp$-FUNCTION. Def $\wp(z) = \frac{1}{z^2} + \sum_{w \in \Lambda \setminus \{0\}} \left( \frac{1}{(z-w)^2} - \frac{1}{w^2} \right)$. Show $\sum_{w \in \Lambda \setminus \{0\}} \frac{1}{|w|^t}$ converges iff $t > 2$, use to show $\wp$ is and elliptic function with periods $\Lambda$, $\wp$ is even degree 2, $\wp'(z) = \sum_{w \in \Lambda} \frac{-2}{(z-w)^3}$ is odd deg 3.

14. $\wp'$ has poles order 3 at each lattice pt, and zeros precisely at half lattice pts. Prop: $\exists$ constants $g_2, g_3$ st $\wp'^2 = 4\wp^3 - g_2\wp - g_3$ (look at series expansions, error term analytic, bounded and elliptic so constant). Note can write $\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)$ with $e_1 + e_2 + e_3 = 0$. Discussion about GAGA.

15. Thm: $f$ elliptic, periods $\Lambda$, then $\exists Q_1, Q_2$ rational st $f(z) = Q_1(\wp(z)) + \wp'(z) Q_2(\wp(z))$, and if $f$ even then $Q_2 = 0$ (define $E = \{z \in \mathbb{C} : z \in \frac{1}{2}\Lambda \text{ or } f'(z) = 0\}$, chose $c, d \notin f(E)$ st $g(z) = \frac{f(z)-c}{f(z)-d}$ has simple roots and poles and is elliptic and even, so can write zeros/poles of $g$ as $\{\pm a_1, .., \pm a_n\}$ and $\{\pm b_1, .., \pm b_n\}$ all distinct, consider $h(z) = \prod_i \frac{\wp(z)-\wp(a_i)}{\wp(z)-\wp(b_i)}$, then $\gamma = g/h$ const, untangle to get $f = Q_1(\wp(z))$, if $f$ odd consider $f/\wp'$ even, $f$ even consider decomp into sum of odd and even parts). Def of group acting properly disctsly on top space, use $\pi : X \to X/G$ to give $X/G$ conf structure on which $\pi$ analytic. UNIFORMIZATION VER 1: $R$ simp connected RS $\Rightarrow R \cong \mathbb{C}_\infty, \mathbb{C}$ or $\mathbb{D}$ . UNIF VER 2: any RS conf isom to $\tilde{R}/G$, where $\tilde{R}$ is one of $\mathbb{C}_\infty, \mathbb{C}$ or $\mathbb{D}$ and $G$ is a subgroup of autom group of $\tilde{R}$ which acts properly disctsly. Use to argue: (i) $R$ univ covered by $\mathbb{C}_\infty \Rightarrow R \cong \mathbb{C}_\infty$ (Aut$(\mathbb{C}_\infty)$ =\{Mobius maps\} fixed pt free) (ii) $R$ univ covered by $\mathbb{C} \Rightarrow R \cong \mathbb{C}, \mathbb{C}^*, \mathbb{C}/\Lambda$ (Auts are $z \mapsto az + b$ so isolated additive subgrps $G = 0, \mathbb{Z}, \mathbb{Z}^2$).

16. $R$ unif covered by $\mathbb{D}$, then $R$ not conf equiv to any of above (conf equiv from $\mathbb{C} \to R$, lift to map to disk, well defined by monodromy, # liouville). Cor: Riemann mapping thm: any simp connected domain $\mathcal{U}$ in $\mathbb{C}$ is $\cong \mathbb{D}$ (non compact so $\mathcal{U} \cong \mathbb{C}$ or $\mathcal{U} \cong \mathbb{D}$, CW to give at worst pole at $\infty$ if $\mathbb{C} \tilde{\to} \mathcal{U}$, the pole mulst be simple, # as $\mathcal{U}$ proper). Cor: Picard's thm: $f : \mathbb{C} \to \mathbb{C} \setminus \{0,1\}$ is analytic, then $f$ is constant (use monodromy).