Algebraic Topology lecture recap

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1 Lectures

1. Intro, state/prove gluing lemma, def cell complex by attaching maps
2. Def homotopy (rel A), show it forms an equiv relation (show by homotopy squares), def homtpy equivalent spaces, contractible ($\simeq *$), show homtpy equiv is equiv reln of spaces.
3. Def deformation retract, paths, loops, concatenation (NB: $\gamma_0 \cdot \gamma_1$ means do γ_0 *then* γ_1, opposite to fn comp), def inverse path $\bar{\gamma}(t) = \gamma(1-t)$ and constant path, def $\pi_0(X)$ as path cpts of X, def induced map $\pi_0(f) : \pi_0(X) \to \pi_0(Y), [x] \mapsto [f(x)]$, show (i) homtpc fns induce same map on $\pi_0(X)$, (ii)induced map of comp is comp of induced maps (iii) ind map of identity is identity on π_0. GOAL: define fund group. Def 'homtpc as paths', show propn: this respects concatenation, is associative, concatenating with constant is identity, inverse paths are inverse.
4. Prove prev propns using important homotopy squares, use propn to def FUND GROUP $\pi_1(X)$ as equiv classes of loops at x_0, with as concatenation as group mult, previous checks show well defined. Def based space, map of based spaces, based homotopy. Def induced map of based spaces $\pi_1(f) : \pi_1(X, x_0) \to \pi_1(Y, f(x_0)), [\gamma] \mapsto [f \circ \gamma]$ (also write f_* for ind hom). Show
   (a) ind map $\pi_1(f)$ is group hom
   (b) homptc maps induce same ind hom, ie $f \simeq f'$ rel $\{x_0\} \Rightarrow \pi_1(f) = \pi_1(f')$
   (c) induced hom of comp is comp of induced homs, ie $\pi_1(g \circ f) = \pi_1(g) \circ \pi_1(f)$
   (d) ind hom of identity is identity hom on π_1, ie $\pi_1(\mathrm{id}_{(X,x_0)}) = \mathrm{id}_{\pi_1(X,x_0)}$

   Def CHANGE OF BASEPOINT HOM. Given $u : x_0 \rightsquigarrow x_1$, define $u_\# : \pi_1(X, x_0) \to \pi_1(X, x_1), [\gamma] \mapsto [\bar{u} \cdot \gamma \cdot u]$. Show: (i) homtpc paths define same CoBpt hom. (ii) constant CoBpt hom is id (iii) if $v : x_1 \rightsquigarrow x_2$ then $(u \cdot v)_\# = v_\# \circ u_\#$ (NB order reversing) (iv) CoBpt hom commutes with ind hom (v) if u is a loop then $u_\#$ is conjugation by $[u]$
5. Cor: X path connected $\Rightarrow \forall x_0, x_1 \in X, \pi_1(X, x_0) \cong \pi_1(X, x_1)$ (NB this iso is non-canonical, depends highly on choice of homtpy class of path!) GOAL: show fund group is invariant of homotopy equivalent spaces. First show: for $f, g : X \to Y, x_0 \in X, f \simeq_H g$, we have $u(t) = H(x_0, t) : f(x_0) \rightsquigarrow g(x_0)$ is a path in Y, and $u_\# \circ f_* = g_*$ (prf: set up map from $I \times I$, then straight line homtpy). Thm: if $f : X \to Y$ is a hompty equiv then $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isom. Cor: Contractible spaces are simply connected.
6. COVERING SPACES. Def a covering map $p : \tilde{X} \to X$ as a surj map where each $x \in X$ has a nbhd U with $p^{-1}(U) \cong \coprod_{\alpha \in \Lambda} V_\alpha$ with $p|_{V_\alpha} : V_\alpha \xrightarrow{\sim} U$ homeo. Examples: $\exp : \mathbb{R} \to S^1$, $z^n : S^1 \to S^1$, antipodal: $S^2 \to \mathbb{RP}^2$. Def a lift $\tilde{f}$ of f along p as by $p \circ \tilde{f} = f$. Show uniqueness of lifts, in particular set where they agree is clopen (so agree everywhere or nowhere) Proof: point set top.
7. PATH LIFTING LEMMA. Given $p : \tilde{X} \to X$ covering and $\gamma : I \to X$ path, if $\tilde{x}_0 \in p^{-1}(\gamma(0))$ then $\exists! \tilde{\gamma} : I \to \tilde{X}$ lift st $\tilde{y}(0) = \tilde{x}_0$. Prf: consider sup of set where they agree, use local homs. Generalise to HOMOTOPY LIFTING LEMMA. Think of a homotopy $f_0 \simeq_H f_1$ as a bunch of paths $\gamma_x(t) = H(x, t)$, then path lifting applies to each of these paths simultaneously in a cts way.
8. Cor of HLL (Hompty lifting for paths): if $\gamma \simeq \gamma'$ as paths, then $\tilde{\gamma} \simeq \tilde{\gamma'}$ as paths and in particular $\tilde{\gamma}(1) = \tilde{\gamma}'(1)$. Cor: have for each $\gamma : x_0 \rightsquigarrow x_1$, a bijection $\gamma_* : p^{-1}(x_0) \to p^{-1}(x_1)$ given by $\gamma_* : \tilde{x}_0 \mapsto \tilde{\gamma}_{\tilde{x}_0}(1)$. Def: p has well defined *degree*, say p is n-sheeted, $n \in \mathbb{N} \cup \{\infty\}$. Def right action of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$ by $(\tilde{x}_0, [\gamma]) \mapsto \tilde{\gamma}_{\tilde{x}_0}(1) =: \tilde{x}_0 \bullet [\gamma]$ (well defined by Homtpy lifting for paths). Lemma: $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0), [\tilde{\gamma}] \mapsto [p \circ \tilde{\gamma}]$ is injective, hence (prf: HLL on $p \circ \tilde{\gamma}$).

9. ORBIT STABILISER BIJECTION.
   (a) $\tilde{X}$ path connected iff $\pi_1(X, x_0)$ acts transitively on $p^{-1}(x_0)$ ($\underline{\Rightarrow}$: for $y, y' \in p^{-1}(x_0)$, find $\gamma : y \rightsquigarrow y'$, then $y \bullet [p \circ \gamma] = y'$. $\underline{\Leftarrow}$: spse not, use transtve action to find path, #)
   (b) $\mathrm{Stab}_{\pi_1(X,x_0)}(\tilde{x}_0) = \{[\gamma] \in \pi_1(X, x_0) : \exists \tilde{\gamma} : \tilde{x}_0 \rightsquigarrow \tilde{x}_0, p \circ \tilde{\gamma} = \gamma\} = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$
   (c) Orb-stab gives bijection $\ell : \frac{\pi_1(X,x_0)}{p_*(\pi_1(\tilde{X},\tilde{x}_0))} \to p^{-1}(x_0)$, $\ell([(p_*\pi_1(\tilde{X}, \tilde{x}_0)) \cdot \gamma]) = \tilde{x}_0 \bullet [\gamma]$ (preserves group action)

   Def universal cover, cor: if $\tilde{X}$ is univ cover, have equivariant (group action preserving) bijection $\ell : \pi_1(\tilde{X}, \tilde{x}_0) \to p^{-1}(x_0), [\gamma] \mapsto \tilde{x}_0 \bullet [\gamma]$. Use to show $\pi_1(S^1) \cong \mathbb{Z}$ and then Brower's Fixed Point thm (cts maps $D^2 \to D^2$ have fixed pt)

10. Construction of univ cover. Observe {pts in $\tilde{X}$} $\leftrightarrow$ {homtpy classes of paths in X from x_0} so def $\tilde{X} = \{$cts maps $\gamma : I \to X | \gamma(0) = x_0\}/(\simeq$ of paths), $p : [\gamma] \mapsto \gamma(1)$, thm this exists when X is pc,lpc,semilocally sc. GALOIS CORRESPONDENCE.

11. Lifting criterion: given X pc, locally pc, semilocally sc, map $f : Y \to X$, cover $p : \tilde{X} \to X$, $\exists$ lift $\tilde{f} : Y \to \tilde{X}$ iff $f_*\pi_1(Y, y_0) \subseteq p_*\pi_1(\tilde{X}, \tilde{x}_0)$ (ie iff every loop in image of f is a loop in image of p). Def deck transformation (lift p along itself, lifting criterion). Given $H \leq \pi_1(X, x_0)$, def X^H =no one cares.

12. Based uniqueness, unbased uniqueness. GALOIS CORRESPONDENCE.

13. Def alphabet, set of words, elementary reduction, free group on S as $F(S)$ set of reduced words with *concatenate then perform elementary reductions* as group operation (show well defined using homotopy later). State univeral property of free groups, def presentation, state univ prop of presentations, example: stupid presentation, use to be clever.

14. Another view of free groups? Def wedge $\bigvee_i X_i$, def graph as 1-diml cell complex, show univ cover is a $2|S|$-reg tree. Def pushout of groups along common subgroup.

15. State univ property of pushouts, def free product with amalgamation (operation is generally $G_1 \amalg_H G_2$, if inclusions are injective, then write $G_1 *_H G_2$). SEIFERT VAN KAMPEN. VER 1: X space, $A, B \subseteq X$ open cover, $A \cap B = X$, $x_0 \in A \cap B$, then $\pi_1(A, x_0) \amalg_{\pi_1(A\cap B, x_0)} \pi_1(B, x_0) \xrightarrow{\sim} \pi_1(X, x_0)$. VER 2: X cell complex, $A, B \subseteq X$ subcomplexes, $A \cup B = X$, $A \cap B$ path connected, $x_0 \in A \cap B$, then same conclusion. Return to attaching cells. Have $\imath : X \hookrightarrow X \cup_f D^n$, ind map on π_1? VERY USEFUL THEOREM: if $n \geq 3$, $\imath_*$ is an isom, if $n = 2$ then $\imath_*$ is surj, $\mathrm{Ker}\imath_* = \langle\langle [f] \rangle\rangle$ where f is thought of as defining a based loop in X, so $\pi_1(X \cup_f D^2) \cong \pi_1(X)/\langle\langle [f] \rangle\rangle$.

16. Cor: for arb finitely presented group can construct a cell complex with corresponding fund group. CLASSIFICATION OF SURFACES. Def surface, write Σ_g for g holed torus (orientable) and S_g for genus g non orientable surface, eg $S_0 = \mathbb{RP}^2$, S_1 =Klein bottle. Construct using attaching maps to find presentations of fund groups: $\pi_1\Sigma_g = \langle a_1, .., a_g, b_1, .., b_g | [a_1, b_1]...[a_g, b_g] \rangle$ (quotient $\bigvee_{i=1}^g S^1$, or start with $4g$-gon), and $\pi_1 S_g = \langle c_0, .., c_g | \prod_{i=0}^g c_i^2 \rangle$ (quotient $g + 1$ petals, each one twice before moving round). SIMPLICIAL COMPLEXES. Def affine independence of pts, def n-simplex $\sigma = \langle a_0, .., a_n \rangle = \{\sum t_i a_i | \sum t_i = 1, t_i \geq 0\}$.

17. Def barycentric coords as the t_i, def (geometric) simplicial complex as finite set k of simplicies st (i) $\sigma \in K, \tau \leq \sigma \Rightarrow \tau \in K$ (ii) $\forall \sigma, \tau \in K, \sigma \cap \tau \in K$ (possibly empty). Def polyhedron of K as $|K| = \bigcap_{\sigma \in K} \sigma \subseteq \mathbb{R}^m$, def $\dim K = \max_\sigma(\dim \sigma)$, def d-skeleton. Def trianglulation on X as a homeo $h : |K| \to X$ (nb thus X compact Haus). Def standard n-simplex, simplicial $(n - 1)$-sphere. Show any point is in interior of a unique face. Def V_K verticies of K, def simplicial map $f : V_K \to V_L$ so image of simplex is simlplex. Def ind cts map $|f| : |K| \to |L|, \sum t_i a_i \mapsto \sum t_i f(a_i)$, note $|g \circ f| = |g| \circ |f|$. Goal: simp approx thm! Def star $\mathrm{St}_K(v)$ and link.

18. Say $g : V_k \to V_L$ is *simp approx* to $f : |K| \to |L|$ if $\forall v \in V_K, f(\mathrm{St}_K(v)) \subseteq \mathrm{St}_L(g(v))$. Show in this case have $|g| \simeq f$ (rel $\{x \in |K| : |g|(x) = f(x)\}$) (important/long proof!). Def barycentre of simplex, then r-th barycentric subdivision. State SIMPLICIAL APPROXIMATION THM: $\exists r \geq 0$ st $g : V_{K^{(r)}} \to V_L$ is simp approx of f cts, and if $\exists N$ subcomplex of K st $f|_N$ simplicial, then can take g to agree with f on V_N. Def mesh $\mu(K) = \max\{||v_0 - v_1|| : \langle v_0, v_1 \rangle \in K\}$, aim to show $\mu(K^{(r)}) \to 0$ as $r \to \infty$.

19. Show $\mu(K^{(r)}) \leq \left(\frac{n}{n+1}\right)^r \mu(K)$ (use triangle ineq). Use to prove simp approx thm, state lebesgue number lemma along the way. HOMOLOGY. Def oriented simplex, then fix orientation on each simplex and def nth chain group $C_n(K) = \bigoplus_{\sigma \in K} \mathbb{Z}\sigma$. Def boundary hom $d_n : C_n \to C_{n-1}, (a_0, .., a_n) \mapsto \sum_{i=0}^n (-1)^i (a_0, .., \hat{a_i}, .., a_n)$. Lemma: $d_{n-1} \circ d_n = 0$ (two line computation using def).

20. Def n-boundaries $B_n(k) = \operatorname{Im} d_{n+1}$, n-zycles $Z_n(K) = \ker d_n$, nth homology $H_n(K) = \frac{Z_n(K)}{B_n(K)}$ "cycles mod boundaries". Note we have short exact sequences $0 \to B_n \hookrightarrow Z_n \twoheadrightarrow H_n \to 0$ and $0 \to Z_n \hookrightarrow C_n \overset{d_n}{\twoheadrightarrow} B_{n-1} \to 0$. Examples. Def chain complex $C_\bullet$ as sequence of free ab grps $C_0, C_1, ..$ with homs $d_n : C_n \to C_{n-1}$ st $d_{n-1} \circ d_n = 0$, call D_n *differentials*. Def chain map $f_\bullet : C_\bullet \to D_\bullet$ as sequence of homs $f_n : C_n \to D_n$ st commutes with d_ns. Def chain homotopy of chain maps as sequence of homs $h_n : C_n \to D_{n+1}$ st $g_n - f_n = d_{n+1} \circ h_n + h_{n-1} \circ d_n$ (see dexter for explanation and problem sheets for examples). Chain map $f_\bullet : C_\bullet \to D_\bullet$ induces well defd hom on homologies $f_* : H_n(C_\bullet) \to H_n(D_\bullet), [x] \mapsto [f_n(x)]$, and homotopic chain maps $f_\bullet \simeq g_\bullet$ induce same hom $f_* = g_*$ (good exercise in defn chasing to prove).

21. Def chain homotopy equiv if both compns are homtpc to id, cor: chain homtpy equiv maps induce isoms on all homologies. How do we get chain maps? Induce it from simplicial map! Given $f : K \to L$ simp map, def $f_n : C_n(K) \to D_n(K), [a_0, .., a_n] \mapsto [f(a_0), .., f(a_n)]$ gives $f_\bullet : C_\bullet \to D_\bullet$, hence hom $f_* : H_n(K) \to H_n(L)$. Def cone of vertex as star $\cup$ link. Show $\imath : \{v_0\} \to K$ inclusion into cone gives chain homtpy equiv so $H_0(K) = \mathbb{Z}, H_n(K) = 0, n > 0$. Cor: homology of simplicial $(n-1)$-sphere. Lemma: $H_0(k) \cong \mathbb{Z}^{\#\text{ path cpts of }|K|}$. Def exact sequence as $\operatorname{Im} f = \ker g$. State MAYER-VIETORIS.

22. State SNAKE LEMMA: given $0 \to A_\bullet \overset{i_\bullet}{\hookrightarrow} B_\bullet \overset{j_\bullet}{\twoheadrightarrow} C_\bullet \to 0$, have snake homs $\partial_* : H_n(C_\bullet) \to H_{n-1}(A_\bullet)$ st $... \xrightarrow{\partial_*} H_n(A_\bullet) \overset{i_*}{\to} H_n(B_\bullet) \overset{j_*}{\to} H_n(C_\bullet) \xrightarrow{\partial_*} H_{n-1}(A_\bullet) \overset{i_*}{\to} H_{n-1}(B_\bullet) \overset{j_*}{\to} H_{n-1}(C_\bullet) \xrightarrow{\partial_*} ...$ is exact (long boi). Use to prove MV by diagram chasing, prove snek by diagram chasing on 9-lemma lookin grid. GOAL: homtpc cts maps induce same hom on homologies. Def contiguous simplicial maps, show simp approxes to the same cts map are contiguous, then show ctgs maps induce homotopic chain maps so same homology hom. Show $\{\text{functions } a : V_{K'} \to V_K\} \leftrightarrow \{\text{simp approxes } g : K' \to K \text{ of id}\}$.

23. Lemma: any simp approx to identity $a : K' \to K$ defines same isom $\nu_K = a_* : H_n(K') \to H_n(K)$ (prf: induct on number of simplicies, for ind step get chain complexes from MV, then use five lemma). Show cts maps induce well define homology homs, with $(\mathrm{id}_{|K|})_* = \mathrm{id}_{H_n(K)}$, and $(g \circ f)_* = g_* \circ f_*$, cor: $|K| \cong |L| \Rightarrow H_n(K) \cong H_n(L) \forall n$. Then gross mesh boundy thing to get $f \simeq_H g : |K| \to |L| \Rightarrow f_* = g_*$ (ie homology is invariant up to homotopy!!).

24. Def h-triangulation as a homotopy equivalence $h : |K| \to X$ (the h is for homotopy!) so def $H_n(X) = H_n(K)$, well defined by prev work. Now show $H_k(S^n) = 1_{\{k=0,n-1\}} \mathbb{Z}$, so S^n is not contractible (so $\mathbb{R}^n \cong \mathbb{R}^m \Rightarrow n = m$!!). Cor: Brower's fixed point thm: $f : D^n \to D^n$ has fixed pt (prf same as for π_1). Homology of surfaces: def $\Sigma_{g,1} = \Sigma_g - \{\text{small open disc}\}$, note $\Sigma_{g,1} \simeq R_{2g} = \bigvee_1^{2g} S^1$, use $R_k = R_{k-1} \cup_* S^1$ for MV. Note if $K = M \cup_L N$ all connected then bottom line of MV detaches. Also note $\Sigma_g = \Sigma_{g-1,1} \cup_{S^1} \Sigma_{1,1}$, so we can get $H_1(\Sigma_g) = \mathbb{Z}^{2g}, H_1(S_g) = \mathbb{Z}^g \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}$.