

An Introduction to Quantum Computing

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Abstract

Quantum computing is a potentially revolutionary principle which will be continued to be researched and studied for the foreseeable future as the importance of efficiency and the limit of binary computing is approached. This paper aims to provide an overview of the field of quantum computing for individuals with a minor understanding of physics, computer science, and mathematics. An introduction to quantum computing will leave the reader with a comfortable overview of the field and insight into which topic in particular they find most interesting.

This paper will talk briefly about the recent history of quantum computing as well as a small subset of quantum mechanics as it relates to quantum computations and the cornerstones which currently make quantum computing possible. It aims to establish the differences between conventional and quantum computing with a goal to speak about how certain algorithms will run more efficiently and what applications in the field this can be used for. Near the end, we will look at the current issues within the field and its future importance.

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1 History

Quantum computing is a relatively new field in relation to computer science as a discipline with the informal start originating in the late 1970's and early 1980's as Richard Feynman speculated that quantum mechanics could not be effectively modeled through a classical computer. In accordance with Moore's law, the size of a silicon chip would continue to shrink until the individual elements were no larger than several atoms and would be subject to quantum effects at that scale. Feynman published an abstract model in 1982 in which he analyzed the outcome of using a quantum simulator in order to avoid the exponential slowdown which is common with classical computers. (3)

In 1985, David Deutsch published a paper proving that any physical process could be, in theory, effectively rendered on a quantum computer. As a result, a quantum computer, which is able to operate in an exponential time, could provide a wide array of values for heavy data crunching, modelling of complex systems, or in the general solving NP-Complete classical problems in polynomial time.¹ Deutsch proved a basic algorithm which will be worked through later in the paper.

Until 1994, the quantum computing field remained relatively unchanged until Shor was able to prove and set a method for a common NP-Hard factorization problem which could call on the benefits allowed through quantum computers, which would run in a time much shorter than what will be ever possible on classical computers. (3) As a field, this momentous finding was able to push the field of research for quantum computing out of the view of the select who were performing research on the project to the public eye. Shor's algorithm will be explored later in the paper as well.

2 Quantum Concepts and Theories

As a quick note, the material that will only be covered consists of a very small section of quantum mechanics encompassing finite dimensional quantum mechanic where the vector spaces which represent the states of the dimension are finite in size.

¹In computational complexity theory, a decision problem is NP-complete when it is both in NP and NP-hard. The set of NP-complete problems is often denoted by NP-C or NPC. The abbreviation NP refers to "nondeterministic polynomial time".

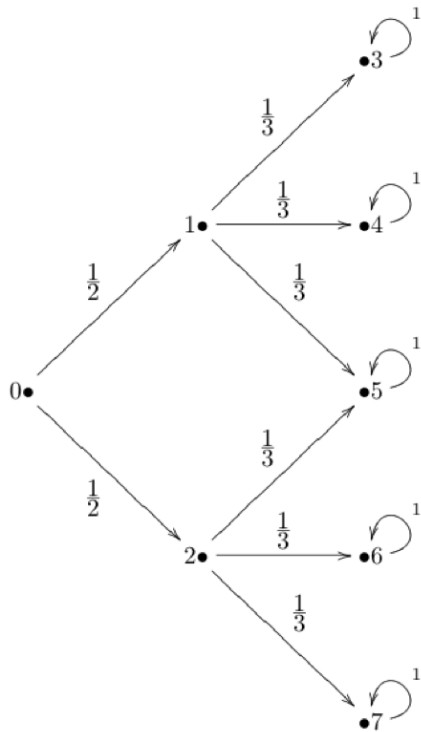
Although any given solution to an NP-complete problem can be verified quickly (in polynomial time), there is no known efficient way to locate a solution in the first place.

Figure 2: Matrix representing the progression after one time click

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

2.1 Double Slit Experiment

Young's double slit experiment is one of the most foundational experiments related to the field of quantum mechanics and demonstrates the wave-particle duality of photons when conducted. Before approaching the quantum model, it is interesting to explore a classical model and the probabilities associated with it before moving on.



Pretend for a moment, that there is an experiment where there is a sharpshooter who is guaranteed to always shoot through one or the other open windows, with equal probability. Once the bullet passes through the window, it has an equal probability of hitting three targets. There is one target which is shared between both open windows.

The probability matrix associated with this scene is shown.

By representing the data points as a matrix, it is possible to identify the probability where the bullet might be found on the next time click by simply using matrix multiplication. (4). The

Figure 1: Corresponding graph to scenario

Figure 3: Matrix representing the progression of two time clicks

$$B \star B = B^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{3} & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{3} & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 1 & 0 & 0 \\ \frac{1}{6} & 0 & \frac{1}{3} & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{6} & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

matrix shown above, B , represents the state of the system after one time click. By multiplying the matrix by itself you are able to represent the state after two time clicks.

The takeaway from this example is to show that after two time clicks the bullets will be in

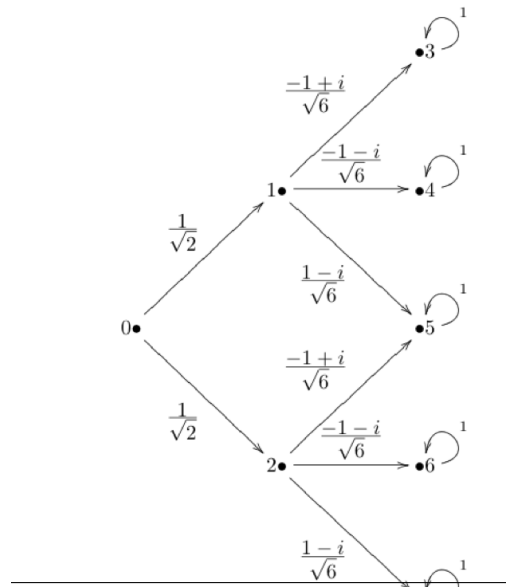
the state

$$B^2 X = [0, 0, 0, \frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}]^T$$

Which means that $B^2[5, 0]$ is equivalent to $\frac{1}{3}$. Which is the two states $\frac{1}{6} + \frac{1}{6}$.

Pretend for a moment that the shooter has now been changed to a flashlight which can spread light into both of the windows with a similar setting. Once the light has passed through the windows, it again travels randomly to one of the three respective target locations. Represented in this graph is the modulus, where the modulus squared represents the probability of the specific event taking place. $\frac{1}{\sqrt{2}}^2$ is $\frac{1}{2}$ and more importantly $\left| \frac{\pm 1 \pm i}{\sqrt{6}} \right|^2 = \frac{1}{3}$ ²

The above matrices represented the state of the experiment using the moduli of the components. In order to interpret this information



²The complex number weights represented here are not to represent the actual quantum probability weights as this would require acquiring the distance of the slit spacing, the width of the individual slits. Rather the numbers given are to represent the point of quantum interference which will be highlighted later on

Figure 4: Corresponding quantum modulus

graph to scenario

Figure 5: Matrix representing the state after one time click

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-1+i}{\sqrt{6}} & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{-1-i}{\sqrt{6}} & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1-i}{\sqrt{6}} & \frac{-1+i}{\sqrt{6}} & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-1-i}{\sqrt{6}} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1-i}{\sqrt{6}} & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Figure 6: Matrix representing the state after two time clicks

$$P^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-1+i}{\sqrt{12}} & \frac{-1+i}{\sqrt{6}} & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{-1-i}{\sqrt{12}} & \frac{-1-i}{\sqrt{6}} & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{-1+i}{\sqrt{6}} & \frac{1-i}{\sqrt{6}} & 0 & 0 & 1 & 0 & 0 \\ \frac{-1-i}{\sqrt{12}} & 0 & \frac{-1-i}{\sqrt{6}} & 0 & 0 & 0 & 1 & 0 \\ \frac{-1+i}{\sqrt{12}} & 0 & \frac{1-i}{\sqrt{6}} & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Figure 7: Probability matrix of the modulus squared after two time clicks

$$|P^2[i, j]|^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{3} & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{3} & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{6} & 0 & \frac{1}{3} & 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{1}{6} & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

in reference to the classical scenario it is helpful to consider the probability of the individual locations. This can be shown by squaring the individual components of the P^2 matrix.

For the most part, the probability matrix for P^2 is the same as the probability matrix for B^2 ; however, there is one important distinction to be made. Which is that while $B^2[5, 0] = \frac{1}{3}$ in the quantum simulation $P^2[5, 0] = 0$. On a mathematical basis, this is trivially written as

$$\frac{1}{\sqrt{2}} \left(\frac{-1+i}{\sqrt{6}} \right) + \frac{1}{\sqrt{2}} \left(\frac{1-i}{\sqrt{6}} \right) = \frac{-1+i}{\sqrt{12}} + \frac{1-i}{\sqrt{12}} = 0$$

This may seem troubling at first, but it must be remembered that photons are subject to particle interference and thus, on the shared target of the windows, the probability drops to 0. Furthermore, one may try to argue that if the experiment was carried out using only one photon, then the probability would again return to $\frac{1}{3}$, but this assumption would also be incorrect. A single photon is said to have a superposition in being in every possible position simultaneously. This is not mimicked in the classical world; however, due to the photon residing in every position at once it is prone to interference when the shared target is reached. As with most quantum phenomena, the particle is only determined to be in a certain state with a certain probability when a measurement is taken. In quantum mechanics, a measurement causes the superposition state to collapse to a certain pure state. (4)

This phenomena is central to the success of quantum computing, as it allows for an exponential number of computations or simulations to be run in parallel thus becoming exponentially more efficient.

2.2 Unitary Transformations

The complex numbers which represent the quantum position are commonly known as the amplitudes of the wave function. A core theory in regards to quantum systems is the idea of a unitary transformation. Provided that the entire amplitude function Ψ can be identified through a vector, a unitary transformation is a multiplication of those vectors by a transformation matrix whose inverse equals its conjugate transpose. (5) The inherent properties of unitary transformations indicate that the total probability of the set always remains the same where the sum of the squares of the amplitudes is equivalent to 1 and that all changes are preserve the information across all states. Once a quantum state is observed in an isolated system, the quantum state is determined for all past and future times.

Furthermore, unitary transformations are fully reversible. This property can be shown to be true because $|\psi\rangle \mapsto U|\psi\rangle$ preserves the normal standardization across states it can then be set that $1 = \langle\psi|\psi\rangle = \langle\psi|U^\dagger U|\psi\rangle$ for all unit vectors $|\psi\rangle$. By linearity it follows that $U^\dagger U = I$ which indicates invertibility.³

2.3 Bracket Notation

An n dimensional quantum system indicates that a particle can be in one of n states or positions. The quantum system could also represent the energy level of the photon polarization direction; however, for the purpose of defining the bracket notation consider just the position of the particle.⁴

The state

$$|\psi\rangle = [0, 1, 0, 0, \dots, 0]^T$$

is to represent that the particle is found at position 1. Similarly the state

$$|\psi'\rangle = [0, \dots, 1, \dots, 0]^T$$

is said to be found at the position j where the particle can be found. These states, where the particle can be certainly found are known as pure states. A superposition of the general form

$$|\phi\rangle = [c_0, c_1, \dots, c_j, \dots, c_{n-1}]^T$$

³The format used is known as bracket notation and will be understood in the following subsection. Where the $\langle x|$ represents the “bra” section and the $|x\rangle$ represents the “ket” section and the entire representation can be shown by $\langle x|x\rangle$

⁴This section focuses on the “ket” of the bracket notation. The matching “bra” $\langle x|$ denotes the conjugate transpose of $|x\rangle$. This choice is arbitrary, but is the convention which is widely used when talking about quantum computing (2)

can be added to another superposition state simply by adding elements individually. Adding the initial to

$$|\phi'\rangle = [c'_0, c'_1, \dots, c'_j, \dots, c'_{n-1}]^T$$

yields

$$|\phi\rangle + |\phi'\rangle = [c_0 + c'_0, c_1 + c'_1, \dots, c_j + c'_j, \dots, c_{n-1} + c'_{n-1}]^T$$

This process of adding the complex vector spaces are valid and yield accurate results.

The only component which matters is not the length $|\phi\rangle$, but the direction of the component. Working with these vectors, it makes more sense to work with a normalized version of the vector

$$\frac{|\phi\rangle}{||\phi\rangle|}$$

While this works for adding the superposition states together in order to combine quantum systems it becomes necessary to calculate the tensor product ⁵. If we take $|\phi\rangle$ to be the first quantum system and take $|\phi'\rangle$ to be the second quantum system we represent the combined system as

$$|\phi\rangle \otimes |\phi'\rangle = |\phi, \phi'\rangle = |\phi\phi'\rangle$$

2.4 Observations

As mentioned before, when a quantum superposition state it condenses to a single pure state in order for the experiment to show the particle at a single position. In an attempt to predict which state the particle will condense to we look at the sum of the squares of the modulus (4)

$$S = |c_0|^2 + |c_1|^2 + |c_j|^2 + |c_{n-1}|^2$$

Therefore there is a $|c_0|^2/S$ chance of the superposition collapsing to the 0th pure state and a $|c_1|^2/S$ chance of collapsing to the 1st pure state as the way that the quantum state collapses is random, but can be represented as a hermitian. ⁶

Eigenvalue is a real number which can be found in a hermitian matrix if for a matrix A in $M^{n \times ns}$, there is a number m in M and a vector $|\phi\rangle$ in M^n such that $A|\phi\rangle = m|\phi\rangle$. m in this case is an Eigenvalue, where $|\phi\rangle$ is known as a **Eigenvector** of A associated with m .

All eigenvalues in a hermitian matrix are all real numbers. Furthermore, distinct eigenvectors

⁵SOME INFORMATION ON THE TENSOR PRODUCT PLACED IN HERE

⁶an $n \times n$ matrix is considered hermitian if $A = A^\dagger$. In other words, only if $A^T = \overline{A}$

which have distinct eigenvalues of a hermitian matrix are orthogonal. It follows that the set of eigenvectors form a basis for the entire complex vector space which represents the quantum of interest. (4). Taking $A|\phi\rangle = m|\phi\rangle$, it becomes obvious that, as stated before, the only part of the state that matters is the direction rather than the length. This means that $m|\phi\rangle = |\phi\rangle$. A critical assumption which can be made following this statement is that if the current state of the quantum system is based on the eigenvector basis, then the system will not change.

3 Qubit

Fundamentally, a bit is the state of any system. A classical bit represents one of two distinct positions for a scenario as in a 1 or 0, on or off, true or false. In classical computers all data is stored, shuttled, and interpreted through 1's or 0's. Due to the binary state of data, this limits the number of computations that can be done at any single time on a classical machine.

Quantum bits (qubits), rather, is a unit vector in a two dimensional complex vector space. When observed a quantum bit will settle into either a $|0\rangle$ or $|1\rangle$ binary state. The implementation of a qubit could correspond to the polarization of the photon or the spin-up or spin-down components of an electron. (2) A quantum bit can be represented as a superposition of $|0\rangle$ or $|1\rangle$ such that $a|0\rangle + b|1\rangle$ where a and b are complex numbers such that $|a|^2 + |b|^2 = 1$.

An important consideration is that the state can only be measured once. Though it may seem possible to clone a qubit so that it may be measured in two possible ways, this is impossible. Because quantum states are subject only to unitary transformations, cloning is not allowed. The proof, established in 1982 by Wootters and Zurek, is an example of the linearity of unitary transformations. (2)

Assume that U is a unitary transformation which is possible to clone so that in all quantum states $|a\rangle$, $U(|a0\rangle) = |aa\rangle$. Let $|a\rangle$ and $|b\rangle$ be orthogonal quantum states. And $U(|a0\rangle) = |aa\rangle$ and $U(|b0\rangle) = |bb\rangle$. Consider the case $|c\rangle = (1/\sqrt{2})(|a\rangle + |b\rangle)$.

$$\begin{aligned} U(|c0\rangle) &= \frac{1}{\sqrt{2}}(U(|a0\rangle) + U(|b0\rangle)) \\ &= \frac{1}{\sqrt{2}}(|aa\rangle + |bb\rangle) \end{aligned}$$

But since U was defined as a cloning transformation

$$\begin{aligned} U(|c0\rangle) &= |cc\rangle \\ &= \left(\frac{1}{\sqrt{2}}(|aa\rangle + |bb\rangle)\right)^2 \\ &= \frac{1}{2}(|aa\rangle + |ab\rangle + |ba\rangle + |bb\rangle) \end{aligned}$$

which is not equal to $1/\sqrt{2}(|aa\rangle + |bb\rangle)$. This proves that there is no unitary operation to clone unknown quantum states. This *no cloning* principle is only valid for unknown quantum states, as it is in fact possible to clone known quantum states. Though it is possible to obtain particles in an entangled state.

3.1 Notation

3.2 Implementation

4 Applications

4.1 Implementation

5 Algorithms

5.1 Shor's Algorithm

5.2 Grover's Algorithm

5.3 Deutsch Algorithm

6 Error Correction and Measurement

7 Future of Quantum Computing



References and Notes

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