Spring 2024: Final Project
COM-308: Quantum Computing

Notations

We denote $|i\rangle$ the state in the computational basis corresponding to the binary decomposition of i. For example, with 4 qubits, $|5\rangle = |0101\rangle$.

The HHL Algorithm

The HHL (Harrow-Hassidim-Lloyd) algorithm is a quantum algorithm designed to solve systems of linear equations. Given a matrix $A \in \mathbb{C}^{2^n \times 2^n}$ and a vector $\mathbf{b} \in \mathbb{C}^{2^n}$, we want to find a vector $\mathbf{x} \in \mathbb{C}^{2^n}$ that verifies

$$A\mathbf{x} = \mathbf{b}. \tag{1}$$

The HHL algorithm is exponentially faster than classical methods when the matrix A is sparse. If s is the number of non-zero elements per row in A, then the quantum circuit can be constructed with $O(ns^2)$ gates whereas the best classical algorithm runs in $O(2^n s)$.

To solve the linear system $A\mathbf{x} = \mathbf{b}$ with a quantum circuit, we need to represent $\mathbf{b}$ and $\mathbf{x}$ by quantum states; thus, we need to scale them to unit length $\|\mathbf{b}\| = \|\mathbf{x}\| = 1$. Then $\mathbf{b}$ can be represented by a state $|b\rangle$ using n qubit such that $|b\rangle = \sum_{i=0}^{2^n-1} b_i |i\rangle$. Here, the b_i are the components of $\mathbf{b}$. The vector solution $\mathbf{x}$ can be then represented by the state $|x\rangle$ that verifies

$$|x\rangle = cA^{-1}|b\rangle, \quad c^{-1} = \|A^{-1}|b\rangle\| \tag{2}$$

where c ensures that the state is normalized.

The HHL algorithm uses quantum phase estimation to encode the solution $\mathbf{x}$ into a quantum state. To do so, the algorithm requires that the matrix A be Hermitian.

Question Theory 1: Show that if A is not Hermitian, we can still find a solution to the system by running the HHL algorithm on the larger system:

$$\tilde{A} = \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix}, \quad \tilde{b} = \begin{pmatrix} b \\ 0 \end{pmatrix}. \tag{3}$$

So, from now on, we will assume that A is a Hermitian matrix, i.e. $A = A^\dagger$. Thus, by the spectral theorem (see Homework 1 Ex 3), there exists a set of orthogonal states $(|u_i\rangle)_{i=0,\dots,2^n-1}$ such that A can be written

$$A = \sum_{i=0}^{2^n-1} \lambda_i |u_i\rangle\langle u_i| \tag{4}$$

where the $\lambda_i \in \mathbb{R}$ are the eigenvalues of A. The $(|u_i\rangle)_{i=0,\dots,2^n-1}$ form an eigenbasis of A. The state $|b\rangle$ can also be written in the $(|u_i\rangle)_{i=0,\dots,2^n-1}$ basis and we denote

$$|b\rangle = \sum_{i=0}^{2^n-1} \beta_i |u_i\rangle. \tag{5}$$

Question Theory 2: Check that $|x\rangle = c\sum_{i=0}^{2^n-1} \frac{\beta_i}{\lambda_i} |u_i\rangle$ is solution to the system.

The circuit

The HHL circuit is represented in Figure 1. The circuit uses three registers:

- The top register is 1 ancilla qubit initialized to $|0\rangle$.
- The middle register is a memory register that stores the eigenvalues λ_i of A. More precisely, we will store the binary representation of λ_i. The number of qubits m needed for this register will therefore depend on λ_i. This register is initialized to $|0\rangle^{\otimes m}$.
- The bottom register uses n qubits and is initialized with the state $|b\rangle$.

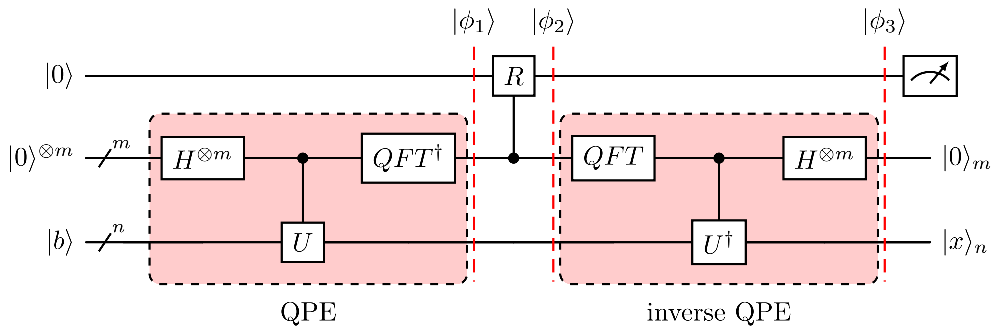

Figure 1: Quantum Circuit for HHL Algorithm

The circuit is composed of 4 steps, a quantum phase estimation (QPE), a controlled rotation, an inverse quantum phase estimation and a measurement. Let us detail the gates appearing in each part:

Quantum phase estimation: This part of the circuit is detailed in Figure 2. The circuit starts with a Hadamard gate on each qubit of the memory register. The unitary U is

$$U = e^{i2\pi \frac{A}{2^m}}. \tag{6}$$

Then we apply an inverse quantum Fourier transform on the memory register.

$$QFT^\dagger |k\rangle = \frac{1}{\sqrt{2^m}} \sum_{j=0}^{2^m-1} e^{-i2\pi \frac{kj}{2^m}} |j\rangle . \tag{7}$$

Controlled rotation: The gate R realizes the transformation

$$R(|0\rangle \otimes |\lambda\rangle) = \left(\sqrt{1 - \frac{1}{\lambda^2}} |0\rangle + \frac{1}{\lambda} |1\rangle \right) \otimes |\lambda\rangle . \tag{8}$$

Inverse QPE: We apply the **inverse** gates of the QPE in **reverse** order to set back the memory register to $|0\rangle^{\otimes m}$. The memory register is no longer entangled with the output register.

Measurement: The algorithm outputs $|x\rangle$ if the ancilla qubit is measured in state $|1\rangle$.

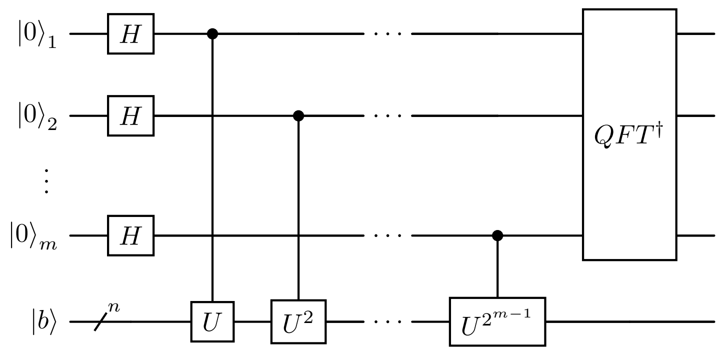

Figure 2: Detailed QPE

Analysis

We assume that the eigenvalues of A are positive integers, i.e. $\lambda_i \in \mathbb{N}^*$ and $\max_i(\lambda_i) < 2^m$. The circuit starts with a quantum phase estimation that stores the eigenvalues in the memory register.

Question Theory 3: Show that the state $|\phi_1\rangle$ defined in Figure 1 is

$$|\phi_1\rangle = |0\rangle \otimes \left(\sum_i \beta_i |\lambda_i\rangle \otimes |u_i\rangle\right). \tag{9}$$

Hint: Use the lecture notes on QPE. You can start by answering these questions:

- *What are the eigenvalues and eigenvectors of U ?*
- *What happens if $|b\rangle = |u_i\rangle$?*

Then we apply a controlled rotation R to create the $\frac{1}{\lambda}$ factor.

Question Theory 4: Compute $|\phi_2\rangle$.

We want to disentangle the memory register from the output state. Thus, we apply the inverse QPE.

Question Theory 5: Show that the state $|\phi_3\rangle$ defined in Figure 1 is

$$|\phi_3\rangle = \sum_i \beta_i \left(\sqrt{1 - \frac{1}{\lambda_i^2}}\,|0\rangle + \frac{1}{\lambda_i}|1\rangle\right) \otimes |0\rangle^{\otimes m} \otimes |u_i\rangle. \tag{10}$$

Hint: Start with $|b\rangle = |u_i\rangle$ and use the fact that gates are unitary.

Question Theory 6: Show that the output of the circuit is a solution of the linear system if the result of the measurement is "1". What is the probability of obtaining this result? Use $\max_i(\lambda_i) < 2^m$ to lower bound this result.

Question Implementation 1: Implement the HHL circuit on a simulator. Consider only the case where the eigenvalues of A are powers of 2. More detailed instructions are given in the notebook.

Measuring the solution

We want to measure $|x\rangle$ to learn its value. We are used to measuring states with projectors. The probability of a state $|\phi\rangle$ to be in state $|\psi\rangle$ is

$$P(|\psi\rangle) = \langle\phi| \left(|\psi\rangle\langle\psi|\right) |\phi\rangle = |\langle\phi\,|\psi\rangle\,|^2. \tag{11}$$

Question Implementation 2: On a simulator and a real device, run the HHL circuit to measure $|x\rangle$, the solution of the system

$$A = \frac{1}{2}\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \text{ and } |b\rangle = \frac{1}{\sqrt{3}}(2\,|0\rangle - |1\rangle). \tag{12}$$

Compare it to the expected solution $\frac{A^{-1}\mathbf{b}}{\|A^{-1}\mathbf{b}\|}$. What is missing to fully reconstruct the solution?

Limitations

Although the HHL algorithm offers exponential speedup over classical methods, it has several limitations:

- It requires that the matrix A be sparse to reach the announced complexity. In our implementation, we do not pay attention to this requirement.
- We need to have some a priori knowledge of the eigenvalues λ_i to have an exact and efficient algorithm.
- The algorithm only gives a solution with some probability of error.
- The preparation of the quantum state for $|b\rangle$ is often difficult and requires additional resources.
- We do not have direct access to the result. We still need to measure $|x\rangle$ to learn the solution.
- On a real quantum noisy device, we will only have access to an approximate solution.

Bonus - Another measurement

If we measure our state $|x\rangle$ only in the computational basis, which means that we choose the set of projectors $|i\rangle\langle i|$, we will only obtain the norm of each amplitude in the computational basis. If the amplitude is complex, we will not be able to reconstruct the full state $|x\rangle$. Thus, we need another kind of measurement.

An observable O is a Hermitian matrix that represents a physical property we want to measure in our system. It can be the position, momentum, spin ... We are interested to know the average value of that property. As a Hermitian matrix, O can be decomposed into its eigenbasis, $O = \sum_k \mu_k |v_k\rangle\langle v_k|$. The expected value of O observed in the state $|\phi\rangle$ is then $\langle O\rangle = \langle\phi| O |\phi\rangle = \sum_k \mu_k \langle\phi| \left(|v_k\rangle\langle v_k| \right) |\phi\rangle = \sum_k \mu_k P(|v_k\rangle)$.

Remark that if we choose O to be a projector $|\psi\rangle\langle\psi|$, then O has only one non-zero eigenvalue which is 1 associated with the eigenvector $|\psi\rangle$. Therefore, the expected value of O is $P(|\psi\rangle)$.

We focus on the case where A is a 2×2 matrix and $\mathbf{b}$ is a vector of size 2. We denote by ρ the matrix $\rho = |x\rangle\langle x|$. The set of operators $P = \frac{1}{\sqrt{2}}\{I, X, Y, Z\}$ is a basis of the space of the 2×2 matrices. Thus, we can decompose ρ in this basis:

$$\rho = \frac{c_I I + c_X X + c_Y Y + c_Z Z}{\sqrt{2}} \tag{13}$$

Question Theory 7: Show that $c_\sigma = \mathrm{Tr}(\rho\sigma)$ for all $\sigma \in P$.

Hint: Show that for all $\sigma_i\sigma_j \in P$, $\mathrm{Tr}(\sigma_i\sigma_j) = 0$ *if* $j \neq i$ *and* 1 *if* $i = j$.

Pauli matrices are Hermitian and can be interpreted as observables. Therefore, by measuring each operator $\sigma \in P$, we can have access to their expected value $\langle\sigma\rangle = \langle x| \sigma |x\rangle = \mathrm{Tr}(\langle x| \sigma |x\rangle) = \mathrm{Tr}(\rho\sigma)$, and we can reconstruct the density matrix ρ

$$\rho = \frac{\langle I\rangle I + \langle X\rangle X + \langle Y\rangle Y + \langle Z\rangle Z}{\sqrt{2}}. \tag{14}$$

Question Implementation 3: Run your HHL circuit on a simulator and a real quantum device and reconstruct completely the solution state for the system defined as

$$A = \frac{1}{9}\begin{pmatrix} 13 & 2 + i4 \\ 2 - i4 & 14 \end{pmatrix} \text{ and } |b\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle) \tag{15}$$

Observable measurements will be detailed in the notebook.