Question Theory 1

To apply HHL Algorithm, the Matrix must be hermitian. If A is not hermitian, we construct a new system with $\tilde{A}$ and $\tilde{b}$ where $\tilde{A}$ appears to be hermitian.

$\tilde{A}^{\dagger} = \begin{bmatrix} 0 & (A^{\dagger})^{\dagger} \\ A^{\dagger} & 0 \end{bmatrix} = \begin{bmatrix} 0 & A \\ A^{\dagger} & 0 \end{bmatrix} = \tilde{A}$ $\quad \tilde{A} = \tilde{A}^{\dagger} \Rightarrow \tilde{A}$ is hermitian, we can apply HHL Algorithm.

$\tilde{A}\tilde{x} = \tilde{b} \iff \begin{bmatrix} 0 & A \\ A^{\dagger} & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix} \iff \begin{cases} Ax_2 = b \\ A^{\dagger}x_1 = 0 \end{cases}$ if A is invertible then $A^{\dagger}x_1 = 0$ implies $x_1 = 0$ (because $A^{\dagger}$ is also invertible)

$\Rightarrow \tilde{x} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$ and $x_2 = x$. such that $Ax_2 = b \iff x_2 = A^{-1}b = x$ which is the solution of the original system.

Running HHL on $\tilde{A}\tilde{x} = \tilde{b}$ leads to a state proportional to $|\tilde{x}\rangle$ st.: $|\tilde{x}\rangle \propto |0\rangle \otimes |x\rangle$. Measuring the solution extracts $|x\rangle$, solving the original system. Thus the HHL applied to the extand system allows to recover x solving $Ax = b$.

We can recover $\tilde{x}$ with HHL and deduce that the second half of parameters found for $\tilde{x}$ correspond to the initially searched x

Question Theory 2:

$A = \sum_{i=0}^{2^n-1} \lambda_i |u_i\rangle\langle u_i|$ $\qquad |b\rangle = \sum_{i=0}^{2^n-1} \beta_i |u_i\rangle$ $\qquad$ Is $|x\rangle = \sum_{i=0}^{2^n-1} \frac{\beta_i}{\lambda_i} |u_i\rangle$ solution?

$$A|x\rangle = c\sum_{i=0}^{2^n-1}\sum_{j=0}^{2^n-1}(\lambda_i |u_i\rangle\langle u_i|)\left(\frac{\beta_j}{\lambda_j}|u_j\rangle\right) = c\sum_{i,j=0}^{2^n-1}\frac{\lambda_i}{\lambda_j}\beta_j |u_i\rangle\underbrace{\langle u_i|u_j\rangle}_{\delta_{ij}} = c\sum_{i=0}^{2^n-1}\frac{\cancel{\lambda_i}}{\cancel{\lambda_i}}\beta_i |u_i\rangle = c\sum_{i=0}^{2^n-1}\beta_i |u_i\rangle = c|b\rangle \Rightarrow |x\rangle \text{ satisfies } A|x\rangle = c|b\rangle.$$

HHL Algorithm requires normalized state: $A|x\rangle = c|b\rangle \iff |x\rangle = cA^{-1}|b\rangle$ with $c^{-1} = \|A^{-1}|b\rangle\|$

In this case we have indeed $x = A^{-1}b$ and x is solution to the system $Ax = b$ in the quantum state form.

Question Theory 3:

$U = e^{i2\pi\frac{A}{2^m}} = \sum_{i=0}^{2^n-1} e^{i2\pi\frac{\lambda_i}{2^m}} |u_i\rangle\langle u_i|$ $\qquad |\phi_1\rangle = |0\rangle \otimes \sum_i \beta_i |\lambda_i\rangle \otimes |u_i\rangle$

$|\phi_1\rangle = |0\rangle \otimes QPE\, |0\rangle^{\otimes m} \otimes |b\rangle$ $\qquad U|u_i\rangle = e^{i2\pi\frac{\lambda_i}{2^m}} |u_i\rangle$

$$|\phi_1\rangle = |0\rangle \otimes QPE\left(|0\rangle^{\otimes m} \otimes \sum_{i=0}^{2^n-1}\beta_i |u_i\rangle\right)$$
$$= |0\rangle \otimes \sum_{i=0}^{2^n-1}\beta_i\, QPE\left(|0\rangle^{\otimes m} \otimes |u_i\rangle\right)$$

$$= |0\rangle \otimes \sum_{i=0}^{2^n-1}\beta_i\, QFT^{\dagger} \otimes \mathbb{1}^{\otimes m}\left\{\frac{1}{\sqrt{2}}\left(|0\rangle + e^{i2\pi\frac{\lambda_i}{2^m}}|1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i2\pi\frac{\lambda_i}{2^m}2}|1\rangle\right) \otimes \dots \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i2\pi\frac{\lambda_i}{2^m}2^{m-1}}|1\rangle\right) \otimes |u_i\rangle.\right\}$$

$$= |0\rangle \otimes \sum_{i=0}^{2^n-1}\beta_i\left(QFT^{\dagger} \otimes \mathbb{1}^{\otimes n}\right)\left(\frac{1}{2^{m/2}}\sum_{z=0}^{2^m-1} e^{2\pi i\frac{\lambda_i}{2^m}z}|z\rangle \otimes |u_i\rangle\right)$$ But $\lambda_i \in \mathbb{N}^*_i$ and $\max_i(\lambda_i) \leq 2^m$, so we can write:

$$|\phi_1\rangle = |0\rangle \otimes \sum_{i=0}^{2^n-1}\beta_i\, \underbrace{QFT^{\dagger}QFT}_{\mathbb{1}}|\lambda_i\rangle \otimes |u_i\rangle$$

$$|\phi_1\rangle = |0\rangle \otimes \left(\sum_{i=0}^{2^n-1}\beta_i |\lambda_i\rangle \otimes |u_i\rangle\right) \quad ☒$$

Question Theory 4:

$$R(|0\rangle \otimes |\lambda\rangle) = \left(\sqrt{1-\frac{1}{\lambda^2}}\,|0\rangle + \frac{1}{\lambda}|1\rangle\right) \otimes |\lambda\rangle$$

$$R_{12} \otimes \mathbb{1}^{\otimes n} |\phi_1\rangle = \cancel{R\otimes\mathbb{1}}\left(|0\rangle \otimes \sum_{i=0}^{2^n-1} \beta_i |\lambda_i\rangle \otimes |u_i\rangle\right)$$

$$= \sum_{i=0}^{2^n-1} \beta_i\, R(|0\rangle \otimes |\lambda_i\rangle) \otimes |u_i\rangle$$

$$|\phi_2\rangle = \sum_{i=0}^{2^n-1} \beta_i \left(\sqrt{1-\frac{1}{\lambda_i^2}}\,|0\rangle + \frac{1}{\lambda_i}|1\rangle\right) \otimes |\lambda_i\rangle \otimes |u_i\rangle$$

Question Theory 5:

We have previously shown that: $|0\rangle \otimes QPE\left(|0\rangle^{\otimes m} \otimes \sum_{i=0}^{2^n-1} \beta_i |u_i\rangle\right) = |0\rangle \otimes \left(\sum_{i=0}^{2^n-1} \beta_i |\lambda_i\rangle \otimes |u_i\rangle\right)$

$$\Leftrightarrow \quad QPE\left(|0\rangle^{\otimes m} \otimes \sum_{i=0}^{2^n-1} \beta_i |u_i\rangle\right) = \left(\sum_{i=0}^{2^n-1} \beta_i |\lambda_i\rangle \otimes |u_i\rangle\right)$$

Since $U = QPE$ is unitary, we have: $QPE \cdot QPE^\dagger = \mathbb{1}$. Therefore we can rewrite the above equation as:

$$\underbrace{QPE^\dagger\, QPE}_{\mathbb{1}}\left(|0\rangle^{\otimes m} \otimes \sum_{i=0}^{2^n-1} \beta_i |u_i\rangle\right) = QPE^\dagger\left(\sum_{i=0}^{2^n-1} \beta_i |\lambda_i\rangle \otimes |u_i\rangle\right)$$

$$|0\rangle^{\otimes m} \otimes \sum_{i=0}^{2^n-1} \beta_i |u_i\rangle = \sum_{i=0}^{2^n-1} \beta_i\, QPE^\dagger\left(|\lambda_i\rangle \otimes |u_i\rangle\right) \quad (*)$$

Let's come back to $|\phi_3\rangle = (\mathbb{1} \otimes QPE)|\phi_2\rangle = (\mathbb{1} \otimes QPE) \sum_{i=0}^{2^n-1} \beta_i \left(\sqrt{1-\frac{1}{\lambda_i}}\,|0\rangle + \frac{1}{\lambda_i}|1\rangle\right) \otimes |\lambda_i\rangle \otimes |u_i\rangle$

By linearity: $|\phi_3\rangle = \sum_{i=0}^{2^n-1} \left(\sqrt{1-\frac{1}{\lambda_i^2}}\,|0\rangle + \frac{1}{\lambda_i}|1\rangle\right) \otimes \beta_i\, QPE^\dagger\left(|\lambda_i\rangle \otimes |u_i\rangle\right)$

$$|\phi_3\rangle = \sum_{i=0}^{2^n-1}\left\{ \underbrace{\beta_i\sqrt{1-\frac{1}{\lambda_i^2}}}_{\beta_i'}\,|0\rangle \otimes QPE^\dagger\left(|\lambda_i\rangle \otimes |u_i\rangle\right)\right\} + \sum_{i=0}^{2^n-1}\left\{ \underbrace{\frac{1}{\lambda_i}\beta_i}_{\beta_i''}\,|1\rangle \otimes QPE^\dagger\left(|\lambda_i\rangle \otimes |u_i\rangle\right)\right\}$$

$$|\phi_3\rangle = |0\rangle \otimes \sum_{i=0}^{2^n-1}\left\{\beta_i'\, QPE^\dagger\left(|\lambda_i\rangle \otimes |u_i\rangle\right)\right\} + |1\rangle \otimes \sum_{i=0}^{2^n-1}\left\{\beta_i''\, QPE^\dagger\left(|\lambda_i\rangle \otimes |u_i\rangle\right)\right\} \quad (*)$$

$$|\phi_3\rangle = |0\rangle \otimes |0\rangle^{\otimes m} \otimes \sum_{i=0}^{2^n-1} \beta_i' |u_i\rangle + |1\rangle \otimes |0\rangle^{\otimes m} \otimes \sum_{i=0}^{2^n-1} \beta_i'' |u_i\rangle$$

$$|\phi_3\rangle = \sum_{i=0}^{2^n-1}\left\{\sqrt{1-\frac{1}{\lambda_i^2}}\,\beta_i |0\rangle \otimes |0\rangle^{\otimes m} \otimes |u_i\rangle + \beta_i \frac{1}{\lambda_i}|1\rangle \otimes |0\rangle^{\otimes m} \otimes |u_i\rangle\right.$$

$$|\phi_3\rangle = \sum_{i=0}^{2^n-1} \beta_i \left(\sqrt{1-\frac{1}{\lambda_i^2}}\,|0\rangle + \frac{1}{\lambda_i}|1\rangle\right) \otimes |0\rangle^{\otimes m} \otimes |u_i\rangle \quad \boxtimes$$

Question Theory 6:

$$P_1 = |1\rangle\langle 1| \otimes \mathbb{1}^{\otimes m} \otimes \mathbb{1}^{\otimes n}$$

$$|\psi_{after}\rangle \propto P_1 |\phi_3\rangle = \sum_{i=0}^{2^n-1} \beta_i \frac{1}{\lambda_i} |1\rangle \otimes |0\rangle^{\otimes m} \otimes |u_i\rangle = |1\rangle \otimes |0\rangle^{\otimes m} \otimes \sum_{i=0}^{2^n-1} \frac{\beta_i}{\lambda_i} |u_i\rangle$$

$$Prob(|1\rangle) = \langle \Phi_3 | P_1 | \Phi_3 \rangle = \sum_{i,j=0}^{2^n-1} \left(\frac{\beta_j}{\lambda_j}\right)^* \left(\frac{\beta_i}{\lambda_i}\right) \underbrace{\langle u_j | u_i \rangle}_{\delta_{ij}} = \sum_{i=0}^{2^n-1} \frac{|\beta_i|^2}{\lambda_i^2}$$

$$|\Psi_{after}\rangle = |1\rangle \otimes |0\rangle^{\otimes m} \frac{1}{\sqrt{\sum_{i=0}^{2^n-1} \frac{|\beta_i|^2}{\lambda_i^2}}} \sum_{i=0}^{2^n-1} \frac{\beta_i}{\lambda_i} |u_i\rangle$$

In the bottom register we have indeed $|x\rangle = c \sum_{i=0}^{2^n-1} \frac{\beta_i}{\lambda_i} |u_i\rangle$ with $c^{-1} = \sqrt{\sum_{i=0}^{2^n-1} \frac{|\beta_i|^2}{\lambda_i^2}}$ which is solution of $|x\rangle = c A^{-1} |b\rangle$

$$\| A^{-1} |b\rangle \|^2 = \left(\sum_{k,\ell}^{2^n-1} \frac{1}{\lambda_k^*} \beta_\ell^* \langle u_\ell | u_k \rangle \langle u_k | \right) \left(\sum_{i,j=0}^{2^n-1} \left(\frac{1}{\lambda_i} |u_i\rangle\langle u_i|\right) (\beta_j |u_j\rangle) \right) = \sum_{k,i=0}^{2^n-1} \frac{\beta_k^*}{\lambda_k} \frac{\beta_i}{\lambda_i} \langle u_k | u_i \rangle = \sum_{i=0}^{2^n-1} \frac{|\beta_i|^2}{\lambda_i^2}$$

This coincides with the normalization constant s.t. $c^{-1} = \sqrt{Prob(|1\rangle)} = \| A^{-1} |b\rangle \|$. We can conclude that x is solution of $Ax = b$.

Let's find a bound for the probability of collapsing on $|1\rangle$. $Prob(|1\rangle) = \sum_{i=0}^{2^n-1} \frac{|\beta_i|^2}{\lambda_i^2}$

$$\sum_{i=0}^{2^n-1} |\beta_i|^2 = 1 \qquad \max_i \{\lambda_i\} \leq 2^m \iff \max_i \{\lambda_i\}^2 \leq 2^{2m} \iff \max_i \{\lambda_i^2\} \leq 2^{2m}$$

$$\iff \frac{1}{\max_i\{\lambda_i^2\}} \geq \frac{1}{2^{2m}} \qquad \frac{\sum_{i=0}^{2^n-1} |\beta_i|^2}{2^{2m}} \leq \sum_{i=0}^{2^n-1} \frac{|\beta_i|^2}{\max_i\{\lambda_i^2\}} \leq \sum_{i=0}^{2^n-1} \frac{|\beta_i|^2}{\lambda_i^2} = Prob(|1\rangle)$$

$$\frac{1}{2^{2m}} \leq \frac{1}{\max_i\{\lambda_i^2\}} \leq Prob(|1\rangle)$$

Question Theory 7.

We know that: $\sigma_i \sigma_j = \delta_{ij} \mathbb{1} + i \varepsilon_{ijk} \sigma_k$

$$\iff Tr(\sigma_i \sigma_j) = \delta_{ij} Tr(\mathbb{1}) + i \varepsilon_{ijk} \underbrace{Tr(\sigma_k)}_{=0}$$

$$\iff Tr(\sigma_i \sigma_j) = 2\delta_{ij} \quad \text{for } \sigma_i \in \{\mathbb{1}, \hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z\}$$

So $Tr(\sigma_i \sigma_j) = \delta_{ij}$ for $\sigma_i \in P = \frac{1}{\sqrt{2}} \{\mathbb{1}; \hat{x}; \hat{y}; \hat{z}\}$

$$\rho = \sum_{\sigma_i \in P} c_{\sigma_i} \sigma_i \qquad \sigma_i \in P = \frac{1}{\sqrt{2}} \{\mathbb{1}; \hat{X}; \hat{Y}; Z\}$$

$$Tr(\rho \sigma_j) = Tr\left[\sum_{\sigma_i \in P} c_{\sigma_i} \sigma_i \sigma_j\right] = \sum_{\sigma_i \in P} c_{\sigma_i} Tr[\sigma_i \sigma_j] = \sum_{\sigma_i \in P} c_{\sigma_i} \delta_{ij} = c_{\sigma_j}$$

But $\rho = |x\rangle\langle x|$ so $Tr(\rho \sigma_j) = Tr(|x\rangle\langle x| \sigma_j) = Tr(\langle x | \sigma_j | x \rangle) = \langle x | \sigma_j | x \rangle = \langle \sigma_j \rangle = c_{\sigma_j}$ for $\sigma_j \in P = \frac{1}{\sqrt{2}} \{\mathbb{1}; X, Y, Z\}$

We conclude $\rho = \dfrac{\langle \mathbb{1} \rangle \mathbb{1} + \langle X \rangle X + \langle Y \rangle Y + \langle Z \rangle Z}{\sqrt{2}}$

$$|\phi_3\rangle = \sum_{i=0}^{2^m-1} \beta_i \left(\sqrt{1-\frac{1}{\lambda_i^2}}|0\rangle + \frac{1}{\lambda_i}|1\rangle\right)\otimes|0\rangle^{\otimes m}\otimes|u_i\rangle$$

$$\rho_{out} = \sum_{i=0}^{2^m-1}\sum_{j=0}^{2^m-1} \beta_j\beta_i\left(\sqrt{1-\frac{1}{\lambda_i^2}}|0\rangle + \frac{1}{\lambda_i}|1\rangle\right)\left(\sqrt{1-\frac{1}{\lambda_j^2}}\langle 0| + \frac{1}{\lambda_j}\langle 1|\right)\otimes|0\rangle\langle 0|^{\otimes m}\otimes|u_i\rangle\langle u_j|$$

$$Tr\left(P_1\otimes \mathbb{1}^{\otimes m}\otimes o\,\rho_{out}\right) = \sum_{i,j=0}^{2^m-1}\beta_i\beta_j\left(\sqrt{1-\frac{1}{\lambda_j^2}}\langle 0| + \frac{1}{\lambda_j}\langle 1|\right)\overbrace{P_1}^{|1\rangle\langle 1|}\left(\sqrt{1-\frac{1}{\lambda_i^2}}|0\rangle + \frac{1}{\lambda_i}|1\rangle\right)\underbrace{{}^{\otimes m}\langle 0|\mathbb{1}^{\otimes m}|0\rangle^{\otimes m}}_{1}\cdot\langle u_j|o|u_i\rangle$$

$$= \sum_{i,j=0}^{2^m-1}\frac{\beta_i\beta_j}{\lambda_i\lambda_j}\langle u_j|o|u_i\rangle$$

$$\rho_{P_1} = \frac{P_1\rho_{out}P_1}{Tr(P_1\rho_{out})} = \frac{P_1\rho_{out}P_1}{P\text{-}succes}$$

$$P_o = \mathbb{1}\otimes\mathbb{1}^{\otimes m}\otimes o$$

$$P_1 = |1\rangle\langle 1|\otimes\mathbb{1}^{\otimes m}\otimes\mathbb{1}$$

$$P_{tot} = |1\rangle\langle 1|\otimes\mathbb{1}^{\otimes m}\otimes P_o.$$

$$P_{tot} = P_o\cdot P_1 = |1\rangle\langle 1|\otimes\mathbb{1}^{\otimes m}\otimes P_o$$

$$\langle o_{q3}\rangle_{|P_1} = Tr(P_o\,\rho_{P_1})$$

$$= Tr\left(P_o\,\frac{P_1\rho_{out}P_1}{P_{succes}}\right)$$

$$= \frac{1}{P_{succes}}Tr\left(P_oP_1\rho_{out}P_1\right)$$

$$\rho_{after} = \frac{P_{tot}\,\rho_{out}\,P_{tot}}{Tr(P_{tot}\rho_{out})} = \frac{P_oP_1\,\rho_{out}\,P_oP_1}{Tr(P_oP_1\rho_{out})}$$

$$= \frac{1}{P_{succes}}Tr\left(P_1P_oP_1\rho_{out}\right)$$

P_o et P_1 commutent

$$= \frac{1}{P_{succes}}Tr\left(P_oP_1^2\rho_{out}\right)$$

$$= \frac{1}{P_{succes}}Tr\left(P_oP_1\rho_{out}\right) = \boxed{\frac{1}{P_{succes}}Tr\left(|1\rangle\langle 1|\otimes\mathbb{1}^{\otimes m}\otimes o\,\rho_{out}\right) = \langle o_{q3}\rangle_{|P_1}}$$

$$Tr(o\rho) = Tr\left(o\left(c_i\mathbb{1} + c_xX + c_yY + c_zZ\right)\right)$$

$$\langle o\rangle = 2c_o$$

$$c_o = \frac{\langle o\rangle}{2}$$

$$\rho_{rec} = |x\rangle\langle x| = \begin{bmatrix}x_0\\x_1\end{bmatrix}\begin{bmatrix}x_0^* & x_1^*\end{bmatrix} = \begin{bmatrix}|x_0|^2 & x_0x_1^*\\ x_0^*x_1 & |x_1|^2\end{bmatrix} = \begin{bmatrix}|x_0|^2 & |x_0|e^{i\varphi_0}|x_1|e^{-i\varphi_1}\\ |x_0|e^{-i\varphi_0}|x_1|e^{i\varphi_1} & |x_1|^2\end{bmatrix}$$

$$= \begin{bmatrix}|x_0|^2 & |x_0x_1|e^{i(\varphi_0-\varphi_1)}\\ |x_0x_1|e^{-i(\varphi_0-\varphi_1)} & |x_1|^2\end{bmatrix} =$$

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[[ 0.52818035+0.j          -0.11272142-0.4863124j]
 [-0.11272142+0.4863124j   0.47181965+0.j        ]]
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$$|x_0|^2 = |\rho_{00}|^2 \qquad |x_0x_1|e^{i(\varphi_0-\varphi_1)} = |\rho_{01}|e^{i\varphi_{01}}$$

$$|x_0x_1|e^{-i(\varphi_0-\varphi_1)} = |\rho_{10}|e^{-i\varphi_{01}} \qquad |x_1|^2 = |\rho_{11}|^2$$

Question Theory 5:

We have previously shown that: $|0\rangle \otimes QPE\left(|0\rangle^{\otimes m} \otimes \sum_{i=0}^{2^n-1} \beta_i |u_i\rangle\right) = |0\rangle \otimes \left(\sum_{i=0}^{2^n-1} \beta_i |\lambda_i\rangle \otimes |u_i\rangle\right)$

$$\iff \quad QPE\left(|0\rangle^{\otimes m} \otimes \sum_{i=0}^{2^n-1} \beta_i |u_i\rangle\right) = \left(\sum_{i=0}^{2^n-1} \beta_i |\lambda_i\rangle \otimes |u_i\rangle\right)$$

Since $U = QPE$ is unitary, we have: $QPE \cdot QPE^\dagger = \mathbb{1}$ Therefore we can rewrite the above equation as:

$$\underbrace{QPE^\dagger QPE}_{\mathbb{1}}\left(|0\rangle^{\otimes m} \otimes \sum_{i=0}^{2^n-1} \beta_i |u_i\rangle\right) = QPE^\dagger\left(\sum_{i=0}^{2^n-1} \beta_i |\lambda_i\rangle \otimes |u_i\rangle\right)$$

$$|0\rangle^{\otimes m} \otimes \sum_{i=0}^{2^n-1} \beta_i |u_i\rangle = \sum_{i=0}^{2^n-1} \beta_i \; QPE^\dagger\left(|\lambda_i\rangle \otimes |u_i\rangle\right) \qquad (*)$$

Let's come back to $|\phi_3\rangle = (\mathbb{1} \otimes QPE)\,|\phi_2\rangle = (\mathbb{1} \otimes QPE) \sum_{i=0}^{2^n-1} \beta_i \left(\sqrt{1 - \frac{1}{\lambda_i}}\,|0\rangle + \frac{1}{\lambda_i}|1\rangle\right) \otimes |\lambda_i\rangle \otimes |u_i\rangle$

By linearity: $|\phi_3\rangle = \sum_{i=0}^{2^n-1} \left(\sqrt{1 - \frac{1}{\lambda_i^2}}\,|0\rangle + \frac{1}{\lambda_i}|1\rangle\right) \otimes \beta_i\, QPE^\dagger\left(|\lambda_i\rangle \otimes |u_i\rangle\right)$

$$|\phi_3\rangle = \sum_{i=0}^{2^n-1}\left\{ \underbrace{\beta_i \sqrt{1 - \frac{1}{\lambda_i^2}}}_{\beta_i'}\,|0\rangle \otimes QPE^\dagger\left(|\lambda_i\rangle \otimes |u_i\rangle\right)\right\} + \sum_{i=0}^{2^n-1}\left\{ \underbrace{\frac{1}{\lambda_i}\beta_i}_{\beta_i''}\,|1\rangle \otimes QPE^\dagger\left(|\lambda_i\rangle \otimes |u_i\rangle\right)\right\}$$

$$|\phi_3\rangle = |0\rangle \otimes \sum_{i=0}^{2^n-1}\left\{\beta_i'\, QPE^\dagger\left(|\lambda_i\rangle \otimes |u_i\rangle\right)\right\} + |1\rangle \otimes \sum_{i=0}^{2^n-1}\left\{\beta_i''\, QPE^\dagger\left(|\lambda_i\rangle \otimes |u_i\rangle\right)\right\} \qquad (*)$$

$$|\phi_3\rangle = |0\rangle \otimes |0\rangle^{\otimes m} \otimes \sum_{i=0}^{2^n-1} \beta_i' |u_i\rangle + |1\rangle \otimes |0\rangle^{\otimes m} \otimes \sum_{i=0}^{2^n-1} \beta_i'' |u_i\rangle$$

$$|\phi_3\rangle = \sum_{i=0}^{2^n-1}\left\{ \sqrt{1 - \frac{1}{\lambda_i^2}}\,\beta_i |0\rangle \otimes |0\rangle^{\otimes m} \otimes |u_i\rangle + \beta_i \frac{1}{\lambda_i} |1\rangle \otimes |0\rangle^{\otimes m} \otimes |u_i\rangle\right.$$

$$|\phi_3\rangle = \sum_{i=0}^{2^n-1} \beta_i \left(\sqrt{1 - \frac{1}{\lambda_i^2}}\,|0\rangle + \frac{1}{\lambda_i}|1\rangle\right) \otimes |0\rangle^{\otimes m} \otimes |u_i\rangle \qquad \blacksquare$$