Exercise Ex1

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Student: Romain Couyoumtzélis Sciper: 340933

Please use this template to submit your answers.
If you had to modify code from the notebook, please include the modified code in your submission either as screenshot or in a

```
\begin{lstlisting}[language=Python]
\end{lstlisting}
```

environment.

You only need to include the code cells that you modified.

Note, that references to other parts of the documents aren't resolved in this template and will show as ??. Check the text of the exercises on website for the reference

Exercise 1
Calculate the vector product $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ with

$$\mathbf{a} = \begin{pmatrix} 2 \\ 6 \\ 4 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 5 \\ 1 \\ 7 \end{pmatrix} \tag{1}$$

and, for the same $\mathbf{a}, \mathbf{b}$, the scalar product

$$d = \mathbf{a} \cdot \mathbf{b} \tag{2}$$

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = \begin{pmatrix} 38 \\ 6 \\ -28 \end{pmatrix}$$

$$\mathbf{d} = \mathbf{a} \cdot \mathbf{b} = 44$$

Exercise 2
Evaluate the matrix product $\mathbf{C} = \mathbf{AB}$.

$$\mathbf{A} = \begin{pmatrix} 6 & 8 & 2 \\ 9 & 1 & 5 \\ 7 & 4 & 3 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 9 & 6 & 7 \\ 5 & 4 & 4 \\ 3 & 2 & 8 \end{pmatrix}. \tag{3}$$

$$\mathbf{A} = \begin{pmatrix} 100 & 72 & 90 \\ 101 & 68 & 107 \\ 92 & 64 & 89 \end{pmatrix}$$

Exercise 3
Evaluate the determinant for the matrix $\mathbf{A}$.

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 2 \\ 2 & 2 & 3 \end{pmatrix} \tag{4}$$

$$\det(\mathbf{A}) = 1 \tag{5}$$

Exercise 4
Does the exponent of an operator always satisfy the relation $e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}$? Start from the definition of the matrix exponential.

$$e^{\hat{A}+\hat{B}} \overset{?}{=} e^{\hat{A}}e^{\hat{B}}$$

$$e^{\hat{A}+\hat{B}} = \sum_{n=0}^{\infty} \frac{(\hat{A}+\hat{B})^n}{n!} = \mathbb{1} + \hat{A}+\hat{B} + \frac{1}{2!}(\hat{A}+\hat{B})^2 + \dots = \mathbb{1} + \hat{A} + \hat{B} + \frac{1}{2}(\hat{A}^2 + \hat{A}\hat{B} + \hat{B}\hat{A} + \hat{B}^2) + \dots$$

$$e^{\hat{A}} \cdot e^{\hat{B}} = \sum_{n=0}^{\infty} \frac{\hat{A}^n}{n!} \sum_{m=0} \frac{\hat{B}^m}{m!} = (\mathbb{1} + \hat{A} + \frac{1}{2!}\hat{A}^2 + \dots)(\mathbb{1} + \hat{B} + \frac{1}{2!}\hat{B}^2)$$
$$= \mathbb{1} + \hat{B} + \frac{1}{2!}\hat{B}^2 + \hat{A} + \hat{A}\hat{B} + \frac{1}{2}\hat{A}\hat{B}^2 + \frac{1}{2}\hat{A}^2 + \frac{1}{2}\hat{A}^2\hat{B} + \frac{1}{4}\hat{A}^2\hat{B}^2)$$
$$= \mathbb{1} + \hat{A} + \hat{B} + \frac{1}{2!}(\hat{A}^2 + \hat{B}^2) + \hat{A}\hat{B} + \frac{1}{2}(\hat{A}\hat{B}^2 + \hat{A}^2\hat{B}) + \frac{1}{4}(\hat{A}^2\hat{B}^2).$$

Let's compare terms of different orders.

Order 1: $e^{\hat{A}+\hat{B}}$: $\hat{A}+\hat{B}$ $\qquad$ $e^{\hat{A}} \cdot e^{\hat{B}}$: $\hat{A}+\hat{B}$ $\qquad$ order 1 terms are equal.

Order 2: $e^{\hat{A}+\hat{B}}$: $\frac{1}{2}(\hat{A}^2 + AB + BA + \hat{B}^2)$ $\qquad$ $e^{\hat{A}} \cdot e^{\hat{B}}$: $\frac{1}{2}(\hat{A}^2 + \hat{B}^2) + \hat{A}\hat{B}$ $\qquad$ order 2 terms are already not equal.

If $[\hat{A}, \hat{B}] = AB - BA = 0$: $\quad e^{\hat{A}+\hat{B}} = \sum_{n=0}^{\infty} \frac{(\hat{A}+\hat{B})^n}{n!}$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \hat{A}^k \hat{B}^{n-k}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{\cancel{n!}} \frac{\cancel{n!}}{k!(n-k)!} \hat{A}^k \cdot \hat{B}^{n-k} \qquad n = m+k.$$

$$= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{\hat{A}^k}{k!} \frac{\hat{B}^m}{m!} \qquad m = n-k. \quad 0 \leq k \leq n \quad \text{so} \quad m \geq 0.$$

$$= e^{\hat{A}} \cdot e^{\hat{B}}$$

Exercise 5
Find the eigenvalues and eigenvectors of the matrix $\mathbf{A}$.

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} \tag{6}$$

$$\begin{aligned} \text{eigenvalue} : \ & \lambda = 3 \quad (\text{multiplicity} = 2) \\ \text{associated eigenvector} : \ & \mathbf{v} = \begin{pmatrix} 1 & 1 \end{pmatrix} \end{aligned}$$

Exercise 6
Show that if the product $\mathbf{C} = \mathbf{AB}$ of two Hermitian matrices is also Hermitian, then $\mathbf{A}$ and $\mathbf{B}$ commute.

$$\begin{aligned} C = AB =& C^\dagger = (AB)^\dagger \\ \Leftrightarrow AB =& B^\dagger A^\dagger = BA \quad \text{since } A^\dagger = A \quad \text{and } B = B^\dagger \\ \Leftrightarrow AB - BA =& 0 = [A, B] \end{aligned}$$

Since $[A, B] = 0$, we have that A and B commute.

Exercise 7
Explain the connection between the Heisenberg uncertainty principle and the commutation relation.

If we take for example the momentum and the position operators, we have the following commutation relation :

$$[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar$$

Then the Heisenberg Uncertainty Principle states that for two conjugate variables such as momentum and position, the following inequality holds :

$$\Delta x \cdot \Delta p \geq \hbar/2$$

This non-zero commutator implies that precise simultaneous measurements are impossible, as derived from the general uncertainty relation :

$$(\Delta A)^2 \cdot (\Delta B)^2 \geq (\frac{1}{2i}|\langle[\hat{A}, \hat{B}]\rangle|)^2$$

Exercise 8
What is the meaning of a multiplication of a bra and a ket

$$\langle a|b\rangle\,, \tag{7}$$

and, conversely, an operator formed by a ket and a bra?

$$\hat{\mathrm{O}} = |a\rangle\langle b| \tag{8}$$

$\langle a|b\rangle$ corresponds to a hermitian scalar product of two vectors living in the same Hilbert space.

$|a\rangle\langle b|$ is an outer product forming a linear operator, which becomes, when represented in a basis, a matrix with elements determined by the components of $|a\rangle$ and $\langle b|$, acting as a projection operator onto $|a\rangle$ weighted by $\langle b|$.

Exercise 9
Given a basis $\{\psi\}$ for which

$$\langle\psi_i|\psi_j\rangle = \delta_{ij}, \tag{9}$$

where δ_{ij} is the Kronecker delta, for any state Ψ

$$|\Psi\rangle = \sum_j c_j\,|\psi_j\rangle\,, \tag{10}$$

the inner product is defined as

$$\langle\Psi|\Psi\rangle = 1. \tag{11}$$

Show that this holds only as long as

$$\sum_j c_j^2 = 1, \tag{12}$$

where the c_j are the aforementioned expansion coefficients.

$$\begin{aligned}
\langle\psi|\psi\rangle &= \sum_j\sum_i c_i^* c_j\,\langle\psi_i|\psi_j\rangle \\
&= \sum_j\sum_i c_i c_j \delta_{ij} \\
&= \sum_j c_j^2 \\
&= 1
\end{aligned}$$

Therefore we have indeed that $\langle\psi|\psi\rangle = 1$ if $\sum_j c_j^2 = 1$

Bonus Exercise 10
Prove that, given the above conditions, $c_j = \langle\psi_j|\Psi\rangle$.

Since $\langle\psi_i|\psi_j\rangle = \delta_{ij}$, this means that the set $\{|\psi_j\rangle\}$ forms an orthonormal basis and therefore $\sum_j = |\psi_j\rangle\langle\psi_j| = \mathbb{I}$. Consequently :

$$\begin{aligned}
|\Psi\rangle &= \mathbb{I}\,|\Psi\rangle \\
&= \sum_j |\psi_j\rangle \underbrace{\langle\psi_j|\,|\psi\rangle}_{c_j} \\
&= \sum_j c_j\,|\psi_j\rangle \\
\Leftrightarrow \langle\psi_k|\Psi\rangle &= \sum_j c_j \underbrace{\langle\psi_k|\psi_j\rangle}_{\delta_{jk}} \\
&= c_k
\end{aligned}$$

Exercise 11
Diagonalise the matrices $\mathbf{A}$ and $\mathbf{B}$. Specify which one is Hermitian.

$$\mathbf{A} = \begin{pmatrix} 1 & 1-i \\ 1+i & 2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{13}$$

$$A^\dagger = \begin{pmatrix} \overline{1} & \overline{1+i} \\ \overline{1-i} & \overline{2} \end{pmatrix} = \begin{pmatrix} 1 & 1-i \\ 1+i & 2 \end{pmatrix} = A$$

$$\begin{aligned}
A - \lambda I &= \begin{pmatrix} 1-\lambda & 1-i \\ 1+i & 2-\lambda \end{pmatrix} \\
\det(A-\lambda I) &= (1-\lambda)(2-\lambda) - (1-i)(1+i) \\
&= (1-\lambda)(2-\lambda) - (1-i^2) = (1-\lambda)(2-\lambda) - (1-(-1)) \\
&= (1-\lambda)(2-\lambda) - 2 = 2 - \lambda - 2\lambda + \lambda^2 - 2 \\
&= \lambda^2 - 3\lambda \\
&= \lambda^2 - 3\lambda = 0 \implies \lambda(\lambda-3) = 0
\end{aligned}$$

Eigenvalues: $\lambda_1 = 0$, $\lambda_2 = 3$.
For $\lambda_1 = 0$:

$$\begin{pmatrix} 1 & 1-i \\ 1+i & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x + (1-i)y = 0 \implies x = (i-1)y$$

Eigenvector normalized: $\begin{pmatrix} \frac{i-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$

For $\lambda_2 = 3$:

$$\begin{pmatrix} 1-3 & 1-i \\ 1+i & 2-3 \end{pmatrix} = \begin{pmatrix} -2 & 1-i \\ 1+i & -1 \end{pmatrix}$$

$$-2x + (1-i)y = 0 \implies x = \frac{1-i}{2}y$$

Eigenvector normalized: $\begin{pmatrix} \frac{1-i}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{pmatrix}$.

Therefore A is expressed as $A = PDP^{-1}$ where :

$$P = \begin{pmatrix} \frac{i-1}{\sqrt{3}} & \frac{1-i}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{pmatrix} \quad D = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}$$

$B^\dagger = \begin{pmatrix} \overline{1} & \overline{0} \\ \overline{1} & \overline{1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \neq B$

$$\det(B - \lambda I) = \det \begin{pmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{pmatrix} = (1-\lambda)^2$$

$$(1-\lambda)^2 = 0 \implies \lambda = 1 \quad \text{(algebraic multiplicity 2)}$$

For $\lambda = 1$:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$y = 0$$

Eigenvector: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

We have that $\ker(B - \mathbb{I}) = \text{span}\{(1 \quad 0)\}$. So $\dim\ker(B - \mathbb{I}) = 1$ which is different of 2. Therefore B can not be diagonalized.

A is hermitian since $A = A^\dagger$ but B is not hermitian since $B \neq B^\dagger$

Bonus Exercise 12

Prove that the eigenvalues of a Hermitian operator are real.

Let O be an hermitian operator such that : $O = O^\dagger$ and has a spectral decomposition : $O = \sum_i \lambda_i |\lambda_i\rangle \langle\lambda_i|$,

$$\begin{aligned} O &= \sum_i \lambda_i |\lambda_i\rangle \langle\lambda_i| \\ &= O^\dagger = \sum_i \lambda_i^* (|\lambda_i\rangle \langle\lambda_i|)^\dagger \\ &= \sum_i \lambda_i^* |\lambda_i\rangle \langle\lambda_i| \end{aligned}$$

The latter implies that $\lambda_i = \lambda_i^*$ meaning that the eigenvalues of an hermitian operator are real

Exercise 13
Give the position and the momentum operators (consider only one dimension) in the position representation.

The position operator reads : $\hat{x}\psi(x) = x\psi(x)$, where $\psi(x)$ is the wavefunction. The momentum operator $\hat{p}$ is given by $\hat{p} = -i\hbar\frac{d}{dx}$, acting on $\psi(x)$ as $\hat{p}\psi(x) = -i\hbar\frac{d\psi(x)}{dx}$.

Exercise 14
Give the commutator of the position and linear momentum operators in the position representation (consider one dimension only).

$$[\hat{x}, \hat{p}]\psi(x) = \hat{x}(\hat{p}\psi(x)) - \hat{p}(\hat{x}\psi(x))$$
$$\hat{x}(-i\hbar\frac{d\psi}{dx}) - (-i\hbar\frac{d}{dx}(x\psi(x))) = -i\hbar x\frac{d\psi}{dx} + i\hbar\frac{d}{dx}(x\psi) = i\hbar\psi(x)$$

Therefore $[\hat{x}, \hat{p}] = i\hbar$.

Exercise 15
Is the electronic Hamiltonian $\hat{H}_{el}$ a linear operator and why (not)?

The electronic Hamiltonian $\hat{H}_{el}$ is given by :

$$\hat{H}_{\text{elec}}(\mathbf{R}) = -\frac{1}{2}\sum_{n}\nabla_n^2 + \sum_{I<J}\frac{Z_I Z_J}{|\mathbf{R}_I - \mathbf{R}_J|} - \sum_{I,n}\frac{Z_I}{|\mathbf{R}_I - \mathbf{r}_n|} + \sum_{n<m}\frac{1}{|\mathbf{r}_m - \mathbf{r}_n|}$$

This hamiltonian has first a kinetic part which consists in a linear differential operator (Laplacian). The second term represents the nuclear-nuclear repulsion energy. It is a constant with respect to the electronic coordinates $\mathbf{r}_n$ (since $\mathbf{R}_I$ are fixed nuclear positions) which acts linearly on a wavefunction $\psi_{elec}(\mathbf{r}, \mathbf{R})$. This is the same principle for the electron-nuclear attraction potential term and the electron-electron repulsion potential term which are functions that multiply linearly the wavefunction.

Exercise 16
Show that, if two operators $\hat{\text{A}}$, $\hat{\text{B}}$ commute and if $|\psi\rangle$ is an eigenvector of $\hat{\text{A}}$, $\hat{\text{B}}|\psi\rangle$ is an eigenvector of $\hat{\text{A}}$, too, with the same eigenvalue.

Bonus: If $|\psi\rangle$ is part of a set of degenerate eigenvectors, show that the subspace spanned by the eigenvalues of $\hat{\text{A}}$ is invariant under the action of $\hat{\text{B}}$.

Let a be the eigenvalue associated to $\hat{\text{A}}|\psi\rangle = a|\psi\rangle$

$$
\begin{aligned}
&[\hat{\mathrm{A}}, \hat{\mathrm{B}}] = \hat{\mathrm{A}}\hat{\mathrm{B}} - \hat{\mathrm{B}}\hat{\mathrm{A}} = 0 \\
&\hat{\mathrm{A}} \left|\psi\right\rangle = a \left|\psi\right\rangle \implies \hat{\mathrm{A}}\hat{\mathrm{B}} \left|\psi\right\rangle = \hat{\mathrm{B}} \underbrace{\hat{\mathrm{A}} \left|\psi\right\rangle}_{a|\psi\rangle} \\
\Leftrightarrow &\hat{\mathrm{A}}(\hat{\mathrm{B}} \left|\psi\right\rangle)) = a(\hat{\mathrm{B}} \left|\psi\right\rangle)
\end{aligned}
$$

We see that $\hat{\mathrm{B}} \left|\psi\right\rangle$ is an eigenvector of $\hat{\mathrm{A}}$ with the same eigenvalue a

Bonus :
We consider first $|\phi\rangle$ which can be any vector in S, so $|\phi\rangle = \sum_i c_i|\psi_i\rangle$, where $|\psi_i\rangle$ are the degenerate eigenvectors of $\hat{A}$ with eigenvalue a.
Secondly, we aim to apply $\hat{A}$ to $|\phi\rangle$. Since $|\psi_i\rangle$ are eigenvectors, $\hat{A}|\psi_i\rangle = a|\psi_i\rangle$, so $\hat{A}|\phi\rangle = a|\phi\rangle$, meaning $|\phi\rangle$ is in the eigenspace of $\hat{A}$ with eigenvalue a.
Now, compute $\hat{B}|\phi\rangle$. We need to check if $\hat{B}|\phi\rangle$ is also in S. Knowing that $\hat{A}$ and $\hat{B}$ commute, we have: $\hat{A}(\hat{B}|\phi\rangle) = \hat{B}(\hat{A}|\phi\rangle) = \hat{B}(a|\phi\rangle) = a(\hat{B}|\phi\rangle)$. This shows that $\hat{B}|\phi\rangle$ is also an eigenvector of $\hat{A}$ with eigenvalue a, so it must lie in the same eigenspace S (since S is the space of all vectors with eigenvalue a under $\hat{A}$).
We conclude that since $\hat{B}|\phi\rangle$ has the same eigenvalue a under $\hat{A}$, and S is the full eigenspace for a, $\hat{B}|\phi\rangle$ must be in S. This holds for any $|\phi\rangle$ in S, proving S is invariant under $\hat{B}$.

Exercise 17
Demonstrate that, if two hermitian operators $\hat{\mathrm{A}}$, $\hat{\mathrm{B}}$ commute and $|\psi_1\rangle$, $|\psi_2\rangle$ are eigenvectors of $\hat{\mathrm{A}}$ associated to different eigenvalues, then the matrix element $\langle\psi_1| \hat{\mathrm{B}} |\psi_2\rangle$ vanishes.

Let $\hat{A} \left|\psi_1\right\rangle = a_1 \left|\psi_1\right\rangle$ and $\hat{A} \left|\psi_2\right\rangle = a_2 \left|\psi_2\right\rangle$ such that $a_1 \neq a_2$ and $a_1, a_2 \in \mathbb{R}$.

$$
\begin{aligned}
\langle\psi_1| \hat{A}\hat{B} |\psi_2\rangle - \langle\psi_1| \hat{B}\hat{A} |\psi_2\rangle &= 0 \\
a_1 \langle\psi_1| \hat{B} |\psi_2\rangle - a_2 \langle\psi_1| \hat{B} |\psi_2\rangle &= 0 \\
(a_1 - a_2) \langle\psi_1| \hat{B} |\psi_2\rangle &= 0 \\
\Leftrightarrow \langle\psi_1| \hat{B} |\psi_2\rangle &= 0 \quad \text{since } a_1 \neq a_2
\end{aligned}
$$

Bonus Exercise 18
Show that the potential energy operator $\hat{\mathrm{V}}(\mathbf{r})$ is multiplicative when applied to the real-space wavefunction.

We know that the potential operator $\hat{\mathrm{V}}(\mathbf{r})$ is diagonal in the position basis $\{|\mathbf{r}\rangle\}$ such that $\hat{\mathrm{V}}(\mathbf{r}) \left|\mathbf{r}\right\rangle = V(\mathbf{r}) \left|\mathbf{r}\right\rangle$

$$
\begin{aligned}
i\hbar\frac{\partial}{\partial t}\left|\psi(t)\right\rangle &= \hat{H}\left|\psi(t)\right\rangle \\
\Leftrightarrow i\hbar\frac{\partial}{\partial t}\left\langle \mathbf{r}|\psi(t)\right\rangle &= \frac{-\hbar^2}{2m}\nabla^2\left\langle \mathbf{r}|\psi(t)\right\rangle + \left\langle \mathbf{r}\right|\hat{V}(\mathbf{r})\left|\psi(t)\right\rangle \\
i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t) &= \frac{-\hbar^2}{2m}\nabla^2\psi(\mathbf{r},t) + V(\mathbf{r})\left\langle \mathbf{r}|\psi(t)\right\rangle \\
i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t) &= \frac{-\hbar^2}{2m}\nabla^2\psi(\mathbf{r},t) + V(\mathbf{r})\psi(\mathbf{r},t)
\end{aligned}
$$

Since this holds for any state $|\psi\rangle$ and $|\mathbf{r}\rangle$ of the Hilbert space, we have that:

$$\hat{\mathrm{V}}(\mathbf{r})\psi(\mathbf{r}) = V(\mathbf{r})\psi(\mathbf{r})$$

Bonus Exercise 19
The link between position and momentum representation is given by a Fourier transform. Explain how this relates to the Heisenberg uncertainty principle.

The link between position $\psi(\mathbf{r})$ and momentum $\phi(\mathbf{p})$ is via Fourier transform:
$\phi(\mathbf{p}) = \frac{1}{\sqrt{2\pi\hbar}}\int \psi(\mathbf{r})e^{-i\mathbf{p}\cdot\mathbf{r}/\hbar}d\mathbf{r}$.
This implies a trade-off: localized $\psi(\mathbf{r})$ (small Δx) spreads $\phi(\mathbf{p})$ (large Δp) and vice-versa
Hence, Heisenberg's uncertainty principle holds: $\Delta x \Delta p \geq \hbar/2$. Meaning that we can't know with infinite precision the wavefunction in both domain.
Thus, Fourier transform/duality underlies the position-momentum uncertainty.

Exercise 20
In a system that consists of only two states (such as an electron spin in a magnetic field, where the electron spin can be in one of two orientations), the Hamiltonian has the following matrix elements: $H_{11} = a,\ H_{22} = b,\ H_{12} = d,\ H_{21} = d$. How would you determine the energy levels E and the eigenstates $\mathbf{\Psi}$ of the system? (You do not need to solve this problem explicitly, merely outlining the procedure is sufficient.)

We need to diagonalize the Hamiltonian. First we need to find the eigenvalues, which correspond to the eigenenergies of the system. We solve $\det(\hat{H} - E\cdot\mathbb{I})$. When eigenvalues ($E_+$ and E_-) have been found, we can further proceed to find their associated eigenvector ($|E_+\rangle$ and $|E_-\rangle$ called $\mathbf{\Psi}$ in the question) which are the eigenstates of the system.

Exercise 21
Define two vectors, $\phi1$ and $\phi2$, with two elements each, that are normalized, in the sense $\langle\phi_i|\phi_i\rangle = 1$, and orthogonal in the sense that $\langle\phi_i|\phi_j\rangle = 0$.

Hint: In `numpy` a vector `v` with the two elements `1` and `2` is defined through the command

```
v=np.array([1,2])
```

```
ϕ1 = np.array([np.cos(.5), np.sin(.5)]) # replace with your choice for ϕ1
ϕ2 = np.array([np.sin(.5), -np.cos(.5)]) # replace with your choice for ϕ2

print(f'ϕ1: {ϕ1}')
print(f'ϕ2: {ϕ2}')
```

✓ 0.0s

```
ϕ1: [0.87758256 0.47942554]
ϕ2: [ 0.47942554 -0.87758256]
```

Exercise 22
Show that ϕ_1 and ϕ_2 are normalized and orthonormal

Hint: Here are reported some useful `numpy` functions to work with vectors:

- `v.dot(w)` - inner product (scalar product) of two vectors `v`, `w`
- `v.conj()` - complex conjugate of a vector `v`
- `v.conj().dot(w)` - inner product of $v^\dagger w$

```
# Check Normalization

ϕ1_norm = ϕ1.dot(ϕ1) # Replace with vector operation
ϕ2_norm = ϕ2.dot(ϕ2) # Replace with vector operation

print(f'<ϕ1|ϕ1> = {ϕ1_norm}')
print(f'<ϕ2|ϕ2> = {ϕ2_norm}')
```

✓ 0.0s

```
<ϕ1|ϕ1> = 1.0
<ϕ2|ϕ2> = 1.0
```

```python
# Check Orthogonality

ϕ1ϕ2 = ϕ1.conj().dot(ϕ2) # Replace with vector operation

print(f'<ϕ1|ϕ2> = {ϕ1ϕ2}')
```

✓ 0.0s

```
<ϕ1|ϕ2> = 0.0
```