Homework 1 Part 1

Autor

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Here is the google drive link where all the handwritten exercises have been done :
https://drive.google.com/drive/folders/13eTO-HEgfAStOO96SLifJubtNbKFRqfA?usp=drive_link

Quantum Mechanics for Non-Physicists

Table des Matières

1 Homework Part 1

1.1 Exercise 1

By the Spectral Theorem :

$$\widehat{A} = \sum_{i=1}^{n} \lambda_i \left|i\right\rangle\!\left\langle i\right| \tag{1.1}$$

Where λ_i is the i-th eigenvalue associated to the i-th eigenvector $|i\rangle$. Therefore we can rewrite the $\widehat{Z}$ matrix which is the $\widehat{\sigma_z}$ Pauli matrix :

$$\widehat{Z} = (+1)\left|0\right\rangle\!\left\langle 0\right| + (-1)\left|1\right\rangle\!\left\langle 1\right| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{1.2}$$

One can also verify that $|0\rangle$ and $|1\rangle$ are the eigenvectors of this matrix :

$$\widehat{Z}|0\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle \tag{1.3}$$

$$\widehat{Z}|1\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle \tag{1.4}$$

We conclude that :

$$\widehat{Z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{1.5}$$

1.2 Exercise 2

One can use the definition of the projector to find the $\widehat{\sigma_x}$ Pauli matrix such that :

$$\widehat{X} = |0\rangle\langle 1| + |1\rangle\langle 0| = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{1.6}$$

1.3 Exercise 3

$|0\rangle$ and $|1\rangle$ (same for $|+\rangle$ and $|-\rangle$) point sideways because the Bloch Sphere is a geometrical abstraction of an Hilbertian space which is complex in contrast to the 3D-Euclidian space in which we live.

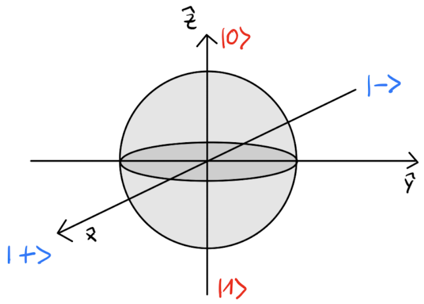


FIGURE 1.1 – Bloch Sphere with states $|0\rangle$, $|1\rangle$, $|+\rangle$, $|-\rangle$

1.4 Exercise 4

$$\langle -|+\rangle = \frac{1}{2}\left(\langle 0| - \langle 1|\right)\left(|0\rangle + |1\rangle\right)$$
$$\langle -|+\rangle = \frac{1}{2}\left(\langle 0|0\rangle + \langle 0|1\rangle - \langle 1|0\rangle - \langle 1|1\rangle\right)$$
$$\langle -|+\rangle = \frac{1}{2}(1-1) = 0$$

$|+\rangle$ et $|-\rangle$ are orthogonal.

$$\langle +|+\rangle = \frac{1}{2}\left(\langle 0| + \langle 1|\right)\left(|0\rangle + |1\rangle\right) = \frac{1}{2}\left(\langle 0|0\rangle + \langle 1|1\rangle\right) = 1$$
$$\langle -|-\rangle = \frac{1}{2}\left(\langle 0| - \langle 1|\right)\left(|0\rangle - |1\rangle\right) = \frac{1}{2}\left(\langle 0|0\rangle + \langle 1|1\rangle\right) = 1$$

$$\sqrt{\langle +|+\rangle}, \sqrt{\langle -|-\rangle} = 1$$

Since $|+\rangle$ and $|-\rangle$, both $\in \mathbb{C}^2$, are orthonormal, they form an orthonormal basis of an Hilbert space of $\mathbb{C}^2$, i.e., $\mathcal{H} \in \mathbb{C}^2$.

- $|0\rangle = \alpha\,|+\rangle + \beta\,|-\rangle$
  $\alpha = \langle +|0\rangle = \frac{1}{\sqrt{2}}\left(\langle 0| + \langle 1|\right)|0\rangle = \frac{1}{\sqrt{2}}$
  $\beta = \langle -|0\rangle = \frac{1}{\sqrt{2}}\left(\langle 0| - \langle 1|\right)|0\rangle = \frac{1}{\sqrt{2}}$
- $|1\rangle = \gamma\,|+\rangle + \epsilon\,|-\rangle$
  $\gamma = \langle +|1\rangle = \frac{1}{\sqrt{2}}\left(\langle 0| + \langle 1|\right)|1\rangle = \frac{1}{\sqrt{2}}$
  $\epsilon = \langle -|1\rangle = \frac{1}{\sqrt{2}}\left(\langle 0| - \langle 1|\right)|1\rangle = -\frac{1}{\sqrt{2}}$

$$|0\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right) \quad \text{et} \quad |1\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle - |-\rangle\right)$$

1.5 Exercice 5

Spectral theorem :

$$\widehat{A} = \sum_{i=1}^{n} \lambda_i \left|i\right\rangle \left\langle i\right|$$

$$\hat{X} = +1\left|+\right\rangle\left\langle+\right| - 1\left|-\right\rangle\left\langle-\right| = \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

$$\hat{X} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \quad \text{“}\widehat{\sigma_x}\text{ Pauli matrix"}$$

1.6 Exercise 6

— $\hat{X}\left|0\right\rangle = \left|1\right\rangle \quad \hat{X} \equiv \text{NOT gate for} \left|0\right\rangle \text{ and } \left|1\right\rangle$
— $\hat{X}\left|1\right\rangle = \left|0\right\rangle$
— $\hat{Z}\left|+\right\rangle = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\end{bmatrix} = \left|-\right\rangle \quad \left\{\hat{Z} \equiv \text{NOT gate for states} \left|+\right\rangle \text{ and } \left|-\right\rangle\right\}$
— $\hat{Z}\left|-\right\rangle = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix} = \left|+\right\rangle$

1.7 Exercise 7

$$\hat{H} = \frac{\hat{Z}}{\sqrt{2}} + \frac{\hat{X}}{\sqrt{2}}$$

— $\hat{H}\left|0\right\rangle = \frac{1}{\sqrt{2}}\left(\left|0\right\rangle + \left|1\right\rangle\right) = \left|+\right\rangle$
— $\hat{H}\left|1\right\rangle = \frac{1}{\sqrt{2}}\left(\left|0\right\rangle - \left|1\right\rangle\right) = \left|-\right\rangle$
— $\hat{H}\left|+\right\rangle = \frac{1}{\sqrt{2}}\hat{H}\left(\left|0\right\rangle + \left|1\right\rangle\right) = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\left(\left|0\right\rangle + \left|1\right\rangle\right) + \frac{1}{\sqrt{2}}\left(\left|0\right\rangle - \left|1\right\rangle\right)\right) = \frac{1}{\sqrt{2}}\left(2\left|0\right\rangle\right) = \left|0\right\rangle$
— $\hat{H}\left|-\right\rangle = \frac{1}{\sqrt{2}}\hat{H}\left(\left|0\right\rangle - \left|1\right\rangle\right) = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\left(\left|0\right\rangle + \left|1\right\rangle\right) - \frac{1}{\sqrt{2}}\left(\left|0\right\rangle - \left|1\right\rangle\right)\right) = \frac{1}{\sqrt{2}}\left(2\left|1\right\rangle\right) = \left|1\right\rangle$

$$\hat{H}^2 = \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix} = \mathbb{I}$$

$\hat{H}$ is unitary.

$$\hat{H}\left|0\right\rangle = \left|+\right\rangle, \quad \hat{H}\left|1\right\rangle = \left|-\right\rangle, \quad \hat{H}\left|+\right\rangle = \left|0\right\rangle, \quad \hat{H}\left|-\right\rangle = \left|1\right\rangle$$

1.8 Exercise 8

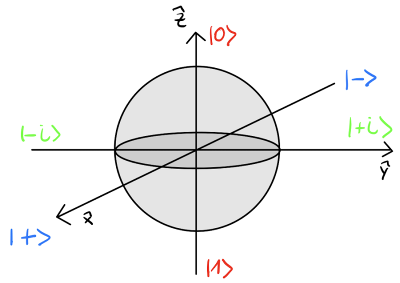


FIGURE 1.2 – Bloch Sphere with states $|0\rangle$, $|1\rangle$, $|+\rangle$, $|-\rangle$, $|+i\rangle$, $|-i\rangle$

1.9 Exercise 9

$$|+i\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\+i\end{bmatrix} \quad \langle +i| = \frac{1}{\sqrt{2}}[1 \;\; -i]$$

$$|-i\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix} \quad \langle -i| = \frac{1}{\sqrt{2}}[1 \;\; +i]$$

1.10 Exercise 10

Using spectral theorem :

$$\hat{Y} = +1\,|+i\rangle\,\langle +i| - 1\,|-i\rangle\,\langle -i| = \frac{1}{2}\begin{bmatrix}1\\i\end{bmatrix}[1 \;\; -i] - \frac{1}{2}\begin{bmatrix}1\\-i\end{bmatrix}[1 \;\; i] = \frac{1}{2}\begin{bmatrix}1 & -i\\i & 1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1 & i\\-i & 1\end{bmatrix} = \begin{bmatrix}0 & -i\\i & 0\end{bmatrix}$$

$$\hat{Y} = \begin{bmatrix}0 & -i\\i & 0\end{bmatrix}, \quad \text{“}\widehat{\sigma_y}\text{ Pauli matrix”}$$

1.11 Exercise 11

$$\langle -i| + i\rangle = \frac{1}{2}\left(\langle 0| + i\langle 1|\right)\left(|0\rangle + i|1\rangle\right)$$

$$\langle -i| + i\rangle = \frac{1}{2}\left(\langle 0|0\rangle + i\langle 0|1\rangle + i\langle 1|0\rangle - \langle 1|1\rangle\right) = \frac{1}{2}(1-1) = 0$$

$$\langle -i| + i\rangle = 0 \quad |-i\rangle \text{ et } |+i\rangle \text{ sont orthogonaux.}$$

$$\langle +i| + i\rangle = \frac{1}{2}\left(\langle 0| + i\langle 1|\right)\left(|0\rangle + i|1\rangle\right) = \frac{1}{2}(1+1) = 1$$

$$\langle -i| - i\rangle = \frac{1}{2}\left(\langle 0| + i\langle 1|\right)\left(|0\rangle - i|1\rangle\right) = \frac{1}{2}(1+1) = 1$$

$$\sqrt{\langle +i| + i\rangle}, \sqrt{\langle -i| - i\rangle} = 1$$

Since $|+i\rangle$ and $|-i\rangle$, both $\in \mathbb{C}^2$, are orthonormal, they form an orthonormal basis of an Hilbert space of $\mathbb{C}^2$, i.e

1.12 Exercise 12

From $|+i\rangle$ and $|-i\rangle$ to $|0\rangle$ and $|1\rangle$:

$$\hat{M} = |0\rangle\langle+i| + |1\rangle\langle-i| = \begin{bmatrix}1\\0\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -i\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & i\end{bmatrix}$$
$$= \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -i\\0 & 0\end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix}0 & 0\\1 & i\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -i\\1 & i\end{bmatrix}$$

From $|0\rangle$ and $|1\rangle$ to $|+i\rangle$ to $|-i\rangle$:

We need to apply the dagger operator to $\hat{M}$, which gives us :

$$\hat{A} = \hat{M}^\dagger = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -i\\1 & i\end{bmatrix}^\dagger = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\i & -i\end{bmatrix}$$

$$\hat{A}\hat{A}^\dagger = \hat{A}^\dagger\hat{A} = \mathbb{I}$$
$$\hat{A} \text{ is unitary, as well as } \hat{M}, \quad \hat{A}^\dagger = \hat{A}^{-1}.$$

1.13 Exercise 13

From $|+\rangle$ and $|-\rangle$ to $|+i\rangle$ and $|-i\rangle$:

$$\hat{B} = |+i\rangle\langle+| + |-i\rangle\langle-| = \frac{1}{2}\begin{bmatrix}1\\i\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\-i\end{bmatrix}\begin{bmatrix}1 & -1\end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix}1 & 1\\i & i\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1 & -1\\-i & i\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & i\end{bmatrix}$$

From $|+i\rangle$ and $|-i\rangle$ to $|+\rangle$ and $|-\rangle$:

We need to apply the dagger operator to $\hat{B}$, which gives us :

$$\hat{N} = \hat{B}^\dagger = \begin{bmatrix}1 & 0\\0 & -i\end{bmatrix}$$

$$\hat{B}\hat{B}^\dagger = \hat{B}\hat{N} = \mathbb{I}$$
$$\hat{B} \text{ is unitary, as well as } \hat{N}, \quad \hat{B}^\dagger = \hat{B}^{-1}.$$

1.14 Exercise 14

We know that $\hat{X}^\dagger = \hat{X}$, $\hat{Y}^\dagger = \hat{Y}$, $\hat{Z}^\dagger = \hat{Z}$, and that the "dagger" operator is linear :

— $\left(\hat{X} \pm \hat{Z}\right)^\dagger = \hat{X}^\dagger \pm \hat{Z}^\dagger = \hat{X} \pm \hat{Z}$

Since $\hat{X} \pm \hat{Z}$ is Hermitian, it is observable.

— $\left(\hat{X} \pm \hat{Y}\right)^\dagger = \hat{X}^\dagger \pm \hat{Y}^\dagger = \hat{X} \pm \hat{Y}$

Since $\hat{X} \pm \hat{Y}$ is Hermitian, it is observable.

— $\left(\hat{X} \pm i\hat{Y}\right)^\dagger = \hat{X}^\dagger \mp i\hat{Y}^\dagger = \hat{X} \mp i\hat{Y} \neq \hat{X} \pm i\hat{Y}$

Since $\hat{X} \pm i\hat{Y}$ is not Hermitian, it is not observable.

1.15 Exercise 15

About state $|+\rangle$:

$$|+\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle + |1\rangle\right)$$

When measuring along the $\hat{Z}$-axis, we express $|+\rangle$ in the $\{|0\rangle, |1\rangle\}$ basis and we get the probability of $|+\rangle$ collapsing on $|0\rangle$ and the one on $|1\rangle$ which are :

$$\text{Prob}(1) = |\langle 0|+\rangle|^2 = \frac{1}{2}, \quad \text{and} \quad \text{Prob}(-1) = |\langle 1|+\rangle|^2 = \frac{1}{2}.$$

If the probability is related to the frequency at which we obtained each value, since +1 and -1 are equiprobable, the mean $\langle\hat{Z}\rangle$ should be 0. Indeed, we got as much +1 as -1 while prepared with the state $|+\rangle$.

$$\langle\hat{Z}\rangle_+ = \langle +|\hat{Z}|+\rangle = \sum_i \lambda_i \, |\langle i|+\rangle|^2 = 1 \cdot |\langle 0|+\rangle|^2 - 1 \cdot |\langle 1|+\rangle|^2 = 1 \cdot \frac{1}{2} - 1 \cdot \frac{1}{2} = 0$$

About state $|-\rangle$:

$$|-\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle - |1\rangle\right)$$

When measuring along the $\hat{Z}$-axis, we express $|-\rangle$ in the $\{|0\rangle, |1\rangle\}$ basis and we get the probability of $|-\rangle$ collapsing on $|0\rangle$ and the one on $|1\rangle$ which are :

$$\text{Prob}(1) = |\langle 0|-\rangle|^2 = \frac{1}{2}, \quad \text{and} \quad \text{Prob}(-1) = |\langle 1|-\rangle|^2 = \frac{1}{2}.$$

If the probability is related to the frequency at which we obtained each value, since +1 and -1 are equiprobable, the mean $\langle\hat{Z}\rangle$ should be 0. Indeed, we got as much +1 as -1 while prepared with the state $|-\rangle$.

$$\langle\hat{Z}\rangle_- = \langle -|\hat{Z}|-\rangle = \sum_i \lambda_i \, |\langle i|-\rangle|^2 = 1 \cdot |\langle 0|-\rangle|^2 - 1 \cdot |\langle 1|-\rangle|^2 = 1 \cdot \frac{1}{2} - 1 \cdot \frac{1}{2} = 0$$

1.16 Exercise 16

We clearly see that $\langle \hat{Z} \rangle = \langle \psi | \hat{Z} | \psi \rangle = \cos(\theta)$. $\phi = 0$ in this graph but does not have any influence.

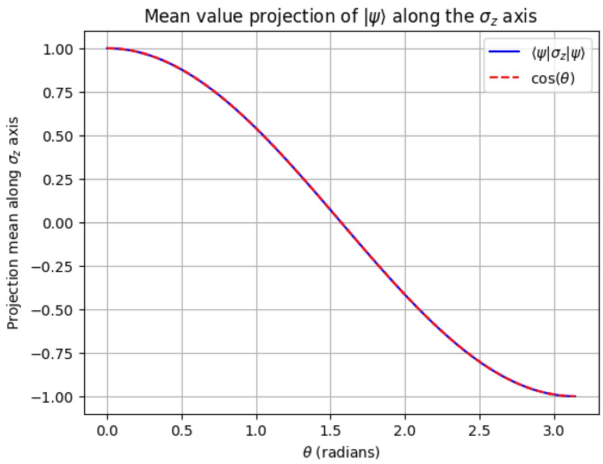


FIGURE 1.3 – Mean value projection of $|\psi\rangle$ along the σ_z axis.

We can see that the projection of the arrow along $\hat{Z}$-axis gives $\cos(\theta)$. It turns out that it would be the same value as a classical arrow taking a continuum of values between -1 and 1. We therefore expect that the mean of repeated measurements of the same quantum state (arrow) will tend to the classical mean value while each outcome of a measurement is discrete (either +1 or -1) and not a continuum.

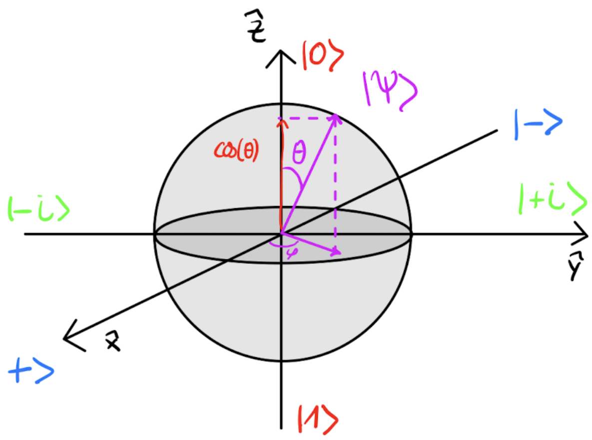


FIGURE 1.4 – Bloch Sphere representation.

1.17 Exercise 17

$$\lambda_i \equiv \text{eigenvalue associated to eigenvector } |i\rangle \text{ of } \hat{Z}$$

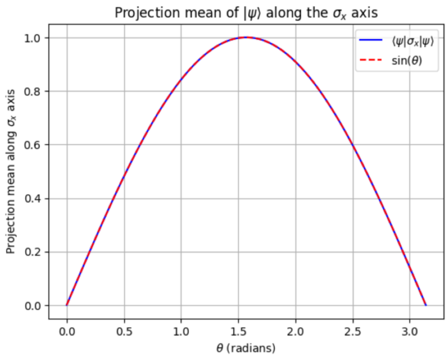
$$\langle \hat{Z} \rangle = \sum_{i=1}^{2} \lambda_i \left| \langle i|\psi\rangle \right|^2 = \sum_{i=1}^{2} \lambda_i \langle \psi | i \rangle \langle i | \psi \rangle = \langle \psi \left| \left(\sum_{i=1}^{2} \lambda_i \, |i\rangle\langle i| \right) \right| \psi \rangle = \langle \psi | \hat{Z} | \psi \rangle$$

$$\langle \psi | \hat{Z} | \psi \rangle = \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\varphi} \sin\left(\frac{\theta}{2}\right) \end{bmatrix}^{\dagger} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\varphi} \sin\left(\frac{\theta}{2}\right) \end{bmatrix}$$

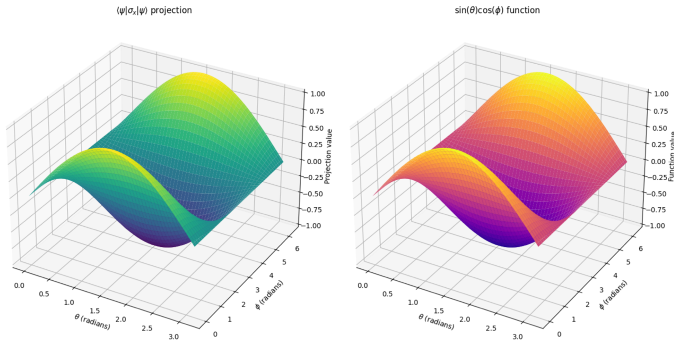
$$= \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) & e^{-i\varphi} \sin\left(\frac{\theta}{2}\right) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\varphi} \sin\left(\frac{\theta}{2}\right) \end{bmatrix}$$

$$= \cos^2\left(\frac{\theta}{2}\right) - e^{-i\varphi} e^{i\varphi} \sin^2\left(\frac{\theta}{2}\right) = \cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right) = \cos(\theta) = \langle \hat{Z} \rangle$$

1.18 Exercise 18



(a) If $\varphi = 0$



(b) For different values of θ and φ

$$\langle\psi|\hat{X}|\psi\rangle = \begin{bmatrix}\cos\left(\frac{\theta}{2}\right)\\ e^{i\varphi}\sin\left(\frac{\theta}{2}\right)\end{bmatrix}^{\dagger}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\cos\left(\frac{\theta}{2}\right)\\ e^{i\varphi}\sin\left(\frac{\theta}{2}\right)\end{bmatrix} = \begin{bmatrix}\cos\left(\frac{\theta}{2}\right) & e^{-i\varphi}\sin\left(\frac{\theta}{2}\right)\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\cos\left(\frac{\theta}{2}\right)\\ e^{i\varphi}\sin\left(\frac{\theta}{2}\right)\end{bmatrix}$$
$$= \begin{bmatrix}\cos\left(\frac{\theta}{2}\right) & e^{-i\varphi}\sin\left(\frac{\theta}{2}\right)\end{bmatrix}\begin{bmatrix}e^{i\varphi}\sin\left(\frac{\theta}{2}\right)\\ \cos\left(\frac{\theta}{2}\right)\end{bmatrix} = 2\sin(\theta)\cos(\varphi)$$

$$\langle\psi|\hat{Z}|\psi\rangle = \cos(\theta)$$

While looking on the Bloch sphere, one can see that the projection of the state $|\psi\rangle$ along $\hat{X}$-axis gives $\sin(\theta)\cos(\varphi)$, which corresponds to the mean value previously calculated. Comparing to $\langle\psi|\hat{Z}|\psi\rangle$, we make the same deduction that the mean of repeated measurements of the quantum state $|\psi\rangle$ along $\hat{X}$ will tend to give the same value as a classical arrow taking any value between -1 and 1, while the "quantum" outcomes are discrete values such as $+1$ and -1.

1.19 Exercise 19

$$\mathrm{Var}(\hat{Z}) = \langle 0|\hat{Z}^2|0\rangle - (\langle 0|\hat{Z}|0\rangle)^2 = \langle 0|\hat{Z}|0\rangle - (\langle 0|\hat{Z}|0\rangle)^2 = 1 - 1^2 = 1 - 1 = 0$$
$$\hat{Z}^2 = \mathbb{I}$$

$$\mathrm{Var}(\hat{X}) = \langle 0|\hat{X}^2|0\rangle - (\langle 0|\hat{X}|0\rangle)^2 = \langle 0|\hat{X}|0\rangle - 0 = 1$$

Since the variance can represent the disparity of measurement, one can say :

- For $\hat{Z}$, since $|0\rangle$ is an eigenstate, the outcome will always be $+1$, and the variance will be 0 as demonstrated above.
- For $\hat{X}$, $|0\rangle$ is not an eigenstate ; the outcome is either $+1$ or -1 with a mean 0. Therefore, the disparity from the mean will be 1 (standard deviation $\sigma = \sqrt{\mathrm{Var}(\hat{X})}$), since we go in a positive or negative way to 1, which corresponds to the value calculated.

1.20 Exercise 20

One measurement :

$$\langle +i|+\rangle = \frac{1}{2}\begin{bmatrix}1 & -i\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1-i}{2}, \quad |\langle +i|+\rangle|^2 = \frac{1-i}{2}\cdot\frac{1+i}{2} = \frac{1}{2}.$$

$$\langle -i|+\rangle = \frac{1}{2}\begin{bmatrix}1 & i\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1+i}{2}, \quad |\langle -i|+\rangle|^2 = \frac{1+i}{2}\cdot\frac{1-i}{2} = \frac{1}{2}.$$

One can see, if measuring state $|+\rangle$ with $\hat{Y}$ operator, we get after the first measurement $+1$ or -1 with equal probability of $\frac{1}{2}$. We therefore cannot predict the outcome but we only can know the probability of collapsing on $|+i\rangle$ or $|-i\rangle$ since they are the eigenstates of operator $\hat{Y}$.

Mean of many measurements :

$$\langle +|\hat{Y}|+\rangle = \frac{1}{2}\begin{bmatrix}1 & i\end{bmatrix}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{bmatrix}1\\i\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & i\end{bmatrix}\begin{bmatrix}-i\\i\end{bmatrix} = 0.$$

The mean is 0, which corresponds to our expectation. Indeed, outcomes $+1$ and -1 are equiprobable, therefore after many measurements we will have as many $+1$ than -1 giving a mean of 0.

1.21 Exercise 21

This question translates a problem of non-commutativity of observables. The sequence alternates since the probability to collapse on a state is always $\frac{1}{2}$.

Indeed :

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right), \quad |-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right), \quad |0\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right), \quad |1\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle - |-\rangle\right)$$

It is non-deterministic (random sequence).

We conclude that alternate measurements between $\hat{Z}$ and $\hat{X}$ cannot form a coherent sequence which would allow us to determine in the mean time $\hat{Z}$-projection and $\hat{X}$-projection.

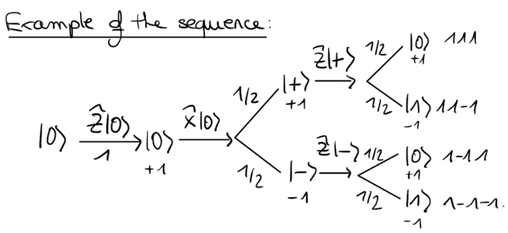


FIGURE 1.6 – Example of the sequence

1.22 Exercise 22

Mean value along $\hat{X}$-axis :

$$\langle\psi|\hat{X}|\psi\rangle = \sin(\theta)\cos(\varphi)$$

corresponds to our previous assumptions.

Mean value along $\hat{Z}$-axis :

$$\langle\psi|\hat{Z}|\psi\rangle = \cos(\theta)$$

also matches our assumptions.

Since we prepared for each measurement $|\psi\rangle$, then alternate $\hat{X}$ or $\hat{Z}$ operator measurement gives us independent results (each measurement is independent from the previous one). In comparison to the previous exercise, measurements were not independent since the qubit is not prepared into its original state before measurement. The experience of this exercise is therefore more consistent to determine $\hat{X}$ and $\hat{Z}$ projection, meaning that getting $\hat{X}$ and $\hat{Z}$ information in the mean time is impossible.

1.23 Exercise 23

$$\left|\psi'\right\rangle = \hat{U}\left|\psi\right\rangle = \hat{U}\cos\left(\frac{\theta}{2}\right)\left|0\right\rangle + \hat{U}e^{i\varphi}\sin\left(\frac{\theta}{2}\right)\left|1\right\rangle.$$

$$\left|\psi'\right\rangle = \hat{U}\left|\psi\right\rangle = \cos\left(\frac{\theta'}{2}\right)\left|0\right\rangle + e^{i\varphi'}\sin\left(\frac{\theta'}{2}\right)\left|1\right\rangle.$$

$$\langle\psi'|\psi'\rangle = \langle\psi'|\hat{U}^\dagger\hat{U}|\psi'\rangle = \langle\psi|\psi\rangle = 1.$$

Les coefficients :

$$\alpha_0 = \cos\left(\frac{\theta'}{2}\right), \quad \alpha_1 = e^{i\varphi'}\sin\left(\frac{\theta'}{2}\right)$$

vérifient :

$$|\alpha_0|^2 + |\alpha_1|^2 = \cos^2\left(\frac{\theta'}{2}\right) + \left|e^{i\varphi'}\right|^2\sin^2\left(\frac{\theta'}{2}\right) = \cos^2\left(\frac{\theta'}{2}\right) + \sin^2\left(\frac{\theta'}{2}\right) = 1.$$

Therefore, $\hat{U}\left|\psi\right\rangle$ creates a state $\left|\psi'\right\rangle = \cos\left(\frac{\theta'}{2}\right)\left|0\right\rangle + \sin\left(\frac{\theta'}{2}\right)e^{i\varphi'}\left|1\right\rangle$ of norm 1, which can be represented on the Bloch sphere.

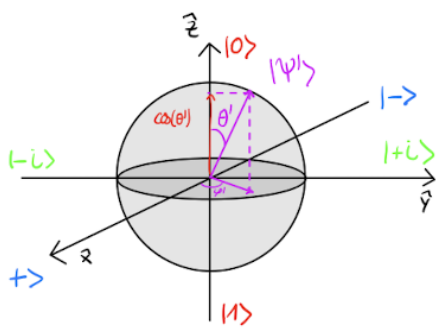


FIGURE 1.7 – Bloch Sphere representation of the new state $|\psi'\rangle$

1.24 Exercise 24

$$\hat{X}^\dagger = \begin{bmatrix} 0 & 1^* \\ 1^* & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \hat{X} \quad \Rightarrow \text{Hermitian}$$

$$\hat{Y}^\dagger = \begin{bmatrix} 0 & (-i)^* \\ i^* & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} = \hat{Y} \quad \Rightarrow \text{Hermitian}$$

$$\hat{Z}^\dagger = \begin{bmatrix} 1^* & 0 \\ 0 & (-1)^* \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \hat{Z} \quad \Rightarrow \text{Hermitian}$$

$$\hat{X}^\dagger \hat{X} = \hat{X}^2 = \mathbb{I}, \quad \hat{Y}^\dagger \hat{Y} = \hat{Y}^2 = \mathbb{I}, \quad \hat{Z}^\dagger \hat{Z} = \hat{Z}^2 = \mathbb{I} \quad \Rightarrow \text{Unitary}$$

$$\begin{aligned} \hat{X}\left|+\right\rangle &= \left|+\right\rangle, \quad \hat{X}\left|-\right\rangle = -\left|-\right\rangle \\ \hat{Y}\left|+i\right\rangle &= \left|+i\right\rangle, \quad \hat{Y}\left|-i\right\rangle = -\left|-i\right\rangle \\ \hat{Z}\left|0\right\rangle &= \left|0\right\rangle, \quad \hat{Z}\left|1\right\rangle = -\left|1\right\rangle \end{aligned}$$

All Pauli matrices have real eigenvalues and preserve the vector length.

$$\hat{X} \equiv \text{X-axis on the Bloch Sphere}$$

$$\hat{Y} \equiv \text{Y-axis on the Bloch Sphere}$$

$$\hat{Z} \equiv \text{Z-axis on the Bloch Sphere}$$

1.25 Exercise 25

$$\hat{X} = (+1)\left|+\right\rangle\left\langle+\right| + (-1)\left|-\right\rangle\left\langle-\right| \quad (\text{projector on } \left|+\right\rangle \text{ and } \left|-\right\rangle)$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

1.26 Exercise 26

$$\hat{Y} = (+1)\left|+i\right\rangle\left\langle+i\right| + (-1)\left|-i\right\rangle\left\langle-i\right| \quad (\text{projector on } \left|+i\right\rangle \text{ and } \left|-i\right\rangle)$$

$$= \frac{1}{2}\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

1.27 Exercise 27

Initialization :

$$\hat{H}^2 = \hat{H} \cdot \hat{H} = \left(\sum_i h_i \left| h_i \right\rangle \left\langle h_i \right| \right) \cdot \left(\sum_j h_j \left| h_j \right\rangle \left\langle h_j \right| \right)$$

$$= \sum_i \sum_j h_i h_j \left| h_i \right\rangle \left\langle h_i | h_j \right\rangle \left\langle h_j \right| = \sum_i h_i^2 \left| h_i \right\rangle \left\langle h_i \right|$$

The property is true for $n = 2$ since $n = 0, 1$ are trivial.

Recurrence :
Suppose that the property $\hat{H}^m = \sum_i h_i^m \left| h_i \right\rangle \left\langle h_i \right|$ is true for rank $n = m \in \mathbb{N}$, we will show that it is also true for $n = m + 1$:

$$\hat{H}^{m+1} = \hat{H}^m \cdot \hat{H} = \left(\sum_i h_i^m \left| h_i \right\rangle \left\langle h_i \right| \right) \cdot \left(\sum_j h_j \left| h_j \right\rangle \left\langle h_j \right| \right)$$

$$= \sum_i \sum_j h_i^m h_j \left| h_i \right\rangle \left\langle h_i | h_j \right\rangle \left\langle h_j \right| = \sum_i h_i^{m+1} \left| h_i \right\rangle \left\langle h_i \right| \quad \forall m \in \mathbb{N}$$

$$\hat{H}^{m+1} = \sum_i h_i^{m+1} \left| h_i \right\rangle \left\langle h_i \right| \quad \forall m \in \mathbb{N}$$

The property is true for $m + 1$, $m \in \mathbb{N}$.

Conclusion :
Since the property is true for any rank $m \in \mathbb{N}$, by recurrence, we conclude that $\hat{H}^m = \sum_i h_i^m \left| h_i \right\rangle \left\langle h_i \right|$.

1.28 Exercise 28

$$|\lambda_i|^2 = \lambda_i^* \lambda_i \implies \langle v_i | \lambda_i | v_i \rangle = \langle v_i | \lambda_i^* \lambda_i | v_i \rangle = \langle v_i | U^\dagger U | v_i \rangle = \langle v_i | v_i \rangle = 1.$$

$$\Rightarrow |\lambda_i|^2 = 1 \implies |\lambda_i| = 1.$$

$$\lambda_i = e^{i\gamma_i} \quad \text{by definition we can write :} \quad \hat{U} = \sum_i \lambda_i |v_i\rangle\langle v_i| = \sum_i e^{i\gamma_i} |v_i\rangle\langle v_i| = f(\hat{O}).$$

$\hat{O} =$ Hermitian operator of eigenvalues γ_i. By identification : $\gamma_i = \alpha h_i$ where h_i are the eigenvalues of $\hat{H}$.

Therefore $\hat{U}$ can be written as $\hat{U} = e^{i\alpha\hat{H}} = f(\hat{H})$.

1.29 Exercise 29

$$\hat{X}, \quad \hat{X}^2 = \mathbb{I}, \quad \hat{X}^3 = \hat{X}, \quad \hat{X}^{2m} = \mathbb{I}, \quad \hat{X}^{2m+1} = \hat{X}.$$

$$e^{-i\frac{\alpha}{2}\hat{X}} = \sum_{n=0}^{\infty} \frac{(-i\frac{\alpha}{2}\hat{X})^n}{n!} = \mathbb{I} - i\frac{\alpha}{2}\hat{X} - \frac{(\alpha/2)^2\mathbb{I}}{2!} + \frac{i(\alpha/2)^3\hat{X}}{3!} + \frac{(\alpha/2)^4\mathbb{I}}{4!} - \dots$$

$$= \left(1 - \frac{(\alpha/2)^2}{2!} + \frac{(\alpha/2)^4}{4!} - \dots\right)\mathbb{I} - i\left(\frac{\alpha}{2} - \frac{(\alpha/2)^3}{3!} + \dots\right)\hat{X}$$

$$e^{-i\frac{\alpha}{2}\hat{X}} = \cos\left(\frac{\alpha}{2}\right)\mathbb{I} - i\sin\left(\frac{\alpha}{2}\right)\hat{X}$$

$$e^{-i\frac{\alpha}{2}\hat{X}} = \begin{bmatrix} \cos(\alpha/2) & -i\sin(\alpha/2) \\ -i\sin(\alpha/2) & \cos(\alpha/2) \end{bmatrix}$$

$$e^{-i\frac{\alpha}{2}\hat{X}}|\psi\rangle = \cos\left(\frac{\alpha}{2}\right)|\psi\rangle - i\sin\left(\frac{\alpha}{2}\right)\hat{X}|\psi\rangle$$

$$e^{i(a_0\mathbb{I}+|a|(\hat{m}\cdot\vec{\sigma}))} = e^{ia_0\mathbb{I}}\left[\cos(|a|)\mathbb{I} + i\sin(|a|)(\hat{m}\cdot\vec{\sigma})\right]$$

This corresponds to a rotation of $2|a|$ around a unitary vector $\hat{m}$. $\vec{\sigma}$ is a vector containing Pauli-matrices (ie. Bloch Sphere axis).

$$e^{-i\frac{\alpha}{2}\hat{X}}|0\rangle = \begin{bmatrix} \cos(\alpha/2) \\ e^{i3\pi/2}\sin(\alpha/2) \end{bmatrix}, \quad e^{i\frac{\alpha}{2}\hat{X}}|1\rangle = \begin{bmatrix} \cos(\alpha/2) \\ e^{i\pi/2}\sin(\alpha/2) \end{bmatrix}$$

$\theta = \alpha$ et $\varphi = \frac{3\pi}{2}, \frac{\pi}{2}$. This represents a rotation of angle α in the YZ plane around X axis.

Finally, with $e^{-i\frac{\alpha}{2}\hat{X}}$, we have $|a| = \frac{\alpha}{2}$ et $\hat{m} = (1, 0, 0)$, this corresponds to rotation of angle α around $\hat{X}$ axis.

1.30 Exercise 30

$$e^{-i\frac{\alpha}{2}\hat{Y}} = \sum_i f(\hat{Y})|\lambda_i\rangle\langle\lambda_i| = e^{-i\frac{\alpha}{2}}|+i\rangle\langle+i| + e^{i\frac{\alpha}{2}}|-i\rangle\langle-i|$$

$$= \frac{1}{2}\begin{bmatrix} e^{-i\frac{\alpha}{2}} & -ie^{-i\frac{\alpha}{2}} \\ ie^{-i\frac{\alpha}{2}} & e^{i\frac{\alpha}{2}} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} e^{i\frac{\alpha}{2}} & ie^{i\frac{\alpha}{2}} \\ -ie^{i\frac{\alpha}{2}} & e^{-i\frac{\alpha}{2}} \end{bmatrix} = \begin{bmatrix} \cos\left(\frac{\alpha}{2}\right) & -\sin\left(\frac{\alpha}{2}\right) \\ \sin\left(\frac{\alpha}{2}\right) & \cos\left(\frac{\alpha}{2}\right) \end{bmatrix}$$

$$e^{-i\frac{\alpha}{2}\hat{Y}}|0\rangle = \begin{bmatrix} \cos\left(\frac{\alpha}{2}\right) \\ \sin\left(\frac{\alpha}{2}\right) \end{bmatrix} \quad \text{and} \quad e^{-i\frac{\alpha}{2}\hat{Y}}|1\rangle = \begin{bmatrix} -\sin\left(\frac{\alpha}{2}\right) \\ \cos\left(\frac{\alpha}{2}\right) \end{bmatrix}$$

We stay in the XZ-plane, so on the Bloch sphere, this is a rotation of angle α around the Y-axis.

Also, with the form of the matrix and as explained in the previous question, $e^{-i\frac{\alpha}{2}\hat{Y}}$ leads to a rotation of angle α around the $\hat{Y}$-axis.

1.31 Exercise 31

$$e^{-i\frac{\theta}{2}\hat{Y}}|0\rangle = \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) \end{bmatrix}$$

$$e^{-i\frac{\phi}{2}\hat{Z}}\left(e^{-i\frac{\theta}{2}\hat{Y}}|0\rangle\right) = e^{-i\frac{\phi}{2}}\begin{bmatrix}1 & 0\\0 & e^{i\phi}\end{bmatrix}\begin{bmatrix}\cos\left(\frac{\theta}{2}\right)\\\sin\left(\frac{\theta}{2}\right)\end{bmatrix} = e^{-i\frac{\phi}{2}}\begin{bmatrix}\cos\left(\frac{\theta}{2}\right)\\e^{i\phi}\sin\left(\frac{\theta}{2}\right)\end{bmatrix} = e^{-i\frac{\phi}{2}}|\psi\rangle$$

$$|\psi\rangle = \begin{bmatrix}\cos\left(\frac{\theta}{2}\right)\\\sin\left(\frac{\theta}{2}\right)e^{i\phi}\end{bmatrix}$$

Since two systems are equal to a global phase,

$$e^{-i\frac{\phi}{2}\hat{Z}}\left(e^{-i\frac{\theta}{2}\hat{Y}}|0\rangle\right)$$

is the same state as

$$|\psi\rangle = \begin{bmatrix}\cos\left(\frac{\theta}{2}\right)\\\sin\left(\frac{\theta}{2}\right)e^{i\phi}\end{bmatrix}$$

1.32 Exercise 32

$$e^{-i\frac{\phi}{2}\hat{Z}}|0\rangle = e^{-i\frac{\phi}{2}}\begin{bmatrix}1 & 0\\0 & e^{i\phi}\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = e^{-i\frac{\phi}{2}}\begin{bmatrix}1\\0\end{bmatrix}$$

$$e^{-i\frac{\theta}{2}\hat{Y}}e^{-i\frac{\phi}{2}\hat{Z}}|0\rangle = e^{-i\frac{\phi}{2}}\begin{bmatrix}\cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right)\\\sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right)\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = e^{-i\frac{\phi}{2}}\begin{bmatrix}\cos\left(\frac{\theta}{2}\right)\\\sin\left(\frac{\theta}{2}\right)\end{bmatrix}$$

$$\neq |\psi\rangle \quad \text{The order of rotations is important.}$$

To demonstrate, we can check if these two operators commute :

$$e^{-i\frac{\phi}{2}\hat{Z}}e^{-i\frac{\theta}{2}\hat{Y}} = e^{-i\frac{\phi}{2}}\begin{bmatrix}1 & 0\\0 & e^{i\phi}\end{bmatrix}\begin{bmatrix}\cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right)\\\sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right)\end{bmatrix} = e^{-i\frac{\phi}{2}}\begin{bmatrix}\cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right)\\\sin\left(\frac{\theta}{2}\right)e^{i\phi} & \cos\left(\frac{\theta}{2}\right)e^{i\phi}\end{bmatrix}$$

$$e^{-i\frac{\theta}{2}\hat{Y}}e^{-i\frac{\phi}{2}\hat{Z}} = e^{-i\frac{\phi}{2}}\begin{bmatrix}\cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right)e^{i\phi}\\\sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right)e^{i\phi}\end{bmatrix}$$

$$e^{-i\frac{\phi}{2}\hat{Z}}e^{-i\frac{\theta}{2}\hat{Y}} \neq e^{-i\frac{\theta}{2}\hat{Y}}e^{-i\frac{\phi}{2}\hat{Z}}, \quad \text{these operators don't commute.}$$

Thus, the rotation order is important such that :

$$e^{-i\frac{\phi}{2}\hat{Z}}\left(e^{-i\frac{\theta}{2}\hat{Y}}|0\rangle\right) \neq e^{-i\frac{\theta}{2}\hat{Y}}\left(e^{-i\frac{\phi}{2}\hat{Z}}|0\rangle\right).$$

1.33 Exercise 33

$$\hat{M} = \frac{\hat{X}+\hat{Z}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} = \text{Hadamard gate}$$

Eigenvalues :

$$\lambda_1 = +1 \quad \text{and} \quad \lambda_2 = -1$$

Eigenvectors :

$$|\lambda_1\rangle = \begin{bmatrix}\cos\left(\frac{\pi}{8}\right)\\ \sin\left(\frac{\pi}{8}\right)\end{bmatrix} = \frac{1}{2}\begin{bmatrix}\sqrt{2+\sqrt{2}}\\ \sqrt{2-\sqrt{2}}\end{bmatrix}, \quad |\lambda_2\rangle = \begin{bmatrix}\sin\left(\frac{\pi}{8}\right)\\ -\cos\left(\frac{\pi}{8}\right)\end{bmatrix} = \frac{1}{2}\begin{bmatrix}\sqrt{2-\sqrt{2}}\\ -\sqrt{2+\sqrt{2}}\end{bmatrix}$$

The function of the Hamiltonian $f(H)$:

$$f(H) = e^{-i\frac{\alpha}{2}\hat{M}} = \sum_n e^{-i\frac{\alpha}{2}\lambda_n}|\lambda_n\rangle\langle\lambda_n|$$

Expanded matrix form :

$$e^{-i\frac{\alpha}{2}\hat{M}} = \begin{bmatrix}\cos^2\left(\frac{\pi}{8}\right)e^{-i\frac{\alpha}{2}} + \sin^2\left(\frac{\pi}{8}\right)e^{i\frac{\alpha}{2}} & \sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)\left(e^{-i\frac{\alpha}{2}} - e^{i\frac{\alpha}{2}}\right)\\ \sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)\left(e^{-i\frac{\alpha}{2}} - e^{i\frac{\alpha}{2}}\right) & \cos^2\left(\frac{\pi}{8}\right)e^{i\frac{\alpha}{2}} + \sin^2\left(\frac{\pi}{8}\right)e^{-i\frac{\alpha}{2}}\end{bmatrix}$$

$$e^{-i\frac{\alpha}{2}\hat{M}} = \begin{bmatrix}\cos\left(\frac{\alpha}{2}\right) - i\frac{\sqrt{2}}{2}\sin\left(\frac{\alpha}{2}\right) & -i\frac{\sqrt{2}}{2}\sin\left(\frac{\alpha}{2}\right)\\ -i\frac{\sqrt{2}}{2}\sin\left(\frac{\alpha}{2}\right) & \cos\left(\frac{\alpha}{2}\right) - i\frac{\sqrt{2}}{2}\sin\left(\frac{\alpha}{2}\right)\end{bmatrix} = \cos\left(\frac{\alpha}{2}\right)\mathbb{I} - i\sin\left(\frac{\alpha}{2}\right)\left(\frac{\hat{Z}}{\sqrt{2}} + \frac{\hat{X}}{\sqrt{2}}\right)$$

Simplified form :

$$e^{-i\frac{\alpha}{2}\hat{M}} = \cos\left(\frac{\alpha}{2}\right)\mathbb{I} - i\sin\left(\frac{\alpha}{2}\right)\hat{M}$$

This corresponds to a rotation of angle α around the axis located in the XZ-plane with an angle $\beta = \frac{\pi}{4}$, based on the division of $\hat{Z}$ and $\hat{X}$ by $\sqrt{2}$.

The axis $\vec{n}$ is given by :

$$\vec{n} = (\cos(\beta), 0, \sin(\beta)) = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

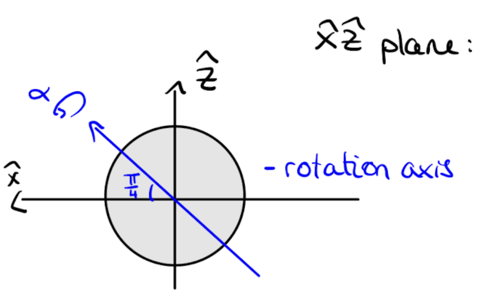


FIGURE 1.8 – Rotation axis represented in the XZ plane of the Bloch Sphere

1.34 Exercise 34

$$e^{-i\alpha\hat{X}}e^{-i\alpha\hat{Z}} = \left(\cos\left(\frac{\alpha}{2}\right)\mathbb{I} - i\sin\left(\frac{\alpha}{2}\right)\hat{X}\right)\left(\cos\left(\frac{\alpha}{2}\right)\mathbb{I} - i\sin\left(\frac{\alpha}{2}\right)\hat{Z}\right)$$

$$= \cos^2\left(\frac{\alpha}{2}\right)\mathbb{I} - \cos\left(\frac{\alpha}{2}\right)i\sin\left(\frac{\alpha}{2}\right)\hat{Z} - \cos\left(\frac{\alpha}{2}\right)i\sin\left(\frac{\alpha}{2}\right)\hat{X} - \sin^2\left(\frac{\alpha}{2}\right)\hat{X}\hat{Z}$$

$$= \cos^2\left(\frac{\alpha}{2}\right)\mathbb{I} - i\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)\hat{X} - i\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)\hat{Z} - \sin^2\left(\frac{\alpha}{2}\right)(-i\hat{Y})$$

$$e^{-i\alpha\hat{X}}e^{-i\alpha\hat{Z}} \neq e^{-i\alpha(\hat{X}+\hat{Z})}$$

Based on the previous exercise, knowing that $\hat{X}$ and $\hat{Z}$ don't commute,

$$e^{-i\alpha\hat{X}}e^{-i\alpha\hat{Z}} \neq e^{-i\alpha(\hat{X}+\hat{Z})}$$

In fact, $\hat{X} + \hat{Z}$ is not unitary.

1.35 Exercise 35

This time we have $\hat{M} = \frac{\hat{X}+\hat{Y}}{\sqrt{2}}$ which is unitary. Based on what we have done in previous sections, we can imagine that $e^{-i\frac{\alpha\hat{M}}{2}}$ shall be a rotation α around the axis located in the $\hat{X}\hat{Y}$ plane and at $\frac{\pi}{4}$ in this plane since we have the coefficient $\frac{1}{\sqrt{2}}$.

Let's prove it.

$$\hat{M} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1-i \\ 1+i & 0 \end{bmatrix}$$

Eigenvalues :

$$\lambda_1 = +1, \quad \lambda_2 = -1$$

Eigenvectors :
For $\lambda_1 = +1$:

$$\text{we find} \quad |v_1\rangle = \frac{1}{2}\begin{bmatrix} 1-i \\ \sqrt{2} \end{bmatrix} \quad \text{and} \quad |v_1\rangle\langle v_1| = \frac{1}{4}\begin{bmatrix} 2 & \sqrt{2}(1-i) \\ \sqrt{2}(1+i) & 2 \end{bmatrix}$$

For $\lambda_2 = +1$:

$$\text{we find} \quad |v_2\rangle = \frac{1}{2}\begin{bmatrix} 1-i \\ \sqrt{2} \end{bmatrix} \quad \text{and} \quad |v_2\rangle\langle v_2| = \frac{1}{4}\begin{bmatrix} 2 & -\sqrt{2}(1-i) \\ -\sqrt{2}(1+i) & 2 \end{bmatrix}$$

Therefore :

$$e^{-i\frac{\alpha}{2}\hat{M}} = e^{-i\frac{\alpha}{2}}\Big(|v_1\rangle\langle v_1| + e^{i\frac{\alpha}{2}}|v_2\rangle\langle v_2|\Big) = \cos\left(\frac{\alpha}{2}\right)\mathbb{I} + \begin{pmatrix} 0 & -\frac{\sqrt{2}(1+i)\sin\left(\frac{\alpha}{2}\right)}{2} \\ -\frac{\sqrt{2}(1+i)\sin\left(\frac{\alpha}{2}\right)}{2} & 0 \end{pmatrix}$$

$$e^{-i\frac{\alpha}{2}\hat{M}} = \cos\left(\frac{\alpha}{2}\right)\mathbb{I} - \frac{1}{\sqrt{2}}i\sin\left(\frac{\alpha}{2}\right)\hat{X} - \frac{1}{\sqrt{2}}i\sin\left(\frac{\alpha}{2}\right)\hat{Y}$$

$$= \cos\left(\frac{\alpha}{2}\right)\mathbb{I} - i\sin\left(\frac{\alpha}{2}\right)\hat{M}$$

Finally, we can conclude that :

$$e^{-i\frac{\alpha\hat{M}}{2}} = \cos\left(\frac{\alpha}{2}\right)\hat{I} - i\sin\left(\frac{\alpha}{2}\right)\hat{M}$$

$e^{-i\frac{\alpha\hat{M}}{2}}$ is indeed a rotation α around the axis located in the $\hat{X}\hat{Y}$ plane and at $\frac{\pi}{4}$ in this plane, since we have the coefficient $\frac{1}{\sqrt{2}}$.

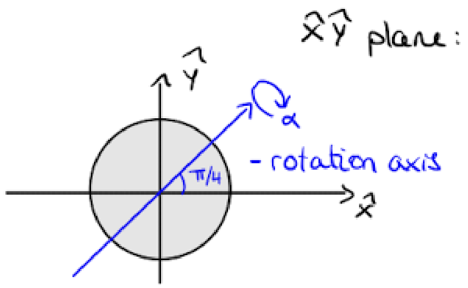


FIGURE 1.9 – Rotation axis represented in the XY plane of the Bloch Sphere

1.36 Exercise 36

$$\langle\psi|\hat{X}|\psi\rangle = \begin{pmatrix}\cos\left(\frac{\theta}{2}\right) & \sin\left(\frac{\theta}{2}\right)e^{-i\phi}\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\cos\left(\frac{\theta}{2}\right)\\ \sin\left(\frac{\theta}{2}\right)e^{i\phi}\end{pmatrix}$$

$$= \left[\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)e^{-i\phi}\right]\begin{pmatrix}\sin\left(\frac{\theta}{2}\right)e^{i\phi}\\ \cos\left(\frac{\theta}{2}\right)\end{pmatrix}$$

$$= \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)e^{i\phi} + \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)e^{-i\phi}$$

$$= \sin(\theta)\cdot\frac{e^{i\phi}+e^{-i\phi}}{2}$$

$$\langle\psi|\hat{X}|\psi\rangle = \sin(\theta)\cos(\phi)$$

$$\phi = \arccos\left(\frac{\langle\psi|\hat{X}|\psi\rangle}{\sin\left[\arccos\left(\langle\psi|\hat{Z}|\psi\rangle\right)\right]}\right)$$

$$\langle\psi|\hat{Z}|\psi\rangle = \begin{pmatrix}\cos\left(\frac{\theta}{2}\right) & \sin\left(\frac{\theta}{2}\right)e^{-i\phi}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\cos\left(\frac{\theta}{2}\right)\\ \sin\left(\frac{\theta}{2}\right)e^{i\phi}\end{pmatrix}$$

$$= \cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)$$

$$\langle\psi|\hat{Z}|\psi\rangle = \cos(\theta)$$

$$\theta = \arccos\Big(\langle\psi|\hat{Z}|\psi\rangle\Big)$$

1.37 Exercise 37

Turning a $|+\rangle$ into $|0\rangle$ and $|-\rangle$ into $|1\rangle$ requires and Hadamard gate.

$$|\Psi'\rangle = \hat{H}|\Psi\rangle = \cos\left(\frac{\theta}{2}\right)\hat{H}|0\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\phi}\hat{H}|1\rangle$$

$$|\Psi'\rangle = \cos\left(\frac{\theta}{2}\right)|+\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\phi}|-\rangle$$

$$|\Psi'\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)e^{i\phi} \\ \cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)e^{i\phi}\end{pmatrix}$$

We need to apply an Hadamard gate to the state $|\Psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\phi}|1\rangle$, so that the coefficients are switched to the $|+\rangle, |-\rangle$ basis such as $|\Psi'\rangle = \cos\left(\frac{\theta}{2}\right)|+\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\phi}|-\rangle$. In order to know the mean along $\hat{Z}$, we need to estimate the projection of this new state $|\Psi'\rangle$ on the $\hat{Z}$ axis on the Bloch Sphere. A way to visualize is just to turn the Bloch Sphere and "imagine" $\hat{X}$-axis took the place of $\hat{Z}$-axis

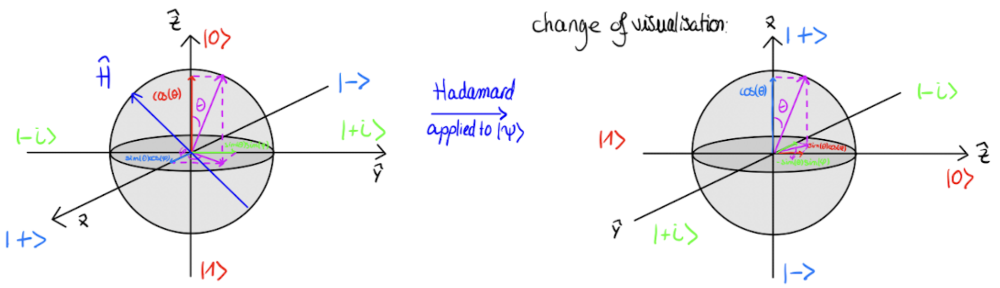


FIGURE 1.10 – Different vizualisation on the Bloch Sphere after applying and Hadamard Gate

We then measure along $\hat{Z}$-axis and as a reminder, the projection on the different axis in the Bloch sphere corresponds to the mean value after several repeated measurements of a state exactly prepared the same way before each experience. We obtain with an Hadamard and then a measurement along $\hat{Z}$: $\langle\Psi'|\hat{Z}|\Psi'\rangle = \sin(\theta)\cos(\phi) = \langle\Psi|\hat{X}|\Psi\rangle$.

Proof:

$$\langle\psi'|\hat{Z}|\psi'\rangle = \frac{1}{2}\left[\cos(\tfrac{\theta}{2}) + \sin(\tfrac{\theta}{2})e^{-i\phi} \quad \cos(\tfrac{\theta}{2}) - \sin(\tfrac{\theta}{2})e^{-i\phi}\right]\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \cos(\tfrac{\theta}{2}) + \sin(\tfrac{\theta}{2})e^{i\phi} \\ \cos(\tfrac{\theta}{2}) - \sin(\tfrac{\theta}{2})e^{i\phi} \end{bmatrix}$$

$$= \frac{1}{2}\left[\cos(\tfrac{\theta}{2}) + \sin(\tfrac{\theta}{2})e^{-i\phi} \quad \cos(\tfrac{\theta}{2}) - \sin(\tfrac{\theta}{2})e^{-i\phi}\right]\begin{bmatrix} \cos(\tfrac{\theta}{2}) + \sin(\tfrac{\theta}{2})e^{i\phi} \\ -\cos(\tfrac{\theta}{2}) + \sin(\tfrac{\theta}{2})e^{i\phi} \end{bmatrix}$$

$$= \frac{1}{2}\left(\cos(\tfrac{\theta}{2}) + \sin(\tfrac{\theta}{2})e^{-i\phi}\right)\left(\cos(\tfrac{\theta}{2}) + \sin(\tfrac{\theta}{2})e^{i\phi}\right) + \frac{1}{2}\left(\cos(\tfrac{\theta}{2}) - \sin(\tfrac{\theta}{2})e^{-i\phi}\right)\left(-\cos(\tfrac{\theta}{2}) + \sin(\tfrac{\theta}{2})e^{i\phi}\right).$$

$$= \frac{1}{2}\left(\cancel{\cos^2(\tfrac{\theta}{2})} + \cos(\tfrac{\theta}{2})\sin(\tfrac{\theta}{2})e^{i\phi} + \cos(\tfrac{\theta}{2})\sin(\tfrac{\theta}{2})e^{-i\phi} + \cancel{\sin^2(\tfrac{\theta}{2})}\right) + \frac{1}{2}\left(-\cancel{\cos^2(\tfrac{\theta}{2})} + \cos(\tfrac{\theta}{2})\sin(\tfrac{\theta}{2})e^{i\phi} + \cos(\tfrac{\theta}{2})\sin(\tfrac{\theta}{2})e^{-i\phi} - \cancel{\sin^2(\tfrac{\theta}{2})}\right)$$

$$= \frac{1}{2}\left(2\cos(\tfrac{\theta}{2})\sin(\tfrac{\theta}{2})e^{i\phi} + 2\cos(\tfrac{\theta}{2})\sin(\tfrac{\theta}{2})e^{-i\phi}\right)$$

$$= \sin(\theta)\,\frac{e^{i\phi} + e^{-i\phi}}{2}$$

$$\boxed{\langle\psi'|\hat{Z}|\psi'\rangle = \langle\psi'|\hat{H}^\dagger\hat{Z}|\hat{H}\psi\rangle = \sin(\theta)\cos(\phi) = \langle\psi|\hat{X}|\psi\rangle}$$

FIGURE 1.11 – Handwritten proof for the mean of $|\psi'\rangle$ along $\hat{Z}$-axis

$$\langle\psi'|\hat{Y}|\psi'\rangle = \frac{1}{2}\left[\cos(\tfrac{\theta}{2}) + \sin(\tfrac{\theta}{2})e^{-i\phi} \quad \cos(\tfrac{\theta}{2}) - \sin(\tfrac{\theta}{2})e^{-i\phi}\right]\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} \cos(\tfrac{\theta}{2}) + \sin(\tfrac{\theta}{2})e^{i\phi} \\ \cos(\tfrac{\theta}{2}) - \sin(\tfrac{\theta}{2})e^{i\phi} \end{bmatrix}$$

$$= \frac{1}{2}\left[\cos(\tfrac{\theta}{2}) + \sin(\tfrac{\theta}{2})e^{-i\phi} \quad \cos(\tfrac{\theta}{2}) - \sin(\tfrac{\theta}{2})e^{-i\phi}\right]\begin{bmatrix} -i\cos(\tfrac{\theta}{2}) + i\sin(\tfrac{\theta}{2})e^{i\phi} \\ i\cos(\tfrac{\theta}{2}) + i\sin(\tfrac{\theta}{2})e^{i\phi} \end{bmatrix}$$

$$= \frac{i}{2}\left[\left(\cos(\tfrac{\theta}{2}) + \sin(\tfrac{\theta}{2})e^{-i\phi}\right)\left(-\cos(\tfrac{\theta}{2}) + \sin(\tfrac{\theta}{2})e^{i\phi}\right) + \frac{i}{2}\left(\cos(\tfrac{\theta}{2}) - \sin(\tfrac{\theta}{2})e^{-i\phi}\right)\left(\cos(\tfrac{\theta}{2}) + \sin(\tfrac{\theta}{2})e^{i\phi}\right).\right.$$

$$= \frac{i}{2}\left[-\cancel{\cos^2(\tfrac{\theta}{2})} + \cos\tfrac{\theta}{2}\sin\tfrac{\theta}{2}e^{i\phi} - \cos\tfrac{\theta}{2}\sin\tfrac{\theta}{2}e^{-i\phi} + \cancel{\sin^2\tfrac{\theta}{2}}\right] + \frac{i}{2}\left(\cancel{\cos^2(\tfrac{\theta}{2})} + \cos\tfrac{\theta}{2}\sin\tfrac{\theta}{2}e^{i\phi} - \cos\tfrac{\theta}{2}\sin\tfrac{\theta}{2}e^{-i\phi} - \cancel{\sin^2\tfrac{\theta}{2}}\right)$$

$$= \frac{i}{2}\left[2\cos\tfrac{\theta}{2}\sin\tfrac{\theta}{2}e^{i\phi} - 2\cos\tfrac{\theta}{2}\sin\tfrac{\theta}{2}e^{-i\phi}\right]$$

$$= -\sin(\theta)\sin(\phi).$$ (In order to understand why there is a minus before $\sin(\theta)\sin(\phi)$ on the sphere above)

FIGURE 1.12 – Handwritten proof for the mean of $|\psi'\rangle$ along $\hat{Y}$-axis