Exercise 1:

$$\frac{\partial}{\partial t}\langle\psi(t)|\hat{L}|\psi(t)\rangle \stackrel{?}{=} \langle\psi(t)|\left(-\frac{i}{\hbar}[\hat{L},\hat{H}]\right)|\psi(t)\rangle$$

$$\frac{\partial}{\partial t}\langle\psi(t)|\hat{L}|\psi(t)\rangle = \frac{\partial\langle\psi(t)|}{\partial t}\cdot\hat{L}|\psi(t)\rangle + \langle\psi(t)|\hat{L}\frac{\partial|\psi(t)\rangle}{\partial t}$$

$$= \langle\psi(t)|\left(\frac{i}{\hbar}\right)\hat{H}\cdot\hat{L}|\psi(t)\rangle + \langle\psi(t)|\hat{L}\left(\frac{-i}{\hbar}\right)\hat{H}|\psi(t)\rangle$$

$$= \langle\psi(t)|\left(\frac{-i}{\hbar}\right)\left[\hat{L}\hat{H}-\hat{H}\hat{L}\right]|\psi(t)\rangle$$

$$\frac{\partial}{\partial t}\langle\psi(t)|\hat{L}|\psi(t)\rangle = \langle\psi(t)|\left(\frac{-i}{\hbar}\right)[\hat{L},H]|\psi(t)\rangle \quad \blacksquare$$

Exercise 2:

$|\psi(t)\rangle = \alpha_0(t)|E_0\rangle + \alpha_1(t)|E_1\rangle.$

Eq. 1: $\frac{\partial}{\partial t}|\psi(t)\rangle = \frac{-i}{\hbar}H|\psi(t)\rangle.$

$H = E_0\cdot|E_0\rangle\langle E_0| + E_1|E_1\rangle\langle E_1|.$

$H\alpha_0(t)|E_0\rangle = E_0\cdot\alpha_0(t)|E_0\rangle = E_0|\tilde{\psi}_0(t)\rangle$

$H\alpha_1(t)|E_1\rangle = E_1\alpha_1(t)|E_1\rangle = E_1|\tilde{\psi}_1(t)\rangle$

$$\frac{\partial}{\partial t}\left[\alpha_0(t)|E_0\rangle + \alpha_1(t)|E_1\rangle\right] = \frac{\partial}{\partial t}\left[\alpha_0(t)|E_0\rangle\right] + \frac{\partial}{\partial t}\left[\alpha_1(t)|E_1\rangle\right]. = \frac{-i}{\hbar}H\,\alpha_0(t)|E_0\rangle - \frac{i}{\hbar}H\alpha_1(t)|E_1\rangle.$$

$$\frac{\partial}{\partial t}\left[|\psi(t)\rangle\right] = \frac{-i}{\hbar}H|\psi(t)\rangle \iff \frac{\partial}{\partial t}\left[|\tilde{\psi}_0(t)\rangle\right] + \frac{\partial}{\partial t}\left[|\tilde{\psi}_1(t)\rangle\right] = -\frac{i}{\hbar}H|\tilde{\psi}_0(t)\rangle - \frac{i}{\hbar}H|\tilde{\psi}_1(t)\rangle$$

$$\frac{\partial}{\partial t}\left[|\tilde{\psi}_0(t)\rangle\right] + \frac{\partial}{\partial t}\left[|\tilde{\psi}_1(t)\rangle\right] = \frac{-i}{\hbar}E_0|\tilde{\psi}_0(t)\rangle - \frac{i}{\hbar}E_1|\tilde{\psi}_1(t)\rangle$$

$$\iff \begin{cases} |\tilde{\psi}_0(t)\rangle = e^{-i\frac{E_0 t}{\hbar}}\cdot|\tilde{\psi}_0(t=0)\rangle = e^{-i\frac{E_0 t}{\hbar}}\cdot\alpha_0(t=0)|E_0\rangle \\ |\tilde{\psi}_1(t)\rangle = e^{-i\frac{E_1 t}{\hbar}}\cdot|\tilde{\psi}_1(t=1)\rangle = e^{-i\frac{E_1 t}{\hbar}}\cdot\alpha_1(t=0)|E_1\rangle \end{cases}$$

Therefore we have $|\psi(t)\rangle = |\tilde{\psi}_0(t)\rangle + |\tilde{\psi}_1(t)\rangle$

$$|\psi(t)\rangle = e^{-i\frac{E_0 t}{\hbar}}\cdot\alpha_0(t=0)|E_0\rangle + e^{-i\frac{E_1 t}{\hbar}}\cdot\alpha_1(t=0)|E_1\rangle. \quad \blacksquare$$

Check:

$$\frac{\partial}{\partial t}|\psi(t)\rangle = -\frac{i}{\hbar}E_0\, e^{-i\frac{E_0 t}{\hbar}}\alpha_0(t=0)|E_0\rangle - \frac{i}{\hbar}E_1\cdot e^{-i\frac{E_1 t}{\hbar}}\cdot\alpha_1(t=0)|E_1\rangle$$

$$\frac{-i}{\hbar}H|\psi(t)\rangle = \frac{-i}{\hbar}e^{-i\frac{E_0 t}{\hbar}}\cdot\alpha_0(t=0)\,H|E_0\rangle - \frac{i}{\hbar}e^{-i\frac{E_1 t}{\hbar}}\cdot\alpha_1(t=0)H|E_1\rangle.$$

$$= \frac{-i}{\hbar}e^{-i\frac{E_0}{\hbar}}\alpha_0(t=0)\,E_0|E_0\rangle - \frac{i}{\hbar}e^{-i\frac{E_1 t}{\hbar}}\cdot\alpha_1(t=0)\,E_1|E_1\rangle$$

We see that $|\psi(t)\rangle = e^{-i\frac{E_0 t}{\hbar}}\cdot\alpha_0(t=0)|E_0\rangle + e^{-i\frac{E_1 t}{\hbar}}\cdot\alpha_1(t=0)|E_1\rangle$. is a solution of the Schroedinger's Equation with its associated Hamiltonian.

Exercise 3:

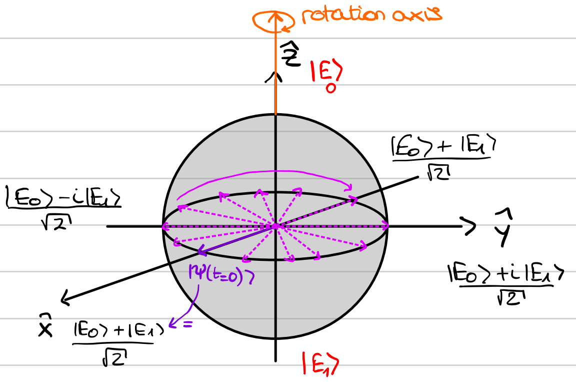
The solution of an arbitrary Qubit is given by: $|\psi(t)\rangle = \alpha_0(t=0)|E_0\rangle + \alpha_1(t=0)e^{-i\frac{E_1-E_0}{\hbar}t}|E_1\rangle$

Here we have either $\alpha_0(t=0)=0$ and $\alpha_1(t=0)=1$ or $\alpha_1(t=1)=0$ and $\alpha_0(t=0)=1$. ie. $|\psi(t=0)\rangle = |E_1\rangle$ or $|\psi(t=0)\rangle = |E_0\rangle$

The solution is therefore given by:

if $\alpha_0(t=0)=0$: $|\Psi(t)\rangle = e^{-i\frac{E_1-E_0 t}{\hbar}} \cdot |E_1\rangle \Rightarrow$ the Qubit stays in state $|E_1\rangle$ with a global phase that does not matter.
if $\alpha_1(t=0)=0$: $|\Psi(t)\rangle = |E_0\rangle \Rightarrow$ the Qubit stays in state $|E_0\rangle$

Exercise 4:



$$|\Psi(t)\rangle = \begin{bmatrix} \cos(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2})e^{i\varphi} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2}\, e^{-i\frac{E_1-E_0}{\hbar}t} \end{bmatrix}$$

$$|\Psi(t=0)\rangle = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \frac{|E_0\rangle + |E_1\rangle}{\sqrt{2}}$$

The Qubit will rotate clockwise at the Larmor frequency $\Omega_L = \frac{E_1-E_0}{\hbar}$ around "$\hat{z}$" axis and beginning in the "$\hat{x}\hat{y}$-plane" aligned with $\hat{x} = \frac{|E_0\rangle+|E_1\rangle}{\sqrt{2}}$.

Exercise 5:

$E_1 - E_0 = 590 \cdot 4200 . = 2{,}478 \cdot 10^6$ [J]

$T_L = \frac{h}{E_1-E_0} = \frac{6\cdot 10^{-34}}{2{,}478\cdot 10^6} = 2{,}42\cdot 10^{-40}$ [s].

Since the period for the Big Mac is nearly 0, it does not make sense to consider it as a quantum object. We understand now why it is considered classical, indeed the energy is macroscopic. Therefore, the energy must be few orders under the one of the Planck's constant in order to be expirimentally accessible. This would corresponds to the energy of a quantum or microscopic system like the energy associated to quantum transitions

Exercise 6:

Eigenvectors of $\hat{H}$ are : $\{|E_0\rangle ; |E_1\rangle\}$.

$H|E_0\rangle = E_0|E_0\rangle$.
$E_0 \mathbb{1}|E_0\rangle = E_0|E_0\rangle$

$\tilde{H} = \hat{H} - \mathbb{1}E_0$

- $\tilde{H}|E_0\rangle = [\hat{H} - \mathbb{1}E_0]|E_0\rangle = \hat{H}|E_0\rangle - E_0|E_0\rangle = E_0|E_0\rangle - E_0|E_0\rangle = 0|E_0\rangle$.
- $\tilde{H}|E_1\rangle = [\hat{H} - \mathbb{1}E_0]|E_1\rangle = \hat{H}|E_1\rangle - E_0|E_1\rangle = (E_1-E_0)|E_1\rangle$.

The eigenvectors are the same of $\hat{H}$ i.e. $\{|E_0\rangle ; |E_1\rangle\}$ with eigenvalues $\{0; E_1-E_0\}$ with resp.

Since the new Hamiltonian has the same eigenvectors than the previous one, |ψ(t)> is also a superposition of the same eigenvectors |E0> and |E1>. We also noticed that the Schroedinger equation gives a solution in which we propagate every state with its associated energy (or eigenvalue), since the solution is a linear combination of exp(-i*t*Ei/ℏ)*|Ei> (propagation of each mode). We can now consider the same solution for

this new Hamiltonian at the exception that the new energy or eigenvalues associated to the eigenvectors are 0 for |E0> and E1-E0 for |E1>. We can now write down the evolution of the system :

$|\Psi(t)\rangle = \alpha_0(t=0)e^{-i\frac{0}{\hbar}t}|E_0\rangle + \alpha_1(t=0)e^{-i\frac{E_1-E_0}{\hbar}t}|E_1\rangle$
$|\Psi(t)\rangle = \alpha_0(t=0)|E_0\rangle + \alpha_1(t=0)e^{-i\frac{E_1-E_0}{\hbar}t}|E_1\rangle$
This is the same form of the solution we obtain for the Hamiltonian $\hat{H}$.

Exercice 7:

$$|\Psi(t)\rangle = \exp\left(-i\frac{\hat{H}}{\hbar}t\right).|\Psi(t=0)\rangle.$$

$$\frac{\partial}{\partial t}[|\Psi(t)\rangle] = -\frac{i}{\hbar}\hat{H}\exp\left(-i\frac{\hat{H}}{\hbar}t\right).|\Psi(t=0)\rangle. = -\frac{i}{\hbar}\hat{H}|\Psi(t)\rangle$$

Therefore $|\Psi(t)\rangle = \exp\left(-i\frac{\hat{H}}{\hbar}t\right)|\Psi(t=0)\rangle$ satisfies Schroedinger Equation

Eq. 5: $|\Psi(t)\rangle = \alpha_0(t=0)|E_0\rangle + \alpha_1(t=0)\exp\left(-i\frac{E_1-E_0t}{\hbar}\right)|E_1\rangle.$

We learned: $f(\hat{H}) = \Sigma_i f(h_i)\,|h_i\rangle\langle h_i|$ $\quad f(x) = e^{-i\frac{xt}{\hbar}}$

$$|\Psi(t)\rangle = \left[f(E_0)|E_0\rangle\langle E_0| + f(E_1)|E_1\rangle\langle E_1|\right]|\Psi(t=0)\rangle$$

$$= e^{-i\frac{E_0t}{\hbar}}|E_0\rangle\langle E_0|\Psi(t=0)\rangle + e^{-i\frac{E_1t}{\hbar}}|E_1\rangle\langle E_1|\Psi(t=0)\rangle$$

$$= \alpha_0(t=0)e^{-i\frac{E_0t}{\hbar}}|E_0\rangle + \alpha_1(t=0)e^{-i\frac{E_1t}{\hbar}}|E_1\rangle.$$

$$|\Psi(t)\rangle = e^{-i\frac{E_0t}{\hbar}}\left[\alpha_0(t=0)|E_0\rangle + \alpha_1(t=0)e^{-i\frac{E_1-E_0t}{\hbar}}|E_1\rangle\right] \equiv Eq.5$$

Since the global phase is irrelevant in our measurements we find that $|\Psi(t)\rangle = e^{-i\frac{\hat{H}}{\hbar}t}|\Psi(t=0)\rangle$ satisfies Eq. 5.

Exercise 8:

$$\hat{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \hat{Y} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \hat{Z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

$\hat{H} = \begin{bmatrix} a & x-iy \\ x-iy & b \end{bmatrix}$ is the form of any 2x2 hermitian matrix.

$$-\omega_I \mathbb{1} - \frac{\omega_x}{2}\hat{X} - \frac{\omega_y}{2}\hat{Y} - \frac{\omega_z}{2}\hat{Z} = -\begin{bmatrix} \omega_I + \frac{\omega_z}{2} & \frac{\omega_x}{2} - i\frac{\omega_y}{2} \\ \frac{\omega_x}{2} - i\frac{\omega_y}{2} & \omega_I - \frac{\omega_z}{2} \end{bmatrix}$$

By identifying coeff. of $\frac{\hat{H}}{\hbar}$ we have: $-a = \omega_I + \frac{\omega_z}{2}$ $\Rightarrow \omega_I = -\frac{a+b}{2}$

$-b = \omega_I - \frac{\omega_z}{2}$

$c - id = -\frac{\omega_x}{2} + i\frac{\omega_y}{2} \Rightarrow \omega_x = -2c \quad \omega_y = -2d.$

we have $\begin{cases} \omega_I = -\frac{a+b}{2} \\ \omega_x = -2c \\ \omega_y = -2d \\ \omega_z = -(a-b) \end{cases}$ we can now write: $\frac{\hat{H}}{\hbar} = \frac{a+b}{2}\mathbb{1} + c\hat{X} + d\hat{Y} + (a-b)\hat{Z}.$

This proves that any 2x2 hermitian matrix, can be written as a linear superposition of Pauli matrices and the identity.

Exercice 9:

set $\omega_I = 0$.

$$\hat{H} = -\frac{\hbar}{2}\begin{bmatrix} \omega_z & \omega_x - i\omega_y \\ \omega_x - \omega_y & -\omega_z \end{bmatrix}$$

$\det\left[\hat{H} - \lambda\mathbb{1}\right] \stackrel{!}{=} 0$ For an hermitian matrix we know that $Tr(\hat{H}) = \lambda_1 + \lambda_2$ & $\det(\hat{H}) = \lambda_1 \cdot \lambda_2.$

$\mathrm{Tr}\,\hat{H} = 0 \Rightarrow \lambda_1 = -\lambda_2$

$$\det(\hat{H}) = \frac{\hbar^2}{4}\left[(-\omega_z^2) - (\omega_x^2 + \omega_y^2)\right]$$
$$= -\frac{\hbar^2}{4}\left[\omega_x^2 + \omega_y^2 + \omega_z^2\right]$$

then $\lambda_1 = \frac{\hbar}{2}\sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}$ & $\lambda_2 = -\frac{\hbar}{2}\sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}$

regarding the question $\lambda = \pm\sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}$ this means that the eigenvalues of $\frac{\hat{H}}{\hbar}$ are $\lambda_1 = +\frac{1}{2}\sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}$ and $\lambda_2 = -\frac{1}{2}\sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}$

Exercice 10:

$$\lambda = \frac{\sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}}{2}$$

$$\frac{\hat{H}}{\hbar} = -\frac{1}{\cancel{2}}\begin{bmatrix} \cancel{2}\lambda\cos(\theta) & \cancel{2}\lambda\sin(\theta)\cos(\phi) - i2\lambda\sin(\theta)\sin(\phi) \\ \cancel{2}\lambda\sin(\theta)\cos(\phi) + i2\lambda\sin(\theta)\sin(\phi) & -\cancel{2}\lambda\cos(\theta)\end{bmatrix}$$

$$\frac{\hat{H}}{\hbar} = -\lambda\begin{bmatrix} \cos(\theta) & \sin(\theta)e^{-i\phi} \\ \sin(\theta)e^{i\phi} & -\cos(\theta)\end{bmatrix}$$

$$\frac{\hat{H}}{\hbar}|\psi_1\rangle = -\lambda\begin{bmatrix} \cos(\theta) & \sin(\theta)e^{-i\phi} \\ \sin(\theta)e^{i\phi} & -\cos(\theta)\end{bmatrix}\begin{bmatrix}\cos(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2})e^{i\phi}\end{bmatrix} = -\lambda\begin{bmatrix}\cos(\theta)\cos(\frac{\theta}{2}) + \sin(\theta)\sin(\frac{\theta}{2}) \\ \cos(\frac{\theta}{2})\sin(\theta)e^{i\phi} - \cos(\theta)\sin(\frac{\theta}{2})e^{i\phi}\end{bmatrix} = -\lambda\begin{bmatrix}\cos(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2})e^{i\phi}\end{bmatrix}$$

$$\Leftrightarrow \quad \hat{H}|\psi_1\rangle = -\lambda|\psi_1\rangle \qquad |\psi_1\rangle = \begin{bmatrix}\cos(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2})e^{i\phi}\end{bmatrix}$$

The vector $|\psi_1\rangle = \begin{bmatrix}\cos(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2})e^{i\phi}\end{bmatrix}$ projected on the different axis of the bloch sphere gives the vector $\vec{n} = \begin{bmatrix}\sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta)\end{bmatrix}$

which is exactly the same vector (with a const $-$) for $(\omega_x;\omega_y;\omega_z) = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}\begin{bmatrix}\sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta)\end{bmatrix}$ meaning that they are aligned against on the bloch sphere.

Another eigenvector would be one orthogonal to $|\psi_1\rangle$:

$$\langle\psi_1|\psi_2\rangle = \left[\cos(\tfrac{\theta}{2}) \quad \sin(\tfrac{\theta}{2})e^{-i\phi}\right]\begin{bmatrix}\alpha \\ \beta\end{bmatrix} = 0$$

$$\Leftrightarrow \cos(\tfrac{\theta}{2})\cdot\alpha + \sin(\tfrac{\theta}{2})e^{-i\phi}\beta = 0$$

$$\Leftrightarrow \begin{cases}\alpha = \sin(\frac{\theta}{2}) \\ \beta = -\cos(\frac{\theta}{2})e^{i\phi}\end{cases}$$

The other eigenvector is $|\psi_2\rangle = \begin{bmatrix}\sin(\frac{\theta}{2}) \\ -\cos(\frac{\theta}{2})e^{i\phi}\end{bmatrix}$.

One can check:

$$\frac{\hat{H}}{\hbar}|\psi_2\rangle = -\lambda\begin{bmatrix} \cos(\theta) & \sin(\theta)e^{-i\phi} \\ \sin(\theta)e^{i\phi} & -\cos(\theta)\end{bmatrix}\begin{bmatrix}\sin(\frac{\theta}{2}) \\ -\cos(\frac{\theta}{2})e^{i\phi}\end{bmatrix} = -\lambda\begin{bmatrix}\sin(\frac{\theta}{2})\cos(\theta) - \sin(\theta)\cos(\frac{\theta}{2}) \\ \sin(\theta)\sin(\frac{\theta}{2})e^{i\phi} + \cos(\theta)\cos(\frac{\theta}{2})e^{i\phi}\end{bmatrix} = \lambda\begin{bmatrix}\sin(\frac{\theta}{2}) \\ -\cos(\frac{\theta}{2})\,e^{i\phi}\end{bmatrix}$$

Since this vector is orthogonal to $|\psi_1\rangle$, it is aligned against $|\psi_1\rangle$ into the Bloch sphere representation and therefore aligned along $(\omega_x;\omega_y;\omega_z)$.

Exercice 11:

rotation of θ in XZ-plane: $U_y = e^{-i\frac{\theta}{2}\vec{Y}} = \begin{bmatrix} \cos(\frac{\theta}{2}) & -\sin(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{bmatrix}$

rotation of ϕ in XY-plane: $U_z = e^{-i\frac{\phi}{2}\hat{z}} = e^{-i\frac{\phi}{2}} \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix}$

$|0\rangle$:

$$U_y \cdot |0\rangle = \begin{bmatrix} \cos(\frac{\theta}{2}) & -\sin(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$U_y |0\rangle = \begin{bmatrix} \cos(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) \end{bmatrix}$$

$$U_z U_y |0\rangle = e^{-i\frac{\phi}{2}} \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix} \begin{bmatrix} \cos(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) \end{bmatrix} = e^{-i\frac{\phi}{2}} \begin{bmatrix} \cos(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) e^{i\phi} \end{bmatrix} \equiv |\psi_1\rangle$$ this is the first eigenvector.

$|1\rangle$:

$$U_y |1\rangle = \begin{bmatrix} \cos(\frac{\theta}{2}) & -\sin(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$U_y |1\rangle = \begin{bmatrix} -\sin(\frac{\theta}{2}) \\ \cos(\frac{\theta}{2}) \end{bmatrix}$$

$$U_z U_y |0\rangle = e^{-i\frac{\phi}{2}} \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix} \begin{bmatrix} -\sin(\frac{\theta}{2}) \\ \cos(\frac{\theta}{2}) \end{bmatrix} = e^{-i\frac{\phi}{2}} \begin{bmatrix} -\sin(\frac{\theta}{2}) \\ \cos(\frac{\theta}{2}) e^{i\phi} \end{bmatrix} \equiv |\psi_2\rangle$$ this is the second eigenvector.

we found the same thing in the previous question.

Please note that the question has been updated after I finished this question, we will continue here by expressing U_y & U_z and see we will obtain the same.

For U_y:

$$U_y = e^{-i\frac{\theta}{2}\vec{Y}} = \begin{bmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{bmatrix} = e^{-i\frac{\vec{H}}{\hbar}t}.$$

$$-i\frac{\vec{H}}{\hbar}t = -i\theta\vec{Y}/2$$

$$\frac{\vec{H}}{\hbar} = \frac{\theta}{t}\vec{Y}/2 = -\omega_y \vec{Y}/2 \Rightarrow \omega_y = -\frac{\theta}{t} \; ; \; \omega_x = 0 \; ; \; \omega_z = 0 \; ; \; \lambda = \frac{\theta}{t}.$$

$$\omega_x = 0 = 2\lambda\sin(\theta)\cos(\phi)$$

$$\omega_y = -\frac{\theta}{t} = 2\lambda\sin(\theta)\sin(\phi) \Rightarrow \theta = -\omega_y t.$$

$$U_y = \begin{bmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{bmatrix} = \begin{bmatrix} \cos(-\omega_y t/2) & -\sin(-\omega_y t/2) \\ \sin(-\omega_y t/2) & \cos(-\omega_y t/2) \end{bmatrix} = \begin{bmatrix} \cos(\omega_y t/2) & \sin(\omega_y t/2) \\ -\sin(\omega_y t/2) & \cos(\omega_y t/2) \end{bmatrix}$$

For U_z:

$$U_z = e^{-i\frac{\phi}{2}} \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix} = e^{-i\frac{\phi}{2}\hat{z}} = e^{-i\frac{\vec{H}}{\hbar}t}.$$

$$-i\frac{\Phi}{2}\vec{z} = -i\frac{\hat{H}}{\hbar}t. \qquad \frac{\hat{H}}{\hbar} = \frac{\Phi}{t}\vec{z}/2$$

$$\omega_z = \frac{-\Phi}{t} = 2\lambda\cos(\theta) \Rightarrow \Phi = -t\omega_z \qquad \lambda = \frac{\Phi}{t}$$

$$\omega_y = 0 = 2\lambda\sin(\theta)\sin(\phi)$$

$$\omega_x = 0 = 2\lambda\sin(\theta)\cos(\phi)$$

donc $U_z = e^{-i\frac{\Phi}{2}\vec{z}} = e^{-i\frac{\Phi}{2}}\begin{bmatrix}1 & 0\\ 0 & e^{i\Phi}\end{bmatrix} = e^{\frac{i\omega_z t}{2}}\begin{bmatrix}1 & 0\\ 0 & e^{-i\omega_z t}\end{bmatrix}$

$|0\rangle$:

$\hookrightarrow U_y|0\rangle = \begin{bmatrix}\cos(\omega_y t/2)\\ -\sin(\omega_y t/2)\end{bmatrix}$

$\hookrightarrow U_z U_y|0\rangle = e^{\frac{i\omega_z t}{2}}\begin{bmatrix}1 & 0\\ 0 & e^{-i\omega_z t}\end{bmatrix}\begin{bmatrix}\cos(\omega_y t/2)\\ -\sin(\omega_y t/2)\end{bmatrix}$

$= e^{\frac{i\omega_z t}{2}}\begin{bmatrix}\cos(\frac{\omega_y}{2}t)\\ -\sin(\frac{\omega_y}{2}t)e^{-i\omega_z t}\end{bmatrix}$

We come back with: $\omega_y = \frac{-\theta}{t}$ $\omega_z = \frac{-\Phi}{t}$

$U_z U_y|0\rangle = e^{-i\frac{\Phi}{2}}\begin{bmatrix}\cos(-\theta/2)\\ -\sin(-\theta/2)e^{i\Phi}\end{bmatrix} = e^{-i\Phi/2}\begin{bmatrix}\cos(\frac{\theta}{2})\\ \sin(\frac{\theta}{2})e^{i\Phi}\end{bmatrix}$

$|1\rangle$:

$\hookrightarrow U_y|1\rangle = \begin{bmatrix}\sin(\omega_y t/2)\\ \cos(\omega_y t/2)\end{bmatrix}$

$\hookrightarrow U_z U_y|0\rangle = e^{\frac{i\omega_z t}{2}}\begin{bmatrix}1 & 0\\ 0 & e^{-i\omega_z t}\end{bmatrix}\begin{bmatrix}\sin(\omega_y t/2)\\ \cos(\omega_y t/2)\end{bmatrix}$

$= e^{\frac{i\omega_z t}{2}}\begin{bmatrix}\sin(\omega_y t/2)\\ \cos(\omega_y t/2)e^{-i\omega_z t}\end{bmatrix}$

We come back with: $\omega_y = \frac{-\theta}{t}$ $\omega_z = \frac{-\Phi}{t}$

$U_z U_y|1\rangle = e^{-\frac{i\Phi}{2}}\begin{bmatrix}\sin(-\frac{\theta}{2})\\ \cos(-\frac{\theta}{2})e^{i\Phi}\end{bmatrix} = e^{-\frac{i\Phi}{2}}\begin{bmatrix}-\sin(\frac{\theta}{2})\\ \cos(\frac{\theta}{2})e^{i\Phi}\end{bmatrix}$

We again find the the same eigenvectors with a global phase which we can drop.

Exercice 12:

$[\hat{Y};\hat{X}] = \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} - \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} = \begin{bmatrix}-i & 0\\ 0 & i\end{bmatrix} - \begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix} = -2i\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = -2i\hat{z}$

$[\hat{z},\hat{Y}] = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} - \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = \begin{bmatrix}0 & -i\\ -i & 0\end{bmatrix} - \begin{bmatrix}0 & i\\ i & 0\end{bmatrix} = -2i\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = -2i\hat{X}$

$[\hat{X};\hat{Y}] = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} - \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix} - \begin{bmatrix}-i & 0\\ 0 & i\end{bmatrix} = 2i\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = 2i\hat{z}$

$[\hat{Y},\hat{z}] = \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} - \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} = \begin{bmatrix}0 & i\\ i & 0\end{bmatrix} - \begin{bmatrix}0 & -i\\ -i & 0\end{bmatrix} = 2i\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = 2i\hat{X}$

$[\hat{z};\hat{x}] = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} - \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} - \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} = 2i\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} = 2i\hat{Y}.$

$[\hat{X};\hat{X}] = \hat{X}\hat{X} - \hat{X}\hat{X} = \hat{X}^2 - \hat{X}^2 = 0$

$[\hat{Y};\hat{Y}] = \hat{Y}\hat{Y} - \hat{Y}\hat{Y} = \hat{Y}^2 - \hat{Y}^2 = 0$

Exercice 13:

$\frac{\hat{H}}{\hbar} = -\omega\frac{\vec{z}}{2}$ $\quad M = \begin{bmatrix}0 & -\omega & 0\\ \omega & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$

$\langle \dot{X} \rangle = -\omega \cdot \langle Y \rangle$

$\langle \dot{Y} \rangle = \omega \cdot \langle X \rangle$

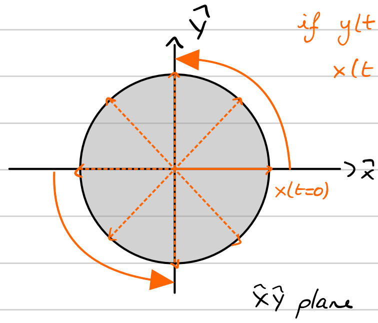
$\langle \dot{Z} \rangle = 0 \implies \frac{d}{dt}\langle Z \rangle = 0 \iff \langle Z \rangle = z(t=0) \iff \boxed{z(t) = z(t=0)}$ $\quad z(t)$ is constant

Exercice 14:

$$\begin{cases} \langle \dot{X} \rangle = -\omega \langle Y \rangle & \iff \langle Y \rangle = -\frac{1}{\omega}\langle \dot{X} \rangle \implies \langle \dot{Y} \rangle = -\frac{1}{\omega}\langle \ddot{X} \rangle \\ \langle \dot{Y} \rangle = \omega \langle X \rangle & \iff \langle X \rangle = \frac{1}{\omega}\langle \dot{Y} \rangle \implies \langle \dot{X} \rangle = \frac{1}{\omega}\langle \ddot{Y} \rangle \end{cases}$$

$$\begin{cases} \langle \ddot{Y} \rangle + \omega^2 \langle Y \rangle = 0 \\ \langle \ddot{X} \rangle + \omega^2 \langle X \rangle = 0 \end{cases} \implies \begin{cases} \langle Y \rangle = y(t=0)\cos(\omega t) + C\sin(\omega t). \\ \langle X \rangle = x(t=0)\cos(\omega t) + K\sin(\omega t) \end{cases}$$

if $y(t=0)=0$ and $x(t=0)=1$



$\langle \dot{Y} \rangle(0) = C\omega = \omega \langle X \rangle(0) \implies C = x(t=0)$

$\langle \dot{X} \rangle(0) = K\omega = -\omega \langle Y \rangle(0) \implies K = -y(t=0)$

Therefore: $\boxed{\begin{aligned} x(t) = \langle X \rangle &= x(t=0)\cos(\omega t) - y(t=0)\sin(\omega t) \\ y(t) = \langle Y \rangle &= x(t=0)\sin(\omega t) + y(t=0)\cos(\omega t) \end{aligned}}$ These are the solution of motion for the harmonic oscillator.

It is a counter clockwise rotation in the xy plane on a circle of radius $\sqrt{x(t=0)^2 + y(t=0)^2}$.

Exercice 15:

$$\hat{M} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \quad \omega_x = \omega_y = \omega_z = 1 \implies \hat{M} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

Thanks to this code we are able to determine the analytical solutions of $\vec{r} = e^{Mt}\vec{r_0}$ with t as a variable and the initial conditions are given by x_0, y_0, z_0.

```python
import numpy as np
import matplotlib.pyplot as plt
import sympy as sp

# Here we define the symbols and the value of omega
t = sp.symbols('t')
x0, y0, z0 = sp.symbols('x0 y0 z0')
omega=1

# We then define the matrix M
M = sp.Matrix([[0, -omega, omega],
               [omega, 0, -omega],
               [-omega, omega, 0]])

# Exponential of the matrix M times the time variable
exp_M_t = sp.exp(M * t)
exp_M_t_simplified = exp_M_t.simplify()

# Definition of initial conditions as variable
initial_conditions = sp.Matrix([x0, y0, z0])

# Analytic solution of the system
result_vector = exp_M_t_simplified * initial_conditions

# Extraction of x(t), y(t), z(t)
x_t, y_t, z_t = result_vector[0].simplify(), result_vector[1].simplify(), result_vector[2].simplify()
```

```python
# define a arbitrary Hamiltonian in 2-D Hilbert space
omega_x = 1
omega_y = 1
omega_z = 1
H = -omega_x * sigmax()/2 - omega_y * sigmay()/2 - omega_z * sigmaz()/2

# diagonalization of Hamiltonian
eigvals, eigstates = H.eigenstates()
print("Eigenvalues:", eigvals)
freq_rotate = (eigvals[1] - eigvals[0]) / (2 * np.pi)  # frequency of rotation scale from rad*Hz to Hz
print("Frequency:", freq_rotate)
period = 1 / freq_rotate
print("Period:", period)

time_list = np.linspace(0, 2 * np.pi, 101)  # [2\pi / omega_z] time points normalized by qubit freq

# definition of the initial state (|0> state)
psi0 = basis(2, 0)

# solve the Schrodinger equation
result = mesolve(H=H, rho0=psi0, tlist=time_list, e_ops=[sigmax(), sigmay(), sigmaz()])

# plot the time evolution of bloch vector
exp_x = result.expect[0]
exp_y = result.expect[1]
exp_z = result.expect[2]

plt.plot(time_list, exp_x, label=r"$\langle\hat{X}\rangle$")
plt.plot(time_list, exp_y, label=r"$\langle\hat{Y}\rangle$")
plt.plot(time_list, exp_z, label=r"$\langle\hat{Z}\rangle$")
```

```python
# plot the vertical line at the period of the rotation
plt.vlines(period, np.min(result.expect), np.max(result.expect), linestyles="dashed", colors="red")

plt.legend(loc=0)
plt.xlabel(r"Time [1/$f_q$]")
plt.ylabel("Expectation value")

# plot on Bloch sphere
b = Bloch()

# Set the desired perspective on the Bloch sphere
b.view = [30, 30]  # Change the numbers to set the azimuth and elevation angles

b.make_sphere()

# plot the time trace of the state
b.add_points(result.expect, meth="l", colors="red")

# plot the rotation angle (eigenstates of the Hamiltonian)
b.add_states(eigstates)
vec = [1,1,1]/np.sqrt(2)
b.add_vectors(vec)

b.render()
b.show()
```

```
Eigenvalues: [-0.8660254  0.8660254]
Frequency: 0.27566444771089604
Period: 3.6275987284684357
```

```python
# Printing exp(M*t)
exp_M_t_simplified
```

$$\begin{bmatrix} \frac{2\cos(\sqrt{3}t)}{3} + \frac{1}{3} & \frac{(1+\sqrt{3}i)^2 e^{\sqrt{3}it}}{12} + \frac{1}{3} + \frac{(1-\sqrt{3}i)^2 e^{-\sqrt{3}it}}{12} & \frac{((-(1+\sqrt{3}i)e^{\sqrt{3}it}+2)e^{\sqrt{3}it}-1+\sqrt{3}i)e^{-\sqrt{3}it}}{6} \\ \frac{(1-\sqrt{3}i)^2 e^{\sqrt{3}it}}{12} + \frac{1}{3} + \frac{(1+\sqrt{3}i)^2 e^{-\sqrt{3}it}}{12} & \frac{2\cos(\sqrt{3}t)}{3} + \frac{1}{3} & \frac{(((-1+\sqrt{3}i)e^{\sqrt{3}it}+2)e^{\sqrt{3}it}-1-\sqrt{3}i)e^{-\sqrt{3}it}}{6} \\ \frac{(((-1+\sqrt{3}i)e^{\sqrt{3}it}+2)e^{\sqrt{3}it}-1-\sqrt{3}i)e^{-\sqrt{3}it}}{6} & \frac{((-(1+\sqrt{3}i)e^{\sqrt{3}it}+2)e^{\sqrt{3}it}-1+\sqrt{3}i)e^{-\sqrt{3}it}}{6} & \frac{2\cos(\sqrt{3}t)}{3} + \frac{1}{3} \end{bmatrix}$$

```python
# Printing x(t)
x_t
```

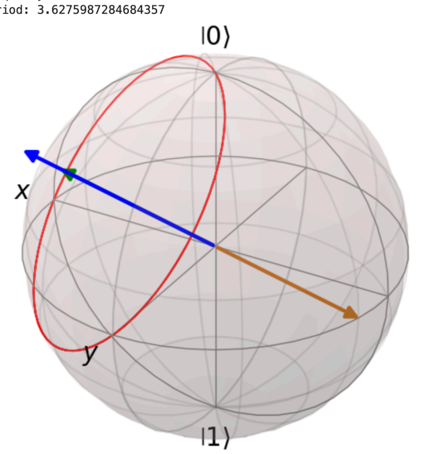
$$\frac{\left(4x_0\left(2\cos\left(\sqrt{3}t\right)+1\right)e^{\sqrt{3}it} + y_0\left(\left(\left(1+\sqrt{3}i\right)^2 e^{\sqrt{3}it}+4\right)e^{\sqrt{3}it} + \left(1-\sqrt{3}i\right)^2\right) - 2z_0\left(\left(\left(1+\sqrt{3}i\right)e^{\sqrt{3}it}-2\right)e^{\sqrt{3}it}+1-\sqrt{3}i\right)\right)e^{-\sqrt{3}it}}{12}$$

```python
# Printing y(t)
y_t
```

$$\frac{\left(x_0\left(\left(\left(1-\sqrt{3}i\right)^2 e^{\sqrt{3}it}+4\right)e^{\sqrt{3}it} + \left(1+\sqrt{3}i\right)^2\right) + 4y_0\left(2\cos\left(\sqrt{3}t\right)+1\right)e^{\sqrt{3}it} - 2z_0\left(\left(\left(1-\sqrt{3}i\right)e^{\sqrt{3}it}-2\right)e^{\sqrt{3}it}+1+\sqrt{3}i\right)\right)e^{-\sqrt{3}it}}{12}$$

```python
# Printing z(t)
z_t
```

$$\frac{\left(-x_0\left(\left(\left(1-\sqrt{3}i\right)e^{\sqrt{3}it}-2\right)e^{\sqrt{3}it}+1+\sqrt{3}i\right) - y_0\left(\left(\left(1+\sqrt{3}i\right)e^{\sqrt{3}it}-2\right)e^{\sqrt{3}it}+1-\sqrt{3}i\right) + 2z_0\left(2\cos\left(\sqrt{3}t\right)+1\right)e^{\sqrt{3}it}\right)e^{-\sqrt{3}it}}{6}$$



→ vector [1;1;1].

Exercice 16:

$\frac{\hat{H}}{\hbar} = -\omega \hat{Z}/2 \implies$ rotation of ωt around $\hat{z}$-axis.

$|\Psi_0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle.$

$$|\Psi(t)\rangle = e^{-i\frac{\hat{H}}{\hbar}t}|\Psi_0\rangle = e^{i\frac{\omega t}{2}\hat{Z}}|\Psi_0\rangle = e^{i\frac{\omega t}{2}}\begin{bmatrix} 1 & 0 \\ 0 & e^{-i\omega t}\end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2}\end{bmatrix} = e^{i\frac{\omega t}{2}}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}e^{-i\omega t}\end{bmatrix}$$

Since the global phase is irrelevant: $|\Psi(t)\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}e^{-i\omega t}|1\rangle.$

we have $\theta = \frac{\pi}{2}$ & $\phi = -\omega t$ on the Bloch sphere. It is therefore a clockwise rotation in $\hat{x}\hat{y}$-plane of angle ωt around $\hat{z}$-axis with initial condition as $|+\rangle$ state.

Exercice 17:

$$\begin{aligned} U|\Psi(t)\rangle &= e^{-i\frac{\omega}{2}\hat{Z}}|\Psi(t)\rangle \\ &= e^{-i\frac{\omega}{2}\hat{Z}}\left[\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}e^{-i\omega t}|1\rangle\right] \\ &= e^{-i\frac{\omega}{2}}\begin{bmatrix} 1 & 0 \\ 0 & e^{i\omega}\end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}e^{-i\omega t}\end{bmatrix} \\ &= e^{-i\frac{\omega}{2}}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2}\,e^{-i(\omega t-\omega)}\end{bmatrix} \end{aligned}$$

Since the global phase is irrelevant we have $U|\Psi(t)\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}e^{-i(\omega t-\omega)}|1\rangle$

$\tilde{H} = U\hat{H}U^\dagger + i\hbar\dot{U}U^\dagger$ since U is time independant, we have $\tilde{H} = U\hat{H}U^\dagger$

$$\frac{\tilde{H}}{\hbar} = U\frac{\hat{H}}{\hbar}U^\dagger = \cancel{e^{-i\frac{\omega}{2}}}\begin{bmatrix} 1 & 0 \\ 0 & e^{i\omega}\end{bmatrix} - \frac{\omega}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1\end{bmatrix}\cancel{e^{i\frac{\omega}{2}}}\begin{bmatrix} 1 & 0 \\ 0 & e^{-i\omega}\end{bmatrix} = -\frac{\omega}{2}\begin{bmatrix} 1 & 0 \\ 0 & e^{i\omega}\end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -e^{-i\omega}\end{bmatrix}$$

$$\boxed{\frac{\tilde{H}}{\hbar} = -\omega\hat{Z}/2.} \qquad = -\frac{\omega}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1\end{bmatrix}$$

The new hamiltonian is the same.

Exercice 18:

rotating frame: $\frac{\tilde{H}}{\hbar} = U\frac{H}{\hbar}U^\dagger + i\dot{U}U^\dagger$. $\qquad \frac{d}{dt}(U|\Psi\rangle) = -\frac{i}{\hbar}\tilde{H}(U|\Psi\rangle) \iff \frac{d}{dt}|\tilde{\Psi}\rangle = -\frac{i}{\hbar}\tilde{H}|\tilde{\Psi}\rangle$

$$U\frac{\hat{H}}{\hbar}U^\dagger = \cancel{e^{\frac{-i\omega t}{2}}}\begin{bmatrix} 1 & 0 \\ 0 & e^{i\omega t}\end{bmatrix}\left[-\frac{\omega_0}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1\end{bmatrix}\right]\cancel{e^{\frac{i\omega t}{2}}}\begin{bmatrix} 1 & 0 \\ 0 & e^{-i\omega t}\end{bmatrix} = -\frac{\omega_0}{2}\begin{bmatrix} 1 & 0 \\ 0 & e^{i\omega t}\end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -e^{-i\omega t}\end{bmatrix} = -\frac{\omega_0}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1\end{bmatrix} = -\frac{\omega_0}{2}\hat{Z}$$

$$i\dot{U}U^\dagger = i\begin{bmatrix} -i\frac{\omega}{2}e^{\frac{-i\omega t}{2}} & 0 \\ 0 & i\frac{\omega}{2}e^{i\omega t}\end{bmatrix}e^{\frac{i\omega t}{2}}\begin{bmatrix} 1 & 0 \\ 0 & e^{-i\omega t}\end{bmatrix} = i\left(-i\frac{\omega}{2}\right)\cancel{e^{\frac{-i\omega t}{2}}}\cdot\cancel{e^{\frac{i\omega t}{2}}}\begin{bmatrix} 1 & 0 \\ 0 & -e^{i\frac{\omega t}{2}}\end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^{-i\frac{\omega t}{2}}\end{bmatrix}$$

$$= \frac{\omega}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1\end{bmatrix} = \frac{\omega}{2}\hat{Z}.$$

Then $\boxed{\frac{\tilde{H}}{\hbar} = [\omega - \omega_0]\,\tilde{Z}/2.}$

To simplify the evolution, choosing ω at resonance i.e. $\omega = \omega_0$, would be a judicious choice since we would align the rotating frame with the natural rotation of the system.

Exercice 19:

$$|\psi(t)\rangle = e^{-i\frac{\tilde{H}}{\hbar}t}|\psi_0\rangle = e^{+i\frac{\omega}{2}\tilde{Y}t}|\psi_0\rangle \qquad |\psi_0\rangle = |0\rangle$$

$$|\psi(t)\rangle = \begin{bmatrix} \cos(\frac{\omega}{2}t) & \sin(\frac{\omega}{2}t) \\ -\sin(\frac{\omega}{2}t) & \cos(\frac{\omega}{2}t) \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(\frac{\omega}{2}t) \\ -\sin(\frac{\omega}{2}t) \end{bmatrix}$$

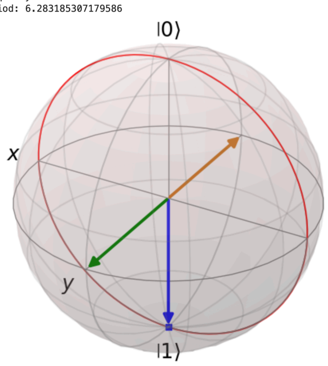
Since $\frac{\theta}{2} = \frac{\omega}{2}t$ is increasing counter-clockwise in ZX-plane, we need to be carefull when solving the equations to find t
$\theta = \omega t$.

If we want $|\psi(t)\rangle = |1\rangle$: $\begin{cases} \cos(\frac{\theta}{2}) = 0 \\ \sin(\frac{\theta}{2})e^{i\phi} = 1 \end{cases} \Rightarrow \omega t = \pi \Rightarrow t = \frac{\pi}{\omega}$ we set $\omega = 0$ at this time

If we want $|\psi(t)\rangle = |+\rangle$: $\begin{cases} \cos(\frac{\theta}{2}) = 1/\sqrt{2} \\ \sin(\frac{\theta}{2})e^{i\phi} = 1/\sqrt{2} \end{cases} \Rightarrow \omega t = \frac{3\pi}{2} \Rightarrow t = \frac{3\pi}{2\omega}$. we set $\omega = 0$ at this time

If we want $|\psi(t)\rangle = |-\rangle$: $\begin{cases} \cos(\frac{\theta}{2}) = 1/\sqrt{2} \\ \sin(\frac{\theta}{2})e^{i\phi} = -1/\sqrt{2} \end{cases}$ $\omega t = \frac{\pi}{2} \Rightarrow t = \frac{\pi}{2\omega}$ we set $\omega = 0$ at this time

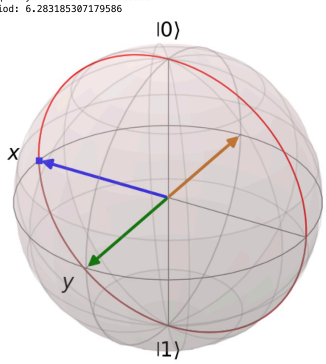
```
transformation(wx=0, wy=1, wz=0, psi0=basis(2, 0),basis_observed='z',value_observed=-1)
Eigenvalues: [-0.5  0.5]
Frequency: 0.15915494309189535
Period: 6.283185307179586
```




```
t_target = 3.1415926535897936
Expectation value for X at the time t = t_target: 1.1411089647160195e-06
Expectation value for Y at the time t = t_target: 0.0
Expectation value for Z at the time t = t_target: -0.999999999999349
```

$= \pi/\omega$

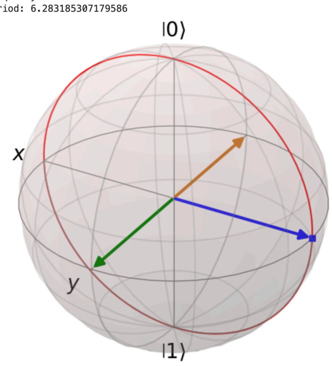
```
transformation(wx=0, wy=1, wz=0, psi0=basis(2, 0),basis_observed='x',value_observed=1)
Eigenvalues: [-0.5  0.5]
Frequency: 0.15915494309189535
Period: 6.283185307179586
```




```
t_target = 4.71238898038469
Expectation value for X at the time t = t_target: 0.9999999999960397
Expectation value for Y at the time t = t_target: 0.0
Expectation value for Z at the time t = t_target: 2.814257600070036e-06
```

$= 3\pi/(2\omega)$

```
transformation(wx=0, wy=1, wz=0, psi0=basis(2, 0),basis_observed='x',value_observed=-1)
Eigenvalues: [-0.5  0.5]
Frequency: 0.15915494309189535
Period: 6.283185307179586
```




```
t_target = 1.5707963267948968
Expectation value for X at the time t = t_target: -0.999999999999962
Expectation value for Y at the time t = t_target: 0.0
Expectation value for Z at the time t = t_target: -2.767894012878358e-07
```

$= \pi/(2\omega)$

In red we can see the evolution of the vector state on the Bloch sphere which is the expected value measured along the different axis. We can see it's a rotation around Y axis in the plane XZ with period 2π/wy and oscillation frequency wy where wy=1 here. The blue points correspond to the expected value at time t_target. We can see that the expected value for t_target=π/wy corresponds to |1>, t_target=3π/(2wy) corresponds to |+> and t_target=π/(2wy) corresponds to |-> as we predicted by calculus.

Exercice 20:

$$|\psi(t)\rangle = e^{-i\frac{\tilde{H}}{\hbar}t}|\psi_0\rangle = e^{+i\frac{\omega}{2}\tilde{X}t}|0\rangle$$

$$|\psi(t)\rangle = \begin{bmatrix} \cos(\frac{\omega}{2}t) & i\sin(\frac{\omega}{2}t) \\ i\sin(\frac{\omega}{2}t) & \cos(\frac{\omega}{2}t) \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(\frac{\omega}{2}t) \\ i\sin(\frac{\omega}{2}t) \end{bmatrix} = \begin{bmatrix} \cos(\frac{\omega}{2}t) \\ e^{i\frac{\pi}{2}}\sin(\frac{\omega}{2}t) \end{bmatrix}$$

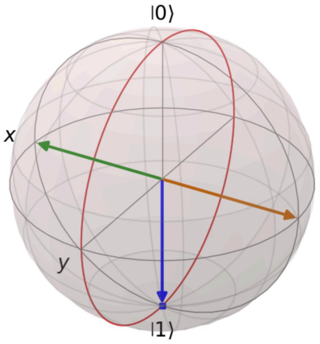
$$\frac{\theta}{2} = \frac{\omega}{2}t \Rightarrow \theta = \omega t.$$

If we want $|1\rangle$: $\begin{cases} \cos(\frac{\theta}{2}) = 0 \\ \sin(\frac{\theta}{2})e^{i\phi} = 1 \end{cases} \Rightarrow \theta = \omega t = \pi \Rightarrow t = \frac{\pi}{\omega}$ we set $\omega = 0$ at this time

If we want $|i\rangle$ $\begin{cases} \cos(\frac{\theta}{2}) = \frac{1}{\sqrt{2}} \\ \sin(\frac{\theta}{2})e^{i\phi} = \frac{1}{\sqrt{2}} i \end{cases}$ $\omega t = \frac{\pi}{2} \Rightarrow t = \frac{\pi}{2\omega}$ we set $\omega = 0$ at this time.

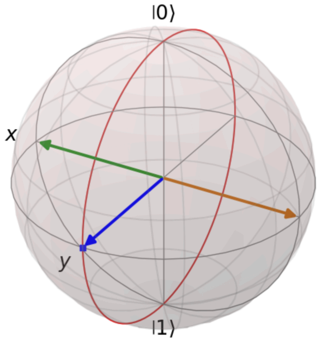
If we want $|-i\rangle$ $\begin{cases} \cos(\frac{\theta}{2}) = \frac{1}{\sqrt{2}} \\ \sin(\frac{\theta}{2})e^{i\phi} = -\frac{1}{\sqrt{2}} i \end{cases}$ $\omega t = \frac{3\pi}{2} \Rightarrow t = \frac{3\pi}{2\omega}$ we set $\omega = 0$ at this time.

```
transformation(wx=1, wy=0, wz=0, psi0=basis(2, 0),basis_observed='z',value_observed=-1)
Eigenvalues: [-0.5  0.5]
Frequency: 0.15915494309189535
Period: 6.283185307179586
```



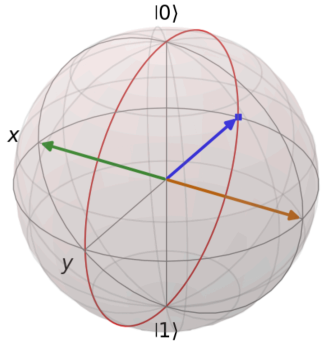

```
t_target = 3.1415926535897936
Expectation value for X at the time t = t_target: 0.0
Expectation value for Y at the time t = t_target: -1.1411089647160195e-06
Expectation value for Z at the time t = t_target: -0.999999999999349
```
$= \pi/\omega$

```
transformation(wx=1, wy=0, wz=0, psi0=basis(2, 0),basis_observed='y',value_observed=1)
Eigenvalues: [-0.5  0.5]
Frequency: 0.15915494309189535
Period: 6.283185307179586
```




```
t_target = 1.5707963267948968
Expectation value for X at the time t = t_target: 0.0
Expectation value for Y at the time t = t_target: 0.999999999999962
Expectation value for Z at the time t = t_target: -2.767894012878358e-07
```
$= \pi/(2\omega)$

```
transformation(wx=1, wy=0, wz=0, psi0=basis(2, 0),basis_observed='y',value_observed=-1)
Eigenvalues: [-0.5  0.5]
Frequency: 0.15915494309189535
Period: 6.283185307179586
```




```
t_target = 4.71238898038469
Expectation value for X at the time t = t_target: 0.0
Expectation value for Y at the time t = t_target: -0.9999999999960397
Expectation value for Z at the time t = t_target: 2.814257600070036e-06
```
$= 3\pi/(2\omega)$

In red we can see the evolution of the vector state on the Bloch sphere which is the expected value measured along the different axis. We can see it's a rotation around X axis in the plane YZ with period 2π/wy and oscillation frequency wy where wy=1 here. The blue points correspond to the expected value at time t_target. We can see that the expected value for t_target=π/wy corresponds to |1>, t_target=π/(2wy) corresponds to |-> and t_target=3π/(2wy) corresponds to |-i> as we predicted by calculus.

Exercice 21:

$\frac{H}{\hbar} = -\omega \vec{X}/2 - \omega \vec{Z}/2.$

eigenvalues: $\lambda_1 = \frac{\sqrt{\omega_x^2 + \omega_x^2}}{2} = \frac{1}{2}\sqrt{\omega^2 + \omega^2} = \frac{\sqrt{2}}{2}\omega \Rightarrow \begin{cases} \lambda_1 = -\frac{\omega}{\sqrt{2}} \\ \lambda_2 = +\frac{\omega}{\sqrt{2}} \end{cases}$ $\omega_q = \lambda_1 - \lambda_2 = \sqrt{2}\,\omega$

eigenvectors: are aligned with: $(\omega_x ; \omega_y ; \omega_z) = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2} \begin{bmatrix} \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta) \end{bmatrix}$ $\omega_y = 0 \Rightarrow \phi = 0.$

here we have : $(\omega ; 0 ; \omega) = \sqrt{2}\,\omega \begin{bmatrix} \sin(\theta) \\ 0 \\ \cos(\theta) \end{bmatrix} \Rightarrow \begin{cases} \sin(\theta) = \frac{1}{\sqrt{2}} \\ \cos(\theta) = \frac{1}{\sqrt{2}} \end{cases}$ $\theta = \frac{\pi}{4}$

$|\lambda_1\rangle = \begin{bmatrix} \cos(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2})e^{i\phi} \end{bmatrix} = \begin{bmatrix} \cos(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) \end{bmatrix} = \begin{bmatrix} \cos(\pi/8) \\ \sin(\pi/8) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2+\sqrt{2}}}{2} \\ \frac{\sqrt{2-\sqrt{2}}}{2} \end{bmatrix}$

$|\lambda_2\rangle = \begin{bmatrix} \sin(\frac{\theta}{2}) \\ -\cos(\frac{\theta}{2})e^{i\phi} \end{bmatrix} = \begin{bmatrix} \sin(\frac{\theta}{2}) \\ -\cos(\frac{\theta}{2}) \end{bmatrix} = \begin{bmatrix} \sin(\pi/8) \\ -\cos(\pi/8) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2-\sqrt{2}}}{2} \\ -\frac{\sqrt{2+\sqrt{2}}}{2} \end{bmatrix}$

This is an Hadamard gate.
Therefore the axis of rotation is at $\frac{\pi}{4}$ on XZ-plane.

We solve Schroedinger equation $|\psi(t)\rangle = e^{-i\frac{H}{\hbar}t}|\psi_0\rangle$

$U = e^{i(\frac{\omega}{2}\vec{X} + \frac{\omega}{2}\vec{Z})t} = f(\frac{H}{\hbar}) = e^{-i\lambda_1 t}|\lambda_1\rangle\langle\lambda_1| + e^{-i\lambda_2 t}|\lambda_2\rangle\langle\lambda_2| = e^{i\frac{\omega}{\sqrt{2}}t}\begin{bmatrix} \cos^2(\pi/8) & \sin(\pi/8)\cos(\pi/8) \\ \sin(\pi/8)\cos(\pi/8) & \sin^2(\pi/8) \end{bmatrix} + e^{-i\frac{\omega}{\sqrt{2}}t}\begin{bmatrix} \sin^2(\pi/8) & -\sin(\pi/8)\cos(\pi/8) \\ -\sin(\pi/8)\cos(\pi/8) & \cos^2(\pi/8) \end{bmatrix}$

$f(x) = e^{-ixt}$

$$\mathcal{U}(t)=\begin{bmatrix} e^{i\frac{\omega}{\sqrt{2}}t}\cos^2(\pi/8)+e^{-i\frac{\omega}{\sqrt{2}}t}\sin^2(\pi/8) & \left(e^{i\frac{\omega}{\sqrt{2}}t}-e^{-i\frac{\omega}{\sqrt{2}}t}\right)\sin(\pi/8)\cos(\pi/8) \\ \left(e^{i\frac{\omega}{\sqrt{2}}t}-e^{-i\frac{\omega}{\sqrt{2}}t}\right)\sin(\pi/8)\cos(\pi/8) & e^{-i\frac{\omega}{\sqrt{2}}t}\cos^2(\pi/8)+e^{i\frac{\omega}{\sqrt{2}}t}\sin^2(\pi/8) \end{bmatrix}$$

$$\mathcal{U}(t)=\begin{bmatrix} e^{i\frac{\omega}{\sqrt{2}}t}\cos^2(\pi/8)+e^{-i\frac{\omega}{\sqrt{2}}t}\sin^2(\pi/8) & 2i\sin(\frac{\omega}{\sqrt{2}}t)\sin(\pi/8)\cos(\pi/8) \\ 2i\sin(\frac{\omega}{\sqrt{2}}t)\sin(\pi/8)\cos(\pi/8) & e^{-i\frac{\omega}{\sqrt{2}}t}\cos^2(\pi/8)+e^{i\frac{\omega}{\sqrt{2}}t}\sin^2(\pi/8) \end{bmatrix}$$

$$|\Psi(t)\rangle=e^{-i\frac{H}{\hbar}t}|\Psi_0\rangle=\mathcal{U}(t)|0\rangle=\begin{bmatrix} e^{i\frac{\omega}{\sqrt{2}}t}\cos^2(\pi/8)+e^{-i\frac{\omega}{\sqrt{2}}t}\sin^2(\pi/8) & 2i\sin(\frac{\omega}{\sqrt{2}}t)\sin(\pi/8)\cos(\pi/8) \\ 2i\sin(\frac{\omega}{\sqrt{2}}t)\sin(\pi/8)\cos(\pi/8) & e^{-i\frac{\omega}{\sqrt{2}}t}\cos^2(\pi/8)+e^{i\frac{\omega}{\sqrt{2}}t}\sin^2(\pi/8) \end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}$$

$$|\Psi(t)\rangle=\begin{bmatrix} e^{i\frac{\omega}{\sqrt{2}}t}\cos^2(\pi/8)+e^{-i\frac{\omega}{\sqrt{2}}t}\sin^2(\pi/8) \\ 2i\sin(\frac{\omega}{\sqrt{2}}t)\sin(\pi/8)\cos(\pi/8) \end{bmatrix}=\begin{bmatrix} e^{i\frac{\omega}{\sqrt{2}}t}\cdot\frac{2+\sqrt{2}}{4}+e^{-i\frac{\omega}{\sqrt{2}}t}\frac{2-\sqrt{2}}{4} \\ i\frac{\sqrt{2}}{2}\sin(\frac{\omega}{\sqrt{2}}t) \end{bmatrix}=\begin{bmatrix} \cos(\frac{\omega}{\sqrt{2}}t)+\frac{\sqrt{2}}{2}i\sin(\frac{\omega}{\sqrt{2}}t) \\ i\frac{\sqrt{2}}{2}\sin(\frac{\omega}{\sqrt{2}}t) \end{bmatrix}$$

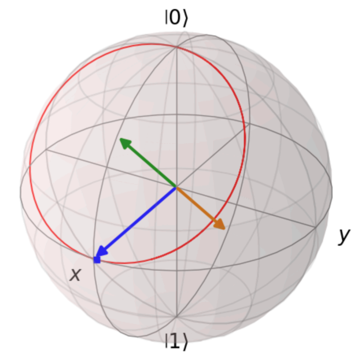
$|\Psi(t_{target})\rangle=|+\rangle$; for that we need $\cos(\frac{\omega}{\sqrt{2}}t)=0$ and $\sin(\frac{\omega}{\sqrt{2}}t)=1$ so $t_{target}=\frac{\pi}{\omega\sqrt{2}}$.

$$\mathcal{U}|0\rangle(t=\frac{\pi}{\omega\sqrt{2}})=\begin{bmatrix} i\,1/\sqrt{2} \\ i\,1/\sqrt{2}\end{bmatrix}=e^{i\pi/2}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}\equiv|+\rangle$$

By calculus we find that we reach $|+\rangle$ while $|\Psi_0\rangle=|0\rangle$ if we let propagates during time $\boxed{t=\frac{\pi}{\sqrt{2}\omega}}$.

Let us verify by computation with taking $\omega_x=\omega_z=1$:

```
transformation(wx=1, wy=0, wz=1, psi0=basis(2, 0),basis_observed='x',value_observed=1)
Eigenvalues: [-0.70710678  0.70710678]
Frequency: 0.22507907903927654
Period: 4.442882938158366
```




```
t_target = 2.199114857512855
Expectation value for X at the time t = t_target: 0.9997508048866227
Expectation value for Y at the time t = t_target: 0.02232187336102123
Expectation value for Z at the time t = t_target: 0.0002491951133772319
```

$\pi/(\sqrt{2}\omega)$

We get indeed $\omega_q=\sqrt{2}\omega=\sqrt{2}$ $(\omega=1)$ and $T_q=\frac{2\pi}{\omega_q}=\frac{2\pi}{\sqrt{2}\omega}=\frac{2\pi}{\sqrt{2}}=\sqrt{2}\pi\simeq4{,}44$

The red curve is the expectation value obtained for the solution of Schroedinger equation while measuring on the different axis.

We see that state $|+\rangle$ is reached (in blue) when $t_{target}=\frac{\pi}{\sqrt{2}\omega}=\frac{\pi}{\sqrt{2}}\simeq2{,}22$. for $\omega=1$. This corresponds to what we have predicted by calculus.

Exercice 22:

$$|\Psi(t)\rangle=e^{-i\frac{H}{\hbar}t}|\Psi_0\rangle=\mathcal{U}(t)|+\rangle=\begin{bmatrix} e^{i\frac{\omega}{\sqrt{2}}t}\cos^2(\pi/8)+e^{-i\frac{\omega}{\sqrt{2}}t}\sin^2(\pi/8) & 2i\sin(\frac{\omega}{\sqrt{2}}t)\sin(\pi/8)\cos(\pi/8) \\ 2i\sin(\frac{\omega}{\sqrt{2}}t)\sin(\pi/8)\cos(\pi/8) & e^{-i\frac{\omega}{\sqrt{2}}t}\cos^2(\pi/8)+e^{i\frac{\omega}{\sqrt{2}}t}\sin^2(\pi/8) \end{bmatrix}\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$$

$$|\Psi(t)\rangle=\begin{bmatrix} \cos(\frac{\omega}{\sqrt{2}}t)+\frac{\sqrt{2}}{2}i\sin(\frac{\omega}{\sqrt{2}}t) & i\frac{\sqrt{2}}{2}\sin(\frac{\omega}{\sqrt{2}}t) \\ i\frac{\sqrt{2}}{2}\sin(\frac{\omega}{\sqrt{2}}t) & \cos(\frac{\omega}{\sqrt{2}}t)-\frac{\sqrt{2}}{2}i\sin(\frac{\omega}{\sqrt{2}}t) \end{bmatrix}\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}=\begin{bmatrix} \frac{1}{\sqrt{2}}\cos(\frac{\omega}{\sqrt{2}}t)+\frac{i}{2}\sin(\frac{\omega}{\sqrt{2}}t)+\frac{i}{2}\sin(\frac{\omega}{\sqrt{2}}t) \\ \frac{1}{\sqrt{2}}\cos(\frac{\omega}{\sqrt{2}}t)+\frac{i}{2}\sin(\frac{\omega}{\sqrt{2}}t)-\frac{i}{2}\sin(\frac{\omega}{\sqrt{2}}t) \end{bmatrix}$$

$$|\psi(t_{target})\rangle = |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}}\cos(\frac{\omega}{\sqrt{2}}t) + \frac{i}{2}\sin(\frac{\omega}{\sqrt{2}}t) + \frac{i}{2}\sin(\frac{\omega}{\sqrt{2}}t) \\ \frac{1}{\sqrt{2}}\cos(\frac{\omega}{\sqrt{2}}t) + \frac{i}{2}\sin(\frac{\omega}{\sqrt{2}}t) - \frac{i}{2}\sin(\frac{\omega}{\sqrt{2}}t) \end{bmatrix}$$

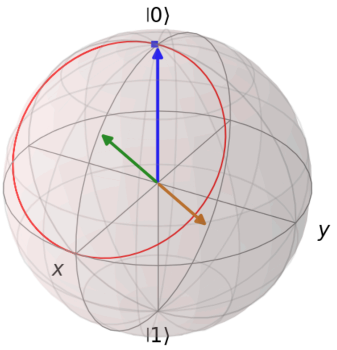
we need $\cos(\frac{\omega}{\sqrt{2}}t) = 0$ and $\sin(\frac{\omega}{\sqrt{2}}t) = 1$.
so again $t = \frac{\pi}{\sqrt{2}\omega}$

$$|\psi(t_{target})\rangle = e^{i\pi/2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \equiv |0\rangle$$ we need $\boxed{t = \frac{\pi}{\sqrt{2}\omega}}$ to go from $|+\rangle$ to $|0\rangle$

Let's check by computation:

```
transformation(wx=1, wy=0, wz=1, psi0=(basis(2,0)+(1+0j)*basis(2,1)).unit(),basis_observed='z',value_observed=1)
```

```
Eigenvalues: [-0.70710678  0.70710678]
Frequency: 0.22507907903927654
Period: 4.442882938158366
```




```
t_target = 2.199114857512855
```
$\pi/(\sqrt{2}\omega)$
```
Expectation value for X at the time t = t_target: 0.0002491907725547011
Expectation value for Y at the time t = t_target: -0.0223216789921l772
Expectation value for Z at the time t = t_target: 0.9997508092274456
```

We see that state $|+\rangle$ is reached (in blue) when have $t_{target} = \frac{\pi}{\sqrt{2}\omega} = \frac{\pi}{\sqrt{2}} \simeq 2{,}22$. for $\omega = 1$. This corresponds to what we have predicted by calculus.

Exercice 23:

$\omega_x = g \ll \omega_z$ for $t = \tau_1$

$$\lambda_1 = -\frac{\sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}}{2} = -\frac{\sqrt{g^2 + \omega_z^2}}{2}$$

$$\lambda_2 = +\frac{\sqrt{g^2 + \omega_z^2}}{2}$$

$$\omega_q = \lambda_2 - \lambda_1 = \sqrt{g^2 + \omega_z^2}$$

eigenvectors: are aligned with: $(\omega_x ; \omega_y ; \omega_z) = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}\begin{bmatrix} \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta) \end{bmatrix}$ $\omega_y = 0 \Rightarrow \phi = 0.$

here we have : $(g ; 0 ; \omega) = \sqrt{g^2 + \omega_z^2}\begin{bmatrix} \sin(\theta) \\ 0 \\ \cos(\theta) \end{bmatrix} \Leftrightarrow \begin{cases} \sin(\theta) = \frac{g}{\sqrt{g^2 + \omega_z^2}} \\ \cos(\theta) = \frac{\omega_z}{\sqrt{g^2 + \omega_z^2}} \end{cases}$

$$|\lambda_1\rangle = \begin{bmatrix} \cos(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2})e^{i\phi} \end{bmatrix} = \begin{bmatrix} \cos(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) \end{bmatrix} = \quad \Leftrightarrow \begin{cases} \theta = \arcsin\left(\frac{g}{\sqrt{g^2 + \omega_z^2}}\right) \\ \theta = \arccos\left(\frac{\omega_z}{\sqrt{g^2 + \omega_z^2}}\right) \end{cases}$$

$$|\lambda_2\rangle = \begin{bmatrix} \sin(\frac{\theta}{2}) \\ -\cos(\frac{\theta}{2})e^{i\phi} \end{bmatrix} = \begin{bmatrix} \sin(\frac{\theta}{2}) \\ -\cos(\frac{\theta}{2}) \end{bmatrix}$$

We solve Schroedinger equation $|\psi(t)\rangle = e^{-i\frac{H}{\hbar}t}|\psi_0\rangle$

$$|\psi(t)\rangle = e^{-i\frac{H}{\hbar}t}|\psi_0\rangle = e^{-i\lambda_1 t}|\lambda_1\rangle\langle\lambda_1|\psi_0\rangle + e^{-i\lambda_2 t}|\lambda_2\rangle\langle\lambda_2|\psi_0\rangle \qquad |\psi_0\rangle = |0\rangle.$$

$$= \cos(\tfrac{\theta}{2})e^{-i\lambda_1 t}|\lambda_1\rangle + \sin(\tfrac{\theta}{2})e^{-i\lambda_2 t}|\lambda_2\rangle.$$

$$|\psi(t)\rangle = \begin{bmatrix} e^{-i\lambda_1 t}\cos^2(\theta/2) \\ e^{-i\lambda_1 t}\cos(\theta/2)\sin(\theta/2) \end{bmatrix} + \begin{bmatrix} e^{-i\lambda_2 t}\sin^2(\theta/2) \\ -e^{-i\lambda_2 t}\cos(\theta/2)\sin(\theta/2) \end{bmatrix}$$

$$|\psi(t)\rangle = \begin{bmatrix} e^{-i\lambda_1 t}\cos^2(\theta/2) + e^{-i\lambda_2 t}\sin^2(\theta/2) \\ (e^{-i\lambda_1 t} - e^{-i\lambda_2 t})\cos(\theta/2)\sin(\theta/2) \end{bmatrix} \qquad \lambda_1 = -\lambda_2.$$

$$|\psi(t)\rangle = \begin{bmatrix} e^{-i\lambda_1 t}\cos^2(\theta/2) + e^{-i\lambda_2 t}\sin^2(\theta/2) \\ (e^{-i\lambda_1} - e^{i\lambda_1 t})\cos(\theta/2)\sin(\theta/2) \end{bmatrix}$$

$$|\psi(t)\rangle = \begin{bmatrix} e^{-i\lambda_1 t}\cos^2(\theta/2) + e^{i\lambda_1 t}\sin^2(\theta/2) \\ -2i\sin(\lambda_1 t)\cos(\theta/2)\sin(\theta/2) \end{bmatrix}$$

- $$\begin{aligned} e^{-i\lambda_1 t}\cos^2(\theta/2) + e^{i\lambda_1 t}\sin^2(\theta/2) &= \cos(\lambda_1 t)\cos^2(\theta/2) - i\sin(\lambda_1 t)\cos^2(\theta/2) + \cos(\lambda_1 t)\sin^2(\theta/2) + i\sin(\lambda_1 t)\sin^2(\theta/2) \\ &= \cos(\lambda_1 t) \quad - i\sin(\lambda_1 t)\left[\cos^2(\theta/2) - \sin^2(\theta/2)\right] \\ &= \cos(\lambda_1 t) - i\sin(\lambda_1 t)\cos(\theta) \\ &= \cos(\lambda_1 t) - i\sin(\lambda_1 t)\frac{\omega_z}{\sqrt{g^2+\omega_z^2}} \end{aligned} \qquad \theta = \arccos\left(\frac{\omega_z}{\sqrt{g^2+\omega_z^2}}\right)$$

- $$\begin{aligned} -2i\sin(\lambda_1 t)\cos(\theta/2)\sin(\theta/2) &= -i\sin(\lambda_1 t)\sin(\theta) \\ &= -i\frac{g}{\sqrt{g^2+\omega_z^2}}\sin(\lambda_1 t). \end{aligned} \qquad \theta = \arcsin\left(\frac{g}{\sqrt{g^2+\omega_z^2}}\right)$$

$$|\psi(t)\rangle = \begin{bmatrix} \cos(\lambda_1 t) - i\sin(\lambda_1 t)\frac{\omega_z}{\sqrt{g^2+\omega_z^2}} \\ -i\frac{g}{\sqrt{g^2+\omega_z^2}}\sin(\lambda_1 t). \end{bmatrix} \qquad \lambda_1 = -\frac{\sqrt{g^2+\omega_z^2}}{2}$$

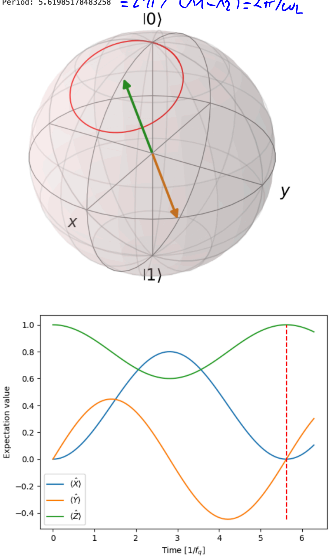
$$\omega_L = \lambda_2 - \lambda_1 = \sqrt{g^2+\omega_z^2}$$

$$|\psi(t)\rangle = \begin{bmatrix} \cos\left(\frac{\sqrt{g^2+\omega_z^2}}{2}t\right) + i\sin\left(\frac{\sqrt{g^2+\omega_z^2}}{2}.t\right)\frac{\omega_z}{\sqrt{g^2+\omega_z^2}} \\ i\frac{g}{\sqrt{g^2+\omega_z^2}}\sin\left(\frac{\sqrt{g^2+\omega_z^2}}{2}t\right) \end{bmatrix} \qquad |\psi(\tau_1)\rangle = \begin{bmatrix} \cos\left(\frac{\sqrt{g^2+\omega_z^2}}{2}\tau_1\right) + i\sin\left(\frac{\sqrt{g^2+\omega_z^2}}{2}.\tau_1\right)\frac{\omega_z}{\sqrt{g^2+\omega_z^2}} \\ i\frac{g}{\sqrt{g^2+\omega_z^2}}\sin\left(\frac{\sqrt{g^2+\omega_z^2}}{2}\tau_1\right) \end{bmatrix}$$

Let's compute for different values of $\omega_x = g \ll \omega_y$:

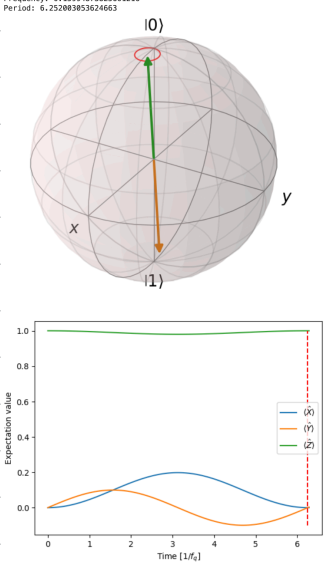
```
transformation(wx=0.5, wy=0, wz=1, psi0=basis(2, 0))
Eigenvalues: [-0.55901699  0.55901699]
Frequency: 0.1779406358542943
Period: 5.61985178483258
```

$= 2\pi/(\lambda_1-\lambda_2) = 2\pi/\omega_L$



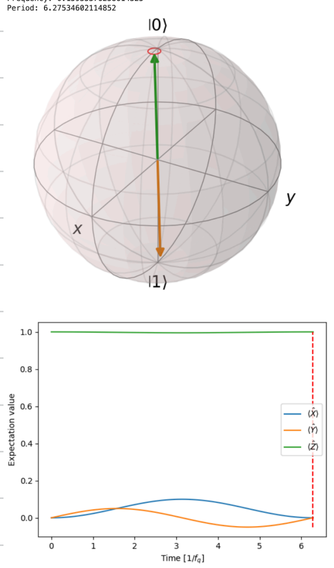
$\omega_x = 0{,}5 \quad \omega_z = 1$

```
transformation(wx=0.1, wy=0, wz=1, psi0=basis(2, 0))
Eigenvalues: [-0.50249378  0.50249378]
Frequency: 0.15994873825601216
Period: 6.252003053624663
```



$\omega_x = 0{,}1 \quad \omega_z = 1$

```
transformation(wx=0.05, wy=0, wz=1, psi0=basis(2, 0))
Eigenvalues: [-0.50062461  0.50062461]
Frequency: 0.15935376258614325
Period: 6.27534602114852
```



$\omega_x = 0{,}05 \quad \omega_z = 1.$

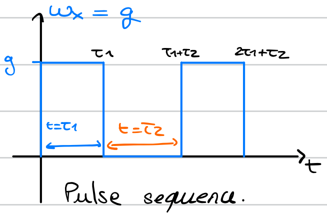
Here we can see that it is indeed a rotation defined around the eigenstates of the Hamiltonian, the oscillation frequence wq is defined by the difference of the eignevalues and therefore the period of rotation by 2π/wq. While decreasing the value of wx, the oscillation is almost fully around Z axis since the wz component is more "powerfull" and this can be seen with the computation of the expectation of the wave vector projected along the different axis with respect to the time t.

Exercice 24:

$\tau_1 = \frac{\pi}{\sqrt{\omega_z^2+g^2}}$ $\tau_2 = \frac{\pi}{\omega_z}$

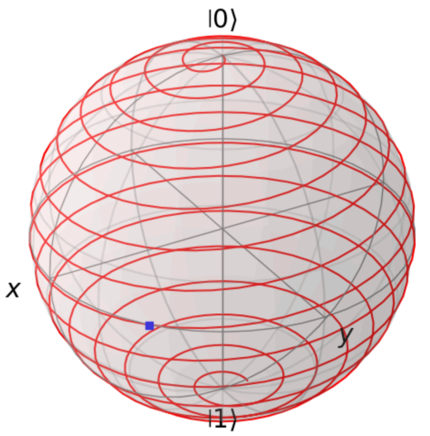
$\frac{\hat{H}}{\hbar} = -\omega_x \hat{X}/2 + -\omega_z \hat{Z}$

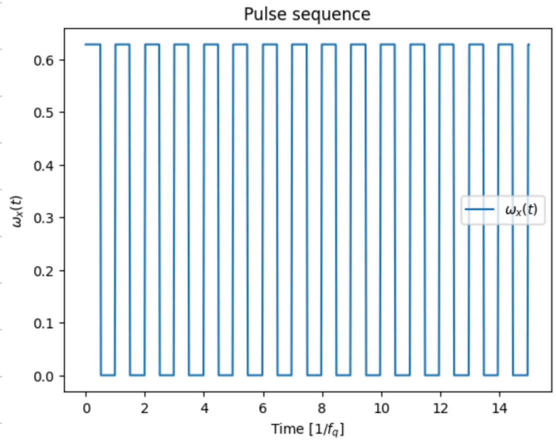
$$\omega_x = \begin{cases} g = \frac{\omega_z}{10} & n(\tau_1+\tau_2) \leq t \leq n(\tau_1+\tau_2)+\tau_1 \\ 0 & n(\tau_1+\tau_2)+\tau_1 \leq t \leq (n+1)(\tau_1+\tau_2) \end{cases}$$

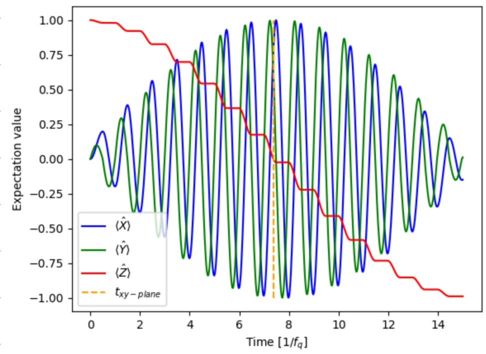



```
omega_z = 2*np.pi
time_list = np.linspace(0, 15, 701)
g = omega_z/10
tau1 = np.pi/np.sqrt(omega_z**2+g**2)
tau2 = np.pi/omega_z
```

```
Le temps pour atteindre le plan XY est :  7.371428571428572
```





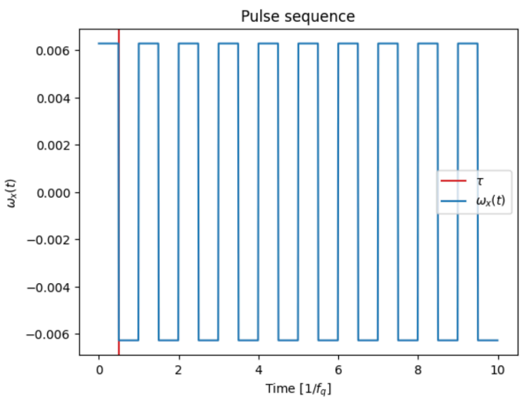


$t = 7{,}36$ for $\omega_z = 2\pi$.

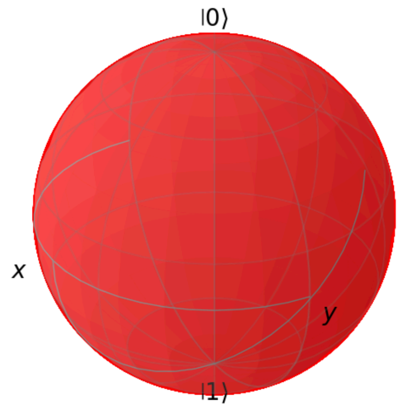
Exercice 25:

```
omega_z = 2*np.pi
time_list = np.linspace(0, 800, 80001)
g = omega_z/1000
tau = np.pi/omega_z
# Definition of the pulse shape (omega_x(t))
def omega_x(t):
    n=np.floor(t/tau)
    if n%2==0 :
        return g
    else:
        return -g
```

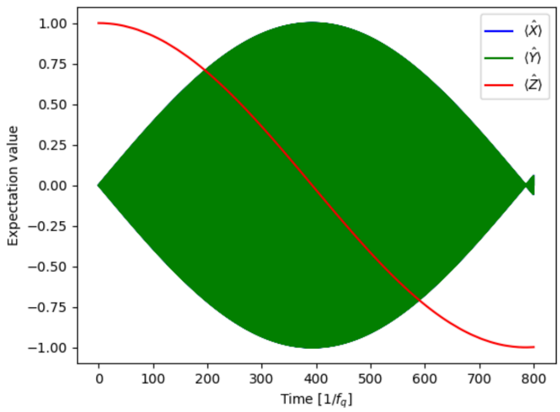
Definition of wz, tau and the pulse sequence of wx.



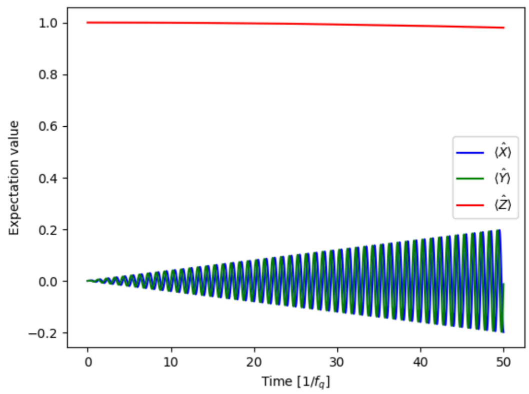
Pulse sequence of wx(t).



Result on the Bloch Sphere. Since the wx(t) is really small, the bloch sphere is fully covered of points because and the time for the expectation to drop is therefore notable.



Expectation value of all the projection along the different axis on the whole time list.



Reduced plot of the expectation value of all the projection along the different axis for only 5'000 points ot of 80'001 points of the time list.

We can notice strong oscillation for the expectation along X and Y, this is due to the pulse sequence along X which is almost irrelevant (wz/1000) compared to the wz of the Z component of the Hamiltonian. If we were able to zoom, we would see that for one sequence tau, we are decreasing the expected value along Z in the clockwise direction and for the next sequence we would decrease on the counterclockwise

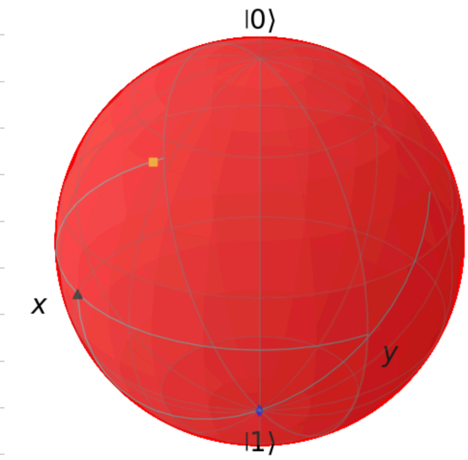
Exercice 26:

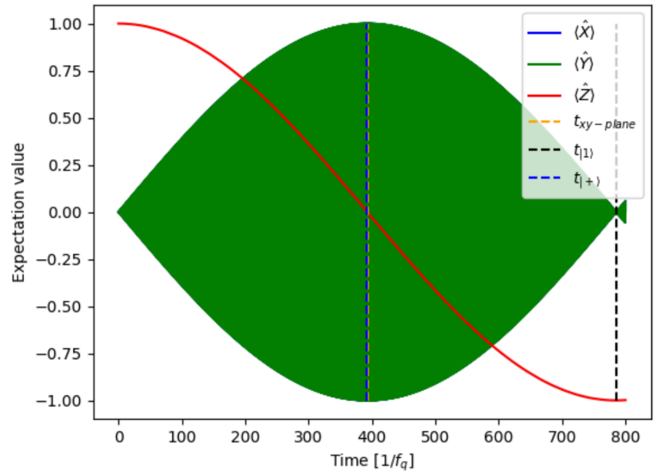
$|+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$ $\quad \langle \hat{X} \rangle = +1$

$|1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$ $\quad \langle \hat{Z} \rangle = 0.$

In black, we observe the time needed to reach the state |1>, in blue the time to reach |+>. Orange corresponds to the time needed to just reach XY-plane but it is not asked in this question. These points are reprsented on the Bloch sphere with their associated color. The corresponding time are written above the Bloch Sphere for the two requested state |1> (t=785.28) and |+> (t =392.5).

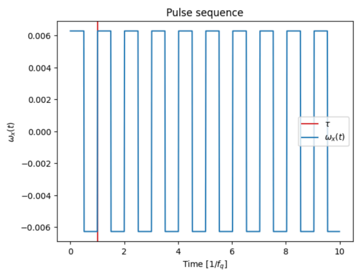
```
Time needed to reach XY plane :  392.74
Time needed to reach |1> state :  785.28
Time needed to reach |+> state :  392.5
```

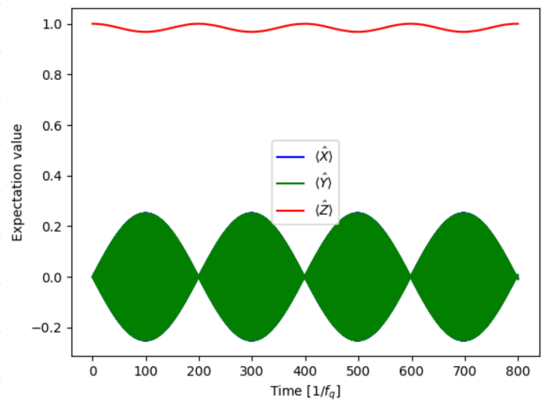


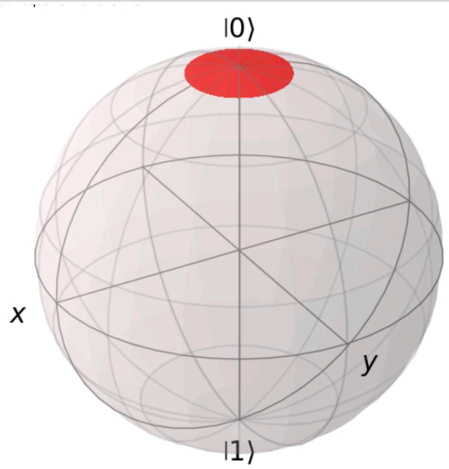


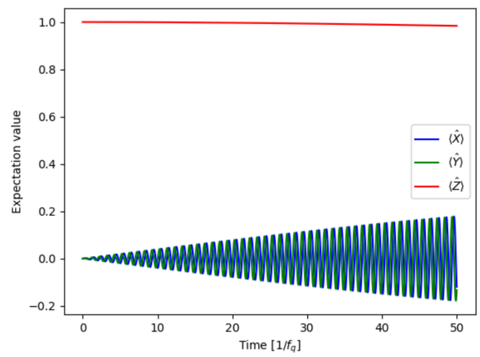
Exercice 27:

```python
omega_z = 2*np.pi
time_list = np.linspace(0, 800, 80001)
g = omega_z/1000
tau = 1.005*np.pi/omega_z
```









With a small variation in tau of 0.5%, we observe a significant change in the evolution of the expectation values compared to the previous results. The <Z> value remains close to a mean of 1, while <X> and<Y> exhibit oscillations with an amplitude slightly exceeding a mean value of 0.2. This indicates that even a minor adjustment in tau can lead to substantial differences in the qubit's dynamics, particularly in the oscillatory behavior of the <X> and <Y> components.

Exercice 28 :

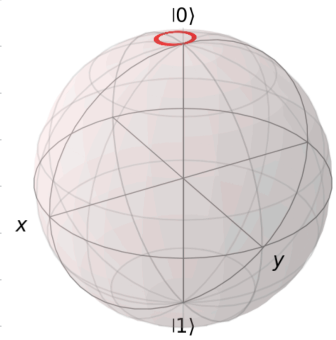
$$\frac{\tilde{H}}{\hbar} = U \frac{H}{\hbar} U^{\dagger} + i \dot{U} U^{\dagger}.$$

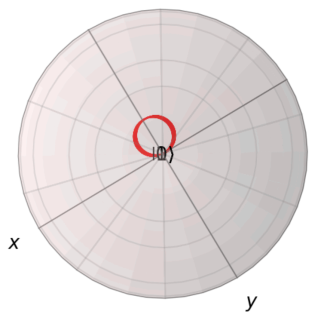
```python
import sympy as sp
g, omega_x, omega_y, omega_z, t = sp.symbols('g omega_x omega_z omega_z t')
I = sp.I
tau = sp.pi/omega_z

sigma_x = sp.Matrix([[0, 1], [1, 0]])
sigma_y = sp.Matrix([[0, -I], [I, 0]])
sigma_z = sp.Matrix([[1, 0], [0, -1]])
psi00 = sp.Matrix([[1], [0]])


H = -g * sigma_x/2 - 0 * omega_y * sigma_y/2 - omega_z * sigma_z/2
U = sp.exp(-I * omega_z * t * sigma_z/2)
U_dagger= sp.Matrix([[sp.exp(I*omega_z*t/2), 0], [0, sp.exp(-I*omega_z*t/2)]])
U_point = sp.Matrix([[-I*omega_z/2 *sp.exp(-I*omega_z*t/2), 0], [0, I*omega_z/2 *sp.exp(I*omega_z*t/2)]])

H_new = U @ H @ U_dagger + I * U_point @ U_dagger
U_new = sp.exp(-I * H_new * t)
eigenvalues = H.eigenvals()
eigenvectors = H.eigenvects()

# Afficher les résultats
print("Eigenvalues:")
print(eigenvalues)
for vect in eigenvectors:
    print(f"Eigenvectors: {vect}")

# Print result
H_new
U_new
```

$$\frac{\tilde{H}}{\hbar} = \begin{bmatrix} 0 & -\frac{g e^{-i\omega_z t}}{2} \\ -\frac{g e^{i\omega_z t}}{2} & 0 \end{bmatrix}$$

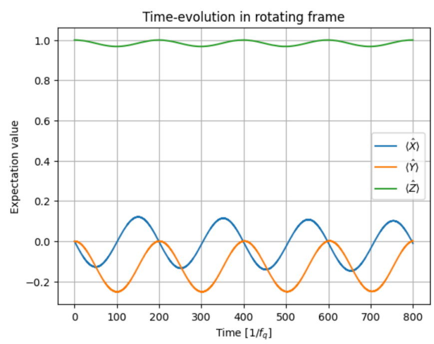
Here $g = \pm g = \pm \frac{\omega_z}{1000}$ that we have in previous question. Here the solution of Schrödinger equation for $g = \pm g$

$$\tilde{U} = e^{-i\frac{\tilde{H}}{\hbar}t} = \begin{bmatrix} \frac{e^{\frac{igt}{2}}}{2} + \frac{e^{-\frac{igt}{2}}}{2} & \frac{e^{\frac{igt}{2}} e^{-i\omega_z t}}{2} - \frac{e^{-\frac{igt}{2}} e^{-i\omega_z t}}{2} \\ \frac{e^{\frac{igt}{2}} e^{i\omega_z t}}{2} - \frac{e^{-\frac{igt}{2}} e^{i\omega_z t}}{2} & \frac{e^{\frac{igt}{2}}}{2} + \frac{e^{-\frac{igt}{2}}}{2} \end{bmatrix} = \begin{bmatrix} \cos(\frac{g}{2}t) & i e^{-i\omega_z t} \sin(\frac{g}{2}t) \\ i e^{i\omega_z t} \sin(\frac{g}{2}t) & \cos(\frac{g}{2}t) \end{bmatrix}$$



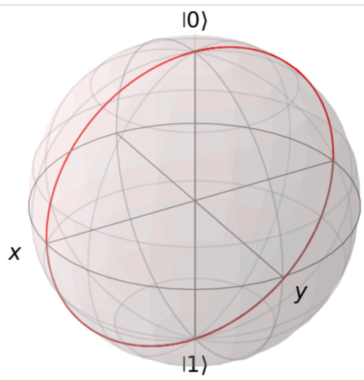


$\tau = 1{,}005\,\tau$

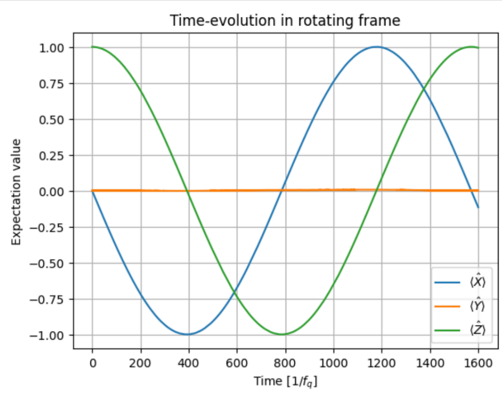


This value have been obtained with $\tau' = 1{,}005\,\tau$. Notice we set the angular frequeny of the rotating frame at ω_z which is the Larmor frequency. But, since $\tau' = 1{,}005 \cdot \frac{\pi}{\omega_z}$, we won't see a pure rotation along $\hat{X}$ or $\hat{Y}$ but our expectations will be slightly shifted as we can see above.

NOW: we come back to $\tau = \frac{\pi}{\omega_z}$ with $\omega_z = 2\pi$ and our rotating frame is set also at $\omega_z = 2\pi$, we clearly observe the fact that setting the rotating frequency at the Larmor frequency will cancel the rotation along $\vec{z}$-axis



$\tau = \frac{\pi}{\omega_z}$



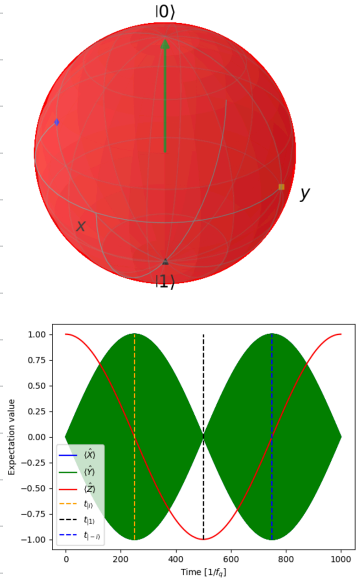
Here we clearly see that the evolution of the expected values is simplified by using a unitary transform in order to switch to a rotating frame. We have an expression which now does not have a rotation along Z axis.

Exercice 29:

$w_d = \omega$

start : $|\Psi_0\rangle = |0\rangle$

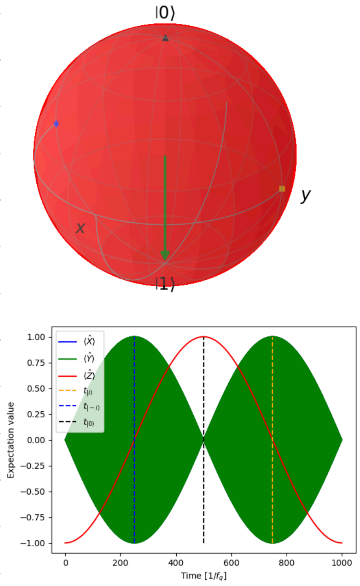
```
Time needed to reach |1> state :  500.0
Time needed to reach |i> state :  250.0
Time needed to reach |-i> state :  750.0
```



$w_d = \omega$

start : $|\Psi_0\rangle = |1\rangle$

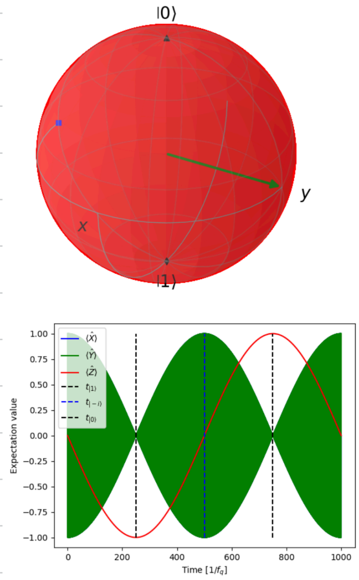
```
Time needed to reach |0> state :  500.0
Time needed to reach |i> state :  750.0
Time needed to reach |-i> state :  250.0
```



$w_d = \omega$

start : $|\Psi_0\rangle = |i\rangle$

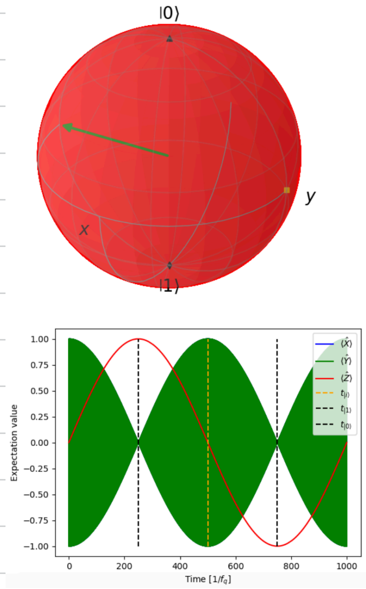
```
Time needed to reach |0> state :  750.0
Time needed to reach |1> state :  250.0
Time needed to reach |-i> state :  500.0
```



$w_d = \omega$

start $|\Psi_0\rangle = |-i\rangle$

```
Time needed to reach |0> state :  250.0
Time needed to reach |1> state :  750.0
Time needed to reach |i> state :  500.0
```

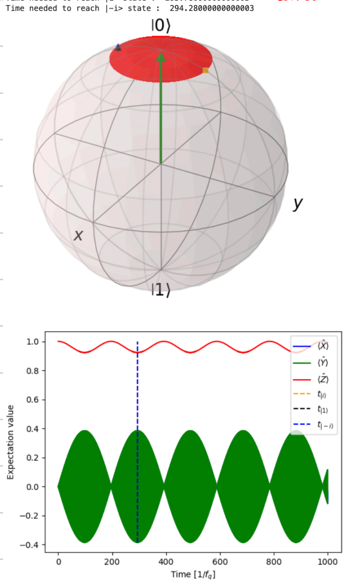


$w_d = 1{,}005\,\omega \quad \omega = 2\pi$

Start : $|\Psi_0\rangle = |0\rangle$

```
omega_q = 6.283185307179586
omega_d = 6.314601233715483
Time needed to reach |1> state :  294.28000000000003
Time needed to reach |i> state :  293.78000000000003
Time needed to reach |-i> state :  294.28000000000003
```

incorrect

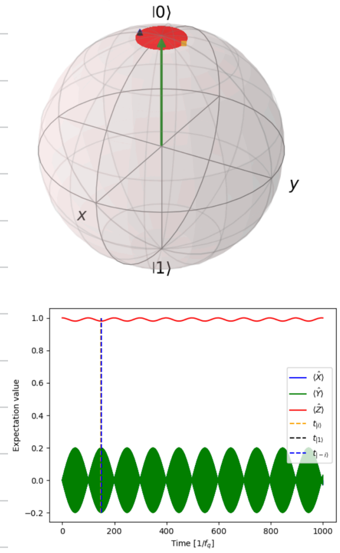


$w_d = 1{,}01\,\omega \quad \omega = 2\pi$

start $|\Psi_0\rangle = |0\rangle$

```
omega_q = 6.283185307179586
omega_d = 6.346017160251382
Time needed to reach |1> state :  149.26
Time needed to reach |i> state :  148.76
Time needed to reach |-i> state :  149.26
```

incorrect

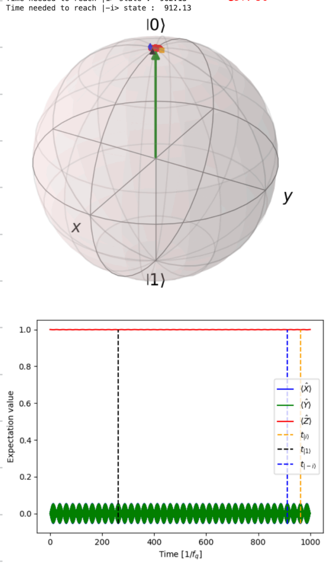


$w_d = 1{,}01 \cdot 2\pi \quad \omega = 2{,}1\pi$

start $|\Psi_0\rangle = |0\rangle$

```
omega_q = 6.5973445725385655
omega_d = 6.346017160251382
Time needed to reach |1> state :  262.38
Time needed to reach |i> state :  962.13
Time needed to reach |-i> state :  912.13
```

incorrect

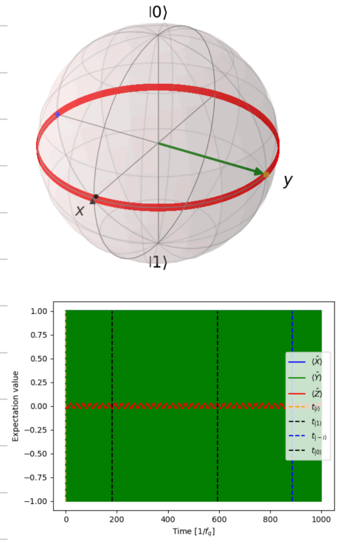


$w_d = 1{,}01 \cdot 2\pi \quad \omega = 2{,}1\pi$

start $|\Psi_0\rangle = |i\rangle$

```
omega_q = 6.5973445725385655
omega_d = 6.346017160251382
Time needed to reach |0> state :  593.5600000000001
Time needed to reach |1> state :  181.19
Time needed to reach |i> state :  0.0
Time needed to reach |-i> state :  887.13
```

incorrect



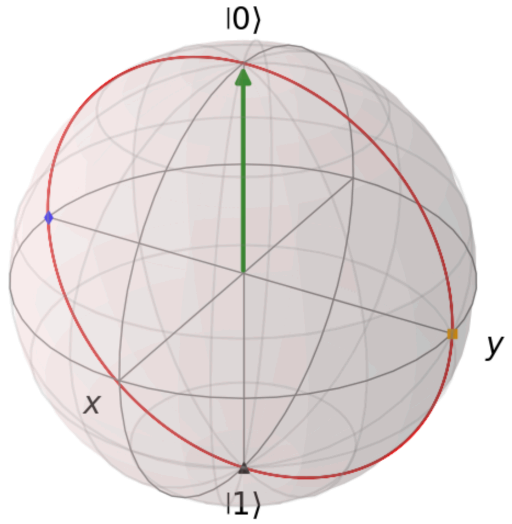
We see that when omega_d is different than omega_z, the states psi0 never ends up unto states |1>, |i> or |-i>. This is due to the fact that we are not exciting at the resonance frequency which allows to reach every state cited previously. We can therefore see that the time and cuve giving when we are reaching the states are erroneous.

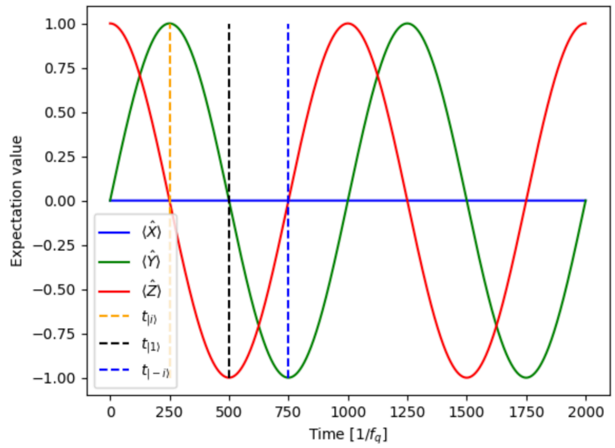
Exercice 30:

$\omega_d = \omega$ start: $|\Psi_0\rangle = |0\rangle$

```
Time needed to reach |1> state :  500.0
Time needed to reach |i> state :  250.0
Time needed to reach |-i> state :  750.0
```

correct!





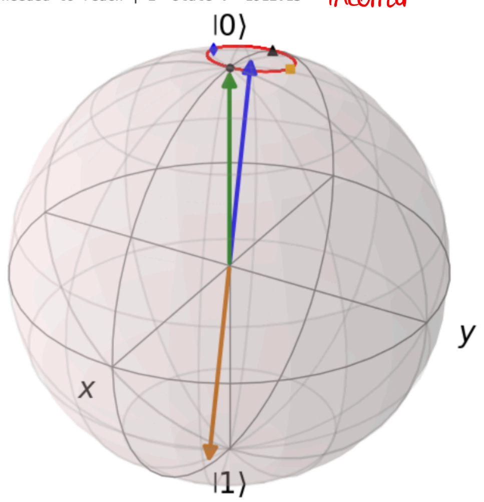
We can see that we are turning our perspective with the same angular frequency than the one of the rotation around Z axis. We also notice that going to a rotating frame does not change the value of the time when we are reaching the other states such as |1>, |i> and |-i>.

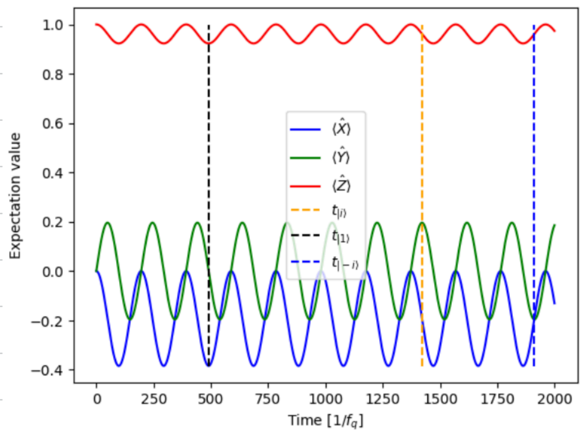
$\omega_d = 1{,}005\ \omega \quad \omega = 2\pi$

start: $|\Psi_0\rangle = |0\rangle$

```
Time needed to reach |1> state :  490.29
Time needed to reach |i> state :  1421.84
Time needed to reach |-i> state :  1912.13
```

incorrect





On the other hand if omega_d is different than omega_q, then in the rotating frame there is still a component associated with rotation around Z axis which leads the global axis of rotation to be shifted in the XY-plane and therefore the solutions for the time needed to reach the states previously cited are erroneous.

Exercice 31:

$$\hat{H}_{df} = -(\omega - \omega_d)\hat{Z}/2 - \exp(-i\omega_d t \hat{Z}/2)\hat{X}\exp(+i\omega_d t\hat{Z}/2) \times g\cos(\omega_d t).$$

$$e^{\pm i\omega_d t \hat{Z}/2} = \cos(\omega_d t)\mathbb{1} \pm i\sin(\omega_d t)\cdot\hat{Z} = \begin{bmatrix} e^{\pm i\frac{\omega_d}{2}t} & 0 \\ 0 & e^{\mp i\frac{\omega_d}{2}t} \end{bmatrix}$$

for $|0\rangle$:

- $e^{i\omega_d t\hat{Z}/2}|0\rangle = \underbrace{e^{\frac{i\omega_d t}{2}}}_{\text{This is a scalar}}|0\rangle.$

- $\hat{X}e^{\frac{i\omega_d t}{2}}|0\rangle = e^{\frac{i\omega_d t}{2}}|1\rangle$

- $e^{-i\omega_d t\hat{Z}/2}|1\rangle = e^{\frac{i\omega_d}{2}t}\cdot e^{i\frac{\omega_d}{2}t}|1\rangle = e^{i\omega_d t}|1\rangle$

Therefore $e^{-i\omega_d t\hat{Z}/2}\hat{X}e^{i\omega_d t\hat{Z}/2}|0\rangle = e^{i\omega_d t}|1\rangle.$

for $|1\rangle$:

$$e^{i\omega_d t \hat{z}/2}|1\rangle = e^{-i\frac{\omega_d t}{2}}|1\rangle$$

$$\hat{x}\, e^{i\frac{\omega_d t}{2}}|1\rangle = e^{i\frac{\omega_d t}{2}}|0\rangle.$$

$$e^{-i\omega_d t\hat{z}/2}\cdot e^{i\frac{\omega_d t}{2}}|0\rangle = e^{-i\frac{\omega_d t}{2}}\cdot e^{-i\frac{\omega_d t}{2}}|0\rangle = e^{-i\omega_d t}|0\rangle.$$

$$|1\rangle\langle 0| = \begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}$$

$$|0\rangle\langle 1| = \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}.$$

There fore, $e^{-i\omega_d t\hat{z}/2}\hat{x}\,e^{i\omega_d t\hat{z}/2}|1\rangle = e^{-i\omega_d t}|0\rangle$.

By reconstruction $e^{-i\omega_d t\hat{z}/2}\hat{x}\,e^{i\omega_d t\hat{z}/2} = \left(e^{-i\omega_d t}|0\rangle\right)\langle 1| + \left(e^{+i\omega_d t}|1\rangle\right)\langle 0|$

$$= e^{-i\omega_d t}|0\rangle\langle 1| + e^{+i\omega_d t}|1\rangle\langle 0|$$

$$e^{-i\omega_d t\hat{z}/2}\hat{x}\,e^{i\omega_d t\hat{z}/2} = \begin{bmatrix}0 & e^{-i\omega_d t}\\ e^{+i\omega_d t} & 0\end{bmatrix}$$

Exercice 32:

We have that $e^{-i\omega_d t\hat{z}/2}\hat{x}\,e^{+i\omega_d t\hat{z}/2} = \begin{bmatrix}0 & e^{-i\omega_d t}\\ e^{+i\omega_d t} & 0\end{bmatrix}$

so $e^{-i\omega_d t\hat{z}/2}\hat{x}\,e^{+i\omega_d t\hat{z}/2}\cdot g\cos(\omega_d t) = g\begin{bmatrix}0 & e^{-i\omega_d t}\cos(\omega_d t)\\ e^{i\omega_d t}\cos(\omega_d t) & 0\end{bmatrix}$

$$= \frac{g}{2}\begin{bmatrix}0 & e^{-i\omega_d t}\left[e^{i\omega_d t}+e^{-i\omega_d t}\right]\\ e^{i\omega_d t}\left[e^{i\omega_d t}+e^{-i\omega_d t}\right] & 0\end{bmatrix}$$

$$= \frac{g}{2}\begin{bmatrix}0 & e^{-2i\omega_d t}+1\\ e^{2i\omega_d t}+1 & 0\end{bmatrix}$$

$$\boxed{e^{-i\omega_d t\hat{z}/2}\hat{x}\,e^{+i\omega_d t\hat{z}/2}\cdot g\cos(\omega_d t) = \frac{g}{2}\hat{x} + \frac{g}{2}\begin{bmatrix}0 & e^{-2i\omega_d t}\\ e^{2i\omega_d t} & 0\end{bmatrix}}$$

Exercice 33:

$$\hat{H}_{df} \simeq -(\omega-\omega_d)\hat{z}/2 - g\hat{x}/2$$

If $\omega=\omega_d \Rightarrow \hat{H}_{df} \simeq -g\hat{x}/2$. we therefore have $\omega_x^{rf} = g \Rightarrow \tau = \frac{2\pi}{\omega_x^{rf}} = \frac{2\pi}{g}$

$$\boxed{\tau = \frac{2\pi}{g} \text{ at resonance}}$$

Exercice 34:

considering $\hat{H}_{df} \approx -g\hat{X}/2 \implies U_{df} = e^{ig\hat{X}t/2} = \begin{bmatrix} \cos(gt/2) & i\sin(gt/2) \\ i\sin(gt/2) & \cos(gt/2) \end{bmatrix}$

Solution of schroedinger equation in rotating frame is given by.

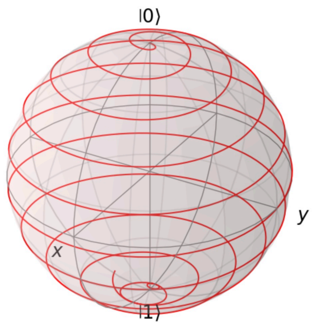
$|\psi^{df}(t)\rangle = U^{df}|\psi_0^{df}\rangle = U^{df}|\psi_0\rangle$ if $|\psi_0\rangle = |0\rangle$ then.

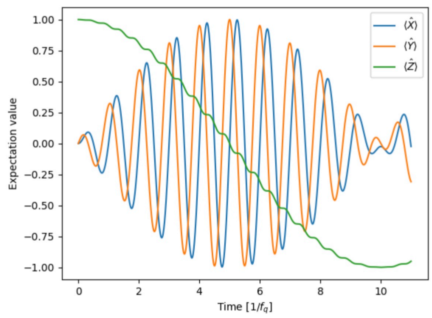
$$|\psi^{df}(t)\rangle = \begin{bmatrix} \cos(gt/2) \\ e^{i\pi/2}\sin(gt/2) \end{bmatrix} . \qquad U = e^{-i\omega_d t \hat{Z}/2}$$

Then $|\psi_{lab}(t)\rangle = U^{-1}|\psi^{df}(t)\rangle = e^{+i\omega_d t\hat{Z}/2}|\psi^{df}(t)\rangle$

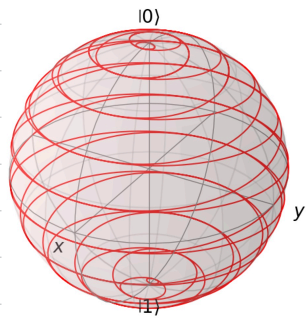
$$|\psi^{df}(t)\rangle = \begin{bmatrix} e^{i\frac{\omega_d}{2}t} & 0 \\ 0 & e^{-i\frac{\omega_d}{2}t} \end{bmatrix} \begin{bmatrix} \cos(gt/2) \\ e^{i\pi/2}\sin(gt/2) \end{bmatrix} = \begin{bmatrix} \cos(gt/2)\, e^{i\omega_d t/2} \\ e^{i\pi/2}\sin(gt/2)\, e^{-i\frac{\omega_d}{2}t} \end{bmatrix} = e^{i\omega_d t/2} \begin{bmatrix} \cos(gt/2) \\ e^{-i(\frac{\omega_d}{2}t - \frac{\pi}{2})}\sin(gt/2) \end{bmatrix}$$

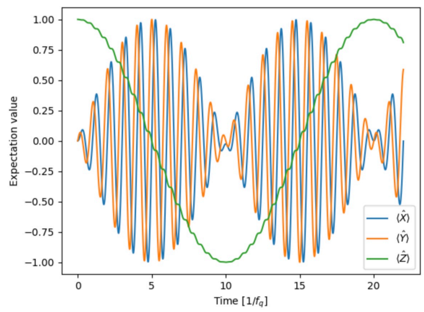
Let's verify by computation:



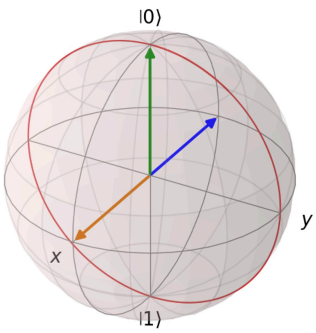


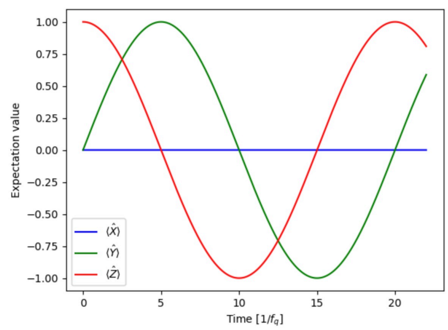
In these graphs, we chose omega_z = 2π, g=omega_z/20, initial state |0> and time going from 0 to 11. The plot of Expactation values and Bloch Sphere are the results obtained in the original lab frame with Hamiltonian given by Equation 9.



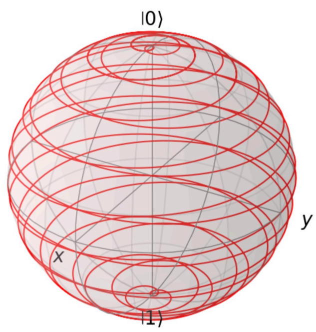


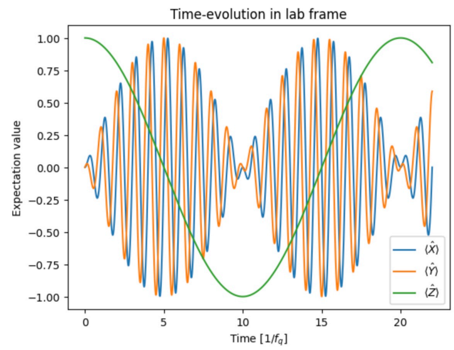
Here we have same settings than the previous graphs but we extented the time observations up to 22 and doubled the number of points equally distriubuted along time axis. We are still in the lab frame.





Here we plotted the Bloch sphere and the exptaction values obtained in the drive frame or rotating frame at resonnance frequency . This was done by applying the unitary transform U. Now, we don't have the rotation around Z axis anymore but only a rotation around X axis. We are in the rotating frame at resonnance frequency.



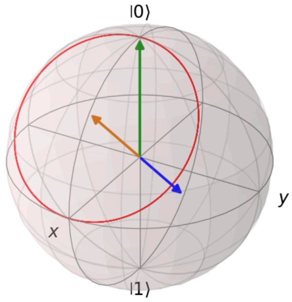


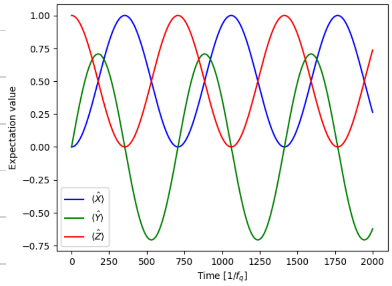
Finally in these graphs, we simplified the Hamiltonian in driven frame by aproximating it by H_driven_frame ≈ -gX/2 as stated in the question knowing also that we are at resonnance. From this hamiltonian in the RF and an inital stae |0>, we are coming back to the lab frame by appliying the inverse of the unitary transform U noted U^(-1). Therefore we obtained almost same results than the original values measured in the lab frame even by making the asumption that H_driven_frame ≈ -gX/2 and omitting the matrix term with exponatials. We came back to lab frame with ≈ same results...

Exercice 35:

$g = \frac{\omega}{1000}$ $\omega = 2\pi$

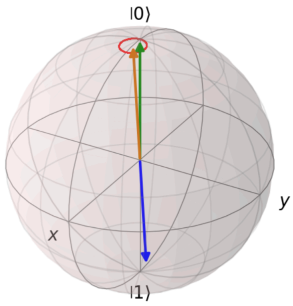
$\omega - \omega_d = g$

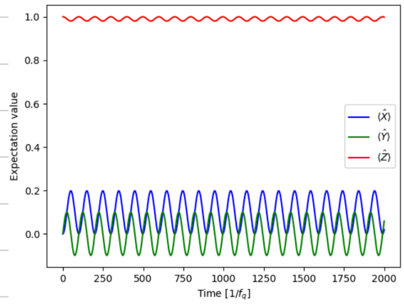




In this case, where we are slightly off-resonance, we can still observe that the drive frequency has a huge effect on the qubit. Therefore, in the rotating frame, while some of the oscillations are canceled out, there remains a component of rotation around the Z-axis with a frequency of ω−ωd.

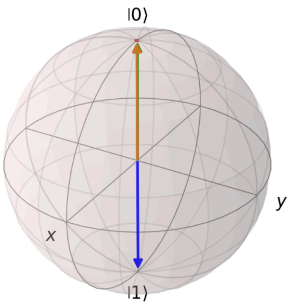
$\omega - \omega_d = 10g$

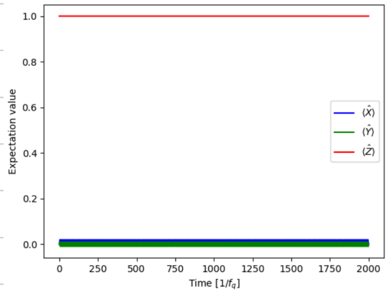




In this case, we are further off-resonance than before, and we observe that the drive frequency has less effect on the qubit. Since ω is larger than ωd, and starting from the ∣0⟩ state, the drive frequency is almost unable to induce a rotation around the X-axis, causing the solution to remain close to the initial state.

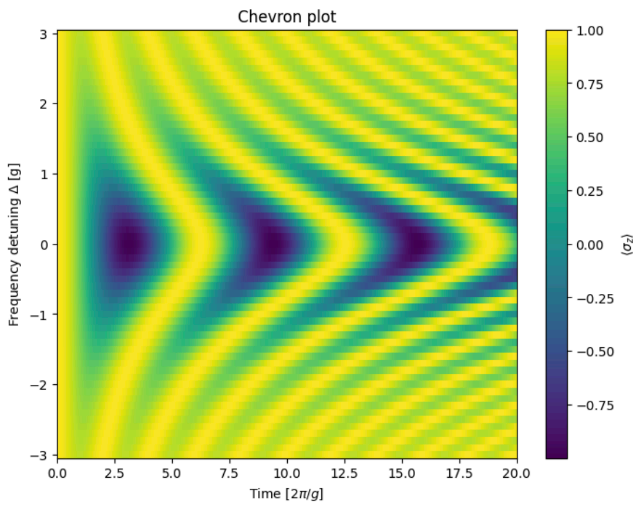
$\omega - \omega_d = 100g$





Finally in this case, we are completely off-resonnance, therefore the drive frequency is unable to induce a rotation around X-axis and only the qubit natural frequency is effective. We will only have a rotation around Z_axis and if we begin with state |0> we will almost stay to the initial condition. With that being said, we can conclude that highly off-resonnace drive of the qubit does not matter

Exercice 36:



Chevron Plot Explanation :

-> X-axis (Time): Represents time in normalized units (2π/g), showing how the qubit's state evolves over time under the influence of an external drive.

-> Y-axis (Frequency detuning Δ): Shows the frequency difference between the qubit's natural frequency (ω) and the drive frequency (ωd), indicating how the qubit reacts to different drive frequencies.

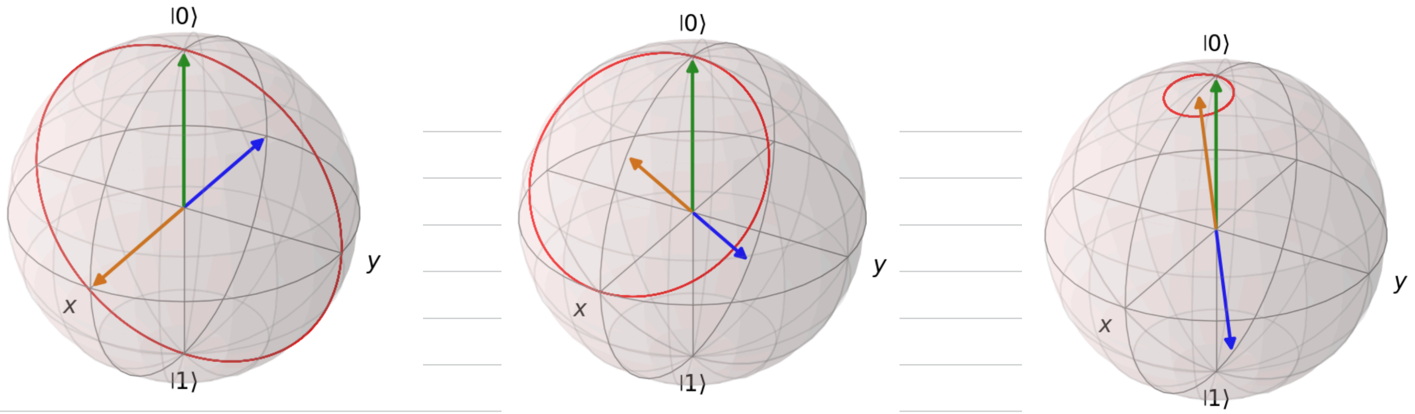
-> Color scale (⟨Z^⟩): Represents the qubit's state projection on the Z-axis of the Bloch sphere. Yellow (light) regions correspond to ⟨Z^⟩≈1 (close to ∣0⟩) and purple (dark) regions to ⟨Z^⟩≈−1 (close to ∣1⟩).

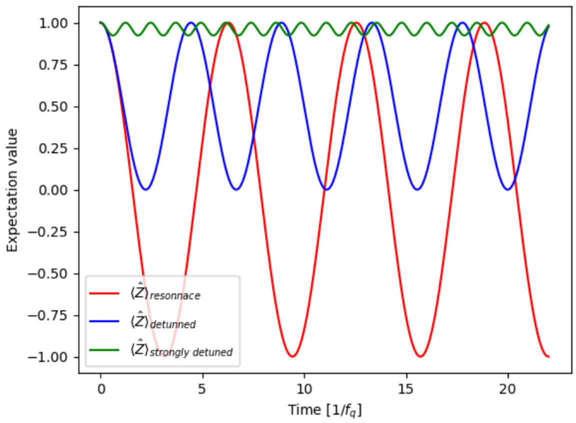
Key Observations :

-> Resonance (Δ=0): When Δ=0, the drive frequency matches the qubit's natural frequency, resulting in full oscillations between ∣0⟩ and ∣1⟩.

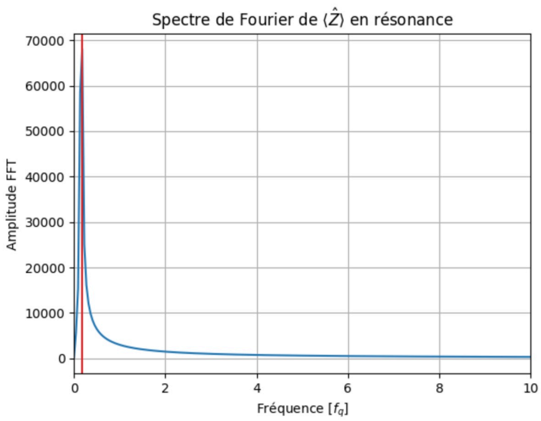
-> Off-resonance: As Δ increases (positive or negative), the qubit oscillations become faster but less pronounced, meaning the qubit remains closer to its initial state.

-> Chevron shape: The V-shaped pattern around Δ=0 illustrates how the qubit oscillates more effectively near resonance, and the farther Δ is from zero, the less effective the drive becomes.





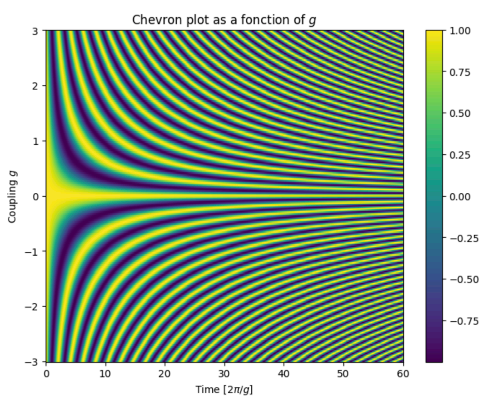
Here we have plotted the Z-exptectation value for different value of Δ (0 ; 1 and 5) for g=1 in drive/rotating frame. When the drive frequency is equal to the natural frequency (ie Δ=ω-ωd=0) then the oscillation to state |0> to |1> are complete. The frequency at which we are going from |0> to again |0> is called the rabi frequency. On the Bloch sphere, we also added the eigenstates of those Hamiltonian with different values of Δ which reprensent the axis of rotation in the rotating frame. Again, when Δ=0 the rotation is fully around X axis.

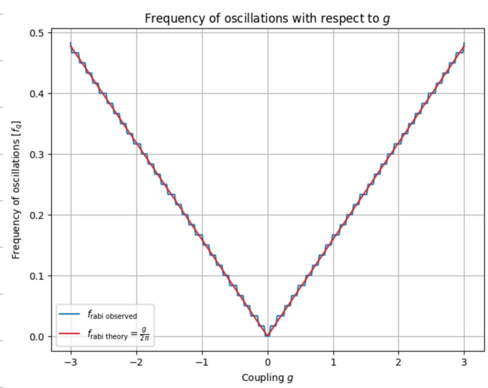


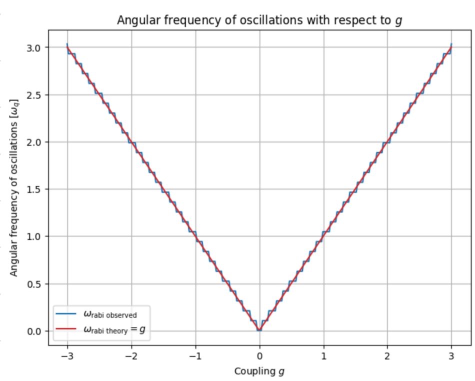

```
Dominant frequency: 0.182 [f_q]
Dominant angular frequency : 1.142
```

We find that that the angular frequency of Rabi is ≈1≈g. Here we used the algorithm of the Fast Fourier Transform (FFT) to find this frequency.

Exercice 37:







In the chevron plot and using the Fourier transform algorithm, we observe that the frequency of oscillations is linearly related to the coupling g. Therefore, at resonance, the Rabi angular frequency corresponds to the value of g (normalized), and the frequency is simply the angular frequency divided by 2π. The results obtained corresponds to the theory.