Exercice 1:

- Dirac's notation: $\langle 00|00\rangle_{AB} = \underbrace{\langle 0|0\rangle_A}_{=1} \underbrace{\langle 0|0\rangle_B}_{=1} = 1.$ Array form: $\langle 00|00\rangle_{AB} = [1\ 0\ 0\ 0]\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = 1+0+0+0 = 0$

- Dirac's notation $\langle 01|00\rangle_{AB} = \underbrace{\langle 0|0\rangle_A}_{=1}\underbrace{\langle 1|0\rangle_B}_{=0} = 0$ Array form: $\langle 01|00\rangle_{AB} = [0\ 1\ 0\ 0]\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = 0$

- Dirac's notation $\langle 10|00\rangle_{AB} = \underbrace{\langle 1|0\rangle_A}_{=0}\underbrace{\langle 0|0\rangle_B}_{=1} = 0$ Array form: $\langle 10|00\rangle_{AB} = [0\ 0\ 1\ 0]\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = 0$

- Dirac's notation $\langle 11|00\rangle_{AB} = \underbrace{\langle 1|0\rangle_A}_{=0}\underbrace{\langle 1|0\rangle_B}_{=0} = 0$ Array form: $\langle 11|00\rangle_{AB} = [0\ 0\ 0\ 1]\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = 0$

- Dirac's notation: $\langle 01|01\rangle_{AB} = \underbrace{\langle 0|0\rangle_A}_{=1}\underbrace{\langle 1|1\rangle_B}_{=1} = 1.$ Array form: $\langle 01|01\rangle_{AB} = [0\ 1\ 0\ 0]\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = 0+1+0+0=0$

- Dirac's notation: $\langle 10|01\rangle_{AB} = \underbrace{\langle 1|0\rangle_A}_{=0}\underbrace{\langle 0|1\rangle_B}_{=0} = 0$ Array form: $\langle 10|01\rangle_{AB} = [0\ 0\ 1\ 0]\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = 0$

- Dirac's notation: $\langle 11|01\rangle_{AB} = \underbrace{\langle 1|0\rangle_A}_{=0}\underbrace{\langle 1|1\rangle_B}_{=1} = 0$ Array form: $\langle 11|01\rangle_{AB} = [0\ 0\ 0\ 1]\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = 0$

- Dirac's notation : $\langle 10|10\rangle_{AB} = \underbrace{\langle 1|1\rangle_A}_{=1}\underbrace{\langle 0|0\rangle_B}_{=1} = 1$ Array form : $\langle 10|10\rangle_{AB} = [0\ 0\ 1\ 0]\begin{bmatrix}0\\0\\1\\0\end{bmatrix} = 1.$

- Dirac's notation : $\langle 11|10\rangle_{AB} = \underbrace{\langle 1|1\rangle_A}_{=1}\underbrace{\langle 1|0\rangle_B}_{=0} = 0$ Array form: $\langle 11|10\rangle_{AB} = [0\ 0\ 0\ 1]\begin{bmatrix}0\\0\\1\\0\end{bmatrix} = 0$

- Dirac's notation: $\langle 11|11\rangle_{AB} = \underbrace{\langle 1|1\rangle_A}_{=1}\underbrace{\langle 1|1\rangle_B}_{=1} = 1$ Array form: $\langle 11|11\rangle_{AB} = [0\ 0\ 0\ 1]\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = 1$

Exercise 2:

$(\hat{X}_A \otimes \mathbb{1}_B)|01\rangle$:

- Dirac's notation: $(\hat{X}_A \otimes \mathbb{1}_B)|01\rangle = \hat{X}_A|0\rangle_A \otimes \mathbb{1}_B|1\rangle_B = |1\rangle_A \otimes |1\rangle_B = |11\rangle_{AB}$

- Matrix form : $(\hat{X}_A \otimes \mathbb{1}_B)|01\rangle = \begin{bmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{bmatrix}\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\1\end{bmatrix} = |11\rangle_{AB}$

$(\hat{X}_A \otimes \mathbb{1}_B)|10\rangle$:

- Dirac's notation: $(\hat{X}_A \otimes \mathbb{1}_B)|10\rangle = \hat{X}_A|1\rangle_A \otimes \mathbb{1}_B|0\rangle_B = |0\rangle_A \otimes |0\rangle_B = |00\rangle_{AB}$

• Matrix form : $(\hat{X}_A \otimes \mathbb{1}_B)|10\rangle = \begin{bmatrix} 0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0 \end{bmatrix}\begin{bmatrix}0\\0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} = |00\rangle_{AB}$

$(\hat{X}_A \otimes \mathbb{1}_B)|11\rangle$:

• Dirac's notation: $(\hat{X}_A \otimes \mathbb{1}_B)|11\rangle = \hat{X}_A|1\rangle_A \otimes \mathbb{1}_B|1\rangle_B = |0\rangle_A \otimes |1\rangle_B = |01\rangle_{AB}$

• Matrix form : $(\hat{X}_A \otimes \mathbb{1}_B)|11\rangle = \begin{bmatrix} 0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0 \end{bmatrix}\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix} = |01\rangle_{AB}$

I clearly prefer Dirac's notation which is much more easier.

Exercise 3:

$(\mathbb{1}_A \otimes \hat{X}_B)|01\rangle$:

• Dirac's notation: $(\mathbb{1}_A \otimes \hat{X}_B)|01\rangle = \mathbb{1}_A|0\rangle_A \otimes \hat{X}_B|1\rangle_B = |0\rangle_A \otimes |0\rangle_B = |00\rangle_{AB}$

• Matrix form : $(\mathbb{1}_A \otimes \hat{X}_B)|01\rangle = \begin{bmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0 \end{bmatrix}\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} = |00\rangle_{AB}$

$(\mathbb{1}_A \otimes \hat{X}_B)|10\rangle$:

• Dirac's notation: $(\mathbb{1}_A \otimes \hat{X}_B)|10\rangle = \mathbb{1}_A|1\rangle_A \otimes \hat{X}_B|0\rangle_B = |1\rangle_A \otimes |1\rangle_B = |11\rangle_{AB}$

• Matrix form : $(\mathbb{1}_A \otimes \hat{X}_B)|10\rangle = \begin{bmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0 \end{bmatrix}\begin{bmatrix}0\\0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\1\end{bmatrix} = |11\rangle_{AB}$

$(\mathbb{1}_A \otimes \hat{X}_B)|11\rangle$:

• Dirac's notation: $(\mathbb{1}_A \otimes \hat{X}_B)|11\rangle = \mathbb{1}_A|1\rangle_A \otimes \hat{X}_B|1\rangle_B = |1\rangle_A \otimes |0\rangle_B = |10\rangle_{AB}$

• Matrix form : $(\mathbb{1}_A \otimes \hat{X}_B)|11\rangle = \begin{bmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0 \end{bmatrix}\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\1\\0\end{bmatrix} = |10\rangle_{AB}$

Exercise 4:

$(\hat{X}_A \otimes \mathbb{1}_B)(\mathbb{1}_A \otimes \hat{X}_B) = \begin{bmatrix} 0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0 \end{bmatrix}\begin{bmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0 \end{bmatrix} = \begin{bmatrix} 0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0 \end{bmatrix} = \hat{X}_A \otimes \hat{X}_B.$

Exercise 5:

Dirac's notation:

• $\hat{X}_A \otimes \hat{X}_B|00\rangle_{AB} = \hat{X}_A|0\rangle_A \otimes \hat{X}_B|0\rangle_B = |1\rangle_A \otimes |1\rangle_B = |11\rangle_{AB}$

- $\hat{X}_A \otimes \hat{X}_B |01\rangle_{AB} = \hat{X}_A|0\rangle \otimes \hat{X}_B|1\rangle = |1\rangle_A \otimes |0\rangle_A = |10\rangle_{AB}$: the quanta of B has been transfered to A.

- $\hat{X}_A \otimes \hat{X}_B |10\rangle_{AB} = \hat{X}_A|1\rangle \otimes \hat{X}_B|0\rangle = |0\rangle_A \otimes |1\rangle_B = |01\rangle_{AB}$: the quanta of A has been transfered to B.

- $\hat{X}_A \otimes \hat{X}_B |11\rangle_{AB} = \hat{X}_A|1\rangle_A \otimes \hat{X}_A|1\rangle_B = |0\rangle_A \otimes |0\rangle_B = |00\rangle_{AB}$.

Matrix form:

- $\hat{X}_A \otimes \hat{X}_B |00\rangle_{AB} = \begin{bmatrix} 0&0&0&1\\ 0&0&1&0\\ 0&1&0&0\\ 1&0&0&0 \end{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\1\end{bmatrix} = |11\rangle_{AB}$

- $\hat{X}_A \otimes \hat{X}_B |01\rangle_{AB} = \begin{bmatrix} 0&0&0&1\\ 0&0&1&0\\ 0&1&0&0\\ 1&0&0&0 \end{bmatrix}\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\1\\0\end{bmatrix} = |10\rangle_{AB}$: the quanta of B has been transfered to A.

- $\hat{X}_A \otimes \hat{X}_B |10\rangle_{AB} = \begin{bmatrix} 0&0&0&1\\ 0&0&1&0\\ 0&1&0&0\\ 1&0&0&0 \end{bmatrix}\begin{bmatrix}0\\0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix} = |01\rangle_{AB}$: the quanta of A has been transfered to B.

- $\hat{X}_A \otimes \hat{X}_B |11\rangle_{AB} = \begin{bmatrix} 0&0&0&1\\ 0&0&1&0\\ 0&1&0&0\\ 1&0&0&0 \end{bmatrix}\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} = |00\rangle_{AB}$

Exercise 6:

- $|++\rangle = |+\rangle_A \otimes |+\rangle_B = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle_B + |1\rangle_B) = \frac{1}{2}(|00\rangle_{AB} + |01\rangle_{AB} + |10\rangle_{AB} + |11\rangle_{AB}) = \boxed{\begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}}$

- $|+-\rangle_{AB} = |+\rangle_A \otimes |-\rangle_B = \frac{1}{\sqrt{2}}(|0\rangle_A + |1\rangle_A) \otimes \frac{1}{\sqrt{2}}(|0\rangle_B - |1\rangle_A) = \frac{1}{2}(|00\rangle_{AB} - |01\rangle_{AB} + |10\rangle_{AB} - |11\rangle_{AB}) = \boxed{\begin{bmatrix}1/2\\-1/2\\1/2\\-1/2\end{bmatrix}}$

- $|-+\rangle_{AB} = |-\rangle_A \otimes |+\rangle_B = \frac{1}{\sqrt{2}}(|0\rangle_A - |1\rangle_A) \otimes \frac{1}{\sqrt{2}}(|0\rangle_B + |1\rangle_B) = \frac{1}{2}(|00\rangle_{AB} + |01\rangle_{AB} - |10\rangle_{AB} - |11\rangle_{AB}) = \boxed{\begin{bmatrix}1/2\\1/2\\-1/2\\-1/2\end{bmatrix}}$

- $|--\rangle_{AB} = |-\rangle_A \otimes |-\rangle_B = \frac{1}{\sqrt{2}}(|0\rangle_A - |1\rangle_A) \otimes \frac{1}{\sqrt{2}}(|0\rangle_B - |1\rangle_B) = \frac{1}{2}(|00\rangle_{AB} - |01\rangle_{AB} - |10\rangle_{AB} + |11\rangle_{AB}) = \boxed{\begin{bmatrix}1/2\\-1/2\\-1/2\\1/2\end{bmatrix}}$

Exercise 7:

- $(\hat{X}_A \otimes \hat{X}_B)|++\rangle_{AB} = \begin{bmatrix} 0&0&0&1\\ 0&0&1&0\\ 0&1&0&0\\ 1&0&0&0 \end{bmatrix}\begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix} = \begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix} = |++\rangle_{AB}$: $|++\rangle_{AB}$ is an eigenvector of $\hat{X}_A \otimes \hat{X}_B$ of eigenvalue $+1$.

- $(\hat{X}_A \otimes \hat{X}_B)|+-\rangle_{AB} = \begin{bmatrix} 0&0&0&1\\ 0&0&1&0\\ 0&1&0&0\\ 1&0&0&0 \end{bmatrix}\begin{bmatrix}1/2\\-1/2\\1/2\\-1/2\end{bmatrix} = \begin{bmatrix}-1/2\\1/2\\-1/2\\1/2\end{bmatrix} = -|+-\rangle_{AB}$: $|+-\rangle_{AB}$ is an eigenvector of $\hat{X}_A \otimes \hat{X}_B$ of eigenvalue -1

$$\cdot(\hat{X}_A \otimes \hat{X}_B)\,|-+\rangle_{AB} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = -|-+\rangle_{AB}$$

: $|-+\rangle_{AB}$ is an eigenvector of $\hat{X}_A \otimes \hat{X}_B$ of eigenvalue -1

$$\cdot(\hat{X}_A \otimes \hat{X}_B)\,|--\rangle_{AB} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} = |--\rangle_{AB}$$

: $|--\rangle_{AB}$ is an eigenvector of $\hat{X}_A \otimes \hat{X}_B$ of eigenvalue $+1$

Exercise 8:

$$\hat{Z}_A + \hat{Z}_B \equiv \hat{Z}_A \otimes \mathbb{1}_B + \mathbb{1}_A \otimes \hat{Z}_B = \begin{bmatrix} 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & -1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} + \begin{bmatrix} 1\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ 0\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & 1\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\hat{Z}_A + \hat{Z}_B = \boxed{\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}} = 2|00\rangle\langle 00| - 2|11\rangle\langle 11|.$$

Here we directly see that the eigenvectors are $|00\rangle$ and $|11\rangle$ $|00\rangle$ has eigenvalue 2 and $|11\rangle$ has eigenvalue -2. The two others can be chosen but must satisfy orthonormality so we choose $|01\rangle_{AB}$ $|10\rangle_{AB}$ both of eigenvalues 0.

Explicit calculation:

$$(\hat{Z}_A + \hat{Z}_B)\,|00\rangle_{AB} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 2|00\rangle_{AB}$$

: $|00\rangle_{AB}$ is an eigenvector of $\hat{Z}_A + \hat{Z}_B$ of eigenvalue 2

$$(\hat{Z}_A + \hat{Z}_B)\,|00\rangle_{AB} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 0\,|01\rangle_{AB}$$

: $|01\rangle_{AB}$ is an eigenvector of $\hat{Z}_A + \hat{Z}_B$ of eigenvalue 0

$$(\hat{Z}_A + \hat{Z}_B)\,|00\rangle_{AB} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = 0\,|10\rangle_{AB}$$

$|10\rangle_{AB}$ is an eigenvector of $\hat{Z}_A + \hat{Z}_B$ of eigenvalue 0

$$(\hat{Z}_A + \hat{Z}_B)\,|00\rangle_{AB} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = -2\,|11\rangle_{AB}$$

$|11\rangle_{AB}$ is an eigenvector of $\hat{Z}_A + \hat{Z}_B$ of eigenvalue -2

Exercise 9:

$|\psi_A\rangle = \alpha_0|0\rangle_A + \alpha_1|1\rangle_A \quad |\psi_B\rangle = \beta_0|0\rangle_B + \beta_1|1\rangle_B \quad |\psi\rangle_{AB} = |\psi_A\rangle \otimes |\psi_B\rangle = \begin{bmatrix} \alpha_0\beta_0 \\ \alpha_0\beta_1 \\ \alpha_1\beta_0 \\ \alpha_1\beta_1 \end{bmatrix}$

Dirac's notation:

$$\langle\psi|\psi\rangle_{AB} = \left(\langle\psi_A|\otimes\langle\psi_B|\right)\left(|\psi_A\rangle\otimes|\psi_B\rangle\right) = \underbrace{\langle\psi_A|\psi_A\rangle}_{=1}\cdot\underbrace{\langle\psi_B|\psi_B\rangle}_{=1} = 1.$$

Matrix form:

$$\langle\psi|\psi\rangle_{AB} = \begin{bmatrix} \alpha_0^*\beta_0^* & \alpha_0^*\beta_1^* & \alpha_1^*\beta_0^* & \alpha_1^*\beta_1^* \end{bmatrix}\begin{bmatrix} \alpha_0\beta_0 \\ \alpha_0\beta_1 \\ \alpha_1\beta_0 \\ \alpha_1\beta_1 \end{bmatrix} = \alpha_0\alpha_0^*\beta_0\beta_0^* + \alpha_0\alpha_0^*\beta_1\beta_1^* + \alpha_1\alpha_1^*\beta_0\beta_0^* + \alpha_1\alpha_1^*\beta_1\beta_1^*$$

$$\langle\psi|\psi\rangle_{AB} = |\alpha_0|^2|\beta_0|^2 + |\alpha_0|^2|\beta_1|^2 + |\alpha_1|^2|\beta_0|^2 + |\alpha_1|^2|\beta_1|^2 = \underbrace{\left[|\alpha_0|^2+|\alpha_1|^2\right]}_{=1}\underbrace{\left[|\beta_0|^2+|\beta_1|^2\right]}_{=1} = 1.$$

Exercise 10:

$$\hat{Z}_A\hat{X}_B \equiv \hat{Z}_A\otimes 1_B \cdot 1_A\otimes\hat{X}_B = \hat{Z}_A\otimes\hat{X}_B = \begin{bmatrix} 1\begin{bmatrix}0&1\\1&0\end{bmatrix} & 0\begin{bmatrix}0&1\\1&0\end{bmatrix} \\ 0\begin{bmatrix}0&1\\1&0\end{bmatrix} & -1\begin{bmatrix}0&1\\1&0\end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&-1&0 \end{bmatrix}$$

$$\left(\hat{Z}_A\otimes\hat{X}_B\right)\left(|\psi\rangle_A\otimes|\psi_B\rangle\right) = \begin{bmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&-1&0 \end{bmatrix}\begin{bmatrix} \alpha_0\beta_0 \\ \alpha_0\beta_1 \\ \alpha_1\beta_0 \\ \alpha_1\beta_1 \end{bmatrix} = \begin{bmatrix} \alpha_0\beta_1 \\ \alpha_0\beta_0 \\ -\alpha_1\beta_1 \\ -\alpha_1\beta_0 \end{bmatrix}$$

$$\hat{Z}_A|\psi_A\rangle\otimes\hat{X}_B|\psi_B\rangle = \begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}\alpha_0\\\alpha_1\end{bmatrix}\otimes\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}\beta_0\\\beta_1\end{bmatrix} = \begin{bmatrix}\alpha_0\\-\alpha_1\end{bmatrix}\otimes\begin{bmatrix}\beta_1\\\beta_0\end{bmatrix} = \begin{bmatrix} \alpha_0\beta_1 \\ \alpha_0\beta_0 \\ -\alpha_1\beta_1 \\ -\alpha_1\beta_0 \end{bmatrix}$$

We see that $\left(\hat{Z}_A\otimes\hat{X}_B\right)\left(|\psi\rangle_A\otimes|\psi_B\rangle\right) = \hat{Z}_A|\psi_A\rangle\otimes\hat{X}_B|\psi_B\rangle$

Exercise 11:

- The length of the resulting state vector is 2^N
- Number of real parameters: each qubit only needs 2 real parameters so N qubits require $2\cdot N$ real parameters.

Exercise 12:

N – Qubits so $2\cdot 2^N$ real numbers, but the norm removes one real number and global phase removes also one. It turns out in this case we need $2\cdot 2^N - 2 = 2(2^N-1)$

Number of real numbers required to describe this state is $2(2^N-1)$

Exercise 13:

$N = 256$ qubits

- Real number for a product state: $\# = 2 \cdot N = 2 \cdot 256 = 512$

- Real parameters for a general quantum state $\# = 2(2^N - 1) = 2(2^{256} - 1) \cong 2{,}3158 \cdot 10^{77}$
  but 10^{80} is slightly greater than the number of real parameters required

We can use all atoms in the universe but this does not really make any sense.

Exercise 14:

- $|\psi_{E1}\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$. If this is a product state, it could be written in the following form:

$$|\psi_A\rangle \otimes |\psi_B\rangle = \begin{array}{c} \begin{bmatrix} \alpha_0\beta_0 \\ \alpha_0\beta_1 \\ \alpha_1\beta_0 \\ \alpha_1\beta_1 \end{bmatrix} \end{array} \begin{array}{c} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array} \quad \text{so:} \quad \left.\begin{array}{l} \alpha_0\beta_0 = \frac{1}{\sqrt{2}} \Rightarrow \alpha_0, \beta_0 \neq 0 \\ \alpha_0\beta_1 = 0 \Rightarrow \beta_1 = 0 \\ \alpha_1\beta_0 = 0 \Rightarrow \alpha_1 = 0 \\ \alpha_1\beta_1 = \frac{1}{\sqrt{2}} \Rightarrow \alpha_1, \beta_1 \neq 0 \end{array}\right\}$$

⚠ We have a contradiction since α_1 and β_1 should be in the mean time 0 and non-zero. $|\psi_{E1}\rangle$ can not be factorized as a product state!

- $|\psi_{E2}\rangle = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle \overset{?}{=} |\psi_A\rangle \otimes |\psi_B\rangle$

$$\left.\begin{array}{l} \alpha_0\beta_0 = 0 \Rightarrow \alpha_0 \text{ or } \beta_0 = 0 \\ \alpha_0\beta_1 = \frac{1}{\sqrt{2}} \Rightarrow \alpha_0, \beta_1 \neq 0 \\ \alpha_1\beta_0 = \frac{1}{\sqrt{2}} \Rightarrow \alpha_1, \beta_0 \neq 0 \\ \alpha_1\beta_1 = 0 \Rightarrow \alpha_1 \text{ or } \beta_1 = 0 \end{array}\right\}$$

⚠ We have a contradiction because we need $\alpha_0, \alpha_1, \beta_0, \beta_1$ to be non zero but $\alpha_0\beta_0$ and $\alpha_1\beta_1$ should be equal to zero which is not possible in the mean time! Therefore $|\psi_{E2}\rangle$ can't be factorized as a product state.

Exercise 15:

Here we have $a = 1$ and measure correspond to a collapse of $|\psi\rangle$ onto state $|11\rangle$:

$$\text{Prob}(M = -2) = |\langle 11|\psi\rangle|^2 = |\psi_{11}|^2 = \left|\langle 11|\left(\tfrac{1}{2}|00\rangle + \tfrac{1}{2}|01\rangle + \tfrac{1}{2}|10\rangle + \tfrac{1}{2}|11\rangle\right)\right|^2 = \left|\tfrac{1}{2}\right|^2 = \tfrac{1}{4}.$$

$\text{Prob}(M = -2) = \frac{1}{4}$

Exercise 16:

$$\text{Prob}(M \neq 2, -2) = |\langle 01|\psi\rangle|^2 + |\langle 10|\psi\rangle|^2 = |\psi_{01}|^2 + |\psi_{10}|^2 = \left|\tfrac{1}{2}\right|^2 + \left|\tfrac{1}{2}\right|^2 = \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2}$$

or $\text{Prob}(M \neq 2, -2) = 1 - |\psi_{00}|^2 + |\psi_{11}|^2 = 1 - \frac{1}{4} - \frac{1}{4} = \frac{1}{2}.$

$\text{Prob}(M \neq 2, -2) = \frac{1}{2}$

Exercise 17:

$$Prob(M=0) = |\langle\psi_0|\psi\rangle|^2$$

$$|\langle\psi_0|\psi\rangle|^2 = \left|\frac{1}{\sqrt{|\psi_{01}|^2+|\psi_{10}|^2}}\left(\psi_{01}^*\langle 01| + \psi_{10}^*\langle 10|\right)|\psi\rangle\right|^2$$

$$= \left|\frac{1}{\sqrt{|\psi_{01}|^2+|\psi_{10}|^2}}\left(|\psi_{01}|^2 + |\psi_{10}|^2\right)\right|^2$$

$$= \left|\sqrt{|\psi_{01}|^2+|\psi_{10}|^2}\right|^2$$

$$= |\psi_{01}|^2 + |\psi_{10}|^2$$

$$Prob(M=0) = |\langle\psi_0|\psi\rangle|^2 = \frac{1}{2} = Prob(M \neq 2,-2)$$

Exercise 18:

We know that $P(M=0) = \frac{1}{2}$ so in order to obtain an entangled state, we have to try the measurement on a product state, on average $N = \frac{1}{P(M=0)} = \frac{1}{1/2} = 2$.

We need to try measurement on average $N = \frac{1}{P(M=0)} = 2$

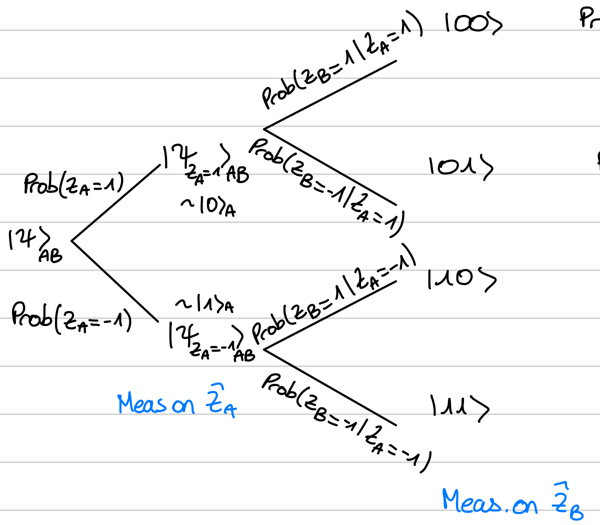
Exercise 19:

$$|\psi\rangle = \psi_{00}|00\rangle + \psi_{01}|01\rangle + \psi_{10}|10\rangle + \psi_{11}|11\rangle$$

$$|\psi_{z_A=1}\rangle = |0\rangle_A \otimes \frac{\psi_{00}|0\rangle_B + \psi_{01}|1\rangle_B}{\sqrt{|\psi_{00}|^2+|\psi_{01}|^2}} \qquad |\psi_{z_A=-1}\rangle = |1\rangle_A \otimes \frac{\psi_{10}|0\rangle_B + \psi_{11}|1\rangle_B}{\sqrt{|\psi_{10}|^2+|\psi_{11}|^2}}$$

First we measure on $\hat{z}_A$. The output is either +1 or −1 and state $|\psi\rangle$ collapses on state $|\psi_{z_A=1}\rangle$ with proba $|\langle\psi_{z_A=1}|\psi\rangle|^2$ or on $|\psi_{z_A=-1}\rangle$ with proba $|\langle\psi_{z_A=-1}|\psi\rangle|^2$. If $z_A=1$, we have $|0\rangle_A$ and if $z_A=-1$ we collapse on Alice's lab on $|1\rangle_A$.

Then we measure on $\hat{z}_B$. The output is either +1 or −1 and state $|\psi_{z_A=1}\rangle$ or $|\psi_{z_A=-1}\rangle$ (depending on outcome of z_A) collapses on $|0\rangle_B$ or $|1\rangle_B$ for Bob partite. We need to use conditionnal probabilities to see each probability of each outcome.



$Prob(z_A=1, z_B=1) = Prob(z_B=1|z_A=1)\cdot Prob(z_A=1)$. $\quad (1;1)$

$= |\langle 00|\psi_{z_A=1}\rangle|^2 |\langle\psi_{z_A=1}|\psi\rangle|^2$.

$Prob(z_A=1, z_B=-1) = Prob(z_B=-1|z_A=1)\cdot Prob(z_A=1)$. $\quad (1;-1)$

$= |\langle 01|\psi_{z_A=1}\rangle|^2 |\langle\psi_{z_A=1}|\psi\rangle|^2$.

$Prob(z_A=-1, z_B=1) = Prob(z_B=1|z_A=-1)\cdot Prob(z_A=-1)$

$= |\langle 10|\psi_{z_A=-1}\rangle|^2 |\langle\psi_{z_A=-1}|\psi\rangle|^2$. $\quad (-1;1)$

$Prob(z_A=-1, z_B=-1) = Prob(z_B=-1|z_A=-1)\cdot Prob(z_A=-1)$

$= |\langle 11|\psi_{z_A=-1}\rangle|^2 |\langle\psi_{z_A=-1}|\psi\rangle|^2$. $\quad (-1;1)$

$(1;1)$:

$$\mathrm{Prob}((1;1)) = \mathrm{Prob}(\hat{z}_B=1, \hat{z}_A=1) = \mathrm{Prob}(\hat{z}_B=1 \mid \hat{z}_A=1)\cdot \mathrm{Prob}(\hat{z}_A=1)$$

$$= |\langle \psi_{z_A=1}|\psi\rangle|^2 \cdot |\langle 00|\psi_{z_A=1}\rangle|^2 = \left[|\psi_{00}|^2+|\psi_{01}|^2\right] \left| \langle 00| \cdot |0\rangle_A \otimes \frac{\psi_{00}|0\rangle_B+\psi_{01}|1\rangle_B}{\sqrt{|\psi_{00}|^2+|\psi_{01}|^2}}\right|^2$$

$$= \left[|\psi_{00}|^2+|\psi_{01}|^2\right] \left| \langle 0|0\rangle_A \cdot \frac{\psi_{00}\langle 0|0\rangle_B+\psi_{01}\langle 0|1\rangle_B}{\sqrt{|\psi_{00}|^2+|\psi_{01}|^2}}\right|^2$$

$$= \left(|\psi_{00}|^2+|\psi_{01}|^2\right) \frac{|\psi_{00}|^2}{|\psi_{00}|^2+|\psi_{01}|^2}$$

$$\mathrm{Prob}((1;1)) = |\psi_{00}|^2$$

$(1;-1)$:

$$\mathrm{Prob}((1;-1)) = \mathrm{Prob}(\hat{z}_B=-1, \hat{z}_A=1) = \mathrm{Prob}(\hat{z}_B=-1 \mid \hat{z}_A=1)\cdot \mathrm{Prob}(\hat{z}_A=1)$$

$$= |\langle \psi_{z_A=1}|\psi\rangle|^2 \cdot |\langle 01|\psi_{z_A=1}\rangle|^2 = \left[|\psi_{00}|^2+|\psi_{01}|^2\right] \left| \langle 01| \cdot |0\rangle_A \otimes \frac{\psi_{00}|0\rangle_B+\psi_{01}|1\rangle_B}{\sqrt{|\psi_{00}|^2+|\psi_{01}|^2}}\right|^2$$

$$= \left[|\psi_{00}|^2+|\psi_{01}|^2\right] \left| \langle 0|0\rangle_A \cdot \frac{\psi_{00}\langle 1|0\rangle_B+\psi_{01}\langle 1|1\rangle_B}{\sqrt{|\psi_{00}|^2+|\psi_{01}|^2}}\right|^2$$

$$= \left(|\psi_{00}|^2+|\psi_{01}|^2\right) \frac{|\psi_{01}|^2}{|\psi_{00}|^2+|\psi_{01}|^2} \qquad \mathrm{Prob}((1;-1)) = |\psi_{01}|^2$$

$(-1;1)$:

$$\mathrm{Prob}((-1;1)) = \mathrm{Prob}(\hat{z}_B=1, \hat{z}_A=-1) = \mathrm{Prob}(\hat{z}_B=1 \mid \hat{z}_A=-1)\cdot \mathrm{Prob}(\hat{z}_A=-1)$$

$$= |\langle \psi_{z_A=-1}|\psi\rangle|^2 \cdot |\langle 10|\psi_{z_A=-1}\rangle|^2 = \left[|\psi_{10}|^2+|\psi_{11}|^2\right] \left| \langle 10| \cdot |1\rangle_A \otimes \frac{\psi_{10}|0\rangle_B+\psi_{11}|1\rangle_B}{\sqrt{|\psi_{10}|^2+|\psi_{11}|^2}}\right|^2$$

$$= \left[|\psi_{10}|^2+|\psi_{11}|^2\right] \left| \langle 1|1\rangle_A \cdot \frac{\psi_{10}\langle 0|0\rangle_B+\psi_{11}\langle 0|1\rangle_B}{\sqrt{|\psi_{10}|^2+|\psi_{11}|^2}}\right|^2$$

$$= \left(|\psi_{10}|^2+|\psi_{11}|^2\right) \frac{|\psi_{10}|^2}{|\psi_{10}|^2+|\psi_{11}|^2}$$

$$\mathrm{Prob}((1;1)) = |\psi_{10}|^2$$

$(-1;-1)$:

$$\mathrm{Prob}((-1;-1)) = \mathrm{Prob}(\hat{z}_B=-1, \hat{z}_A=-1) = \mathrm{Prob}(\hat{z}_B=-1 \mid \hat{z}_A=-1)\cdot \mathrm{Prob}(\hat{z}_A=-1)$$

$$= |\langle \psi_{z_A=-1}|\psi\rangle|^2 \cdot |\langle 11|\psi_{z_A=-1}\rangle|^2 = \left[|\psi_{10}|^2+|\psi_{11}|^2\right] \left| \langle 11| \cdot |1\rangle_A \otimes \frac{\psi_{10}|0\rangle_B+\psi_{11}|1\rangle_B}{\sqrt{|\psi_{10}|^2+|\psi_{11}|^2}}\right|^2$$

$$= \left[|\psi_{10}|^2+|\psi_{11}|^2\right] \left| \langle 1|1\rangle_A \cdot \frac{\psi_{10}\langle 1|0\rangle_B+\psi_{11}\langle 1|1\rangle_B}{\sqrt{|\psi_{10}|^2+|\psi_{11}|^2}}\right|^2$$

$$= \left(|\psi_{10}|^2+|\psi_{11}|^2\right) \frac{|\psi_{11}|^2}{|\psi_{10}|^2+|\psi_{11}|^2}$$

$$\mathrm{Prob}((1;1)) = |\psi_{11}|^2$$

Exercise 20:

Measurement are done locally, so doing before the measurement on Alice's lab or in Bob's lab, should not change the probability of the different output. In any case we compute $P(z_A = i, z_B = j)$ so nothing should change.

$\hat{z}_B|00\rangle = +1|00\rangle$ for a general 2 Qubit-state: $|\psi\rangle = \psi_{00}|00\rangle + \psi_{01}|01\rangle + \psi_{10}|10\rangle + \psi_{11}|11\rangle$.

$\hat{z}_B|01\rangle = -|01\rangle$ We have a partial collapse as previously seen.

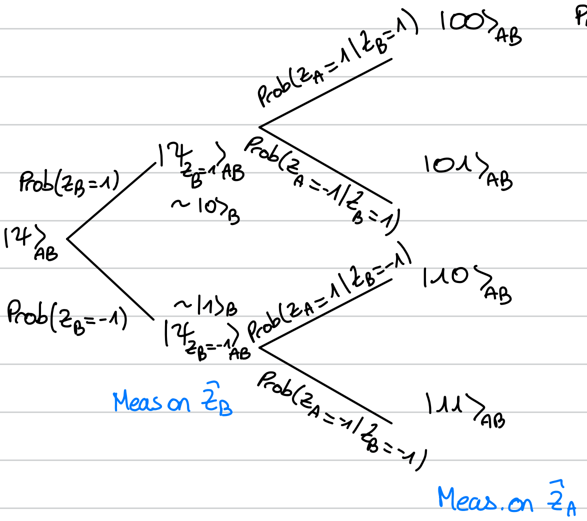
$\hat{z}_B|10\rangle = +|10\rangle$

$\hat{z}_B|11\rangle = -|11\rangle$

$$|\psi_{z_B=1}\rangle = \frac{\psi_{00}|0\rangle_A + \psi_{10}|1\rangle_A}{\sqrt{|\psi_{00}|^2 + |\psi_{10}|^2}} \otimes |0\rangle_B \qquad |\psi_{z_B=-1}\rangle = \frac{\psi_{01}|0\rangle_A + \psi_{11}|1\rangle_A}{\sqrt{|\psi_{01}|^2 + |\psi_{11}|^2}} \times |1\rangle_B$$

First we measure on $\hat{z}_B$. The output is either +1 or −1 and state $|\psi\rangle$ collapses on state $|\psi_{z_B=1}\rangle$ with proba $|\langle\psi_{z_B=1}|\psi\rangle|^2$ or on $|\psi_{z_B=-1}\rangle$ with proba $|\langle\psi_{z_B=-1}|\psi\rangle|^2$. If $z_B = 1$, we have $|0\rangle_B$ and if $z_B=-1$ we collapse on Bob's lab on $|1\rangle_B$.

Then we measure on $\hat{z}_A$. The output is either +1 or −1 and state $|\psi_{z_B=1}\rangle$ or $|\psi_{z_B=-1}\rangle$ (depending on outcome of z_B) collapses on $|0\rangle_A$ or $|1\rangle_A$ for Alice partite. We need to use conditionnal probabilities to see each probability of each outcome.



$$\text{Prob}(z_B=1, z_A=1) = \text{Prob}(z_A=1|z_B=1)\cdot\text{Prob}(z_B=1). \qquad (1;1)$$
$$= |\langle 00|\psi_{z_B=1}\rangle|^2\, |\langle\psi_{z_B=1}|\psi\rangle|^2.$$

$$\text{Prob}(z_B=1, z_A=-1) = \text{Prob}(z_A=-1|z_B=1)\cdot\text{Prob}(z_B=1). \qquad (1;-1)$$
$$= |\langle 01|\psi_{z_B=1}\rangle|^2\, |\langle\psi_{z_B=1}|\psi\rangle|^2.$$

$$\text{Prob}(z_B=-1, z_A=1) = \text{Prob}(z_A=1|z_B=-1)\cdot\text{Prob}(z_B=-1)$$
$$= |\langle 10|\psi_{z_B=-1}\rangle|^2\, |\langle\psi_{z_B=-1}|\psi\rangle|^2. \qquad (-1;1)$$

$$\text{Prob}(z_B=-1, z_A=-1) = \text{Prob}(z_A=-1|z_B=-1)\cdot\text{Prob}(z_B=-1)$$
$$= |\langle 10|\psi_{z_B=-1}\rangle|^2\, |\langle\psi_{z_B=-1}|\psi\rangle|^2. \qquad (-1;1)$$

$$\text{Prob}(z_B=1) = |\langle\psi_{z_B=1}|\psi\rangle|^2 = |\psi_{00}|^2 + |\psi_{10}|^2.$$

$$\text{Prob}(z_B=-1) = |\langle\psi_{z_B=-1}|\psi\rangle|^2 = |\psi_{01}|^2 + |\psi_{11}|^2$$

(1;1):

$$\text{Prob}((1;1)) = \text{Prob}(\hat{z}_B = 1, \hat{z}_A = 1) = \text{Prob}(\hat{z}_A = 1|\hat{z}_B = 1)\cdot\text{Prob}(\hat{z}_B = 1)$$

$$= \left|\langle 00|_{AB}\cdot\frac{\psi_{00}|0\rangle_A + \psi_{10}|1\rangle_A}{\sqrt{|\psi_{00}|^2 + |\psi_{10}|^2}}\otimes|0\rangle_B\right|^2 \left(|\psi_{00}|^2 + |\psi_{10}|^2\right)$$

$$= \left|\frac{\psi_{00}\langle 0|0\rangle_A}{\sqrt{|\psi_{00}|^2 + |\psi_{10}|^2}} + \frac{\psi_{10}\langle 0|1\rangle_A}{\sqrt{|\psi_{00}|^2 + |\psi_{10}|^2}}\right|^2 \left(|\psi_{00}|^2 + |\psi_{10}|^2\right)$$

$$= \frac{|\psi_{00}|^2}{\cancel{|\psi_{00}|^2 + |\psi_{10}|^2}}\cdot\cancel{\left(|\psi_{00}|^2 + |\psi_{10}|^2\right)}$$

$$\text{Prob}((1;1)) = |\psi_{00}|^2$$

$(1;-1)$:

$$\mathrm{Prob}((1;-1)) = \mathrm{Prob}(\hat{z}_B=-1, \hat{z}_A=1) = \mathrm{Prob}(\hat{z}_A=1 \mid \hat{z}_B=-1)\cdot \mathrm{Prob}(\hat{z}_B=-1)$$

$$= |\langle \psi_{z_B=-1}|\psi\rangle|^2 |\langle 01|\psi_{z_B=-1}\rangle|^2 = \left[|\psi_{01}|^2+|\psi_{11}|^2\right]\left|\langle 01|\cdot \frac{\psi_{01}|0\rangle_A+\psi_{11}|1\rangle_A}{\sqrt{|\psi_{01}|^2+|\psi_{11}|^2}}\otimes|1\rangle_B\right|^2$$

$$= \left[|\psi_{01}|^2+|\psi_{11}|^2\right]\left|\frac{\psi_{01}\langle 0|0\rangle_A+\psi_{11}\langle 0|1\rangle_A}{\sqrt{|\psi_{01}|^2+|\psi_{11}|^2}}\cdot\langle 1|1\rangle_B\right|^2$$

$$= \left(\cancel{|\psi_{01}|^2+|\psi_{11}|^2}\right)\frac{|\psi_{01}|^2}{\cancel{|\psi_{01}|^2+|\psi_{11}|^2}}$$

$$\mathrm{Prob}((1;-1)) = |\psi_{01}|^2$$

$(-1;1)$:

$$\mathrm{Prob}((-1;1)) = \mathrm{Prob}(\hat{z}_B=1, \hat{z}_A=-1) = \mathrm{Prob}(\hat{z}_A=-1 \mid \hat{z}_B=1)\cdot \mathrm{Prob}(\hat{z}_B=1)$$

$$= \left|\langle 10|_{AB}\cdot \frac{\psi_{00}|0\rangle_A+\psi_{10}|1\rangle_A}{\sqrt{|\psi_{00}|^2+|\psi_{10}|^2}}\otimes|0\rangle_B\right|^2\left(|\psi_{00}|^2+|\psi_{10}|^2\right)$$

$$= \left|\frac{\psi_{00}\langle 1|0\rangle_A}{\sqrt{|\psi_{00}|^2+|\psi_{10}|^2}}+\frac{\psi_{10}\langle 1|1\rangle_A}{\sqrt{|\psi_{00}|^2+|\psi_{10}|^2}}\right|^2\left(|\psi_{00}|^2+|\psi_{10}|^2\right)$$

$$= \frac{|\psi_{10}|^2}{\cancel{|\psi_{00}|^2+|\psi_{10}|^2}}\cdot\left(\cancel{|\psi_{00}|^2+|\psi_{10}|^2}\right)$$

$$\mathrm{Prob}((1;1)) = |\psi_{10}|^2$$

$(-1;-1)$:

$$\mathrm{Prob}((-1;1)) = \mathrm{Prob}(\hat{z}_B=-1, \hat{z}_A=-1) = \mathrm{Prob}(\hat{z}_A=-1 \mid \hat{z}_B=-1)\cdot \mathrm{Prob}(\hat{z}_B=-1)$$

$$= |\langle \psi_{z_B=-1}|\psi\rangle|^2 |\langle 11|\psi_{z_B=-1}\rangle|^2 = \left[|\psi_{01}|^2+|\psi_{11}|^2\right]\left|\langle 11|\cdot \frac{\psi_{01}|0\rangle_A+\psi_{11}|1\rangle_A}{\sqrt{|\psi_{01}|^2+|\psi_{11}|^2}}\otimes|1\rangle_B\right|^2$$

$$= \left[|\psi_{01}|^2+|\psi_{11}|^2\right]\left|\frac{\psi_{01}\langle 1|0\rangle_A+\psi_{11}\langle 1|1\rangle_A}{\sqrt{|\psi_{01}|^2+|\psi_{11}|^2}}\cdot\langle 1|1\rangle_B\right|^2$$

$$= \left(\cancel{|\psi_{01}|^2+|\psi_{11}|^2}\right)\frac{|\psi_{11}|^2}{\cancel{|\psi_{01}|^2+|\psi_{11}|^2}}$$

$$\mathrm{Prob}((1;1)) = |\psi_{11}|^2$$

We indeed obtain same probabilities.

Exercise 21:

$\hat{Z}_A\otimes\hat{Z}_B|00\rangle_{AB} = \hat{Z}_A|0\rangle_A\otimes\hat{Z}_B|0\rangle_B = (+1)|0\rangle_A\otimes(+1)|0\rangle_B = +|00\rangle_{AB}$ eigenvector: $|00\rangle$; eigenvalue: $+1$

$\hat{Z}_A\otimes\hat{Z}_B|01\rangle_{AB} = \hat{Z}_A|0\rangle_A\otimes\hat{Z}_B|1\rangle_B = (+1)|0\rangle_A\otimes(-1)|0\rangle_B = -|01\rangle_{AB}$ eigenvector: $|01\rangle$; eigenvalue: -1

$\hat{Z}_A\otimes\hat{Z}_B|10\rangle_{AB} = \hat{Z}_A|1\rangle_A\otimes\hat{Z}_B|0\rangle_B = (-1)|1\rangle_A\otimes(+1)|0\rangle_B = -|10\rangle_{AB}$. eigenvector: $|10\rangle$; eigenvalue: -1

$\hat{Z}_A\otimes\hat{Z}_B|11\rangle_{AB} = \hat{Z}_A|1\rangle_A\otimes\hat{Z}_B|1\rangle_B = (-1)|1\rangle_A\otimes(-1)|1\rangle_B = +|11\rangle_{AB}$. eigenvector: $|11\rangle$; eigenvalue: $+1$

Since we cannot make the difference between $|00\rangle$ and $|11\rangle$ ($M=+1$) but also $|01\rangle_{AB}$ and $|10\rangle_{AB}$ ($M=-1$)
One can define the partial collapse states as: $-\ |\psi_{M=1}\rangle = \frac{1}{\sqrt{|\psi_{00}|^2+|\psi_{11}|^2}}\left(\psi_{00}|00\rangle_{AB}+\psi_{11}|11\rangle_{AB}\right)$

These two vectors are the outcomes $-\ |\psi_{M=-1}\rangle = \frac{1}{\sqrt{|\psi_{01}|^2+|\psi_{10}|^2}}\left(\psi_{01}|01\rangle_{AB}+\psi_{10}|10\rangle_{AB}\right)$

The probabilities are for a general 2-Qubit state $|\psi\rangle$:

- $\text{Prob}(M=1) = |\langle\psi_{M=1}|\psi\rangle|^2 = \left| \frac{1}{\sqrt{|\psi_{00}|^2+|\psi_{11}|^2}} \left(\psi_{00}^*\langle 00|_{AB} + \psi_{11}^*\langle 11|_{AB}\right)\left(\psi_{00}|00\rangle + \psi_{01}|01\rangle + \psi_{10}|10\rangle + \psi_{11}|11\rangle\right)\right|^2$

$$= \left| \frac{|\psi_{00}|^2 + |\psi_{11}|^2}{\sqrt{|\psi_{00}|^2+|\psi_{11}|^2}} \right|^2$$

$$\text{Prob}(M=1) = |\psi_{00}|^2 + |\psi_{11}|^2$$

- $\text{Prob}(M=-1) = |\langle\psi_{M=-1}|\psi\rangle|^2 = \left| \frac{1}{\sqrt{|\psi_{01}|^2+|\psi_{10}|^2}} \left(\psi_{01}^*\langle 01|_{AB} + \psi_{10}^*\langle 10|_{AB}\right)\left(\psi_{00}|00\rangle + \psi_{01}|01\rangle + \psi_{10}|10\rangle + \psi_{11}|11\rangle\right)\right|^2$

$$= \left| \frac{|\psi_{01}|^2 + |\psi_{10}|^2}{\sqrt{|\psi_{01}|^2+|\psi_{10}|^2}} \right|^2$$

$$\text{Prob}(M=-1) = |\psi_{01}|^2 + |\psi_{10}|^2$$

Finally for $|\psi\rangle = |+\rangle_A \otimes |+\rangle$ we get:

- $\text{Prob}(M=1) = |\langle\psi_{M=1}|++\rangle_{AB}|^2 = |\psi_{00}|^2 + |\psi_{11}|^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$

↳ outcome $|\psi_{M=1}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle_{AB} + |11\rangle_{AB}\right)$ with Proba: $\text{Prob}(M=1) = \frac{1}{2}$ (Bell state $|B_{00}\rangle$)

- $\text{Prob}(M=1) = |\langle\psi_{M=-1}|++\rangle_{AB}|^2 = |\psi_{01}|^2 + |\psi_{10}|^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$

↳ outcome: $|\psi_{M=-1}\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle_{AB} + |10\rangle_{AB}\right)$ with Proba: $\text{Prob}(M=-1) = \frac{1}{2}$ (Bell state $|B_{01}\rangle$.)

Exercise 22:

$|\psi_p\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$ $\quad |\psi_A\rangle = \alpha_0|0\rangle_A + \alpha_1|1\rangle_A.$ $\quad \alpha_0 = \cos(\frac{\theta}{2})$ $\quad \alpha_1 = \sin(\frac{\theta}{2})e^{i\phi}.$ $\quad \theta \in [0;\pi]$ $\quad \phi \in [0;2\pi]$

$$\langle\psi_A|\hat{X}|\psi_A\rangle = \sin(\theta)\cos(\phi) \Rightarrow |\langle\psi_A|\hat{X}|\psi_A\rangle|^2 = \sin^2(\theta)\cos^2(\phi)$$
$$\langle\psi_A|\hat{Y}|\psi_A\rangle = \sin(\theta)\sin(\phi) \Rightarrow |\langle\psi_A|\hat{Y}|\psi_A\rangle|^2 = \sin^2(\theta)\sin^2(\phi)$$
$$\langle\psi_A|\hat{Z}|\psi_A\rangle = \cos(\theta) \Rightarrow |\langle\psi_A|\hat{Z}|\psi_A\rangle|^2 = \cos^2(\theta)$$

With Bloch representation:

$$|\langle\psi_A|\hat{X}|\psi_A\rangle|^2 + |\langle\psi_A|\hat{Y}|\psi_A\rangle|^2 + |\langle\psi_A|\hat{Z}|\psi_A\rangle|^2 = \sin^2(\theta)\cos^2(\phi) + \sin^2(\theta)\sin^2(\phi) + \cos^2(\phi)$$
$$= \sin^2(\theta)\,\underbrace{\left(\cos^2(\phi) + \sin^2(\phi)\right)}_{=1} + \cos^2(\phi)$$
$$= \sin^2(\theta) + \cos^2(\theta)$$
$$= 1$$

without Bloch representation:

$$\langle\psi_A|\hat{X}|\psi_A\rangle = \left(\alpha_0^*\langle 0| + \alpha_1^*\langle 1|\right)\left(\alpha_0|1\rangle_A + \alpha_1|0\rangle\right) = \alpha_0^*\alpha_1 + \alpha_1^*\alpha_0.$$

$$\langle\psi_A|\hat{Y}|\psi_A\rangle = [\alpha_0^* \ \alpha_1]\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = [\alpha_0^* \ \alpha_1^*]\begin{bmatrix} -i\alpha_1 \\ i\alpha_0 \end{bmatrix} = -i\alpha_0^*\alpha_1 + i\alpha_1^*\alpha_0$$

$\langle i|0\rangle = \frac{1}{\sqrt{2}}$ $\quad \langle i|1\rangle = \frac{-i}{\sqrt{2}}$

$\langle -i|0\rangle = \frac{1}{\sqrt{2}}$ $\quad \langle -i|1\rangle = \frac{+i}{\sqrt{2}}$

$$\langle\psi_A|\hat{Z}|\psi_A\rangle = \left(\alpha_0^*\langle 0| + \alpha_1^*\langle 1|\right)\left(\alpha_0|0\rangle - \alpha_1|1\rangle\right) = |\alpha_0|^2 - |\alpha_1|^2$$

$$|\langle \psi_A|\hat{X}|\psi_A\rangle|^2 = |(a_0 - ib_0)(a_1+ib_1) + (a_0+ib_0)(a_1-ib_1)|^2$$

$$= |a_0a_1 + \cancel{ia_0b_1} - \cancel{ia_1b_0} + b_0b_1 + a_0a_1 - \cancel{ia_0b_1} + \cancel{ib_0a_1} + b_0b_1|^2$$

$$= |2a_0a_1 + 2b_0b_1|^2 = 4a_0^2a_1^2 + 4b_0^2b_1^2 + 8a_0a_1b_0b_1$$

$$|\langle\psi_A|\hat{Y}|\psi_A\rangle|^2 = |-i(a_0-ib_0)(a_1+ib_1) + i(a_0+ib_0)(a_1-ib_1)|^2$$

$$= |i(-\cancel{a_0a_1} - ia_0b_1 + ia_1b_0 - \cancel{b_0b_1}) + i(\cancel{a_0a_1} - ia_0b_1 + ib_0a_1 + \cancel{b_0b_1})|^2$$

$$= |-2ia_0b_1 + 2ia_1b_0|^2$$

$$= |2a_0b_1 - 2a_1b_0|^2 = 4a_0^2b_1^2 + 4a_1^2b_0^2 - 8a_0a_1b_0b_1.$$

$$|\langle\psi_A|\hat{Z}|\psi_A\rangle|^2 = \big||\alpha_0|^2 - |\alpha_1|^2\big|^2 = |\alpha_0|^4 + |\alpha_1|^4 - 2|\alpha_0|^2|\alpha_1|^2.$$

$$|\langle\psi_A|\hat{X}|\psi_A\rangle|^2 + |\langle\psi_A|\hat{Y}|\psi_A\rangle|^2 + |\langle\psi_A|\hat{Z}|\psi_A\rangle|^2 = 4a_0^2a_1^2 + 4b_0^2b_1^2 + \cancel{8a_0a_1b_0b_1}$$
$$+ 4a_0^2b_1^2 + 4a_1^2b_0^2 - \cancel{8a_0a_1b_0b_1}$$
$$+ |\alpha_0|^4 + |\alpha_1|^4 - 2|\alpha_0|^2|\alpha_1|^2.$$

$$= 4a_0^2(a_1^2+b_1^2) + 4b_0^2(a_1^2+b_1^2) + |\alpha_0|^4 + |\alpha_1|^4 - 2|\alpha_0|^2|\alpha_1|^2.$$

$$= 4|\alpha_0|^2 \cdot |\alpha_1|^2 + |\alpha_0|^4 + |\alpha_1|^4 - 2|\alpha_0|^2|\alpha_1|^2$$

$$= |\alpha_0|^4 + |\alpha_1|^4 + 2|\alpha_0|^2|\alpha_1|^2$$

$$= (|\alpha_0|^2 + |\alpha_1|^2)^2$$

$$= 1 \quad \boxtimes \qquad |\langle\psi_A|\hat{X}|\psi_A\rangle|^2 + |\langle\psi_A|\hat{Y}|\psi_A\rangle|^2 + |\langle\psi_A|\hat{Z}|\psi_A\rangle|^2 = 1$$

- $$|\langle\psi_P|\hat{X}_A|\psi_P\rangle|^2 = |\langle\psi_A|\otimes\langle\psi_B|(\hat{X}_A\otimes\mathbb{1}_B)|\psi_A\rangle\otimes|\psi_B\rangle^2$$
$$= |\langle\psi_A|\hat{X}_A|\psi_A\rangle \underbrace{\langle\psi_B|\mathbb{1}|\psi_B\rangle}_{=1}|^2$$
$$= |\langle\psi_A|\hat{X}_A|\psi_A\rangle|^2$$

$$|\langle\psi_P|\hat{Y}_A|\psi_P\rangle|^2 = |\langle\psi_A|\otimes\langle\psi_B|(\hat{Y}_A\otimes\mathbb{1}_B)|\psi_A\rangle\otimes|\psi_B\rangle^2$$
$$= |\langle\psi_A|\hat{Y}_A|\psi_A\rangle \underbrace{\langle\psi_B|\mathbb{1}|\psi_B\rangle}_{=1}|^2$$
$$= |\langle\psi_A|\hat{Y}_A|\psi_A\rangle|^2$$

$$|\langle\psi_P|\hat{Z}_A|\psi_P\rangle|^2 = |\langle\psi_A|\otimes\langle\psi_B|(\hat{Z}_A\otimes\mathbb{1}_B)|\psi_A\rangle\otimes|\psi_B\rangle^2$$
$$= |\langle\psi_A|\hat{Z}_A|\psi_A\rangle \underbrace{\langle\psi_B|\mathbb{1}|\psi_B\rangle}_{=1}|^2$$
$$= |\langle\psi_A|\hat{Z}_A|\psi_A\rangle|^2$$

Therefore $|\langle\psi_P|\hat{X}_A|\psi_P\rangle|^2 + |\langle\psi_P|\hat{Y}_A|\psi_P\rangle|^2 + |\langle\psi_P|\hat{Z}_A|\psi_P\rangle|^2 = |\langle\psi_A|\hat{X}|\psi_A\rangle|^2 + |\langle\psi_A|\hat{Y}|\psi_A\rangle|^2 + |\langle\psi_A|\hat{Z}|\psi_A|^2$

$= 1 \quad \boxtimes$ (see above)

Exercise 23:

$$\bullet\ \langle\psi_p|\hat{x}_A\hat{x}_B|\psi_p\rangle = \langle\psi_A|\otimes\langle\psi_B|\left(\hat{x}_A\otimes\hat{x}_B\right)|\psi_A\rangle\otimes|\psi_B\rangle$$
$$= \langle\psi_A|\hat{x}_A|\psi_A\rangle\cdot\langle\psi_B|\hat{x}_B|\psi_B\rangle$$
$$= \sin(\theta_A)\cos(\phi_A)\cdot\sin(\theta_B)\cos(\phi_B)$$

$$\bullet\ \langle\psi_p|\hat{y}_A\hat{y}_B|\psi_p\rangle = \langle\psi_A|\otimes\langle\psi_B|\left(\hat{y}_A\otimes\hat{y}_B\right)|\psi_A\rangle\otimes|\psi_B\rangle$$
$$= \langle\psi_A|\hat{y}_A|\psi_A\rangle\cdot\langle\psi_B|\hat{y}_B|\psi_B\rangle$$
$$= \sin(\theta_A)\sin(\phi_A)\cdot\sin(\theta_B)\sin(\phi_B)$$

$$\bullet\ \langle\psi_p|\hat{z}_A\hat{z}_B|\psi_p\rangle = \langle\psi_A|\otimes\langle\psi_B|\left(\hat{z}_A\otimes\hat{z}_B\right)|\psi_A\rangle\otimes|\psi_B\rangle$$
$$= \langle\psi_A|\hat{z}_A|\psi_A\rangle\cdot\langle\psi_B|\hat{z}_B|\psi_B\rangle$$
$$= \cos(\theta_A)\cos(\theta_B)$$

$$\langle\psi_p|\hat{x}_A\hat{x}_B|\psi_p\rangle+\langle\psi_p|\hat{y}_A\hat{y}_B|\psi_p\rangle+\langle\psi_p|\hat{z}_A\hat{z}_B|\psi_p\rangle = \sin(\theta_A)\cos(\phi_A)\cdot\sin(\theta_B)\cos(\phi_B)+\sin(\theta_A)\sin(\phi_A)\cdot\sin(\theta_B)\sin(\phi_B) + \cos(\theta_A)\cos(\theta_B)$$

$$= \underbrace{\begin{bmatrix}\sin(\theta_A)\cos(\phi_A)\\ \sin(\theta_A)\sin(\phi_A)\\ \cos(\theta_A)\end{bmatrix}}_{\vec{r}_A}\cdot\underbrace{\begin{bmatrix}\sin(\theta_B)\cos(\phi_B)\\ \sin(\theta_B)\sin(\phi_B)\\ \sin(\theta_B)\end{bmatrix}}_{\vec{r}_B} = \underbrace{\|\vec{r}_A\|}_{=1}\cdot\underbrace{\|\vec{r}_B\|}_{=1}\cdot\underbrace{\cos(\theta)}_{\in[-1;1]}\qquad \theta = \text{angle between } \vec{r}_A \text{ and } \vec{r}_B$$

These are spherical coordinates so the length of these two vector is 1. The scalar product is maximized when $\vec{r}_A$ and $\vec{r}_B$ are aligned in the same direction giving a scalar product of 1 (because $\|\vec{r}_A\| = \|\vec{r}_B\| = 1$). The minimum value is obtained when $\vec{r}_A$ and $\vec{r}_B$ are aligned in the opposite direction giving a scalar product of -1.

Therefore $-1\leq\langle\psi_p|\hat{x}_A\hat{x}_B|\psi_p\rangle+\langle\psi_p|\hat{y}_A\hat{y}_B|\psi_p\rangle+\langle\psi_p|\hat{z}_A\hat{z}_B|\psi_p\rangle\leq 1$ $\square$

Exercise 24:

$|B_0\rangle$:

$$\bullet\ \langle B_0|B_0\rangle = \frac{1}{\sqrt{2}}\left(\langle 01|-\langle 10|\right)\frac{1}{\sqrt{2}}\left(|01\rangle-|10\rangle\right) = \frac{1}{2}\left(\underbrace{\langle 01|01\rangle}_{=1}-\underbrace{\langle 01|10\rangle}_{=0}-\underbrace{\langle 10|01\rangle}_{=0}+\underbrace{\langle 10|10\rangle}_{=1}\right) = \frac{1}{2}(2) = 1.$$

$$\bullet\ \langle B_1|B_0\rangle = \frac{1}{\sqrt{2}}\left(\langle 01|+\langle 10|\right)\frac{1}{\sqrt{2}}\left(|01\rangle-|10\rangle\right) = \frac{1}{2}\left(\underbrace{\langle 01|01\rangle}_{=1}-\underbrace{\langle 01|10\rangle}_{=0}+\underbrace{\langle 10|01\rangle}_{=0}-\underbrace{\langle 10|10\rangle}_{=1}\right) = \frac{1}{2}(1-1) = 0 = \langle B_0|B_1\rangle$$

$$\bullet\ \langle B_2|B_0\rangle = \frac{1}{\sqrt{2}}\left(\langle 00|-\langle 11|\right)\frac{1}{\sqrt{2}}\left(|01\rangle-|10\rangle\right) = \frac{1}{2}\left(\underbrace{\langle 00|01\rangle}_{=0}-\underbrace{\langle 00|10\rangle}_{=0}-\underbrace{\langle 11|01\rangle}_{=0}+\underbrace{\langle 11|10\rangle}_{=0}\right) = 0. = \langle B_0|B_2\rangle$$

$$\bullet\ \langle B_3|B_0\rangle = \frac{1}{\sqrt{2}}\left(\langle 00|+\langle 11|\right)\frac{1}{\sqrt{2}}\left(|01\rangle-|10\rangle\right) = \frac{1}{2}\left(\underbrace{\langle 00|01\rangle}_{=0}-\underbrace{\langle 00|10\rangle}_{=0}+\underbrace{\langle 11|01\rangle}_{=0}-\underbrace{\langle 11|10\rangle}_{=0}\right) = 0 = \langle B_0|B_3\rangle$$

$|B_1\rangle$:

$$\bullet\ \langle B_1|B_1\rangle = \frac{1}{\sqrt{2}}\left(\langle 01|+\langle 10|\right)\frac{1}{\sqrt{2}}\left(|01\rangle+|10\rangle\right) = \frac{1}{2}\left(\underbrace{\langle 01|01\rangle}_{=1}+\underbrace{\langle 01|10\rangle}_{=0}+\underbrace{\langle 10|01\rangle}_{=0}+\underbrace{\langle 10|10\rangle}_{=1}\right) = \frac{1}{2}(1+1) = 1.$$

$$\bullet\ \langle B_2|B_1\rangle = \frac{1}{\sqrt{2}}\left(\langle 00|-\langle 11|\right)\frac{1}{\sqrt{2}}\left(|01\rangle+|10\rangle\right) = \frac{1}{2}\left(\underbrace{\langle 00|01\rangle}_{=0}+\underbrace{\langle 00|10\rangle}_{=0}-\underbrace{\langle 11|01\rangle}_{=0}-\underbrace{\langle 11|10\rangle}_{=0}\right) = 0 = \langle B_1|B_2\rangle$$

$$\bullet\ \langle B_3|B_1\rangle = \frac{1}{\sqrt{2}}\left(\langle 00|+\langle 11|\right)\frac{1}{\sqrt{2}}\left(|01\rangle+|10\rangle\right) = \frac{1}{2}\left(\underbrace{\langle 00|01\rangle}_{=0}+\underbrace{\langle 00|10\rangle}_{=0}+\underbrace{\langle 11|01\rangle}_{=0}+\underbrace{\langle 11|10\rangle}_{=0}\right) = 0 = \langle B_1|B_3\rangle$$

$|B_2\rangle$:

• $\langle B_2|B_2\rangle = \frac{1}{\sqrt{2}}(\langle 00| - \langle 11|)\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) = \frac{1}{2}(\underbrace{\langle 00|00\rangle}_{=1} - \underbrace{\langle 00|11\rangle}_{=0} - \underbrace{\langle 11|00\rangle}_{=0} + \underbrace{\langle 11|11\rangle}_{=1}) = \frac{1}{2}(1+1) = 1.$

• $\langle B_3|B_2\rangle = \frac{1}{\sqrt{2}}(\langle 00| + \langle 11|)\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) = \frac{1}{2}(\underbrace{\langle 00|00\rangle}_{=1} - \underbrace{\langle 00|11\rangle}_{=0} + \underbrace{\langle 11|00\rangle}_{=0} - \underbrace{\langle 11|11\rangle}_{=1}) = \frac{1}{2}(1-1) = 0 = \langle B_2|B_3\rangle$

$|B_3\rangle$:

• $\langle B_3|B_3\rangle = \frac{1}{\sqrt{2}}(\langle 00| + \langle 11|)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{2}(\underbrace{\langle 00|00\rangle}_{=1} + \underbrace{\langle 00|11\rangle}_{=0} + \underbrace{\langle 11|00\rangle}_{=0} + \underbrace{\langle 11|11\rangle}_{=1}) = \frac{1}{2}(1+1) = 1$

Exercise 25:

$|B_1\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$

• $\hat{Z}_A|B_1\rangle = (\hat{Z}_A \otimes \mathbb{1}_B)\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = \frac{1}{\sqrt{2}}(\hat{Z}_A|0\rangle_A \otimes \mathbb{1}_B|1\rangle_B + \hat{Z}_A|1\rangle_A \otimes \mathbb{1}_B|0\rangle_B) = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = |B_0\rangle$

• $(\hat{X}_A\hat{Z}_A)|B_1\rangle = (\hat{X}_A\hat{Z}_A \otimes \mathbb{1}_B)\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = \frac{1}{\sqrt{2}}(\hat{X}_A\hat{Z}_A|0\rangle_A \otimes \mathbb{1}_B|1\rangle_B + \hat{X}_A\hat{Z}_A|1\rangle_A \otimes \mathbb{1}_B|0\rangle_B) = \frac{1}{\sqrt{2}}(\hat{X}_A|0\rangle_A \otimes |1\rangle_B - \hat{X}_A|1\rangle \otimes |0\rangle_B)$

$= \frac{1}{\sqrt{2}}(|11\rangle - |00\rangle)$

$(\hat{X}_A\hat{Z}_A)|B_1\rangle = e^{i\pi}|B_2\rangle \equiv |B_2\rangle$ (global phase is irrelevant).

• $\hat{X}_A|B_1\rangle = (\hat{X}_A \otimes \mathbb{1}_B)\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = \frac{1}{\sqrt{2}}(\hat{X}_A|0\rangle_A \otimes \mathbb{1}_B|1\rangle_B + \hat{X}_A|1\rangle \otimes \mathbb{1}_B|0\rangle_B) = \frac{1}{\sqrt{2}}(|11\rangle + |00\rangle) = |B_3\rangle$

Exercise 26:

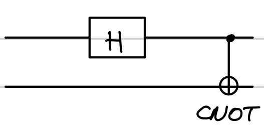
$\hat{U} = |B_0\rangle\langle 11| + |B_1\rangle\langle 01| + |B_2\rangle\langle 10| + |B_3\rangle\langle 00|$

$$\hat{U} = \begin{bmatrix} \langle 00|\hat{U}|00\rangle & \langle 00|\hat{U}|01\rangle & \langle 00|\hat{U}|10\rangle & \langle 00|\hat{U}|11\rangle \\ \langle 01|\hat{U}|00\rangle & \langle 01|\hat{U}|01\rangle & \langle 01|\hat{U}|10\rangle & \langle 01|\hat{U}|11\rangle \\ \langle 10|\hat{U}|00\rangle & \langle 10|\hat{U}|01\rangle & \langle 10|\hat{U}|10\rangle & \langle 10|\hat{U}|11\rangle \\ \langle 11|\hat{U}|00\rangle & \langle 11|\hat{U}|01\rangle & \langle 11|\hat{U}|10\rangle & \langle 11|\hat{U}|11\rangle \end{bmatrix}$$

$$\hat{U} = \begin{bmatrix} \langle 00|B_3\rangle & \langle 00|B_1\rangle & \langle 00|B_2\rangle & \langle 00|B_0\rangle \\ \langle 01|B_3\rangle & \langle 01|B_1\rangle & \langle 01|B_2\rangle & \langle 01|B_0\rangle \\ \langle 10|B_3\rangle & \langle 10|B_1\rangle & \langle 10|B_2\rangle & \langle 10|B_0\rangle \\ \langle 11|B_3\rangle & \langle 11|B_1\rangle & \langle 11|B_2\rangle & \langle 11|B_0\rangle \end{bmatrix}$$

$$\hat{U} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

This is



otherwise, if we consider a change of basis from $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ to $|B_0\rangle, |B_1\rangle, |B_2\rangle, |B_3\rangle$ (with respect to this order):

$\hat{U} = |B_0\rangle\langle 00| + |B_1\rangle\langle 01| + |B_2\rangle\langle 10| + |B_3\rangle\langle 11|.$

$$\hat{U} = \begin{bmatrix} \langle 00|B_0\rangle & \langle 00|B_1\rangle & \langle 00|B_2\rangle & \langle 00|B_3\rangle \\ \langle 01|B_0\rangle & \langle 01|B_1\rangle & \langle 01|B_2\rangle & \langle 01|B_3\rangle \\ \langle 10|B_0\rangle & \langle 10|B_1\rangle & \langle 10|B_2\rangle & \langle 10|B_3\rangle \\ \langle 11|B_0\rangle & \langle 11|B_1\rangle & \langle 11|B_2\rangle & \langle 11|B_3\rangle \end{bmatrix}$$

$$\hat{U} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Exercise 27:

- $\langle B_0|\hat{X}_A|B_0\rangle = \frac{1}{\sqrt{2}}(\langle 01| - \langle 10|)(\hat{X}_A\otimes \mathbb{1}_B)\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$

$$= \frac{1}{2}\left(\langle 01|\hat{X}_A\otimes\mathbb{1}_B|01\rangle - \langle 01|\hat{X}_A\otimes\mathbb{1}_B|10\rangle - \langle 10|\hat{X}_A\otimes\mathbb{1}_B|01\rangle + \langle 10|\hat{X}_A\otimes\mathbb{1}_B|10\rangle\right)$$

$$= \frac{1}{2}\left(\underbrace{\langle 01|11\rangle}_{=0} - \underbrace{\langle 01|00\rangle}_{=0} - \underbrace{\langle 10|11\rangle}_{=0} + \underbrace{\langle 10|00\rangle}_{=0}\right)$$

$\langle B_0|\hat{X}_A|B_0\rangle = 0$ ☒

- $\langle B_0|\hat{Y}_A|B_0\rangle = \langle B_0|\begin{bmatrix} 0 & -i \\ i & 0\end{bmatrix}\otimes\begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix}|B_0\rangle$

$$= \frac{1}{2}\begin{bmatrix} 0 & 1 & -1 & 0\end{bmatrix}\begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0\end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0\end{bmatrix}$$

$$= \frac{i}{2}\begin{bmatrix} 0 & 1 & -1 & 0\end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1\end{bmatrix}$$

$\langle B_0|\hat{Y}_A|B_0\rangle = 0$ ☒

- $\langle B_0|\hat{Z}_A|B_0\rangle = \langle B_0|\hat{Z}_A\otimes\mathbb{1}_B|B_0\rangle = \frac{1}{2}(\langle 01| - \langle 10|)(\hat{Z}_A\otimes\mathbb{1}_B)(|01\rangle - |10\rangle)$

$$= \frac{1}{2}\left(\langle 01|\hat{Z}_A\otimes\mathbb{1}_B|01\rangle - \langle 01|\hat{Z}_A\otimes\mathbb{1}_B|10\rangle - \langle 10|\hat{Z}_A\otimes\mathbb{1}_B|01\rangle + \langle 10|\hat{Z}_A\otimes\mathbb{1}_B|10\rangle\right)$$

$$= \frac{1}{2}\left(\underbrace{\langle 01|01\rangle}_{=1} + \underbrace{\langle 01|10\rangle}_{=0} - \underbrace{\langle 10|01\rangle}_{=0} - \underbrace{\langle 10|10\rangle}_{=1}\right)$$

$$= \frac{1}{2}(1 + 0 - 0 - 1)$$

$\langle B_0|\hat{Z}_A|B_0\rangle = 0$ ☒

This would be a point located at the origin $(0,0,0)$ of the "Bloch Ball", representing the completly mixed state (most entangled here). Since all spin projections are 0, this state corresponds to maximum uncertainty. It is no longer a valid pure state on the Bloch sphere, but instead a mixed state at the center of the ball.

Exercise 28:

- $\langle B_0|\hat{X}_B|B_0\rangle = \frac{1}{2}(\langle 01| - \langle 10|)(\mathbb{1}_A\otimes\hat{X}_B)(|01\rangle - |10\rangle)$

$$= \frac{1}{2}\left(\langle 01|\mathbb{1}_A\otimes\hat{X}_B|01\rangle - \langle 01|\mathbb{1}_A\otimes\hat{X}_B|10\rangle - \langle 10|\mathbb{1}_A\otimes\hat{X}_B|01\rangle + \langle 10|\mathbb{1}_A\otimes\hat{X}_B|10\rangle\right.$$

$$= \frac{1}{2}\left(\underbrace{\langle 01|00\rangle}_{=0} - \underbrace{\langle 01|11\rangle}_{=0} - \underbrace{\langle 10|00\rangle}_{=0} + \underbrace{\langle 10|11\rangle}_{=0}\right)$$

$\langle B_0|\hat{X}_B|B_0\rangle = 0$ ☒

$$\bullet \langle B_0 | \vec{Y}_B | B_0 \rangle = \langle B_0 | \mathbb{1}_A \otimes \vec{Y}_A | B_0 \rangle = \frac{1}{2} \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

$$= \frac{i}{2} \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

$\langle B_0 | \vec{Y}_B | B_0 \rangle = 0$ ⊠

$$\bullet \langle B_0 | \vec{Z}_B | B_0 \rangle = \frac{1}{2} (\langle 01| - \langle 10|) (\mathbb{1}_A \otimes \vec{Z}_B)(|01\rangle - |10\rangle)$$

$$= \frac{1}{2}(\langle 01 | \mathbb{1}_A \otimes \vec{Z}_B | 01 \rangle - \langle 01 | \mathbb{1}_A \otimes \vec{Z}_B | 10 \rangle - \langle 10 | \mathbb{1}_A \otimes \vec{Z}_B | 01 \rangle + \langle 10 | \mathbb{1}_A \otimes \vec{Z}_B | 10 \rangle)$$

$$= \frac{1}{2} \left(-\underbrace{\langle 01 | 01 \rangle}_{=1} - \underbrace{\langle 01 | 10 \rangle}_{=0} + \underbrace{\langle 10 | 01 \rangle}_{=0} + \underbrace{\langle 10 | 10 \rangle}_{=1} \right)$$

$$= \frac{1}{2} (-1 + 1)$$

$\langle B_0 | \vec{Z}_B | B_0 \rangle = 0$ ⊠

Exercise 29:

$$\langle B_0 | \vec{X}_A \otimes \vec{X}_B | B_0 \rangle = \frac{1}{2} (\langle 01| - \langle 10|) (\vec{X}_A \otimes \vec{X}_B)(|01\rangle - |10\rangle)$$

$$= \frac{1}{2} (\langle 01 | \vec{X}_A \otimes \vec{X}_B | 01 \rangle - \langle 01 | \vec{X}_A \otimes \vec{X}_B | 10 \rangle - \langle 10 | \vec{X}_A \otimes \vec{X}_B | 01 \rangle + \langle 10 | \vec{X}_A \otimes \vec{X}_B | 10 \rangle)$$

$$= \frac{1}{2} \left(\underbrace{\langle 01 | 10 \rangle}_{=0} - \underbrace{\langle 01 | 01 \rangle}_{=1} - \underbrace{\langle 10 | 10 \rangle}_{=1} + \underbrace{\langle 10 | 01 \rangle}_{=0} \right)$$

$$= \frac{1}{2} (-1 - 1)$$

$\langle B_0 | \vec{X}_A \otimes \vec{X}_B | B_0 \rangle = -1$ ⊠

$$\bullet \langle B_0 | \vec{Y}_A \otimes \vec{Y}_A | B_0 \rangle = \frac{1}{2} \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{2} (-1 - 1)$$

$\langle B_0 | \vec{Y}_A \otimes \vec{Y}_A | B_0 \rangle = -1$ ⊠

$$\bullet \langle B_0 | \vec{Z}_A \otimes \vec{Z}_B | B_0 \rangle = \frac{1}{2} (\langle 01| - \langle 10|) (\vec{Z}_A \otimes \vec{Z}_B)(|01\rangle - |10\rangle)$$

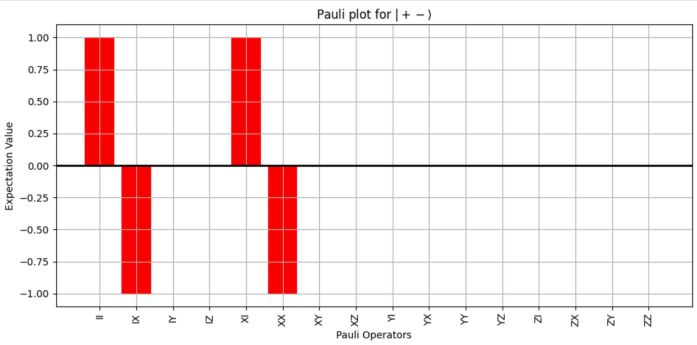
$$= \frac{1}{2}(\langle 01 | \vec{Z}_A \otimes \vec{Z}_B | 01 \rangle - \langle 01 | \vec{Z}_A \otimes \vec{Z}_B | 10 \rangle - \langle 10 | \vec{Z}_A \otimes \vec{Z}_B | 01 \rangle + \langle 10 | \hat{Z}_A \otimes \vec{Z}_B | 10 \rangle)$$

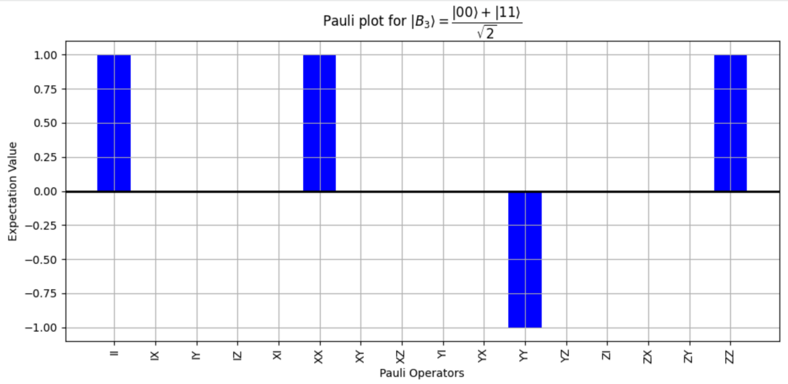
$$= \frac{1}{2}\Big(-\underbrace{\langle 01|01\rangle}_{=1} + \underbrace{\langle 01|10\rangle}_{=0} + \underbrace{\langle 10|01\rangle}_{=0} - \underbrace{\langle 10|10\rangle}_{=1}\Big)$$

$$= \frac{1}{2}(-1-1)$$

$\langle B_0|\vec{Z}_A \otimes \vec{Z}_B|B_0\rangle = -1$ $\boxtimes$

Exercise 30:





Axis not displayed represent a mean of 0.

Exercise 31:

(i) Alice measures +1 in her lab:

If A measures $Z_A = 1$ and B does nothing, the state after measurement is:

$$|\psi\rangle = \frac{(|0\rangle\langle 0|_A \otimes \mathbb{1}_B)}{\underbrace{\sqrt{\langle B_3|\,(|0\rangle\langle 0|_A \otimes \mathbb{1}_B)\,|B_3\rangle}}_{\text{normalization}}}\,|B_3\rangle$$

Proba : $\langle B_3|\,(|0\rangle\langle 0|_A \otimes \mathbb{1}_B)\,|B_3\rangle = \frac{1}{2}(\langle 00| + \langle 11|)\,(|0\rangle\langle 0|_A \otimes \mathbb{1}_B)\,(|00\rangle + |11\rangle) = \frac{1}{2}(\langle 00| + \langle 11|)\big(|0\rangle\underbrace{\langle 0|0\rangle_A}_{=1} \otimes \mathbb{1}_B|0\rangle + |0\rangle\underbrace{\langle 0|1\rangle_B}_{=0} \otimes \mathbb{1}_B|1\rangle_B\big)$

$$= \frac{1}{2}(\langle 00| + \langle 11|)\,(|00\rangle).$$

$\text{Prob}(Z_A = 1) = \langle B_3|\,(|0\rangle\langle 0|_A \otimes \mathbb{1}_B)\,|B_3\rangle = \frac{1}{2}$

normalization: $\sqrt{\langle B_3|\,(|0\rangle\langle 0|_A \otimes \mathbb{1}_B)\,|B_3\rangle} = \frac{1}{\sqrt{2}}$

state after measurement in Alice's lab: $|\psi\rangle = \dfrac{(|0\rangle\langle 0|_A \otimes \mathbb{1}_B)}{1/\sqrt{2}}\,|B_3\rangle$

$$|\psi\rangle = \frac{1/\sqrt{2}}{1/\sqrt{2}}\,(|0\rangle\langle 0|_A \otimes \mathbb{1}_B)\,(|00\rangle + |11\rangle)$$

$$|\psi\rangle = |0\rangle\underbrace{\langle 0|0\rangle}_{=1} \otimes \mathbb{1}_B|0\rangle_B + |0\rangle\underbrace{\langle 0|1\rangle_A}_{=0} \otimes \mathbb{1}_B|1\rangle$$

$$|\psi\rangle = |0\rangle_A \otimes |0\rangle_B$$

Now Bob measure in his lab without knowing the state of his qubit. We need to use state $|\psi\rangle$ since Alice's measurement has projected initial state. Since Bob measures $\vec{Z}_B$, the two possible outcomes are $|0\rangle_B$ or $|1\rangle_B$ for his partite. We again use projector:

- Bob measures -1: $|\psi'\rangle \propto (\mathbb{1}_A \otimes |1\rangle\langle 1|_B)|\psi\rangle = (\mathbb{1}_A \otimes |1\rangle\langle 1|)|00\rangle_{AB} = \mathbb{1}_A|0\rangle_A \otimes |1\rangle\underbrace{\langle 1|0\rangle_B}_{=0} = 0$

Also $\text{Prob}(\hat{z}_B = -1 | z_A = +1) = \langle\psi|(\mathbb{1}_A \otimes |1\rangle\langle 1|_B)|\psi\rangle = \langle 00|(\mathbb{1}_A|0\rangle \otimes |1\rangle\underbrace{\langle 1|0\rangle_B}_{=0}) = 0$

We see that the probability that Bob measures -1 in his lab is 0 if Alice measured $+1$.

- Bob measure $+1$: $|\psi'\rangle \propto (\mathbb{1}_A \otimes |0\rangle\langle 0|_B)|\psi\rangle = (\mathbb{1}_A \otimes |0\rangle\langle 0|_A)|00\rangle = \mathbb{1}_A|0\rangle \otimes |0\rangle\underbrace{\langle 0|0\rangle_B}_{=1}$

$\propto |0\rangle_A \otimes |0\rangle_B$ "already normalized"

$|\psi'\rangle = |0\rangle_A \otimes |0\rangle_B = |00\rangle$

$\text{Prob}(z_B = +1 | z_A = +1) = \langle\psi|(\mathbb{1}_A \otimes |0\rangle\langle 0|)|\psi\rangle = \langle 00|00\rangle = 1.$

$\Leftrightarrow$ normalization $= 1$

State after measurement in Bob's lab is still $|\psi'\rangle = |00\rangle$.

We see that state is unchanged and Bob measures $+1$ with probability 1 if Alice measured $+1$ previously

not asked?

(ii) Alice measures -1 in her lab:

If A measures $z_A = -1$ and B does nothing, the state after measurement is:

$$|\psi\rangle = \frac{(|1\rangle\langle 1|_A \otimes \mathbb{1}_B)}{\underbrace{\sqrt{\langle B_3|(|1\rangle\langle 1|_A \otimes \mathbb{1}_B)|B_3\rangle}}_{\text{normalization}}}|B_3\rangle$$

Proba: $\langle B_3|(|1\rangle\langle 1|_A \otimes \mathbb{1}_B)|B_3\rangle = \frac{1}{2}(\langle 00| + \langle 11|)(|1\rangle\langle 1|_A \otimes \mathbb{1}_B)(|00\rangle + |11\rangle) = \frac{1}{2}(\langle 00| + \langle 11|)(|1\rangle\underbrace{\langle 1|0\rangle_A}_{=0} \otimes \mathbb{1}_B|0\rangle - |1\rangle\underbrace{\langle 1|1\rangle_B}_{=1} \otimes \mathbb{1}_B|1\rangle_B)$

$= \frac{1}{2}(\langle 00| + \langle 11|)(|11\rangle).$

$\text{Prob}(z_A = -1) = \langle B_3|(|1\rangle\langle 1|_A \otimes \mathbb{1}_B)|B_3\rangle = \frac{1}{2}$

normalization: $\sqrt{\langle B_3|(|1\rangle\langle 1|_A \otimes \mathbb{1}_B)|B_3\rangle} = \frac{1}{\sqrt{2}}$

state after measurement in Alice's lab: $|\psi\rangle = \dfrac{(|1\rangle\langle 1|_A \otimes \mathbb{1}_B)|B_3\rangle}{1/\sqrt{2}}$

$|\psi\rangle = \frac{1/\sqrt{2}}{1/\sqrt{2}}(|1\rangle\langle 1|_A \otimes \mathbb{1}_B)(|00\rangle + |11\rangle)$

$|\psi\rangle = |1\rangle\underbrace{\langle 1|0\rangle}_{=0} \otimes \mathbb{1}_B|0\rangle_B + |1\rangle\underbrace{\langle 1|1\rangle_A}_{=1} \otimes \mathbb{1}_B|1\rangle$

$|\psi\rangle = |1\rangle_A \otimes |1\rangle_B$

Now Bob measures in his lab without knowing the state of his qubit. We need to use state $|\psi\rangle$ since Alice's measurement has projected initial state. Since Bob measures $\vec{z}_B$, the two possible outcomes are $|0\rangle_B$ or $|1\rangle_B$ for his partite. We again use projector:

- Bob measures +1: $|\psi'\rangle \propto (\mathbb{1}_A \otimes |0\rangle\langle 0|_B)|\psi\rangle = (\mathbb{1}_A \otimes |0\rangle\langle 0|)|11\rangle_{AB} = \mathbb{1}_A|1\rangle_A \otimes |0\rangle\underbrace{\langle 0|1\rangle_B}_{=0} = 0$

Also $\text{Prob}(\hat{z}_B = +1 | z_A = -1) = \langle\psi|(\mathbb{1}_A \otimes |0\rangle\langle 0|_B)|\psi\rangle = \langle 11|(\mathbb{1}_A|1\rangle \otimes |0\rangle\underbrace{\langle 0|1\rangle_B}_{=0}) = 0$

We see that the probability that Bob measures +1 in his lab is 0 if Alice measured −1.

- Bob measure −1: $|\psi'\rangle \propto (\mathbb{1}_A \otimes |1\rangle\langle 1|_B)|\psi\rangle = (\mathbb{1}_A \otimes |1\rangle\langle 1|_A)|11\rangle = \mathbb{1}_A|1\rangle \otimes |1\rangle\underbrace{\langle 1|1\rangle_B}_{=1}$

$\propto |1\rangle_A \otimes |1\rangle_B$ "already normalized"

$|\psi'\rangle = |1\rangle_A \otimes |1\rangle_B = |11\rangle$

$\text{Prob}(z_B = -1 | z_A = -1) = \langle\psi|(\mathbb{1}_A \otimes |1\rangle\langle 1|)|\psi\rangle = \langle 11|11\rangle = 1.$

$\Leftrightarrow$ normalization $= 1$

State after measurement in Bob's lab $|\psi'\rangle = |11\rangle$.

We see that state is unchanged and Bob measures +1 with probability 1 if Alice measured +1 previously

Conclusion:

- If Alice measured $z_A = +1$, Bob measures $z_B = +1$ with probability 1 so always.
- If Alice measured $z_A = -1$, Bob measures $z_B = -1$ with probability 1 so always. ∎

Exercise 32:

(i) Alice measures $\hat{x}_A = +1$ and then Bob measures $z_B = +1$:

Alice measures +1 so we project state on $|+\rangle\langle +|_A$ but Bob does nothing:

state after measurement: $|\psi\rangle = \dfrac{(|+\rangle\langle +|_A \otimes \mathbb{1}_B)}{\underbrace{\sqrt{\langle B_3|(|+\rangle\langle +|_A \otimes \mathbb{1}_B)|B_3\rangle}}_{\text{normalization:}}}|B_3\rangle.$

$\text{Prob}(\hat{x}_A = 1) = \langle B_3|(|+\rangle\langle +|_A \otimes \mathbb{1}_B)|B_3\rangle = \frac{1}{2}(\langle 00| + \langle 11|)(|+\rangle\langle +|_A \otimes \mathbb{1}_B)(|00\rangle + |11\rangle).$

$= \frac{1}{2}(\langle 00| + \langle 11|)(|+\rangle\langle +|0\rangle_A \otimes \mathbb{1}_B|0\rangle + |+\rangle\langle +|1\rangle_A \otimes \mathbb{1}_B|1\rangle)$

$= \frac{1}{2}(\langle 00| + \langle 11|)(\frac{1}{\sqrt{2}}|+\rangle_A \otimes |0\rangle_B + \frac{1}{\sqrt{2}}|+\rangle_A \otimes |1\rangle_B)$

$= \frac{1}{2\sqrt{2}}(\underbrace{\langle 0|+\rangle_A}_{=1/\sqrt{2}}\underbrace{\langle 0|0\rangle_B}_{=1} + \langle 0|+\rangle_A\underbrace{\langle 0|1\rangle_B}_{=0} + \langle 1|+\rangle_A\underbrace{\langle 1|0\rangle_B}_{=0} + \underbrace{\langle 1|+\rangle_A}_{=1/\sqrt{2}}\underbrace{\langle 1|1\rangle_B}_{=0})$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right.$$

$$Prob(\hat{x}_A = +1) = \frac{1}{2}$$

normalization: $\sqrt{\langle B_3 | (|+\rangle\langle +|_A \otimes 1\!\!1_B) | B_3 \rangle} = \frac{1}{\sqrt{2}}$

state after Alice's measurement: $|\psi\rangle = \frac{|+\rangle\langle +|_A \otimes 1\!\!1_B}{1/\sqrt{2}} \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|+\rangle_A \otimes |0\rangle_B + |+\rangle_A \otimes |1\rangle_B \right) = |+\rangle_A \otimes \frac{|0\rangle_B + |1\rangle_B}{\sqrt{2}}$$

Now Bob measures in his lab without knowing the state of his qubit. We need to use state $|\psi\rangle$ since Alice's measurement has projected initial state. Since Bob measures $\vec{z}_B$, the two possible outcomes are $|0\rangle_B$ or $|1\rangle_B$ for his partite. We again use projector:

- Bob measures $z_B = +1$:

This projects Bob's partite onto $|0\rangle$, we use projector $|0\rangle\langle 0|_B$:

$$|\psi'\rangle = \frac{1\!\!1_A \otimes |0\rangle\langle 0|_B}{\sqrt{\langle \psi | (1\!\!1_A \otimes |0\rangle\langle 0|_B) | \psi \rangle}} |\psi\rangle$$

$$Prob(z_B = +1 | x_A = +1) = \langle \psi | (1\!\!1_A \otimes |0\rangle\langle 0|_B) | \psi \rangle = \left(\langle +|_A \otimes \frac{\langle 0|_B + \langle 1|_B}{\sqrt{2}} \right) \left(1\!\!1_A \otimes |0\rangle\langle 0|_B \right) \left(|+\rangle_A \otimes \frac{|0\rangle + |1\rangle_B}{\sqrt{2}} \right)$$

$$= \left(\langle +|_A \otimes \frac{\langle 0|_B + \langle 1|_B}{\sqrt{2}} \right) \left(|+\rangle_A \otimes \frac{|0\rangle\langle 0|0\rangle_B + |0\rangle\langle 0|1\rangle_B}{\sqrt{2}} \right)$$

$$= \left(\langle +|_A \otimes \frac{\langle 0|_B + \langle 1|_B}{\sqrt{2}} \right) \left(|+\rangle_A \otimes \frac{1}{\sqrt{2}} |0\rangle_B \right)$$

$$= \underbrace{\langle + | + \rangle_A}_{=1} \cdot \frac{1}{2} \left(\underbrace{\langle 0 | 0 \rangle_B}_{=1} + \underbrace{\langle 1 | 0 \rangle_B}_{=0} \right)$$

$$Prob(z_B = +1 | x_A = +1) = \frac{1}{2}$$

normalization: $\sqrt{\langle \psi | (1\!\!1_A \otimes |0\rangle\langle 0|_B) | \psi \rangle} = \frac{1}{\sqrt{2}}$

state after Bob's measurement: $|\psi'\rangle = \frac{1\!\!1_A \otimes |0\rangle\langle 0|_B}{\sqrt{\langle \psi | (1\!\!1_A \otimes |0\rangle\langle 0|_B) | \psi \rangle}} |\psi\rangle = \frac{1\!\!1_A \otimes |0\rangle\langle 0|_B}{1/\sqrt{2}} \left(|+\rangle_A \otimes \frac{|0\rangle + |1\rangle_B}{\sqrt{2}} \right)$

$$|\psi'\rangle = |+\rangle_A \otimes \left(|0\rangle \underbrace{\langle 0 | 0 \rangle_B}_{=1} + |0\rangle \underbrace{\langle 0 | 1 \rangle_B}_{=0} \right)$$

$$|\psi'\rangle = |+\rangle_A \otimes |0\rangle_B = \frac{1}{\sqrt{2}} (|00\rangle + |10\rangle)$$

=> If A measures $\vec{x}_A = +1$ then if B measures on $\vec{z}_A$. He will record $+1$ with Proba $(z_B = +1 | \vec{x}_A = +1) = \frac{1}{2}$ and final state after all measurements is $|\psi'\rangle = |+\rangle_A \otimes |0\rangle_B$

(ii) Alice measures $\hat{X}_A = +1$ and then Bob measures $Z_B = -1$:

• Bob measures $Z_B = -1$:

This projects Bob's partite onto $|1\rangle$, we use projector $|1\rangle\langle 1|_B$:

$$|\psi'\rangle = \frac{\mathbb{1}_A \otimes |1\rangle\langle 1|_B}{\sqrt{\langle\psi|(\mathbb{1}_A \otimes |1\rangle\langle 1|_B)|\psi\rangle}} |\psi\rangle$$

$$\text{Prob}(Z_B = -1 \mid X_A = +1) = \langle\psi|(\mathbb{1}_A \otimes |1\rangle\langle 1|_B)|\psi\rangle = \left(\langle +|_A \otimes \frac{\langle 0|_B + \langle 1|_B}{\sqrt{2}}\right)\left(\mathbb{1}_A \otimes |1\rangle\langle 1|_B\right)\left(|+\rangle_A \otimes \frac{|0\rangle + |1\rangle_B}{\sqrt{2}}\right)$$

$$= \left(\langle +|_A \otimes \frac{\langle 0|_B + \langle 1|_B}{\sqrt{2}}\right)\left(|+\rangle_A \otimes \frac{|1\rangle\langle 1|0\rangle_B + |1\rangle\langle 1|1\rangle_B}{\sqrt{2}}\right)$$

$$= \left(\langle +|_A \otimes \frac{\langle 0|_B + \langle 1|_B}{\sqrt{2}}\right)\left(|+\rangle_A \otimes \frac{1}{\sqrt{2}} |1\rangle_B\right)$$

$$= \underbrace{\langle +|+\rangle_A}_{=1} \cdot \frac{1}{2}\left(\underbrace{\langle 0|1\rangle_B}_{=0} + \underbrace{\langle 1|1\rangle_B}_{=1}\right)$$

$$\text{Prob}(Z_B = -1 \mid X_A = +1) = \frac{1}{2}$$

normalization: $\sqrt{\langle\psi|(\mathbb{1}_A \otimes |1\rangle\langle 1|_B)|\psi\rangle} = \frac{1}{\sqrt{2}}$

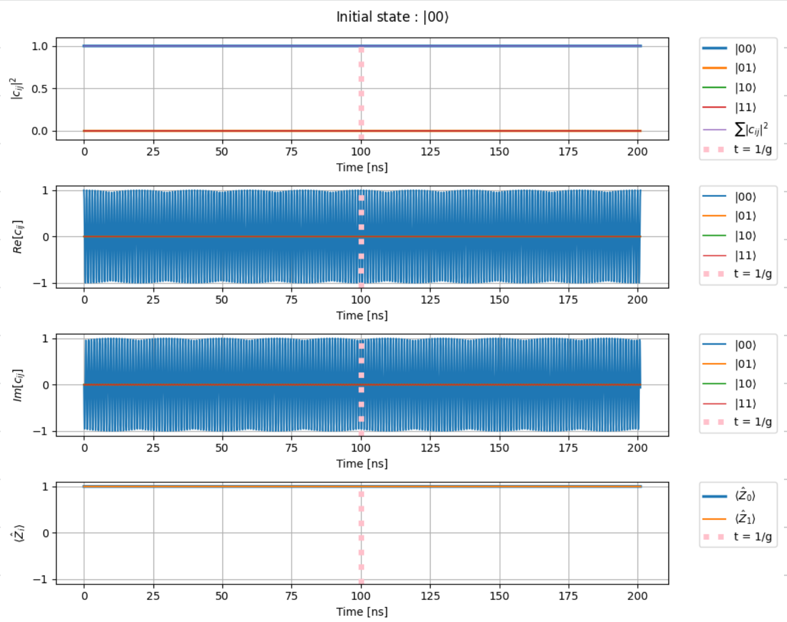
state after Bob's measurement: $|\psi'\rangle = \frac{\mathbb{1}_A \otimes |1\rangle\langle 1|_B}{\sqrt{\langle\psi|(\mathbb{1}_A \otimes |1\rangle\langle 1|_B)|\psi\rangle}} |\psi\rangle = \frac{\mathbb{1}_A \otimes |1\rangle\langle 1|_B}{1/\sqrt{2}}\left(|+\rangle_A \otimes \frac{|0\rangle + |1\rangle_B}{\sqrt{2}}\right)$

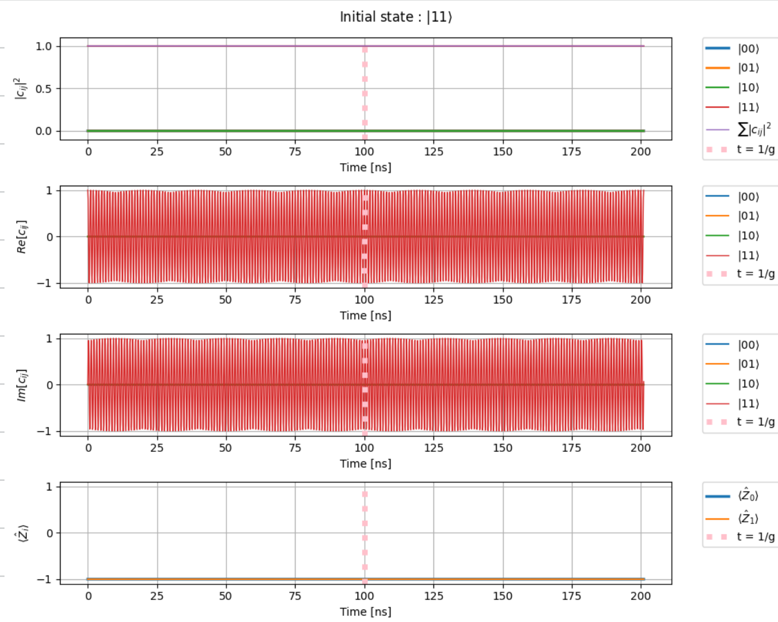
$$|\psi'\rangle = |+\rangle_A \otimes \left(|1\rangle\underbrace{\langle 1|0\rangle_B}_{=0} + |1\rangle\underbrace{\langle 1|1\rangle_B}_{=1}\right)$$

$$|\psi'\rangle = |+\rangle_A \otimes |1\rangle_B . = \frac{1}{\sqrt{2}}(|01\rangle + |11\rangle)$$

$\Rightarrow$ If A measures $\hat{X}_A = +1$ then if B measures on $\hat{Z}_B$, he will record -1 with Proba $(Z_B = -1 \mid \hat{X}_A = +1) = \frac{1}{2}$ and final state after all measurements is $|\psi'\rangle = |+\rangle_A \otimes |1\rangle_B$

Exercise 33:





<u>coeff c_{ij}:</u> we see that only $|c_{00}|=1$ and all the others are 0, meaning that initial state $|00\rangle$ stays $|00\rangle$ during the whole time evolution.

<u>Averages over observables:</u> One can see that averages over Z_1 (Z-spin component of qubit 1) stays at +1 and average over Z_2 (Z-spin component of qubit 2) stays also at +1. This again confirm that initial state $|00\rangle$ stays at $|00\rangle$ since each qubit stays at $|0\rangle$ in its partite.

<u>coeff c_{ij}:</u> we see that only $|c_{11}|=1$ and all the others are 0, meaning that initial state $|11\rangle$ stays $|11\rangle$ during the whole time evolution.

<u>Averages over observables:</u> One can see that averages over Z_1 (Z-spin component of qubit 1) stays at -1 and average over Z_2 (Z-spin component of qubit 2) stays also at -1. This again confirm that initial state $|11\rangle$ stays at $|11\rangle$ since each qubit stays at $|1\rangle$ in its partite.

<u>Conclusion:</u> We can conclude that in both case, the initial state stays the same during time evolution

<u>Why?:</u>

Hamiltonian:
$$H = \begin{bmatrix} -\frac{\omega_1}{2}-\frac{\omega_2}{2} & 0 & 0 & g \\ 0 & -\frac{\omega_1}{2}+\frac{\omega_2}{2} & g & 0 \\ 0 & g & \frac{\omega_1}{2}-\frac{\omega_2}{2} & 0 \\ g & 0 & 0 & \frac{\omega_1}{2}+\frac{\omega_2}{2} \end{bmatrix}$$

$$H = \begin{bmatrix} -2\pi & 0 & 0 & 0.02\pi \\ 0 & 0 & 0.02\pi & 0 \\ 0 & 0.02\pi & 0 & 0 \\ 0.02\pi & 0 & 0 & 2\pi \end{bmatrix}$$

eigenvalues:
$$eigenvalues = \left\{-\frac{\sqrt{4g^2+\omega_1^2-2\omega_1\omega_2+\omega_2^2}}{2} : 1,\ \frac{\sqrt{4g^2+\omega_1^2-2\omega_1\omega_2+\omega_2^2}}{2} : 1,\ -\frac{\sqrt{4g^2+\omega_1^2+2\omega_1\omega_2+\omega_2^2}}{2} : 1,\ \frac{\sqrt{4g^2+\omega_1^2+2\omega_1\omega_2+\omega_2^2}}{2} : 1\right\}$$

$$eigenvalues = \{-6.28349945859136 : 1,\ -0.0628318530717959 : 1,\ 0.0628318530717959 : 1,\ 6.28349945859136 : 1\}$$

$(g \ll \omega_1 = \omega_2)$
$(g = \frac{2\pi}{100}\ ;\ \omega_1 = 2\pi)$

eigenvectors:

$$eigenvectors = \left[\left(-\frac{\sqrt{4g^2+\omega_1^2-2\omega_1\omega_2+\omega_2^2}}{2}, 1, \left[\begin{bmatrix} 0 \\ \frac{-\omega_1+\omega_2}{2g}-\frac{\sqrt{4g^2+\omega_1^2-2\omega_1\omega_2+\omega_2^2}}{2g} \\ 1 \\ 0 \end{bmatrix}\right]\right), \left(\frac{\sqrt{4g^2+\omega_1^2-2\omega_1\omega_2+\omega_2^2}}{2}, 1, \left[\begin{bmatrix} 0 \\ \frac{-\omega_1+\omega_2}{2g}+\frac{\sqrt{4g^2+\omega_1^2-2\omega_1\omega_2+\omega_2^2}}{2g} \\ 1 \\ 0 \end{bmatrix}\right]\right), \left(-\frac{\sqrt{4g^2+\omega_1^2+2\omega_1\omega_2+\omega_2^2}}{2}, 1, \left[\begin{bmatrix} \frac{-\omega_1-\omega_2}{2g}-\frac{\sqrt{4g^2+\omega_1^2+2\omega_1\omega_2+\omega_2^2}}{2g} \\ 0 \\ 0 \\ 1 \end{bmatrix}\right]\right), \left(\frac{\sqrt{4g^2+\omega_1^2+2\omega_1\omega_2+\omega_2^2}}{2}, 1, \left[\begin{bmatrix} \frac{-\omega_1-\omega_2}{2g}+\frac{\sqrt{4g^2+\omega_1^2+2\omega_1\omega_2+\omega_2^2}}{2g} \\ 0 \\ 0 \\ 1 \end{bmatrix}\right]\right)\right]$$

$$eigenvectors = \left[\left(-0.0628318530717959, 1, \left[\begin{bmatrix} 0 \\ -0.707106781186548 \\ 0.707106781186548 \\ 0 \end{bmatrix}\right]\right), \left(0.0628318530717959, 1, \left[\begin{bmatrix} 0 \\ -0.707106781186548 \\ -0.707106781186548 \\ 0 \end{bmatrix}\right]\right), \left(-6.28349945859136, 1, \left[\begin{bmatrix} -0.999987500859308 \\ 0 \\ 0 \\ 0.00499981251210846 \end{bmatrix}\right]\right), \left(6.28349945859136, 1, \left[\begin{bmatrix} 0.00499981251210846 \\ 0 \\ 0 \\ 0.999987500859308 \end{bmatrix}\right]\right)\right]$$

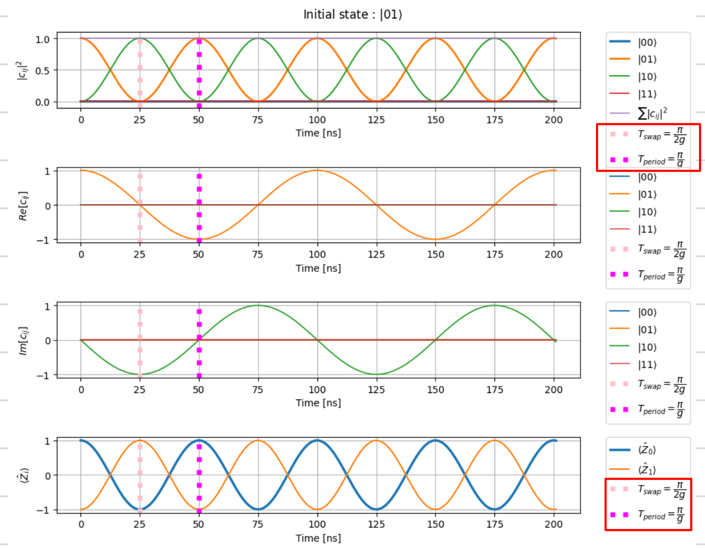
$\hookrightarrow |\lambda_3\rangle \cong -|00\rangle$ $\hookrightarrow |\lambda_4\rangle \cong |11\rangle$.

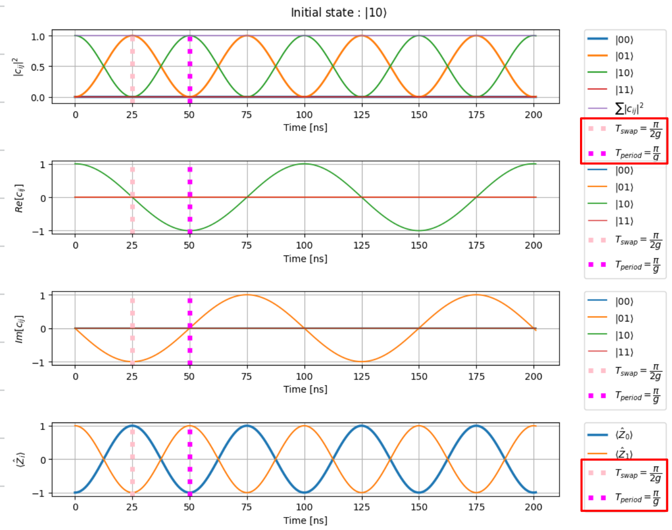
$$H|00\rangle = \begin{bmatrix} -2\pi \\ 0 \\ 0 \\ 0.02\pi \end{bmatrix} \cong -2\pi|00\rangle \qquad H|11\rangle = \begin{bmatrix} 0.02\pi \\ 0 \\ 0 \\ 2\pi \end{bmatrix} \cong 2\pi|11\rangle.$$

We see that for this config. of $g=\frac{2\pi}{100}$ and $\omega_1=\omega_2=2\pi$, $|00\rangle$ and $|11\rangle$ are eigenstates of this Hamiltonian. Therefore, they stay the same under time evolution with a global phase that is irrelevant. (Hamiltonian is time independant, we can solve Schroedinger equation such that $|\psi(t)\rangle = e^{-iHt/\hbar}|\psi_0\rangle = \sum_n e^{-iE_n/\hbar \cdot t}|\lambda_i\rangle\langle\lambda_i|\psi_0\rangle$. where $|11\rangle$ and $|00\rangle$ are two of the 4 eigenstate).

Exercise 34:

$$H = \begin{bmatrix} -\frac{\omega_1}{2}-\frac{\omega_2}{2} & 0 & 0 & g \\ 0 & -\frac{\omega_1}{2}+\frac{\omega_2}{2} & g & 0 \\ 0 & g & \frac{\omega_1}{2}-\frac{\omega_2}{2} & 0 \\ g & 0 & 0 & \frac{\omega_1}{2}+\frac{\omega_2}{2} \end{bmatrix}$$





$$H = \begin{bmatrix} -2\pi & 0 & 0 & 0{,}02\pi \\ 0 & 0 & 0{,}02\pi & 0 \\ 0 & 0{,}02\pi & 0 & 0 \\ 0{,}02\pi & 0 & 0 & 2\pi \end{bmatrix} = -2\pi|00\rangle\langle 00| + 0{,}02\pi|00\rangle\langle 11| + 0{,}02\pi|01\rangle\langle 10| + 0{,}02\pi|10\rangle\langle 01| + 0{,}02\pi|11\rangle\langle 00| + 2\pi|11\rangle\langle 11|.$$

$g = 0{,}02\pi$!

eigenvalue

$$\frac{H}{\hbar}\left(-\frac{1}{\sqrt2}|01\rangle+\frac{1}{\sqrt2}|10\rangle\right) = -\frac{0{,}02\pi}{\sqrt2}|10\rangle + \frac{0{,}02\pi}{\sqrt2}|01\rangle = -0{,}02\pi\left(-\frac{1}{\sqrt2}|01\rangle+\frac{1}{\sqrt2}|10\rangle\right).$$
eigenvector: $-\frac{1}{\sqrt2}|01\rangle + \frac{1}{\sqrt2}|10\rangle$

$$\frac{H}{\hbar}\left(-\frac{1}{\sqrt2}|01\rangle-\frac{1}{\sqrt2}|10\rangle\right) = -\frac{0{,}02\pi}{\sqrt2}|10\rangle - \frac{0{,}02\pi}{\sqrt2}|10\rangle = 0{,}02\pi\left(-\frac{1}{\sqrt2}|01\rangle+\frac{1}{\sqrt2}|10\rangle\right).$$
eigenvector: $-\frac{1}{\sqrt2}|01\rangle - \frac{1}{\sqrt2}|10\rangle$

eigenvalue

$$U = e^{-i\frac{H}{\hbar}t} \cong e^{+i2\pi t}\left[-|00\rangle(-\langle 00|)\right] + e^{-i2\pi t}|11\rangle\langle 11| + e^{+i0{,}02\pi t}\left(-\frac{1}{\sqrt2}|01\rangle+\frac{1}{\sqrt2}|10\rangle\right)\left(-\frac{1}{\sqrt2}\langle 01|+\frac{1}{\sqrt2}\langle 10|\right)$$
$$+ e^{-i0{,}02\pi t}\left(-\frac{1}{\sqrt2}|01\rangle-\frac{1}{\sqrt2}|10\rangle\right)\left(-\frac{1}{\sqrt2}\langle 01|-\frac{1}{\sqrt2}\langle 10|\right)$$

Initial state $|01\rangle$:

- $|\psi_{01}(t)\rangle = U|01\rangle = e^{+i0{,}02\pi t}\left(-\frac{1}{\sqrt2}|01\rangle+\frac{1}{\sqrt2}|10\rangle\right)\left(-\frac{1}{\sqrt2}\right) + e^{-i0{,}02\pi t}\left(-\frac{1}{\sqrt2}|01\rangle-\frac{1}{\sqrt2}|10\rangle\right)\left(-\frac{1}{\sqrt2}\right).$

$$= \frac{1}{2}\left(e^{+i0{,}02\pi t}+e^{-i0{,}02\pi t}\right)|01\rangle - \frac{1}{2}\left(e^{+i0{,}02\pi t}-e^{-i0{,}02\pi t}\right)|10\rangle$$

$$|\psi_{01}(t)\rangle = \cos(\underbrace{0{,}02\pi}_{=g} t)|01\rangle - i\sin(\underbrace{0{,}02\pi}_{=g} t)|10\rangle$$

- $P(|\psi_{01}(t)\rangle = |01\rangle) = |c_{01}|^2 = \cos^2(\underbrace{0{,}02\pi}_{=g} t)$ $\quad T = \frac{\pi}{\omega} = \frac{\pi}{g} = \frac{\pi}{2\pi/100} = \frac{100}{2} = 50$ (see graph $|01\rangle$ above)
- $P(|\psi_{01}(t)\rangle = |10\rangle) = |c_{10}|^2 = \sin^2(\underbrace{0{,}02\pi}_{=g} t).$ $\quad T = \frac{\pi}{\omega} = \frac{\pi}{g} = 50$ (see graph $|01\rangle$ above)

One can see that the proba of being in state $|01\rangle$ is 1 (cosine) every 50 ns and proba of being $|10\rangle$ is also 1 every 50ns also (but sine). The period of alternating between the two states is therefore $T_{swap} = \frac{T}{2} = \frac{\pi}{2g} = 25$ns
As we can see in graph $|01\rangle$ above, $|c_{01}|^2$ (Proba of being in state $|01\rangle$) is 1 but $|c_{10}|^2$ is 0 and 25 ns later $|c_{01}|^2$ is 0 and $|c_{10}|^2$ is 1 (Proba of being in $|10\rangle$).

$$\boxed{T_{swapp} = \frac{\pi}{2g} = 25 \text{ ns}}$$

Initial state $|10\rangle$:

- $|\psi_{10}(t)\rangle = \mathcal{U}|10\rangle = e^{+i\,0{,}02\pi t}\left(-\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle\right)\left(+\frac{1}{\sqrt{2}}\right) + e^{-i\,0{,}02\pi t}\left(-\frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle\right)\left(-\frac{1}{\sqrt{2}}\right).$

$$= -\frac{1}{2}\left(e^{+i0{,}02\pi t} - e^{-i0{,}02\pi t}\right)|01\rangle + \frac{1}{2}\left(e^{+i0{,}02\pi t} + e^{-i0{,}02\pi t}\right)|10\rangle$$

$$|\psi_{10}(t)\rangle = -i\sin(0{,}02\pi t)|01\rangle + \cos(0{,}02\pi t)|10\rangle$$

- $P(|\psi_{10}(t)\rangle = |01\rangle) = |c_{01}|^2 = \sin^2(0{,}02\pi t)$ $\qquad T = \frac{\pi}{\omega} = \frac{\pi}{g} = \frac{\pi}{2\pi/100} = \frac{100}{2} = 50$ (see graph $|10\rangle$ above)
- $P(|\psi_{10}(t)\rangle = |10\rangle) = |c_{10}|^2 = \cos^2(0{,}02\pi t).$ $\qquad T = \frac{\pi}{\omega} = \frac{\pi}{g} = 50$ (see graph $|01\rangle$ above)

One can see that the proba of being in state $|01\rangle$ is 1 (sine) every 50 ns and proba of being $|10\rangle$ is also 1 every 50ns also (but cosine). The period of alternating between the two states is therefore $T_{swap} = \frac{T}{2} = \frac{\pi}{2g} = 25$ns. As we can see in graph $|10\rangle$ above, $|c_{01}|^2$ (Proba of being in state $|01\rangle$) is 1 but $|c_{10}|^2$ is 0 and 25 ns later $|c_{01}|^2$ is 0 and $|c_{10}|^2$ is 1 (Proba of being in $|10\rangle$).

$$\boxed{T_{swap} = \frac{\pi}{2g} = 25\,ns}$$

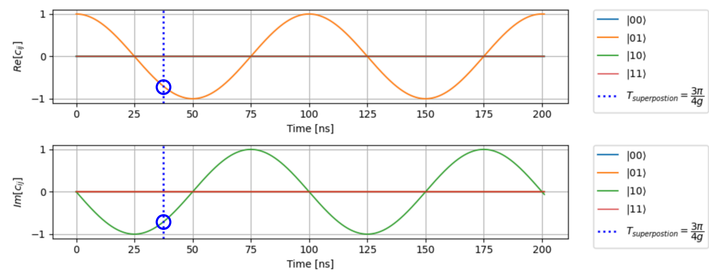
Exercise 35:

Taking what we used in previous question we can rewrite:

- Initial state $|01\rangle$:

$|\psi_{01}(t)\rangle = \mathcal{U}|01\rangle = \cos(gt)|01\rangle - i\sin(gt)|10\rangle$ $\qquad \tau = \frac{3}{8}\cdot\frac{2\pi}{g}.$

$\times(-1)$ what we want but global phase so irrelevant



$|\psi_{01}(t = \frac{3}{8}\cdot\frac{2\pi}{g})\rangle = \cos(\frac{3\pi}{4g}g)|01\rangle - i\sin(\frac{3\pi}{4g}g)|10\rangle$

$|\psi_{01}(\tau)\rangle = \cos(\frac{3\pi}{4})|01\rangle - i\sin(\frac{3\pi}{4})|10\rangle.$

$|\psi_{01}(\tau)\rangle = -\frac{1}{\sqrt{2}}|01\rangle - i\frac{1}{\sqrt{2}}|10\rangle.$

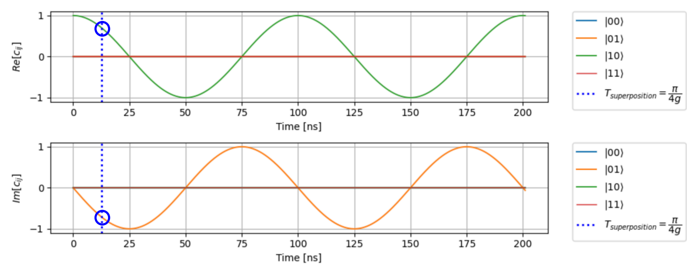
$|\psi_{01}(\tau)\rangle = \frac{e^{i\pi}}{\sqrt{2}}\left(|01\rangle + i|10\rangle\right)$ $\qquad$ (the global phase is irrelevant)

$\Leftrightarrow |\psi_{01}(\tau)\rangle \equiv \frac{1}{\sqrt{2}}\left(|01\rangle + i|10\rangle\right)$ $\qquad$ for $\tau = \frac{3}{8}\cdot\frac{2\pi}{g} = \frac{3\pi}{4g} = \frac{3}{4}T_{period}.$

- Initial state $|10\rangle$:

$|\psi_{10}(t)\rangle = \mathcal{U}|10\rangle = -i\sin(gt)|01\rangle + \cos(gt)|10\rangle$ $\qquad \tau = \frac{1}{4}\frac{\pi}{g}$

$\times(-i)$ what we want but global phase so irrelevant



$|\psi_{10}(t = \frac{1}{8}\frac{2\pi}{g})\rangle = -i\sin(\frac{\pi}{4}g)|01\rangle + \cos(\frac{\pi}{4g}g)|10\rangle$

$|\psi_{10}(\tau)\rangle = -i\sin(\frac{\pi}{4})|01\rangle + \cos(\frac{\pi}{4})|10\rangle.$

$|\psi_{10}(\tau)\rangle = -i\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle.$

$$|\psi_{10}(\tau)\rangle = \frac{e^{i\pi}}{\sqrt{2}} e^{i\pi/2}\left(|01\rangle - i|10\rangle\right) = \frac{e^{i3\pi/2}}{\sqrt{2}}\left(|01\rangle - i|10\rangle\right)$$ (the global phase is irrelevant)

$$\Longleftrightarrow \; |\psi_{10}(\tau)\rangle \equiv \frac{1}{\sqrt{2}}\left(|01\rangle - i|10\rangle\right) \quad \text{for } \tau = \frac{\pi}{4g} = \frac{1}{4} T_{\text{period}}$$

Exercise 36:

$$S = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \qquad S|0\rangle = |0\rangle \qquad S|1\rangle = i|1\rangle$$

$$(S_A \otimes \mathbb{1}_B)\left(|01\rangle + i|10\rangle\right) = S_A|0\rangle_A \otimes \mathbb{1}_B|1\rangle_B + i S_A|1\rangle \otimes \mathbb{1}_B|0\rangle_B$$

$$= |01\rangle + i(i)|10\rangle$$

$$= |01\rangle - |10\rangle$$

$$(S_A \otimes \mathbb{1}_B)\left(|01\rangle + i|10\rangle\right) = |01\rangle - |10\rangle = |B_0\rangle$$

We need to apply a phase gate to Qubit A.

Exercise 37:

We recall that: $\frac{\hat{H}}{\hbar} = -\omega|00\rangle\langle 00| + g|00\rangle\langle 11| + g|01\rangle\langle 10| + g|10\rangle\langle 01| + g|11\rangle\langle 11| + \omega|11\rangle\langle 11|$.

Considering that $|00\rangle$ and $|11\rangle$ don't evoluate, we can troncate our Hamiltonian by considering the evolution of the two state $|01\rangle$ and $|10\rangle$, and we are reduced to a 2-level qubit system. Doing so, we map $|01\rangle \equiv |\uparrow\rangle \equiv |0\rangle$ (north pole of the Bloch Sphere) and $|10\rangle \equiv |\downarrow\rangle \equiv |1\rangle$ (south pole of the Bloch Sphere).

$\frac{\hat{H}^{4\times4}}{\hbar} \simeq g\left(|01\rangle\langle 10| + |10\rangle\langle 01|\right) \Rightarrow$ Plays the role of an $\vec{X}$-gate flipping $|01\rangle$ to $|10\rangle$ and vice versa).

$\Rightarrow \frac{\hat{H}^{2\times2}}{\hbar} \equiv g\left(|0\rangle\langle 1| + |1\rangle\langle 0|\right) = (g)\,\hat{X} = (2g).\hat{X}/2$. with $g = 0{,}02\pi$ and $\omega_1 = \omega_2 = \omega = 2\pi$

$$\exp\left(i\frac{a}{2}\mathbf{n}\cdot\vec{\sigma}\right) = (\cos\frac{a}{2})I + i(\sin\frac{a}{2})\mathbf{n}\cdot\vec{\sigma}$$

$$\mathcal{U}_t = e^{-i(2g)\frac{\hat{X}}{2}t} = \cos(gt)\mathbb{1} - i\sin(gt)\hat{X}.$$

$|\psi(t)_{01}\rangle = \mathcal{U}_t \underset{|01\rangle}{\underset{\equiv}{|0\rangle}} = \cos(gt)|0\rangle - i\sin(gt)|1\rangle$ in order to be in state $|1\rangle$ we need $gT_{\text{Swap}} = \frac{\pi}{2}$

$$\Rightarrow T_{\text{swap}} = \frac{\pi}{2g}.$$

Under time evolution, this will make the qubit $|0\rangle \equiv |01\rangle$ evoluate by doing a rotation around $\hat{X}$ axis and flipping to $|1\rangle \equiv |10\rangle$ with a period of $T = \frac{\pi}{2g}$ as expected in our previous 4x4 Hamiltonian.

The mathematical way of seeing this is the following:

we consider the original Hamiltonian:

$$H = \begin{bmatrix} -\frac{\omega_1}{2} - \frac{\omega_2}{2} & 0 & 0 & g \\ 0 & -\frac{\omega_1}{2} + \frac{\omega_2}{2} & g & 0 \\ 0 & g & \frac{\omega_1}{2} - \frac{\omega_2}{2} & 0 \\ g & 0 & 0 & \frac{\omega_1}{2} + \frac{\omega_2}{2} \end{bmatrix}$$

$$= \left(-\frac{\omega_1}{2} - \frac{\omega_2}{2}\right)|00\rangle\langle 00| + g|00\rangle\langle 11| + \left(-\frac{\omega_1}{2} + \frac{\omega_2}{2}\right)|01\rangle\langle 01| + g|01\rangle\langle 10| + \dots$$
$$+ g|10\rangle\langle 01| + \left(\frac{\omega_1}{2} - \frac{\omega_2}{2}\right)|10\rangle\langle 10| + g|11\rangle\langle 00| + \left(\frac{\omega_1}{2} + \frac{\omega_2}{2}\right)|11\rangle\langle 11| .$$

$$\frac{\hat{H}^{2\times 2}}{\hbar} = \begin{bmatrix} \langle 01|\hat{H}|01\rangle & \langle 01|\hat{H}|10\rangle \\ \langle 10|\hat{H}|01\rangle & \langle 10|\hat{H}|10\rangle \end{bmatrix}$$

$$\frac{\hat{H}^{2\times 2}}{\hbar} = \begin{bmatrix} -\frac{\omega_1}{2} + \frac{\omega_2}{2} & g \\ g & \frac{\omega_1}{2} - \frac{\omega_2}{2} \end{bmatrix} = \left(-\frac{\omega_1}{2} + \frac{\omega_2}{2}\right)\hat{Z} + g\hat{X}$$

If we go back to $\omega_1 = \omega_2$ we find back $\frac{\hat{H}^{2\times 2}}{\hbar} = g\hat{X}$ with $g = 0{,}02\pi$ as explained before. ◼