Exercice 1:

$\hat{x} = x_0\left[\hat{a}+\hat{a}^+\right]$ $\hat{p} = -ip_0\left[\hat{a}-\hat{a}^+\right]$. $x_0 = \sqrt{\frac{\hbar}{2m\omega}}$ $p_0 = \sqrt{\frac{\hbar m\omega}{2}}$

Operator $\hat{x}$:

$|\psi\rangle = |0\rangle$:

$$\langle\psi|\hat{x}|\psi\rangle = \langle 0|\hat{x}|0\rangle = x_0 \langle 0|\hat{a}+\hat{a}^+|0\rangle = x_0\left[\underbrace{\langle 0|\hat{a}|0\rangle}_{=0} + \langle 0|\hat{a}^+|0\rangle\right] = x_0\left[0 + \langle 0|\sqrt{1}|1\rangle\right] = 0$$

$|\psi\rangle = |n\rangle$:

$$\langle\psi|\hat{x}|\psi\rangle = \langle n|\hat{x}|n\rangle = x_0\langle n|\hat{a}+\hat{a}^+|n\rangle = x_0\left[\langle n|\hat{a}|n\rangle + \langle n|\hat{a}^+|n\rangle\right] = x_0\left[\langle n|\sqrt{n}|n-1\rangle + \langle n|\sqrt{n+1}|n+1\rangle\right] = 0$$

Operator $\hat{p}$:

$|\psi\rangle = |0\rangle$:

$$\langle\psi|\hat{p}|\psi\rangle = \langle 0|\hat{p}|0\rangle = -ip_0\langle 0|\hat{a}-\hat{a}^+|0\rangle = -ip_0\left[\underbrace{\langle 0|\hat{a}|0\rangle}_{=0} - \langle 0|\hat{a}^+|0\rangle\right] = ip_0 \langle 0|\sqrt{1}|1\rangle = 0$$

$|\psi\rangle = |n\rangle$:

$$\langle\psi|\hat{p}|\psi\rangle = \langle n|\hat{p}|n\rangle = -ip_0\langle n|\hat{a}-\hat{a}^+|n\rangle = -ip_0\left[\langle n|\sqrt{n}|n\rangle - \langle n|\hat{a}^+|n\rangle\right] = ip_0\langle n|\sqrt{n+1}|n+1\rangle = 0$$

Exercise 2:

x_{rms}:

$|\psi\rangle = |0\rangle$:

$$\hat{x}^2 = x_0^2(\hat{a}+\hat{a}^+)(\hat{a}+\hat{a}^+) = x_0^2\left[\hat{a}\hat{a} + \hat{a}\hat{a}^+ + \hat{a}^+\hat{a} + \hat{a}^+\hat{a}^+\right]$$

$$x_{rms}^2 = \langle 0|\hat{x}^2|0\rangle = x_0^2\underbrace{\langle 0|\hat{a}\hat{a}|0\rangle}_{=0} + x_0^2\langle 0|\hat{a}\hat{a}^+|0\rangle + x_0^2\underbrace{\langle 0|\hat{a}^+\hat{a}|0\rangle}_{=0} + x_0^2\langle 0|a^+a^+|0\rangle.$$

$$x_{rms}^2 = x_0^2\langle 0|1+\underbrace{\hat{a}^+\hat{a}|0\rangle}_{=0} + x_0^2\langle 0|\sqrt{2}\sqrt{1}|2\rangle$$

$$x_{rms} = \sqrt{x_0^2\langle 0|0\rangle} = x_0 = \sqrt{\frac{\hbar}{2m\omega}}$$

$|\psi\rangle = |n\rangle$:

$$x_{rms}^2 = \langle n|\hat{x}^2|n\rangle = x_0^2\langle n|\hat{a}\hat{a}|n\rangle + x_0^2\langle n|\hat{a}\hat{a}^+|n\rangle + x_0^2\langle n|\hat{a}^+\hat{a}|n\rangle + x_0^2\langle n|a^+a^+|n\rangle.$$

$$x_{rms}^2 = x_0^2\langle n|\sqrt{n}\sqrt{n-1}|n-2\rangle + x_0^2\langle n|1+\hat{a}^+\hat{a}|n\rangle + x_0^2\cdot n + x_0^2\langle n|\sqrt{n+1}\cdot\sqrt{n+2}|n+2\rangle$$

$$x_{rms}^2 = x_0^2 + x_0^2\cdot n + x_0^2\cdot n = x_0^2(2n+1)$$

$$x_{rms} = x_0\sqrt{2n+1} = \sqrt{\frac{\hbar}{2m\omega}(2n+1)}$$

Operator $\hat{p}$:

$|\psi\rangle = |0\rangle$:

$$\hat{p}^2 = -p_0^2(\hat{a}-\hat{a}^+)(\hat{a}-\hat{a}^+) = -p_0^2\left[\hat{a}\hat{a} - \hat{a}\hat{a}^+ - \hat{a}^+\hat{a} + \hat{a}^+\hat{a}^+\right].$$

$$p_{rms}^2 = \langle 0|\hat{p}^2|0\rangle = -p_0^2\langle 0|\hat{a}\hat{a}|0\rangle + p_0^2\langle 0|\hat{a}\hat{a}^+|0\rangle + p_0^2\langle 0|\hat{a}^+\hat{a}|0\rangle - p_0\langle 0|\hat{a}^+\hat{a}^+|0\rangle$$

$$p_{rms}^2 = 0 + p_0^2\langle 0|1+\hat{a}^+\hat{a}|0\rangle + p_0^2\cdot 0 + 0 = p_0^2$$

$$p_{rms} = p_0 = \sqrt{\frac{\hbar \omega m}{2}}$$

$\underline{|\psi\rangle = |n\rangle :}$

$$p_{rms}^2 = \langle n|\hat{p}^2|n\rangle = -p_0^2 \langle n|\hat{a}\hat{a}|n\rangle + p_0^2 \langle n|\hat{a}\hat{a}^{+}|n\rangle + p_0^2 \langle n|\hat{a}^{+}\hat{a}|n\rangle - p_0 \langle n|\hat{a}^{+}\hat{a}^{+}|n\rangle$$

$$p_{rms}^2 = 0 + p_0^2 \langle n|1+\hat{a}^{+}\hat{a}|n\rangle + p_0^2 \cdot n - 0. = p_0^2 + p_0^2 \cdot n + p_0^2 \cdot n.$$

$$p_{rms} = p_0\sqrt{(2n+1)} = \sqrt{\frac{\hbar \omega m}{2}(2n+1)}$$

Exercice 3:

$$x_{rms}^{|0\rangle} \cdot p_{rms}^{|0\rangle} = x_0 \cdot p_0 = \sqrt{\frac{\hbar}{2m\omega}} \cdot \sqrt{\frac{\hbar \omega m}{2}} = \frac{\hbar}{2}$$

$$x_{rms}^{|n\rangle} \cdot p_{rms}^{|n\rangle} = x_0 \sqrt{2n+1} \cdot p_0 \sqrt{2n+1} = \frac{\hbar}{2} \cdot (2n+1) \geqslant \frac{\hbar}{2} \quad \forall n \geqslant 0$$

Since we calculated for highest and lowest energy we can say that for any other state or eigenstate that $x_{rms} \cdot p_{rms}$ will be above $\frac{\hbar}{2}$, confirming that our results respect Heisenberg uncertainty relation such that:

$$x_{rms} \cdot p_{rms} \geqslant \frac{\hbar}{2}$$

Exercice 4:

$$\langle E_{pot} \rangle = \langle 0|m\omega^2 \cdot \frac{\hat{x}^2}{2}|0\rangle = \frac{m\omega^2}{2} \langle 0|\hat{x}^2|0\rangle = \frac{m\omega^2}{2} x_0^2 = \frac{m\omega^2}{2} \frac{\hbar}{2m\omega} = \frac{\hbar \omega}{4}$$

$$\langle E_{kin} \rangle = \langle 0|\frac{\hat{p}^2}{2m}|0\rangle = \frac{1}{2m} \langle 0|\hat{p}^2|0\rangle = \frac{1}{2m} \frac{\hbar \omega m}{2} = \frac{\hbar \omega}{4}$$

We see that the mean potential energy is equal to the mean kinetic energy. One can see that our result are consistent since the total mean energy $\langle E \rangle = \langle E_{pot} \rangle + \langle E_{kin} \rangle = 2\frac{\hbar\omega}{4} = \frac{\hbar\omega}{2}$ which again corresponds to the ground state energy E_0 which is equally divided between kinetic and potential energy.

Exercice 5:

(i) $\langle \hat{a} \rangle = \frac{1}{2x_0} \langle \psi|\hat{x}|\psi\rangle + \frac{i}{2p_0} \langle \psi|\hat{p}|\psi\rangle$

$$\langle \psi(t)|\hat{a}|\psi(t)\rangle = \left(\sum_{m=0}^{\infty} \psi_m^* e^{im\omega t} \langle m| \right) \hat{a} \left(\sum_{n=0}^{\infty} \psi_n e^{-in\omega t} |n\rangle \right)$$

$$\langle \psi(t)|\hat{a}|\psi(t)\rangle = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \psi_m^* \psi_n e^{-i\omega t(n-m)} \langle m|\hat{a}|n\rangle$$

$$\langle \psi(t)|\hat{a}|\psi(t)\rangle = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \psi_m^* \psi_n e^{-i\omega t(n-m)} \sqrt{n} \langle m|n-1\rangle \qquad \langle m|n-1\rangle = \delta_{m,n-1} = \begin{cases} 1 & \text{if } m=n-1 \\ 0 & \text{otherwise} \end{cases}$$

$$\langle \psi|\hat{a}|\psi\rangle = \sum_{n=0}^{\infty} \psi_{n-1}^* \psi_n e^{-i\omega t} \sqrt{n}$$

$$\frac{\partial}{\partial t} \langle \psi|\hat{a}|\psi\rangle = \sum_{n=0}^{\infty} \psi_{n-1}^* \psi_n \sqrt{n} \frac{\partial}{\partial t}\left[e^{-i\omega t}\right] = -i\omega \sum_{n=0}^{\infty} \psi_{n-1}^* \psi_n e^{-i\omega t} \sqrt{n} = -i\omega \langle \psi|\hat{a}|\psi\rangle.$$

We indeed have $\frac{\partial}{\partial t} \langle \hat{a} \rangle = -i\omega \langle \hat{a} \rangle.$

(ii) $\langle \psi(t) | \hat{a}^\dagger | \psi(t) \rangle = \left(\sum_{m=0}^{\infty} \psi_m^* e^{imwt} \langle m| \right) \hat{a}^\dagger \left(\sum_{n=0}^{\infty} \psi_n e^{-inwt} |n\rangle \right)$

$$\langle \psi(t) | \hat{a}^\dagger | \psi(t) \rangle = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \psi_m^* \psi_n e^{-i\omega t(n-m)} \langle m|\hat{a}^\dagger|n\rangle$$

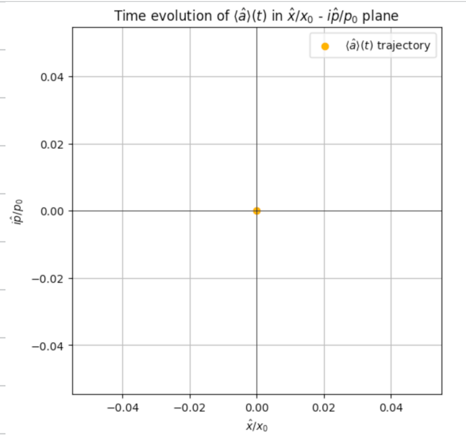
$$\langle \psi(t) | \hat{a}^\dagger | \psi(t) \rangle = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \psi_m^* \psi_n e^{-i\omega t(n-m)} \sqrt{n+1} \langle m|n+1\rangle \qquad \langle m|n+1\rangle = \delta_{m,n+1}$$

$$\langle \psi(t) | \hat{a}^\dagger | \psi(t) \rangle = \sum_{n=0}^{\infty} \psi_{n+1}^* \psi_n e^{i\omega t} \sqrt{n+1} = \langle \hat{a}^\dagger \rangle$$

$$\frac{\partial}{\partial t} \langle \hat{a}^\dagger \rangle = \sum_{n=0}^{\infty} \psi_{n+1}^* \psi_n \sqrt{n+1} \frac{\partial}{\partial t} e^{i\omega t} = i\omega \sum_{n=0}^{\infty} \psi_{n+1}^* \psi_n \sqrt{n+1} e^{i\omega t} = i\omega \langle \hat{a}^\dagger \rangle$$

we indeed have $\frac{\partial}{\partial t} \langle \hat{a}^\dagger \rangle = i\omega \langle \hat{a}^\dagger \rangle$.

Exercice 6:



$|\psi(t=0)\rangle = |2\rangle$:

$$|\psi(t=0)\rangle = \sum_{n=0}^{\infty} |n\rangle\langle n|\psi(t=0)\rangle = |2\rangle.$$

$$|\psi(t)\rangle = e^{-i2\omega t} |2\rangle.$$

$$\frac{\partial}{\partial t} \langle \psi(t)|\hat{a}|\psi(t)\rangle = -i\omega \langle \psi(t)|\hat{a}|\psi(t)\rangle = -i\omega \langle 2| e^{i2\omega t} \hat{a} e^{-2i\omega t} |2\rangle = -i\omega\sqrt{2} \langle 2|1\rangle = 0.$$

Therefore we have: we have $\langle \psi(t)|\hat{a}|\psi(t)\rangle = 0$ which is in the center of plane $\frac{\hat{x}}{x_0}, \frac{i\hat{p}}{p_0}$ – plane.

$|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$:

$$|\psi(t=0)\rangle = \sum_{n=0}^{\infty} |n\rangle\langle n|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle.$$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}} e^{-i\omega t} |1\rangle.$$

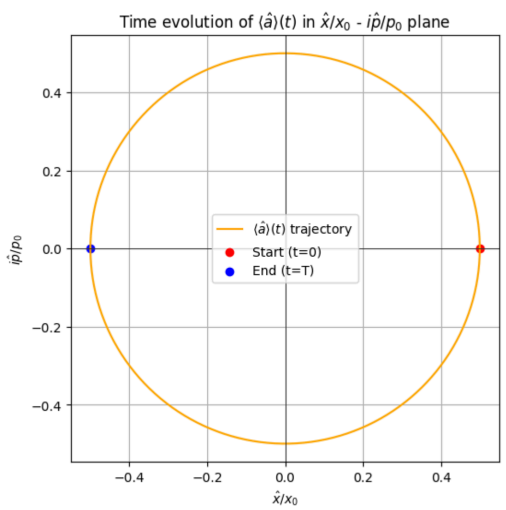
$$\frac{\partial}{\partial t} \langle \psi(t)|\hat{a}|\psi(t)\rangle = -i\omega \langle \psi(t)|\hat{a}|\psi(t)\rangle = -i\omega \frac{1}{2} \left[\langle 0| + e^{i\omega t} \langle 1| \right] \hat{a} \left[|0\rangle + e^{-i\omega t} |1\rangle \right] = -\frac{i\omega}{2} e^{-i\omega t} \langle 0|\hat{a}|1\rangle = -\frac{i\omega}{2} e^{-i\omega t}$$

$$-i\omega \langle \hat{a} \rangle(t) = -\frac{i\omega}{2} e^{-i\omega t} \Rightarrow \langle \hat{a} \rangle(t) = \frac{1}{2} e^{-i\omega t}.$$

$$\langle \hat{a} \rangle(t) = \frac{1}{2}\cos(\omega t) - i\frac{1}{2}\sin(\omega t).$$

since $\hat{a} = \frac{1}{2}\frac{\hat{x}}{x_0} + \frac{i}{2}\frac{\hat{p}}{p_0}$, then $\langle \hat{a} \rangle(t) = \frac{1}{2} \langle \frac{\hat{x}}{x_0} \rangle(t) + \frac{1}{2} \langle \frac{i\hat{p}}{p_0} \rangle(t)$.

By identification we have $\langle \frac{\hat{x}}{x_0} \rangle(t) = \cos(\omega t)$ and $\langle \frac{i\hat{p}}{p_0} \rangle(t) = -i\sin(\omega t)$, meaning that in $\frac{\hat{x}}{x_0}, \frac{i\hat{p}}{p_0}$ – plane, we have a circle of radius $\frac{1}{2}$ evolving counter clockwise.



$\omega = 1$ here.

Exercice 7:

$$|\psi(t=0)\rangle = \sqrt{\tfrac{1}{3}}\,|0\rangle + \sqrt{\tfrac{2}{3}}\,|n\rangle$$

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} \psi_n e^{-in\omega t}|n\rangle = \sqrt{\tfrac{1}{3}}\,|0\rangle + \sqrt{\tfrac{2}{3}}\, e^{-in\omega t}|n\rangle .$$

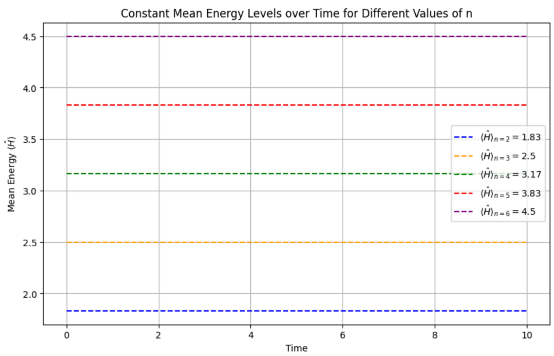
$$\langle\psi(t)|\hat{H}|\psi(t)\rangle = \hbar\omega\left[\sqrt{\tfrac{1}{3}}\langle 0| + \sqrt{\tfrac{2}{3}}\, e^{in\omega t}\langle n|\right]\left[\hat{a}^{+}\hat{a} + \tfrac{1}{2}\right]\left[\sqrt{\tfrac{1}{3}}\,|0\rangle + \sqrt{\tfrac{2}{3}}\, e^{-in\omega t}|n\rangle\right] \qquad \hat{a}^{+}\hat{a}|n\rangle = \hat{a}^{+}\sqrt{n}\,|n-1\rangle = n|n\rangle$$

$$= \hbar\omega\left[\sqrt{\tfrac{1}{3}}\langle 0| + \sqrt{\tfrac{2}{3}}\, e^{in\omega t}\langle n|\right]\sqrt{\tfrac{2}{3}}\, e^{-in\omega t} n|n\rangle + \frac{\hbar\omega}{2}\left[\sqrt{\tfrac{1}{3}}\langle 0| + \sqrt{\tfrac{2}{3}}\, e^{in\omega t}\langle n|\right]\left[\sqrt{\tfrac{1}{3}}\,|0\rangle + \sqrt{\tfrac{2}{3}}\, e^{-in\omega t}|n\rangle\right]$$

$$= \hbar\omega \cdot \frac{2}{3} n + \frac{\hbar\omega}{2}\cdot\frac{1}{3} + \frac{\hbar\omega}{2}\cdot\frac{2}{3}$$

$$= \hbar\omega\left[\frac{2}{3}n + \frac{1}{2}\right]$$

$$\langle\psi(t)|\hat{H}|\psi(t)\rangle = \frac{\hbar\omega}{6}\left[4n+3\right].$$



for $n = 2$, we have $\langle\psi(t)|\hat{H}|\psi(t)\rangle$ which corresponds to the theoritical value of $\frac{\hbar\omega}{6}[4n+3] = \frac{11}{6} = 1{,}833$ for $\omega = 1$ and an $\hbar$-normalization.

Conclusion:

In this scenario, energy is not "really quantized" in the strict sense, because the expectation value of the energy does not match the quantized eigenvalues of the harmonic oscillator. Superposition of energy eigenstates can yield expectation values that are not confined to the quantized energy levels, even though the system's underlying energy eigenvalues remain quantized.

Exercice 8:

$$\hat{a}|\alpha\rangle = \hat{a}\, e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

$$\hat{a}|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\,\hat{a}|n\rangle$$

$$\hat{a}|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\sqrt{n}\,|n-1\rangle$$

$$\hat{a}|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=1}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\sqrt{n}\,|n-1\rangle$$

$$\hat{a}|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=1}^{\infty} \frac{\alpha^n}{\sqrt{(n-1)!}}\,|n-1\rangle \qquad n = m+1$$

$$\hat{a}|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{m=0}^{\infty} \frac{\alpha^{m+1}}{\sqrt{m!}}\,|m\rangle .$$

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \qquad \alpha \in \mathbb{C}$$

Exercice 9:

$$\hat{a}^+ |\psi\rangle = \sum_{n=0}^{\infty} \psi_n \hat{a}^+ |n\rangle$$

$$\hat{a}^+ |\psi\rangle = \sum_{n=0}^{\infty} \psi_n \sqrt{n+1}\, |n+1\rangle.$$

$$\hat{a}^+ |\psi\rangle = \sum_{m=1}^{\infty} \psi_{m-1} \sqrt{m}\, |m\rangle.$$

Left side: $a^+ [\psi_0 |0\rangle + \psi_1 |1\rangle + \dots + \psi_n |n\rangle + \dots]$

Right side: $\psi_0 \sqrt{1} |1\rangle + \psi_1 \sqrt{2} |2\rangle + \dots + \psi_{n-1} \sqrt{n} |n\rangle$

To find eigenstates/values of $\hat{a}^+$ we should have Left side equal to $\lambda |\psi\rangle = \lambda \psi_0 |0\rangle + \dots \lambda \psi_n |n\rangle + \dots$ but comparing with Right side, we lack the term in $|0\rangle$.

The state $|n=0\rangle$ is not represented in the linear combination and can not be recovered by the superposition of the other states since the set of eigenstates $|n\rangle$ are orthonormal.

$\hat{a}^+$ does not have any eigenstates.

Exercice 10:

$$\langle \alpha | \hat{H} | \alpha \rangle = \langle \alpha | \hbar\omega (\hat{a}^+ \hat{a} + \tfrac{1}{2}) | \alpha \rangle = \frac{\hbar\omega}{2} \langle \alpha | \alpha \rangle + \hbar\omega \langle \alpha | \hat{a}^+ \hat{a} | \alpha \rangle$$

$$\langle \alpha | \hat{H} | \alpha \rangle = \frac{\hbar\omega}{2} + \hbar\omega\, \alpha\alpha^*$$

$\langle \alpha | \hat{H} | \alpha \rangle = \hbar\omega |\alpha|^2 + \frac{\hbar\omega}{2}$ where we used Eq. 32 such that $\langle \alpha | \hat{a}^+ a | \alpha \rangle = \alpha\alpha^*$.

Exercice 11:

$$\hat{a}^+ [1 + \hat{a}^+ \hat{a}] \hat{a}$$

$$\hat{a}^+ \hat{a} + (\hat{a}^+)^2 (\hat{a})^2$$

$$\epsilon_{RMS}^2 = \langle \alpha | \hat{H}^2 | \alpha \rangle - \langle \alpha | \hat{H} | \alpha \rangle^2.$$

$$\hat{H}^2 = (\hbar\omega)^2 (\hat{a}^+ \hat{a} + \tfrac{1}{2})(\hat{a}^+ \hat{a} + \tfrac{1}{2}) = (\hbar\omega)^2 (\hat{a}^+ \hat{a} \hat{a}^+ \hat{a} + \hat{a}^+ \hat{a} + \tfrac{1}{4}).$$

$$\hat{a}\hat{a}^+ = 1 + \hat{a}^+ \hat{a}.$$

$$= (\hbar\omega)^2 (2\hat{a}^+ \hat{a} + (\hat{a}^+)^2 (\hat{a})^2 + \tfrac{1}{4})$$

$$\langle \alpha | \hat{H}^2 | \alpha \rangle = (\hbar\omega)^2 \left[2 \langle \alpha | \hat{a}^+ \hat{a} | \alpha \rangle + \langle \alpha | (\hat{a}^+)^2 (\hat{a})^2 | \alpha \rangle + \tfrac{1}{4} \langle \alpha | \alpha \rangle \right]$$

$$= (\hbar\omega)^2 \left[2|\alpha|^2 + \langle \alpha | \hat{a}^+ \hat{a}^+ \alpha^2 | \alpha \rangle. + \tfrac{1}{4} \right].$$

$$= (\hbar\omega)^2 \left[2|\alpha|^2 + \alpha^2 (\alpha^*)^2 + \tfrac{1}{4} \right]$$

$$= (\hbar\omega)^2 \left[|\alpha|^4 + 2|\alpha|^2 + \tfrac{1}{4} \right]$$

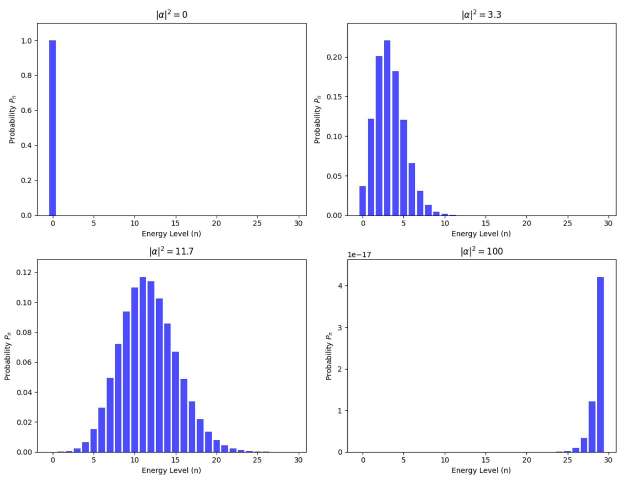
$$(\langle \alpha | \hat{H} | \alpha \rangle)^2 = \left(\hbar\omega |\alpha|^2 + \frac{\hbar\omega}{2} \right)^2 = (\hbar\omega)^2 \left[|\alpha|^4 + |\alpha|^2 + \tfrac{1}{4} \right].$$

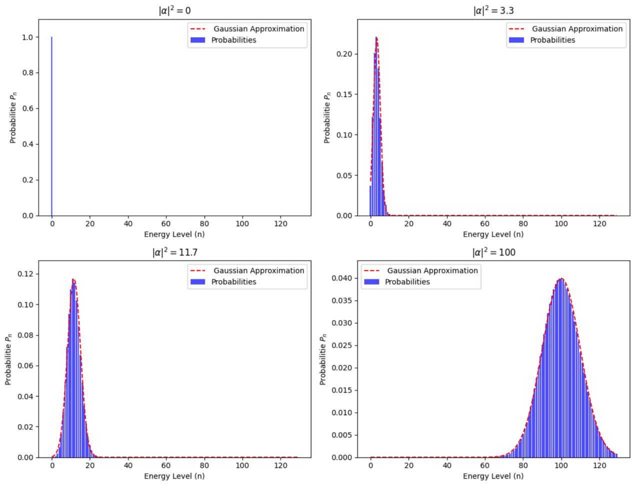
$$\epsilon_{RMS}^2 = \langle \alpha | \hat{H}^2 | \alpha \rangle - \langle \alpha | \hat{H} | \alpha \rangle^2.$$

$$\epsilon_{RMS}^2 = (\hbar\omega)^2 \left[|\alpha|^4 + 2|\alpha|^2 + \tfrac{1}{4} \right] - (\hbar\omega)^2 \left[|\alpha|^4 + |\alpha|^2 + \tfrac{1}{4} \right]. = \hbar^2 \omega^2 |\alpha|^2$$

$$E_{RMS}^2 = \hbar^2\omega^2|\alpha|^2 \qquad E_{RMS} = \hbar\omega|\alpha|$$

Exercice 12:



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For |alpha|^2 = 100:
Mean Energy = 100.5
E_RMS = 10.0
Ratio of Mean Energy to E_RMS = 10.05
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Exercice 13:

$$\cdot \langle\alpha|\hat{x}|\alpha\rangle = \langle\alpha|\, x_0(\hat{a}+\hat{a}^\dagger)|\alpha\rangle = x_0\langle\alpha|\hat{a}|\alpha\rangle + x_0\langle\alpha|\hat{a}^\dagger|\alpha\rangle = x_0\cdot\alpha + x_0\cdot\alpha^* = x_0(\alpha+\alpha^*)$$

$$\langle\hat{x}\rangle^2 = \langle\alpha|\hat{x}|\alpha\rangle^2 = x_0^2\left[\alpha^2 + (\alpha^*)^2 + 2|\alpha|^2\right].$$

$$\langle\alpha|\hat{x}^2|\alpha\rangle = \langle\alpha|\, x_0^2\left[\hat{a}\hat{a} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a}^\dagger\right]|\alpha\rangle = x_0^2\left[\alpha^2 + \alpha\alpha^* + 1 + \alpha\alpha^* + (\alpha^*)^2\right] = x_0^2\left[\alpha^2 + (\alpha^*)^2 + 2|\alpha|^2 + 1\right].$$

$$x_{RMS}^2 = \langle\hat{x}^2\rangle - \langle\hat{x}\rangle^2 = x_0^2\left[\alpha^2 + (\alpha^*)^2 + 2|\alpha|^2 + 1\right] - x_0^2\left[\alpha^2 + (\alpha^*)^2 + 2|\alpha|^2\right]$$

$$x_{RMS}^2 = x_0^2 = \frac{\hbar}{2m\omega} \qquad x_{RMS} = x_0 = \sqrt{\frac{\hbar}{2m\omega}}$$

<u>If we let $|\alpha\rangle$ evolve we have :</u>

$$\hat{a}\,|\alpha(t)\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n(t)}{\sqrt{n!}}\,\hat{a}|n\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=1}^{\infty}\frac{\alpha^n(t)\sqrt{n}}{\sqrt{n!}}|n-1\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{m=0}^{\infty}\frac{\alpha^{m+1}(t)\sqrt{m+1}}{\sqrt{(m+1)!}}|m\rangle = \alpha(t)\,|\alpha(t)\rangle.$$

$$\alpha(t) = \alpha e^{-i\omega t}$$

We have the same properties for $|\alpha(t)\rangle$ than those of $|\alpha\rangle$.

$$\langle\hat{x}\rangle^2 = x_0^2\left[(\alpha(t))^2 + (\alpha^*(t))^2 + 2|\alpha(t)|^2\right] \qquad \langle x^2\rangle = x_0^2\left[(\alpha(t))^2 + (\alpha^*(t))^2 + 2|\alpha(t)|^2 + 1\right.$$

then $x_{RMS}^2(t) = \langle\alpha(t)|\hat{x}^2|\alpha(t)\rangle - \langle\alpha(t)|\hat{x}|\alpha(t)\rangle^2 = x_0^2$

$$x_{RMS}^2(t) = x_0^2 = \frac{\hbar}{2\omega m} \qquad x_{RMS}^2 \text{ does not change over time.}$$

$$x_{RMS} = x_0 = \sqrt{\frac{\hbar}{2m\omega}}$$

Exercice 14:

$\cdot \langle\alpha|\hat{p}|\alpha\rangle = -ip_0 \langle\alpha|\hat{a}-\hat{a}^\dagger|\alpha\rangle = -ip_0\left[\alpha - \alpha^*\right]$

$\langle\alpha|\hat{p}|\alpha\rangle^2 = -p_0^2\left[\alpha^2 + (\alpha^*)^2 - 2|\alpha|^2\right].$

$\cdot \langle\alpha|\hat{p}^2|\alpha\rangle = -p_0^2 \langle\alpha|\left[\hat{a}\hat{a} - \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a}^\dagger\right]|\alpha\rangle = -p_0^2\left[\alpha^2 - \alpha\alpha^* - 1 - \alpha\alpha^* + (\alpha^*)^2\right]$

$= -p_0^2\left[\alpha^2 + (\alpha^*)^2 - 2|\alpha|^2 - 1\right]$

$p_{RMS}^2 = \langle\alpha|\hat{p}^2|\alpha\rangle - \langle\alpha|\hat{p}|\alpha\rangle^2 = -p_0^2\left[\alpha^2 + (\alpha^*)^2 - 2|\alpha|^2 - 1\right] + p_0^2\left[\alpha^2 + (\alpha^*)^2 - 2|\alpha|^2\right].$

$$p_{RMS}^2 = p_0^2 = \frac{\hbar m\omega}{2}.$$

Then $p_{RMS} = p_0 = \sqrt{\frac{\hbar m\omega}{2}}$

$x_{RMS} \cdot p_{RMS} = \sqrt{\frac{\hbar}{2m\omega}} \cdot \sqrt{\frac{\hbar m\omega}{2}} = \frac{\hbar}{2}$ With coherent state, we have $x_{RMS} \cdot p_{RMS} = \frac{\hbar}{2}$ which does not depend on the value of α and respects Heisenberg incertitude principle of Eq (25)

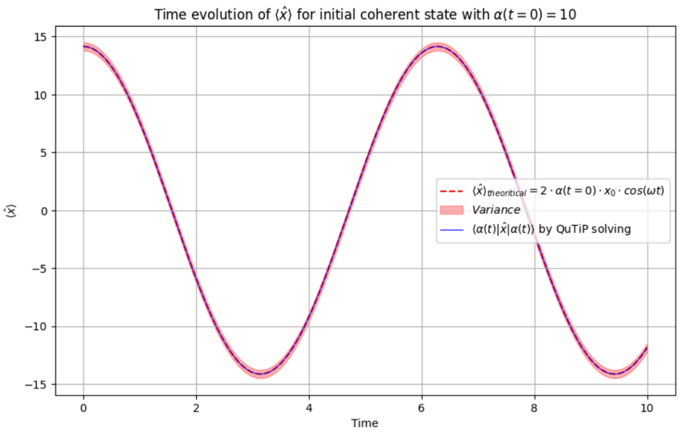
Exercice 15:

$\alpha(t=0) = 10$ then $\alpha(t=0) = \alpha e^{-i\omega 0} = 10 \Rightarrow \alpha = 10.$

We have $|\alpha(t)\rangle = e^{-50} \sum_{n=0}^{\infty} \frac{10^n e^{-in\omega t}}{\sqrt{n!}} |n\rangle$ $\qquad \alpha(t) = 10e^{-i\omega t}$

$\langle\hat{x}\rangle = \langle\alpha(t)|\hat{x}|\alpha(t)\rangle = x_0\left[\alpha(t) + \alpha^*(t)\right] = 20\, x_0\left[\frac{e^{-i\omega t} + e^{i\omega t}}{2}\right]$

$\langle\hat{x}\rangle = 20\, x_0 \cos(\omega t) = 20\sqrt{\frac{\hbar}{2m\omega}} \cos(\omega t).$



By setting the linewidth to x_{RMS}, we visually represent the uncertainty in $\langle\hat{x}\rangle$ due to quantum fluctuations, giving an indication of the spatial spread of the coherent state. If x_{RMS} were 0, we would have a linewidth of 0 and no curve would have been plotted.

Exercice 16:

(i) RHS: $|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle.$

LHS: $e^{-\frac{|\alpha|^2}{2}} e^{\alpha \vec{a}^+} |0\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n (\vec{a}^+)^n}{n!} |0\rangle$

$$= e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \frac{(\vec{a}^+)^n}{\sqrt{n!}} |0\rangle$$

$$= e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$$e^{-\frac{|\alpha|^2}{2}} e^{\alpha \vec{a}^+} |0\rangle = |\alpha\rangle$$

(ii) RHS: $|0\rangle$.

LHS: $e^{-\alpha^* \vec{a}} |0\rangle = \sum_{n=0}^{\infty} \frac{(\alpha^*)^n \vec{a}^n}{n!} |0\rangle$ since $\vec{a}^n |0\rangle = 0 \ \forall n \in \mathbb{N}^*$ but for $n=0$ we have $\vec{a}^0 |0\rangle = |0\rangle$, we only keep $n=0$ in the sum.

$$e^{-\alpha^* \vec{a}} |0\rangle = \frac{(\alpha^*)^0 \vec{a}^0}{0!} |0\rangle = |0\rangle$$

$$e^{-\alpha^* \vec{a}} |0\rangle = |0\rangle$$

Exercice 17:

We need to show that: $e^{-\frac{|\alpha|^2}{2}} e^{\alpha \vec{a}^+} . e^{-\alpha^* \vec{a}} = e^{\alpha \vec{a}^+ - \alpha^* \vec{a}}$

Using BCH formula: $e^{-\frac{|\alpha|^2}{2}} e^{\alpha \vec{a}^+} . e^{-\alpha^* \vec{a}} = e^{-\frac{|\alpha|^2}{2}} e^{\alpha \vec{a}^+ - \alpha^* \vec{a}} . e^{\frac{1}{2}[\alpha \vec{a}^+ ; -\alpha^* \vec{a}]}$

Using $[\alpha^* \vec{a}, \alpha \vec{a}^+] = |\alpha|^2$ or $[\alpha \vec{a}^+ ; \alpha^* \vec{a}] = -|\alpha|^2$. we can write:

$$e^{-\frac{|\alpha|^2}{2}} e^{\alpha \vec{a}^+} . e^{-\alpha^* \vec{a}} = e^{-\frac{|\alpha|^2}{2}} e^{\alpha \vec{a}^+ - \alpha^* \vec{a}} . e^{\frac{|\alpha|^2}{2}} = e^{\alpha \vec{a}^+ - \alpha^* \vec{a}}$$

We indeed have $e^{-\frac{|\alpha|^2}{2}} e^{\alpha \vec{a}^+} . e^{-\alpha^* \vec{a}} = e^{\alpha \vec{a}^+ - \alpha^* \vec{a}}$

Exercice 18:

$|\alpha\rangle = \hat{\mathcal{D}}(\alpha) |0\rangle = e^{\alpha \vec{a}^+ - \alpha^* \vec{a}} |0\rangle$ Is $\hat{\mathcal{D}}(\alpha)$ unitary?

(i) $\hat{\mathcal{D}}(\alpha)^+ = (e^{\alpha \vec{a}^+ - \alpha^* \vec{a}})^+ = e^{\alpha^* \vec{a} - \alpha \vec{a}^+} = e^{-\alpha \vec{a}^+ + \alpha^* \vec{a}} = \hat{\mathcal{D}}(-\alpha)$

- $\hat{\mathcal{D}}(\alpha) \hat{\mathcal{D}}^+(\alpha) = e^{\alpha \vec{a}^+ - \alpha^* \vec{a}} . e^{-\alpha \vec{a}^+ + \alpha^* \vec{a}} = e^{\alpha \vec{a}^+ - \alpha^* \vec{a} - \alpha \vec{a}^+ + \alpha^* \vec{a}} e^{\frac{1}{2}[\alpha \vec{a}^+ - \alpha^* \vec{a} ; -\alpha \vec{a}^+ + \alpha^* \vec{a}]}$

$$= \mathbb{1} . e^{-\frac{1}{2}[\alpha \vec{a}^+ - \alpha^* \vec{a} ; \alpha \vec{a}^+ - \alpha^* \vec{a}]}$$ we used $[\vec{A} ; \vec{A}] = 0$. $\vec{A} = \alpha \vec{a}^+ - \alpha^* \vec{a}$.

$$\hat{\mathcal{D}}(\alpha) \hat{\mathcal{D}}^+(\alpha) = \mathbb{1}$$

- $\hat{\mathcal{D}}^+(\alpha) \hat{\mathcal{D}}(\alpha) = e^{-\alpha \vec{a}^+ + \alpha^* \vec{a}} e^{\alpha \vec{a}^+ - \alpha^* \vec{a}} \ e^{-\alpha \vec{a}^+ + \alpha^* \vec{a} + \alpha \vec{a}^+ - \alpha^* \vec{a}} \ e^{\frac{1}{2}[-\alpha \vec{a}^+ + \alpha^* \vec{a} ; \alpha \vec{a}^+ - \alpha^* \vec{a}]}$

$$= \mathbb{1} \ e^{-\frac{1}{2}[\alpha \vec{a}^+ - \alpha^* \vec{a} ; \alpha \vec{a}^+ - \alpha^* \vec{a}]}$$

$$\hat{D}^\dagger(\alpha)\hat{D}(\alpha) = \mathbb{1}$$

Since $\hat{D}(\alpha)\hat{D}^\dagger(\alpha) = \hat{D}^\dagger(\alpha)\hat{D}(\alpha) = \mathbb{1}$, $\hat{D}(\alpha)$ is unitary.

Since we have $\hat{D}(\alpha)\hat{D}^\dagger(\alpha) = \mathbb{1}$ we must have $\hat{D}^\dagger(\alpha) = \hat{D}^{-1}(\alpha)$ but we have also $\hat{D}^\dagger(\alpha) = \hat{D}(-\alpha)$ so finally $\hat{D}^\dagger(\alpha) = \hat{D}(-\alpha) = \hat{D}^{-1}(\alpha)$.

Exercice 19:

$|\beta\rangle = \hat{D}(\beta)|0\rangle$ so $\hat{D}(\alpha)|\beta\rangle = \hat{D}(\alpha)\hat{D}(\beta)|0\rangle$

$$\hat{D}(\alpha)\hat{D}(\beta) = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}} e^{\beta\hat{a}^\dagger - \beta^*\hat{a}} = e^{(\alpha+\beta)\hat{a}^\dagger - (\alpha+\beta)^*\hat{a}} e^{\frac{1}{2}[\alpha\hat{a}^\dagger - \alpha^*\hat{a};\, \beta\hat{a}^\dagger - \beta^*\hat{a}]}$$

$$(\alpha\hat{a}^\dagger - \alpha^*\hat{a})(\beta\hat{a}^\dagger - \beta^*\hat{a}) = \alpha\beta(\hat{a}^\dagger)^2 - \alpha\beta^*\hat{a}^\dagger\hat{a} - \alpha^*\beta\hat{a}\hat{a}^\dagger + \alpha^*\beta^*\hat{a}^2$$

$$(\beta\hat{a}^\dagger - \beta^*\hat{a})(\alpha\hat{a}^\dagger - \alpha^*\hat{a}) = \alpha\beta(\hat{a}^\dagger)^2 - \alpha^*\beta\hat{a}^\dagger\hat{a} - \alpha\beta^*\hat{a}\hat{a}^\dagger + \alpha^*\beta^*\hat{a}^2$$

$$\begin{aligned}
[\alpha\hat{a}^\dagger - \alpha^*\hat{a};\, \beta\hat{a}^\dagger - \beta^*\hat{a}] &= \hat{a}^\dagger\hat{a}[\alpha^*\beta - \alpha\beta^*] + \hat{a}\hat{a}^\dagger[\alpha\beta^* - \alpha^*\beta].\\
&= [\hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger][\alpha^*\beta - \alpha\beta^*]\\
&= \alpha\beta^* - \alpha^*\beta.\\
&= |\alpha||\beta|\left[e^{i(\arg[\alpha] - \arg[\beta])} - e^{-i(\arg[\alpha] - \arg[\beta])}\right]\\
&= 2i|\alpha||\beta|\sin(\arg[\alpha] - \arg[\beta]) \in \mathbb{C}.
\end{aligned}$$

Therefore $\hat{D}(\alpha)\hat{D}(\beta) = \hat{D}(\alpha+\beta).\exp\left(i.|\alpha||\beta|\sin(\arg[\alpha]-\arg[\beta])\right) = \hat{D}(\alpha+\beta)\exp\left(\frac{1}{2}(\alpha\beta^* - \alpha^*\beta.)\right)$

$\exp\left(i.2|\alpha||\beta|\sin(\arg[\alpha]-\arg[\beta])\right)$ is a global phase so it can be dropped.

$\hat{D}(\alpha)\hat{D}(\beta) \equiv \hat{D}(\alpha+\beta)$.

(i) $\hat{D}^\dagger(\alpha)\hat{a} = (\hat{a} + \alpha)\hat{D}^\dagger(\alpha)$.

$\hat{D}^\dagger(\alpha)\hat{a}|\beta\rangle = \hat{D}(-\alpha)\beta|\beta\rangle = \beta|\beta-\alpha\rangle.$

$\hat{a}\hat{D}^\dagger(\alpha)|\beta\rangle = \hat{a}|\beta-\alpha\rangle = (\beta-\alpha)|\beta-\alpha\rangle$

$\hat{D}^\dagger(\alpha)\hat{a}|\beta\rangle - \hat{a}\hat{D}^\dagger(\alpha)|\beta\rangle = [\beta - (\beta-\alpha)]|\beta-\alpha\rangle = \alpha\hat{D}^\dagger(\alpha)|\beta\rangle.$

$\hat{D}^\dagger(\alpha)\hat{a}|\beta\rangle = [\hat{a}\hat{D}^\dagger(\alpha) + \alpha\hat{D}^\dagger(\alpha)]|\beta\rangle$

$\hat{D}^\dagger(\alpha)\hat{a}|\beta\rangle = (\hat{a}+\alpha)\hat{D}^\dagger(\alpha)|\beta\rangle$ since $|\beta\rangle$ is unique in the Hilbert space, we have that:

$\hat{D}^\dagger(\alpha)\hat{a} = (\hat{a}+\alpha)\hat{D}^\dagger(\alpha)$

(ii) $\hat{D}^\dagger(\alpha)\hat{a}^\dagger = (\hat{a}^\dagger + \alpha^*)\hat{D}^\dagger(\alpha)$. $\quad (\hat{D}(\alpha)\hat{a}|\beta\rangle)^\dagger = (\hat{a}|\beta\rangle)^\dagger\hat{D}^\dagger(\alpha) = \langle\beta|\hat{a}^\dagger\hat{D}^\dagger(\alpha).$

$$\langle\beta|\vec{a}^{\dagger}\vec{\mathcal{D}}^{\dagger}(\alpha) = \left(\vec{\mathcal{D}}(\alpha)\,\vec{a}\,|\beta\rangle\right)^{\dagger} = \left(\vec{\mathcal{D}}(\alpha)\,\beta|\beta\rangle\right)^{\dagger} = \beta^{*}\langle\beta|\vec{\mathcal{D}}^{\dagger}(\alpha).$$

$$\langle\beta|\vec{\mathcal{D}}^{\dagger}(\alpha)\vec{a}^{\dagger} = \left(\vec{a}\,\vec{\mathcal{D}}(\alpha)|\beta\rangle\right)^{\dagger} = \left(\vec{a}\,|\beta+\alpha\rangle\right)^{\dagger} = \left((\beta+\alpha)|\beta+\alpha\rangle\right)^{\dagger} = (\beta+\alpha)^{*}\left(\vec{\mathcal{D}}(\alpha)|\beta\rangle\right)^{\dagger} = (\alpha+\beta)^{*}\langle\beta|\vec{\mathcal{D}}^{\dagger}(\alpha).$$

$$\langle\beta|\vec{a}^{\dagger}\vec{\mathcal{D}}^{\dagger}(\alpha) - \langle\beta|\vec{\mathcal{D}}^{\dagger}(\alpha)\vec{a}^{\dagger} = \beta^{*}\langle\beta|\vec{\mathcal{D}}^{\dagger}(\alpha) - (\beta^{*}+\alpha^{*})\langle\beta|\vec{\mathcal{D}}^{\dagger}(\alpha).$$

$$\langle\beta|\vec{\mathcal{D}}^{\dagger}(\alpha)\vec{a}^{\dagger} = \langle\beta|\vec{a}^{\dagger}\vec{\mathcal{D}}^{\dagger}(\alpha) + \alpha^{*}\langle\beta|\vec{\mathcal{D}}^{\dagger}(\alpha).$$

$$\langle\beta|\vec{\mathcal{D}}^{\dagger}(\alpha)\vec{a}^{\dagger} = \langle\beta|\left(\vec{a}^{\dagger}+\alpha^{*}\right)\vec{\mathcal{D}}^{\dagger}(\alpha).$$

since $|\beta\rangle$ is unique in the Hilbert space, we have that: $\boxed{\vec{\mathcal{D}}^{\dagger}(\alpha)\vec{a}^{\dagger} = (\vec{a}^{\dagger}+\alpha^{*})\vec{\mathcal{D}}^{\dagger}(\alpha)}$

Exercice 20:

We have to show that $\vec{\mathcal{D}}(\alpha) = e^{-i\hat{p}\frac{2\alpha x_0}{\hbar}}$

we know: $\vec{\mathcal{D}}(\alpha) = e^{\alpha\vec{a}^{\dagger}-\alpha^{*}\vec{a}} = e^{-\alpha(\vec{a}-\vec{a}^{\dagger})}$ if $\alpha\in\mathbb{R}$.

$$\vec{p} = -i\sqrt{\frac{\hbar m\omega}{2}}\,(\vec{a}-\vec{a}^{\dagger})$$

$$-i\vec{p}\cdot\frac{2x_0}{\hbar} = -\sqrt{\frac{\hbar m\omega}{2}}\cdot\frac{2}{\hbar}\cdot\sqrt{\frac{\hbar}{2m\omega}}\,(\vec{a}-\vec{a}^{\dagger}) = -(\vec{a}-\vec{a}^{\dagger}).$$

By identification we can rewrite $\vec{\mathcal{D}}(\alpha)$:

$$\boxed{\vec{\mathcal{D}}(\alpha) = e^{-i\vec{p}\cdot\frac{2\alpha x_0}{\hbar}}}$$

Exercice 21:

$$e^{+i\hat{p}\frac{2\alpha x_0}{\hbar}}\,\hat{x}\,e^{-i\hat{p}\frac{2\alpha x_0}{\hbar}} = \vec{\mathcal{D}}(-\alpha)\,\hat{x}\,\vec{\mathcal{D}}(\alpha) = \vec{\mathcal{D}}^{\dagger}(\alpha)\,\hat{x}\,\vec{\mathcal{D}}(\alpha) = \vec{\mathcal{D}}^{\dagger}(\alpha).\,x_0(\vec{a}+\vec{a}^{\dagger})\vec{\mathcal{D}}(\alpha).$$

$$= x_0\,\vec{\mathcal{D}}^{\dagger}(\alpha)\vec{a}\,\vec{\mathcal{D}}(\alpha) + x_0\,\vec{\mathcal{D}}^{\dagger}(\alpha)\vec{a}^{\dagger}\vec{\mathcal{D}}(\alpha)$$

$$= x_0(\vec{a}+\alpha)\vec{\mathcal{D}}^{\dagger}(\alpha)\vec{\mathcal{D}}(\alpha) + x_0(\vec{a}^{\dagger}+\alpha^{*})\vec{\mathcal{D}}^{\dagger}(\alpha)\vec{\mathcal{D}}(\alpha)$$

$$= x_0\left(\vec{a}+\vec{a}^{\dagger}+2\alpha\right) \quad (\alpha\in\mathbb{R}).$$

$$\vec{\mathcal{D}}^{\dagger}(\alpha)\vec{a} = (\vec{a}+\alpha)\vec{\mathcal{D}}^{\dagger}(\alpha)$$
$$\vec{\mathcal{D}}^{\dagger}(\alpha)\vec{a}^{\dagger} = (\vec{a}^{\dagger}+\alpha^{*})\vec{\mathcal{D}}^{\dagger}(\alpha)$$

$$\boxed{e^{+i\hat{p}\frac{2\alpha x_0}{\hbar}}\,\hat{x}\,e^{-i\hat{p}\frac{2\alpha x_0}{\hbar}} = \vec{x} + 2\alpha x_0.}$$

Exercice 22:

$$|\psi\rangle = \sum_n \psi_n|n\rangle = \sum_n |n\rangle\langle n|\psi\rangle.$$

$$\mathbb{1}|\psi\rangle = \left(\sum_m |m\rangle\langle m|\right)|\psi\rangle$$

since $|\psi\rangle$ is unique in $\mathcal{H}=\mathbb{C}^n$ we have:

$$\mathbb{1} = \sum_n |n\rangle\langle n|$$

also we notice that $|0\rangle\langle 0|$ is a matrix of 0 but there is a 1 in position 1×1. Similarly $|n\rangle\langle n|$ is null but there is a 1 in position $(n+1)\times(n+1)$. Therefore the sum of the matrix $|n\rangle\langle n|$ gives the identity.

Exercice 23:

For $N_{max} = 4$:

$$\hat{a} = \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{4} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \hat{a}^{+} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \sqrt{1} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{4} & 0 \end{bmatrix}$$

$$\hat{x} = x_0(\hat{a} + \hat{a}^{+}) = x_0 \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{4} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + x_0 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \sqrt{1} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{4} & 0 \end{bmatrix} = x_0 \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & 0 \\ \sqrt{1} & 0 & \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} \\ 0 & 0 & 0 & \sqrt{4} & 0 \end{bmatrix}$$

we notice that $\hat{x} = \hat{x}^{+}$, thefore $\hat{x}_{N=4}$ is hermitcan.

$$\hat{p} = -ip_0(\hat{a} - \hat{a}^{+}) = p_0 \begin{bmatrix} 0 & -i\sqrt{1} & 0 & 0 & 0 \\ 0 & 0 & -i\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & -i\sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & -i\sqrt{4} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + p_0 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ i\sqrt{1} & 0 & 0 & 0 & 0 \\ 0 & i\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & i\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & i\sqrt{4} & 0 \end{bmatrix} = p_0 \begin{bmatrix} 0 & -i\sqrt{1} & 0 & 0 & 0 \\ i\sqrt{1} & 0 & -i\sqrt{2} & 0 & 0 \\ 0 & i\sqrt{2} & 0 & -i\sqrt{3} & 0 \\ 0 & 0 & i\sqrt{3} & 0 & -i\sqrt{4} \\ 0 & 0 & 0 & i\sqrt{4} & 0 \end{bmatrix}$$

we also notice that $\hat{p} = \hat{p}^{+}$, therfore $\hat{p}_{N=4}$ is hermitcan.

Exercice 24:

$$\hat{a}^{+}\hat{a} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \sqrt{1} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{4} & 0 \end{bmatrix} \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{4} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

Knowing that $[\hat{a}; \hat{a}^{+}] = \mathbb{1} \iff \hat{a}\hat{a}^{+} = \mathbb{1} + \hat{a}^{+}\hat{a}$, we can write:

$$\hat{a}\hat{a}^{+} = \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{4} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \sqrt{1} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{4} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\hat{a}^{+}\hat{a}$ and $\hat{a}\hat{a}^{+}$ are not identical

Exercice 25:

$$[\hat{a};\hat{a}^{+}]=\hat{a}\hat{a}^{+}-\hat{a}^{+}\hat{a}=\begin{bmatrix}1&0&0&0&0\\0&2&0&0&0\\0&0&3&0&0\\0&0&0&4&0\\0&0&0&0&0\end{bmatrix}-\begin{bmatrix}0&0&0&0&0\\0&1&0&0&0\\0&0&2&0&0\\0&0&0&3&0\\0&0&0&0&4\end{bmatrix}=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&-4\end{bmatrix}\neq \mathbb{1}.$$

The commutator $[\hat{a};\hat{a}^{+}]$ does not match with identity.
In order to fix this we should set $\hat{a}\hat{a}^{+}$ such that $\hat{a}\hat{a}^{+}=[\hat{a};\hat{a}^{+}]+\hat{a}^{+}\hat{a}=\mathbb{1}+\hat{a}^{+}\hat{a}$ if we are in finite dimension. Otherwise, we need infinite dimension in order to get exactly the same matrix.

Therefore we have: $\hat{a}\hat{a}^{+}=$
$$\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}+\begin{bmatrix}0&0&0&0&0\\0&1&0&0&0\\0&0&2&0&0\\0&0&0&3&0\\0&0&0&0&4\end{bmatrix}=\begin{bmatrix}1&0&0&0&0\\0&2&0&0&0\\0&0&3&0&0\\0&0&0&4&0\\0&0&0&0&5\end{bmatrix}$$

Exercice 26:

$$\psi_{n+1}(x)=\frac{1}{\sqrt{n+1}}\left[\frac{x}{x_0}.\psi_n(x)-\sqrt{n}\,\psi_{n-1}(x)\right].$$

- $\psi_1(x)=\psi_{n+1}(x)\Big|_{n=0}=\frac{1}{\sqrt{0+1}}\left[\frac{x}{x_0}\psi_0(x)-\sqrt{0}\,\psi_{-1}(x)\right]=\frac{x}{x_0}\psi_0(x)$ since $\psi_{-1}(x)=0$
- $\psi_2(x)=\psi_{n+1}(x)\Big|_{n=1}=\frac{1}{\sqrt{2}}\left[\frac{x}{x_0}\psi_1(x)-\psi_0(x)\right]=\frac{1}{\sqrt{2}}\left[\left(\frac{x}{x_0}\right)^2-1\right]\psi_0(x).$
- $\psi_3(x)=\psi_{n+1}(x)\Big|_{n=2}=\frac{1}{\sqrt{3}}\left[\frac{x}{x_0}\psi_2(x)-\sqrt{2}\,\psi_1(x)\right]=\frac{1}{\sqrt{3}}\left[\frac{x}{x_0}.\frac{1}{\sqrt{2}}\left[\left(\frac{x}{x_0}\right)^2-1\right].\psi_0(x)-\sqrt{2}.\frac{x}{x_0}\psi_0(x)\right]$

$$=\frac{1}{\sqrt{3}}\left[\frac{1}{\sqrt{2}}\left(\frac{x}{x_0}\right)^3-\frac{1}{\sqrt{2}}\frac{x}{x_0}-\sqrt{2}\,\frac{x}{x_0}\right]\psi_0(x)$$

$$\psi_3(x)=\frac{1}{\sqrt{6}}\frac{x}{x_0}\left[\left(\frac{x}{x_0}\right)^2-3\right]\psi_0(x)$$

etc ...

Exercice 27:

Eq (62): $x\psi_0(x)+2x_0^2\frac{\partial\psi_0(x)}{\partial x}=0$

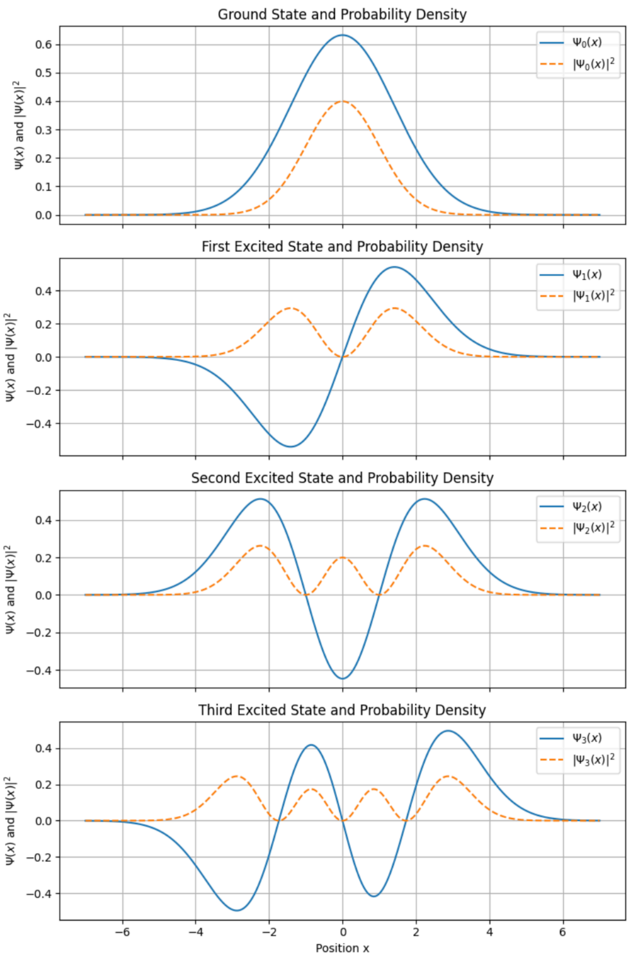
$$\psi_0(x)=\frac{1}{(2\pi x_0^2)^{1/4}}e^{-\left(\frac{x}{2x_0}\right)^2}$$

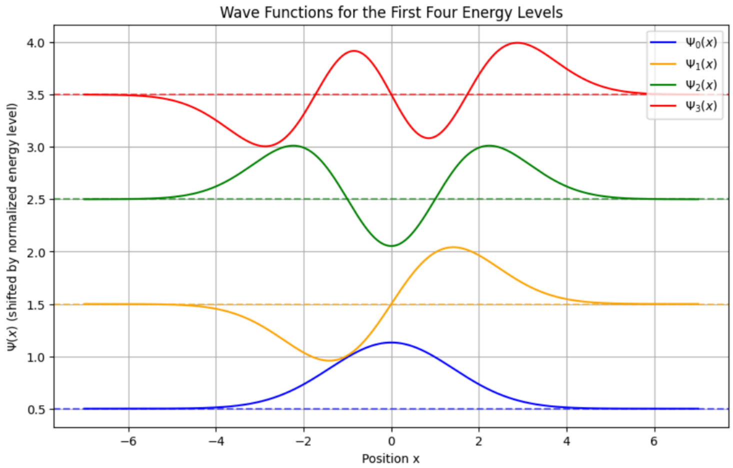
- $\frac{\partial}{\partial x}\psi_0(x)=\frac{1}{(2\pi x_0^2)^{1/4}}\cdot\left(-\frac{1}{x_0}\right)\frac{x}{2x_0}\,e^{-\left(\frac{x}{2x_0}\right)^2}=\frac{-x}{2x_0^2(2\pi x_0^2)^{1/4}}\,e^{-\left(\frac{x}{2x_0}\right)^2}$
- $2x_0^2.\frac{\partial\psi_0(x)}{\partial x}=\frac{-x}{(2\pi x_0^2)^{1/4}}e^{-\left(\frac{x}{2x_0}\right)^2}=-x\,\psi_0(x).$

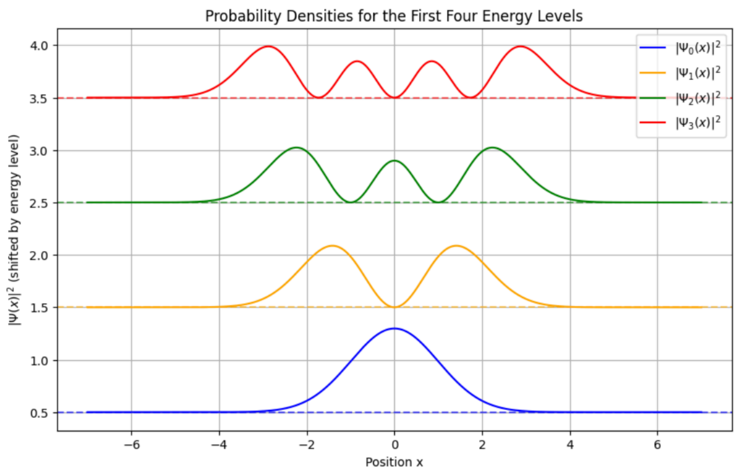
- Finally $x\Psi_0(x) + 2x_0^2 \frac{\partial}{\partial x}\Psi_0(x) = x\Psi_0(x) - x_0\Psi_0(x) = 0.$

$\Psi_0 = \frac{1}{(2\pi x_0^2)^{1/4}} e^{-\left(\frac{x}{2x_0}\right)^2}$ satisfies equation $x\Psi_0(x) + 2x_0^2 \frac{\partial \Psi_0(x)}{\partial x} = 0.$

Exercice 28 :







From the graph:

- $\Psi_0(x)$, the ground states has no nodes.
- $\Psi_1(x)$, the first excited state, has one node (one crossing with the x-axis)
- $\Psi_1(x)$, the second excited state, has two node (two crossing with the x-axis)
- $\Psi_1(x)$, the third excited state, has three node (three crossing with the x-axis)

This shows that the number of nodes for the state $\Psi_n(x)$ is equal to n, which is a characteristic of the energy eigenstate of a quantum harmonic oscillator

Probability $|\Psi_m(x)|^2$ of finding the oscillator in x:

The density represents the spatial distribution of the field in space, indicating where the energy/oscillator is likely to be located. This field is enduced by m bosons/quanta of energy in the same energy state. As the energy level increases, the oscillation of the field can reach farther from the center, meaning the field is more likely to be found farther from its equilibrium position.
With higher energy, the field oscillates with greater amplitude, causing the spatial distribution to extend over larger area. Therefore, it is not suprising that in higher energy states, the oscillator's amplitude is distributed farther from the center. This aligns with the classical analogy where, as energy increases, an oscillator (like a mass on a spring) oscillates with

greater amplitude, moving farther from its equilibrium position. Indeed kinetic energy goes to 0 because it is converted to potential energy when getting closer to the classical turning point. Therefore it spends more time near the boundaries since it is slowing down.

Exercice 29:

We have to check: $\int_{-\infty}^{+\infty} dx\, |\psi_n(x)|^2 = 1.$ $\qquad |\psi_0(x)|^2 = \frac{1}{\sqrt{2\pi x_0^2}} \cdot e^{-\frac{1}{2x_0^2}x^2}$

• $\int_{-\infty}^{+\infty} dx\, |\psi_0(x)|^2 = \int_{-\infty}^{+\infty} dx \left| \frac{1}{(2\pi x_0^2)^{1/4}} e^{-\left(\frac{x}{2x_0}\right)^2} \right|^2 = \frac{1}{\sqrt{2\pi x_0^2}} \cdot \int_{-\infty}^{+\infty} dx\, e^{-\frac{1}{2x_0^2}x^2}$

using gaussian integral: $\int_{-\infty}^{+\infty} dx \exp(-\alpha x^2 + \beta x) = \sqrt{\frac{\pi}{\alpha}}\, e^{\frac{\beta^2}{4\alpha}}$ with $\alpha = \frac{1}{2x_0^2}$ and $\beta = 0$:

$\int_{-\infty}^{+\infty} dx\, |\psi_0(x)|^2 = \frac{1}{\sqrt{2\pi x_0^2}} \cdot \sqrt{2\pi x_0^2}\, e^0 = 1.$ We indeed have $\int_{-\infty}^{+\infty} dx\, |\psi_0(x)|^2 = 1.$

• $\int_{-\infty}^{+\infty} dx\, |\psi_1(x)|^2 = \int_{-\infty}^{+\infty} (x)\frac{x}{x_0^2}|\psi_0(x)|^2 dx = \frac{1}{\sqrt{2\pi x_0^2}} \cdot \frac{1}{x_0^2} \int_{-\infty}^{+\infty} dx\, (x).\left(x e^{-\frac{1}{2x_0^2}x^2}\right)$ $\qquad |\psi_0(x)|^2 = \frac{1}{\sqrt{2\pi x_0^2}} \cdot e^{-\frac{1}{2x_0^2}x^2}$

$\int uv' = uv - \int u'v$ $\qquad u = x \qquad u' = 1$

$\qquad v = -x_0^2 e^{-\frac{1}{2x_0^2}x^2} \qquad v' = x e^{-\frac{1}{2x_0^2}x^2}$

$$\frac{1}{\sqrt{2\pi x_0^2}} \cdot \frac{1}{x_0^2} \int_{-\infty}^{+\infty} dx\, (x).\left(x e^{-\frac{1}{2x_0^2}x^2}\right) = \frac{1}{\sqrt{2\pi x_0^2}} \cdot \frac{1}{x_0^2}\left(\left[-x_0^2 x e^{-\frac{1}{2x_0^2}x^2} \right]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} dx\, -x_0^2 . e^{-\frac{1}{2x_0^2}x^2} \right)$$

$$= \frac{1}{\sqrt{2\pi x_0^2}} \cdot \frac{1}{x_0^2}\, x_0^2 \sqrt{\frac{\pi}{1/2x_0^2}}$$

$$= \sqrt{\frac{2x_0^2\pi}{2\pi x_0^2}} \cdot \frac{x_0^2}{x_0^2}$$

$$\int_{-\infty}^{+\infty} dx\, |\psi_1(x)|^2 = 1.$$

• $\int_{-\infty}^{+\infty} |\psi_2(x)|^2 dx = \int_{-\infty}^{+\infty} dx \left| \frac{1}{\sqrt{2}} \left[\left(\frac{x}{x_0}\right)^2 - 1 \right] \psi_0(x) \right|^2 = \frac{1}{2} \int_{-\infty}^{+\infty} dx\, \frac{x^4}{x_0^4} \psi_0(x)^2 - 2\frac{x^2}{x_0^2}\psi_0(x)^2 + \psi_0(x)^2$

$$= \frac{1}{2} \frac{1}{x_0^4} \cdot \frac{1}{\sqrt{2\pi x_0^2}} \int_{-\infty}^{+\infty} x^4 e^{-\frac{x^2}{2x_0^2}} dx - \frac{1}{x_0^2} \cdot \frac{1}{\sqrt{2\pi x_0^2}} \int_{-\infty}^{+\infty} x^2 e^{-\frac{x^2}{2x_0^2}} dx + \frac{1}{\sqrt{2\pi x_0^2}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2x_0^2}} dx.$$

$$= \frac{1}{2x_0^4} \frac{1}{\sqrt{2\pi x_0^2}} \int_{-\infty}^{+\infty} dx\, x^4 e^{-\frac{x^2}{2x_0^2}} - \underbrace{\int_{-\infty}^{+\infty} dx\, |\psi_1(x)|^2}_{=1} + \frac{1}{2} \underbrace{\int_{-\infty}^{+\infty} dx\, |\psi_0(x)|^2}_{=1}$$

$$= \frac{1}{2x_0^4} \cdot \frac{1}{\sqrt{2\pi x_0^2}} \int_{-\infty}^{+\infty} dx\, x^4 e^{-\frac{x^2}{2x_0^2}} - \frac{1}{2}$$

$u = x^3 \qquad u' = 3x$

$v = -x_0^2 e^{-\frac{x^2}{2x_0^2}} \qquad v' = x e^{-\frac{x^2}{2x_0^2}}$

$$\frac{1}{2x_0^4} \cdot \frac{1}{\sqrt{2\pi x_0^2}} \int_{-\infty}^{+\infty} dx\, x^4 e^{-\frac{x^2}{2x_0^2}} = \frac{1}{2x_0^4} \cdot \frac{1}{\sqrt{2\pi x_0^2}} \left(\left[-x_0 x^3 e^{-\frac{x^2}{2x_0^2}} \right]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} -x_0^2 . 3x^2 e^{-\frac{x^2}{2x_0^2}} dx \right.$$

$$= \frac{1}{2x_0^4} \cdot \frac{1}{\sqrt{2\pi x_0^2}} (3x_0^2) \cdot \int_{-\infty}^{+\infty} x^2 e^{-\frac{x^2}{2x_0^2}} dx.$$

$u = x \qquad u' = 1$

$v = -x_0^2 e^{-\frac{x^2}{2x_0^2}} \qquad v' = x e^{-\frac{x^2}{2x_0^2}}$

$$\frac{3}{2x_0^2} \cdot \frac{1}{\sqrt{2\pi x_0^2}} \int_{-\infty}^{+\infty} x^2 e^{-\frac{x^2}{2x_0^2}} dx. = \frac{3}{2x_0^2} \cdot \frac{1}{\sqrt{2\pi x_0^2}} \left(\left[-x_0^2 x e^{-\frac{x^2}{2x_0^2}} \right]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} -x_0^2 e^{-\frac{x^2}{2x_0^2}} dx \right)$$

$$= \frac{3}{2x_0^2} \cdot \frac{x_0^2}{\sqrt{2\pi x_0^2}} \sqrt{\frac{\pi}{1/2x_0^2}}$$

$$= \frac{3}{2}$$

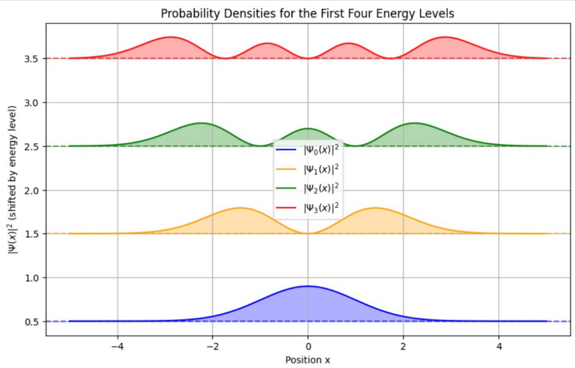
We finally get $\int_{-\infty}^{+\infty} dx |\Psi_2(x)|^2 = \frac{3}{2} - \frac{1}{2} = 1.$

we calculate the integral of each $|\Psi_n(x)|^2$ numerically:

```python
def normalize_check(n,x_0):
    integral, error = quad(lambda x: prob_psi_n(x,n,x_0), -np.inf, np.inf)
    return integral
```

```python
# Verification of the normalization
print("Normalisation of Psi_0:", normalize_check(0,x_0))
print("Normalisation of Psi_1:", normalize_check(1,x_0))
print("Normalisation of Psi_2:", normalize_check(2,x_0))
```

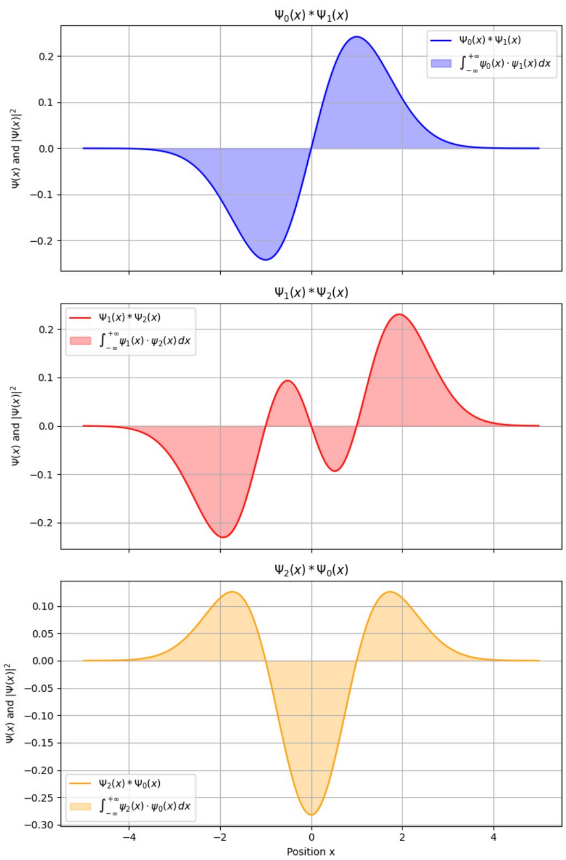
```
Normalisation of Psi_0: 0.9999999999999997
Normalisation of Psi_1: 1.000000000000001
Normalisation of Psi_2: 1.0000000000000024
```



The integrals of $|\Psi_n(x)|^2 = 1$, the stats are well normalized.

Exercice 30:

We see that for every overlap between stats, there is as much area under the curves in positive and negative meaning that the integral will be 0. This can be checked by numerical integration with quad module giving indeed 0 for all functions. Thus, overlap between state is 0 and they are normalized meaning that we built orthonormality of the wave functions.



Also, one can see that:

- $\Psi_0(x)$ is even and $\Psi_1(x)$ is odd, the product is then odd and $\int_{-\infty}^{+\infty} dx\, \Psi_0(x)\Psi_1(x) = 0$
- $\Psi_2(x)$ is even and $\Psi_1(x)$ is odd, the product is then odd and $\int_{-\infty}^{+\infty} dx\, \Psi_1(x)\Psi_2(x) = 0$

Numerical result:

```
Overlap integral of Psi_0 * Psi_1: 0.0
Overlap integral of Psi_1 * Psi_2: 0.0
Overlap integral of Psi_2 * Psi_0: -5.458619125285613e-17
```

$$\cdot \int_{-\infty}^{+\infty} dx\ \psi_0(x)\cdot\psi_2(x) = \int_{-\infty}^{+\infty} dx \frac{1}{\sqrt{2}}\left[\left(\frac{x}{x_0}\right)^2 - 1\right]\psi_0(x)^2 = \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2\pi x_0^2}}\int_{-\infty}^{+\infty}\left(\frac{x}{x_0}\right)^2 e^{-\frac{x^2}{2x_0^2}}dx - \frac{1}{\sqrt{2}}\int_{-\infty}^{+\infty}|\psi_0(x)|^2 dx$$

$$= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2\pi x_0^2}}\frac{1}{x_0^2}\int_{-\infty}^{+\infty} x^2 e^{-\frac{x^2}{2x_0^2}}dx - \frac{1}{\sqrt{2}}$$

$$\int_{-\infty}^{+\infty} x^2 e^{-\frac{x^2}{2x_0^2}}dx = x_0^2\sqrt{\frac{\pi}{1/2x_0^2}}$$ from previous question.

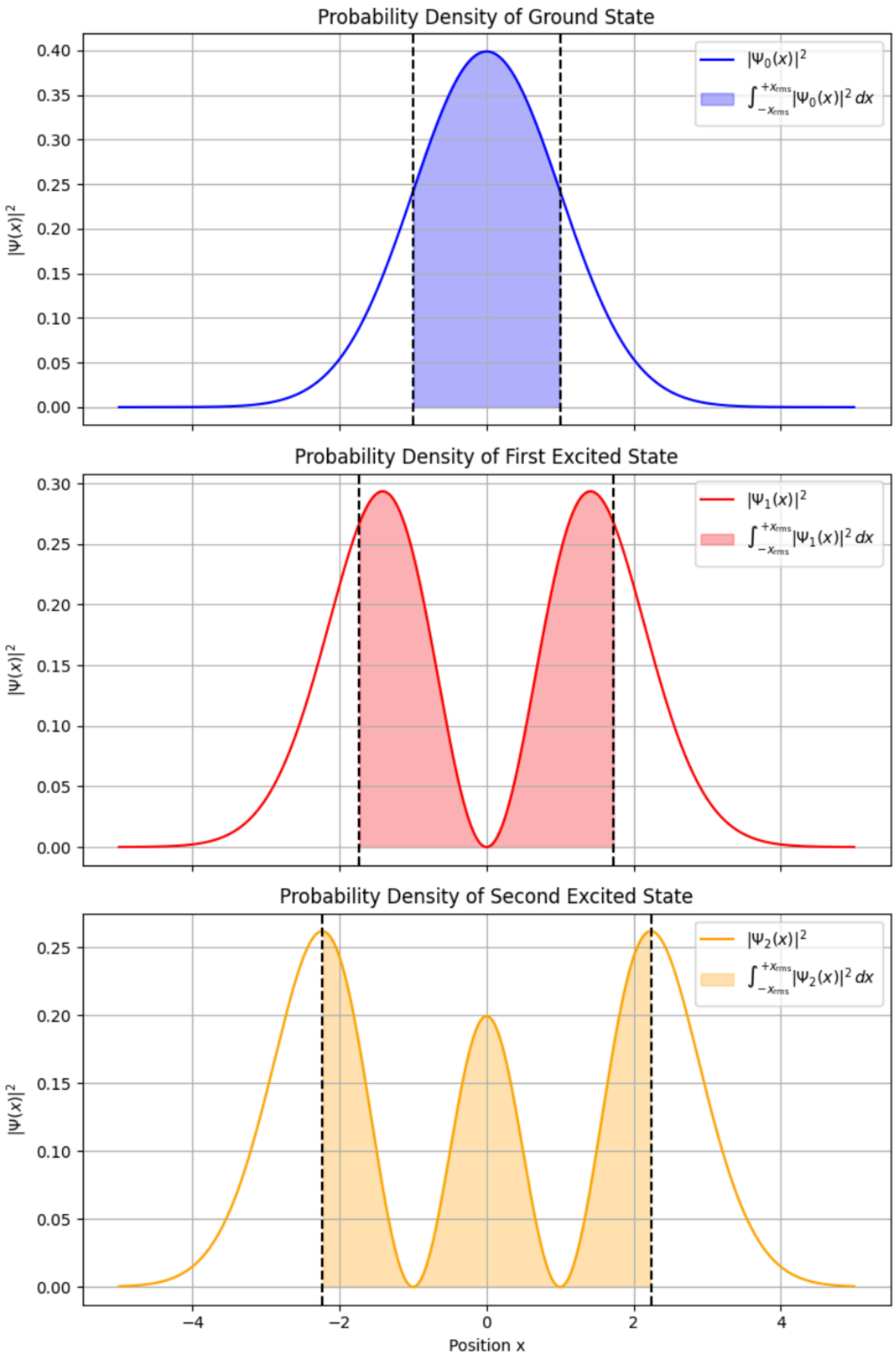
$$= \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{\cancel{2\pi x_0^2}}}\cdot\frac{1}{\cancel{x_0^2}}\cdot\cancel{x_0^2}\sqrt{\cancel{\frac{\pi}{1/2x_0^2}}} - \frac{1}{\sqrt{2}}$$

$$\int_{-\infty}^{+\infty} dx\ \psi_0(x)\cdot\psi_2(x) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$$

Exercice 31:

$|x| < x_{RMS} \iff -x_{RMS} \leq x \leq x_{RMS}$ As a reminder: $x_{rms} = x_0\sqrt{2n+1} = \sqrt{\frac{\hbar}{2m\omega}(2n+1)}$

we took $x_0 = 1$.




```
Probability of finding the oscillator between ±xrms when in Ground State: 0.6826894921370859
Probability of finding the oscillator between ±xrms when in First Excited State: 0.608374823728911
Probability of finding the oscillator between ±xrms when in Second Excited State: 0.535303204464737
```

Here we have $x_{RMS} = \sqrt{2n+1}$
Probability of finding the oscillator around its equilibrium position decreases as n increases. Indeed, the field has higher amplitude of oscillation because it has more energy. It is now less likely to find the energy of the field around x_0.

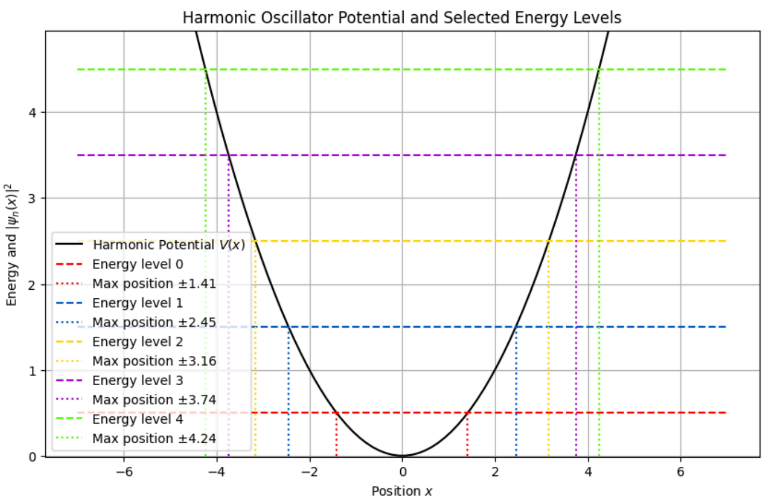
Exercice 32:

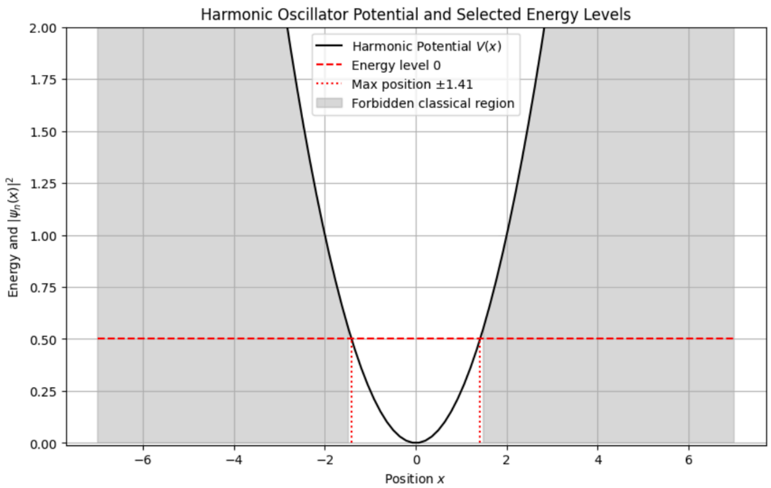
We need to find: $E_n = V(x) = \frac{m\omega^2}{2}x^2$ for each level n.

The supposed forbidden region of the oscillator in a certain energy level E_n should be the value of x outside the following bond: $-\sqrt{\frac{2E_n}{m\omega^2}} \leq x_c \leq \sqrt{\frac{2E_n}{m\omega^2}}$

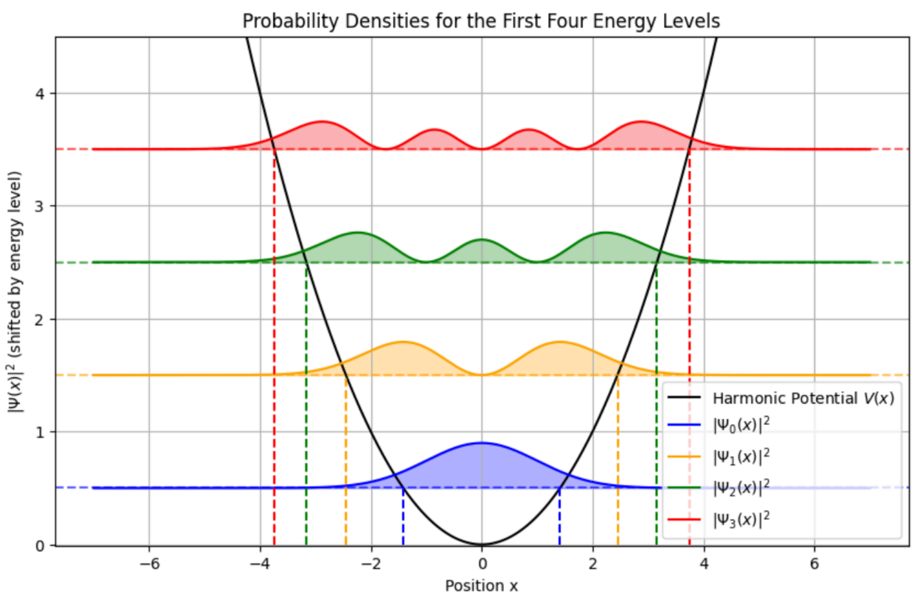
$|x_c| \leq \sqrt{\frac{2\hbar\omega(n+\frac{1}{2})}{m\omega^2}} \iff |x_c| < \sqrt{4(n+\frac{1}{2})}\sqrt{\frac{\hbar}{2m\omega}}$ $\quad |x_c| < x_0\sqrt{4(n+\frac{1}{2})}$

Let's plot this forbidden region for some states ($\Psi_0(x)$ included):



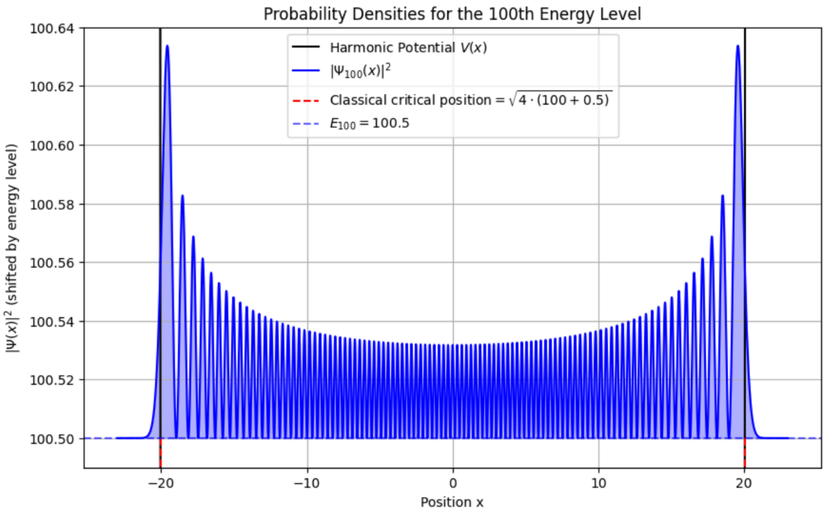


Exercice 33:



As we can see here, the area under the curve after the turning point is non negligeable. Therefore, there is a probability of finding the oscillator after this point.

Exercise 34:



- As we can see here, the area under the curve is clearly greater after the turning point (red dashed line).

- One can therefore say, that this result aligns with classical intuition in a way. In classical mechanics, a particle in a harmonic oscillator spends more time near the turning points because it slows down there before reversing direction. Indeed, it has less and less kinetic energy (it slows down) which is converted to potential energy. Quantum mechanically, the probability density $|\Psi_{100}(x)|^2$ peaks near these points, indicating a higher likelihood of finding the particle near the classical boundaries.
However, due to the quantum nature, there is also a probability of finding the oscillator in the "classically forbidden region". (beyond the turning point), which is not possible in classical mechanics. This is a purely quantum phenomenon and highlights the wave-like nature of particle.

Exercise 35:

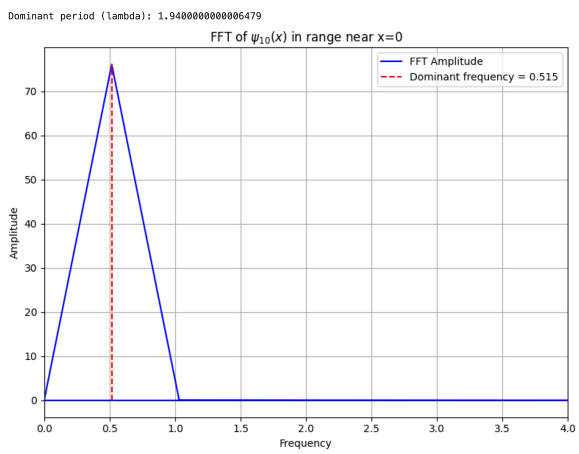
$$\lambda = \frac{2\pi\hbar}{p} \cong \frac{2\pi\hbar}{\sqrt{2mE}} = \frac{2\pi\hbar}{\sqrt{2m\cdot\frac{1}{2}mv^2}} = \frac{h}{mv}$$

assuming $m = 4$ [kg] and $v = 5$ [m/s] we have:

$$\lambda = \frac{h}{mv} = \frac{6{,}626\cdot10^{-34}}{4\cdot20} = 3{,}313\cdot10^{-35}$$

So the De Broglie wavelength of the cat is approximately $3{,}313\cdot10^{-35}$ meters. This wavelength is far below the scale of atomic or molecular structures, meaning that quantum wave properties are insignificant and undetectable at this macroscopic scale. Thus, classical physics adequately describes the motion of the cat in this context, and quantum effects do not manifest in observable ways due to their short De Broglie wavelength.

Exercise 36:

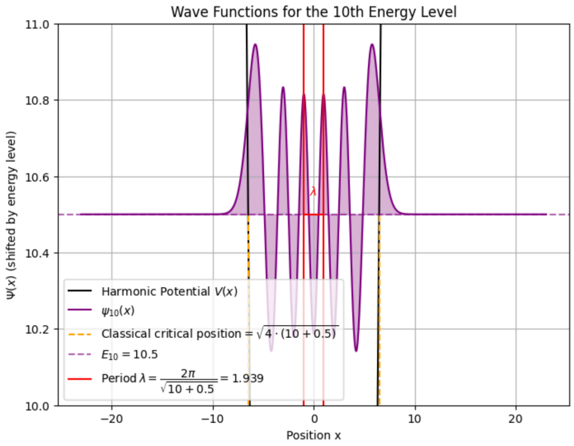


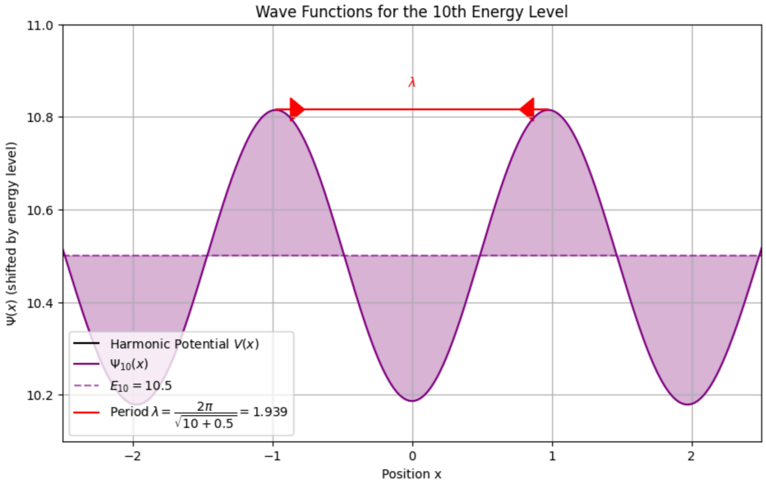
$$p = \hbar K = \sqrt{2mE_{10}}$$

$$\Leftrightarrow \hbar\frac{2\pi}{\lambda} = \sqrt{2mE_{10}}$$

$$\Leftrightarrow \lambda = \frac{2\pi\hbar}{\sqrt{2mE_{10}}} = \frac{2\pi\hbar}{\sqrt{2m\,\hbar\omega(n+\frac{1}{2})}} = \frac{2\pi\sqrt{\hbar}\sqrt{\hbar}}{\sqrt{2m\omega}\sqrt{\hbar}\sqrt{n+\frac{1}{2}}} = \frac{2\pi x_0}{\sqrt{n+\frac{1}{2}}}$$

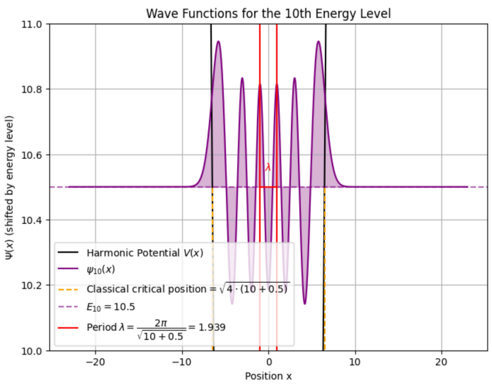
We chose $x_0 = 1$ then λ should be : $\lambda = \frac{2\pi}{\sqrt{10+\frac{1}{2}}}$ for $|10\rangle$.

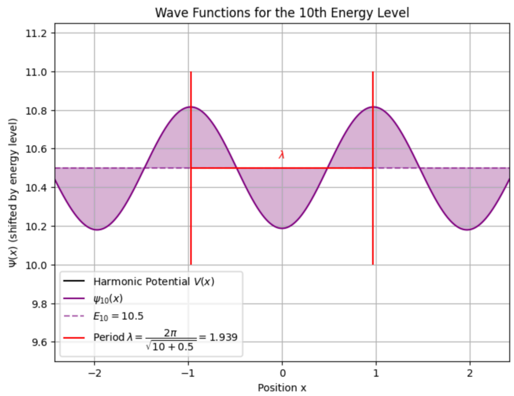




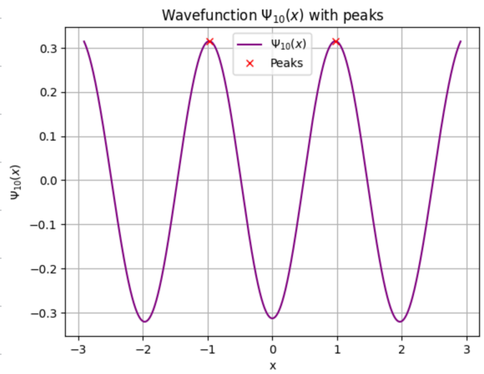
We can see that the dominant frequency is $\lambda \cong 1{,}94$ with an FFT transform. This corresponds to what we geometrically obtained and theoretically obtained with $\lambda = \frac{2\pi x_0}{\sqrt{10+0{,}5}} \cong 1{,}939$ for $x_0 = 1$.

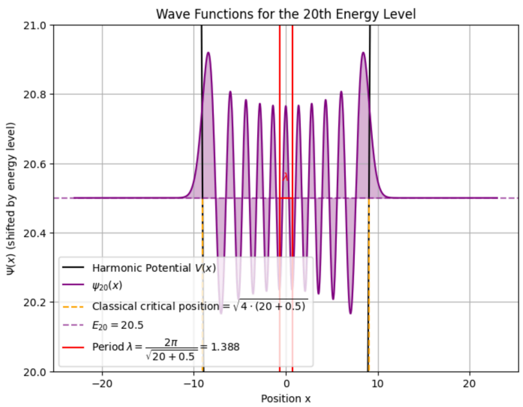
Exercice 37:

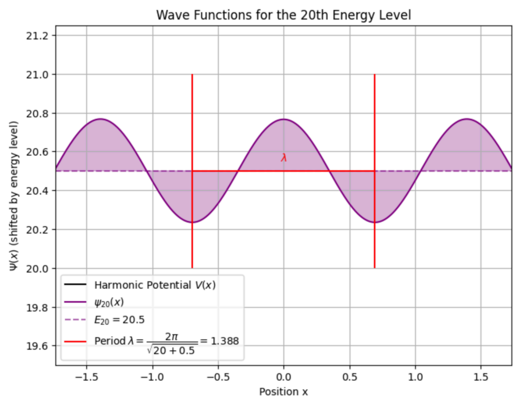




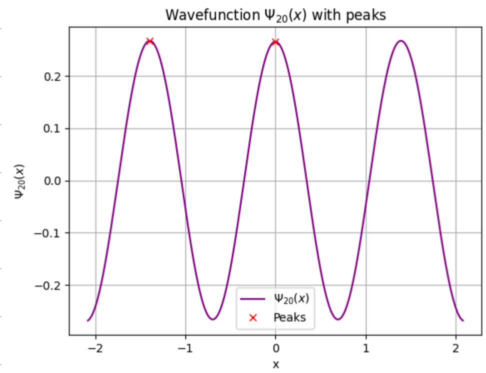
`Dominant period (lambda): 1.9440000000000026`

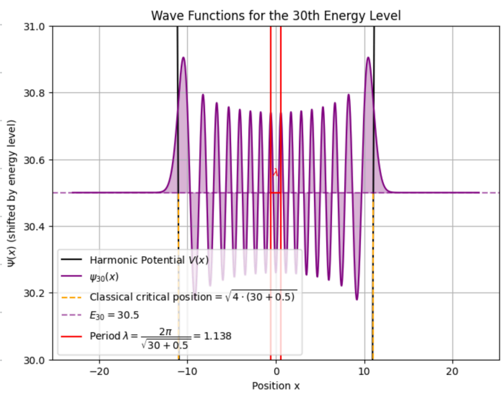


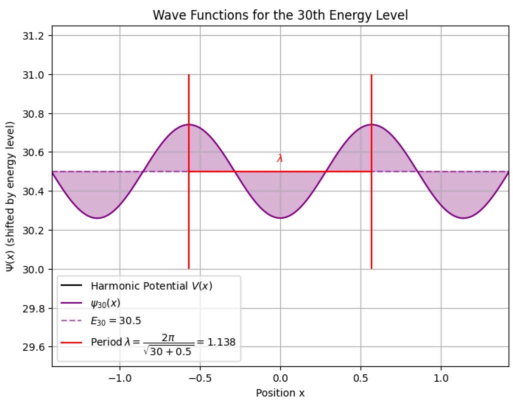




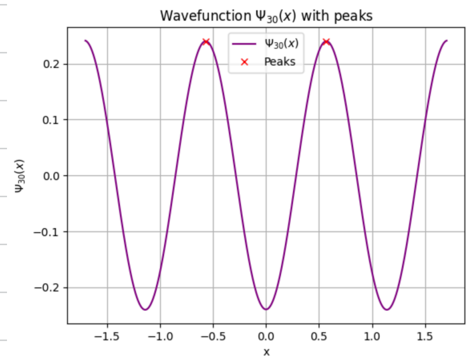
`Dominant period (lambda): 1.3919999999999995`

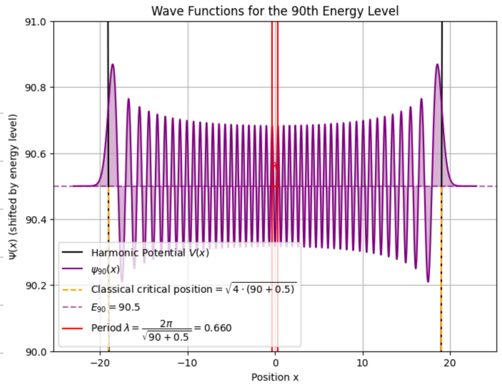


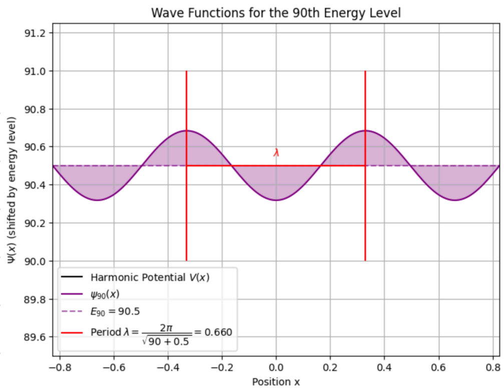


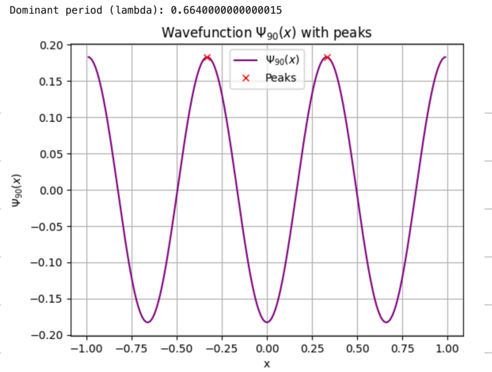


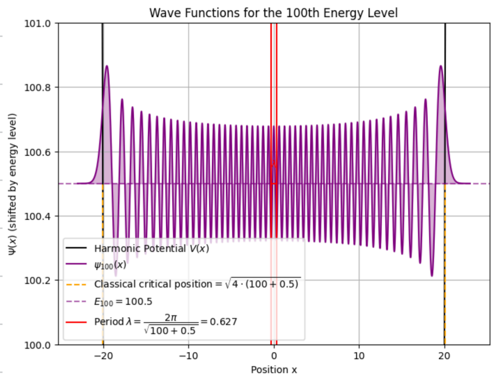
`Dominant period (lambda): 1.1359999999999992`

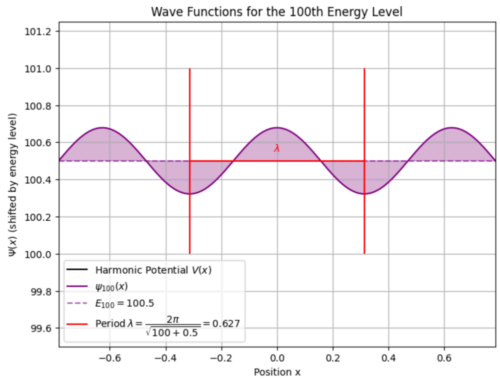


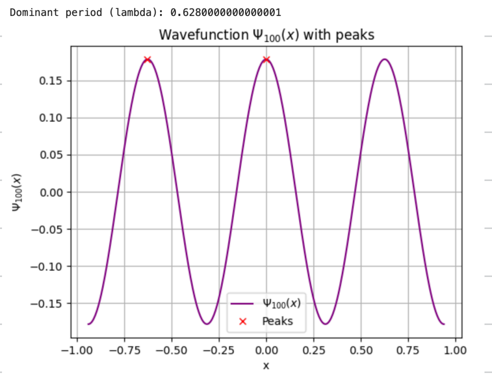


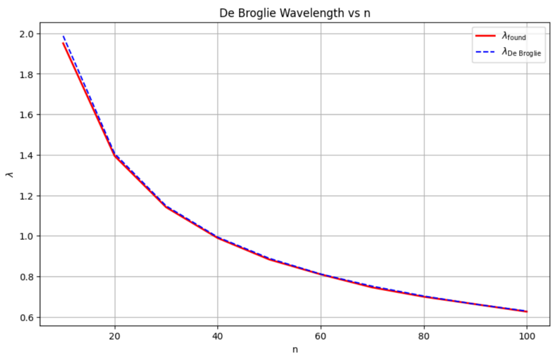












As we can see, the λ_{found} as in the purple above corresponds well to the prediction of the prediction $\lambda_n = \frac{2\pi x_0}{\sqrt{n}}$. The two curves are slighty different in the begining since the potential is not directly negligeable for low n's located down inside V(x). Also for example, $\psi_0(x)$ is a gaussian and the period λ has not a true meaning since there is only one peak.

Exercice 38:

Gaussian initial state. See the following link:

https://drive.google.com/file/d/1xBMc4BVwg_NiMq4R7nTty38XURFmKp-U/view?usp=drive_link

Exercice 39:

Coherent initial state: See the following link:

https://drive.google.com/file/d/1M9lmi302z-evt-ZlCJFYi3FjpTfC4vHC/view?usp=sharing

- The time evolution of a coherent state $|\phi\rangle = |\alpha\rangle$ as an initial state in the quantum harmonic oscillator resembles the behaviour of a classical harmonic oscillator. Observing for $\alpha = 1, 2, 10$, we see that the wave packet oscillatis back and forth around an equilibrium position, maintaining a relatively stable and localized shape. This behavior reflects well the classical motion of an oscillator, where position and momentum periodically.

- For larger values of α, the amplitude of the oscillation increases, and the motion becomes even more reminiscent of a classical trajectory, as the relative uncertainty decreases. Indeed we see that the oscillator can be found farther from its equilibrium position since it has greater energy (α is in factor in front of the mean position $\langle \alpha(t) | \hat{x} | \alpha(t) \rangle$).

Thus, the coherent state effectively captures the classical dynamics within a quantum system, providing a visual representation of the correspondence between quantum and classical.

Exercice 40:

Superposition initial state: See the following link:

https://drive.google.com/file/d/1-wvRYg7IA4UM-jpUwc7rdimFDQxL7v0g/view?usp=sharing

(1) Quantum superposition vs. Classical motion

↳ The video shows a superposition of the states $|0\rangle$ and $|10\rangle$. In quantum mechanics, such a superposition leads to probability interferences patterns. In contrast, a classical oscillator has a well-defined trajectory with a fixed position at any point in time.

(2) Amplitude Modulation and phases.

↳ The animation displays rapid oscillations with changing amplitudes due to the interference between quantum states. This occurs because the states $|0\rangle$ and $|10\rangle$ evolve with different phases, creating complex modulation. A classical oscillator maintains a constant amplitude and lacks such phase-driven interference effect.

(3) Position distribution

↳ $|\Phi(x,t)|^2$ is concentrated near the center with oscillations features due to interference. This corresponds to the mean position $\langle \hat{x} \rangle$ calculated before and being 0. A classical oscillator, however, typically has a high probability of being found near the turning points, where its speed is minimal.

(4) Effect of Superposition and phases.

↳ The interference between the two quantum states leads to regions of high and low probability which we don't encounter classically. The phase evolution of each component state causes constructive and destructive interference resulting in regions of nearly 0 prob.

(5) Conclusion:

↳ The behaviour observed in the video is clearly non classical due to the interference patterns and phase modulation seen in $|\Phi(x,t)|^2$. Thus a classical oscillator with a predictable periodic motion would not move in this way.

If the links do not work above because of handwriting, I post again the links here on a blank page :

Gaussian evolution :
https://drive.google.com/file/d/1xBMc4BVwg_NiMq4R7nTty38XURFmKp-U/view?usp=drive_link

Coherent state evolution :
https://drive.google.com/file/d/1M9lmi302z-evt-ZlCJFYi3FjpTfC4vHC/view?usp=drive_link

Superposition evolution :
https://drive.google.com/file/d/1-wvRYg7IA4UM-jpUwc7rdimFDQxL7v0g/view?usp=drive_link