Homework 1, part 1/3

Version as of Sept 20, 2024, a few typos and errors fixed

Due date for entire HW1 Part 1 (all three parts are due same time) is Oct 20. Start now!

- Each exercise is worth 1 point. No partial credit granted, no detailed solution requested. Please write down your answer (with possibly a brief commentary) to each exercise in the space underneath it. If you need more space, please attach a separate sheet.

- The total grade recorded for this part of the homework is #earned points/total # points.

- We encourage you to use QuTiP in Python to do some of these exercises, at least to verify your answers or just to explore the problem without having to do the math by hands.

Suggested Literature:
L. Susskind Theoretical Minimum book, Ch. 2; Ch. 3
Kay, Laflamme, Mosca Quantum Computing book, Ch 2.1-2.5; Ch 3.1, 3.4; Ch 4.1-4.2

A. Qubit states and their representation in the Bloch sphere.

The state of a qubit is usually parametrized using two angles, θ and ϕ:

$$|\Psi\rangle = \cos\frac{\theta}{2}|0\rangle + \exp(i\phi)\sin\frac{\theta}{2}|1\rangle, \tag{1}$$

and can be represented as a point on a sphere with a unit radius. The "longitude" angle θ varies from 0 to π and the latitude angle ϕ varies from 0 to 2π or from $-\pi$ to π. The angle θ is measured with respect to the direction of the Z-axis which goes from the "South pole" to the "North pole". The angle ϕ is defined in the "Equatorial plane" and measured with respect to the X-axis (the direction of which can be chosen arbitrary). For points $\theta = 0$ we get $|\Psi\rangle = |0\rangle$ (north pole) and for $\theta = \pi$ we get $|\Psi\rangle = |1\rangle$ (south pole) irrespective of the value of ϕ.

Matrix form representation. In order to express states in the "matrix form", we define two-component column vectors

$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

The corresponding dual vectors are obtained by taking the transpose

$\langle 0| = (1, 0)$
$\langle 1| = (0, 1)$

The qubit states $|0\rangle$ and $|1\rangle$ are orthogonal, because the corresponding vectors satisfy the orthogonality condition $\langle 0|1\rangle = (1, 0) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \times 0 + 0 \times 1 = 0$ and hence they form a basis in the space of 2-component complex vectors. These two states are also normalized to a unit length, that is $\langle 0|0\rangle = \langle 1|1\rangle = 1$. The basis formed by $|0\rangle$ and $|1\rangle$ is often called "computational basis". Curiously, even though the states $|0\rangle$ and $|1\rangle$ are orthogonal, their corresponding Bloch vectors are obviously parallel to each other (pointing in opposite directions). There is a difference between the Bloch sphere space, which kind of reflects our 3D world and the 2D complex vector space of qubit states, which is an abstraction required to formulate the rules of quantum mechanics.

Irrelevance of the global phase-factor. We can multiply $|\Psi\rangle$ by $\exp(i\alpha)$, α is any real number, and this operation would not change the state. For example, $-|0\rangle$ is physically no different from $|0\rangle$ (multiplying by $\exp(i\pi)$). Or $(1/\sqrt{2})|0\rangle - (1/\sqrt{2}|1\rangle$ is the same state as $(1/\sqrt{2})|1\rangle - (1/\sqrt{2}|0\rangle$. Or $|0\rangle + i|1\rangle$ is the same state vector as $i|0\rangle - |1\rangle$. To summarize, in order to find the Bloch sphere vector from a given quantum state, we should first eliminate the global phase factor by making the probability in front of $|0\rangle$ a real number and adjust the accordingly the phase of the amplitude in from of $|1\rangle$.

Exercise 1: Construct 2×2 matrix $\hat{Z}$ the eigenvectors of which are $|0\rangle$ and $|1\rangle$ with the corresponding eigenvalues +1 and -1. That is $\hat{Z}|0\rangle = +1|0\rangle$ and $\hat{Z}|1\rangle = -1|1\rangle$.
Hint: calculate matrix elements $\langle 0|\hat{Z}|0\rangle$, $\langle 0|\hat{Z}|1\rangle$, ...

Exercise 2: Find matrix $\hat{X}$ which turns $|0\rangle$ into $|1\rangle$ and $|1\rangle$ into $|0\rangle$ (quantum NOT-gate).

Since states $|0\rangle$ and $|1\rangle$ form a basis, any other qubit state can be presented as their superposition, as defined in Eq. (1). For example, two other common states form a different basis:

$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$
$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

Exercise 3: Mark the states $|+\rangle$ and $|-\rangle$ together with states $|0\rangle$ and $|1\rangle$ on the Bloch sphere. Note, the superposition of "up" and "down" apparently points sideways!

Exercise 4: Show that states $|+\rangle$ and $|-\rangle$ also form a basis. Basis means any other states can be expressed as a linear superposition of the basis states. What would be the states $|0\rangle$ and $|1\rangle$ in this new basis?

Exercise 5: Find a 2x2 matrix $\hat{X}$, the eigenvectors of which are $|+\rangle$ and $|-\rangle$ with eigenvalues $+1$ and -1, respectively.

Exercise 6: Find out states $\hat{X}|0\rangle$, $\hat{X}|1\rangle$, $\hat{Z}|+\rangle$, and $\hat{Z}|-\rangle$.

We can convert from the computational basis $|0\rangle$, $|1\rangle$ to the basis $|+\rangle$, $|-\rangle$ AND back using the Hadamard operator, defined by the matrix $\hat{H} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$. Once can equivalently write $\hat{H} = \hat{Z}/\sqrt{2} + \hat{X}/\sqrt{2}$. Also once can check that $\hat{H}^2 = \hat{I}$ (identity).

Exercise 7: Apply $\hat{H}$ to states $|0\rangle$, $|1\rangle$, $|+\rangle$, $|-\rangle$. Check that $\hat{H}^2 = \hat{I}$

There is yet another commonly used basis of qubit states, this time involving complex numbers:

$|+i\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$
$|-i\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle$

Exercise 8: Mark the states $|+i\rangle$, $|-i\rangle$ on the Bloch sphere with respect to the states $|0\rangle$, $|1\rangle$, $|+\rangle$, $|-\rangle$.

We remind that a dual vector $\langle\Psi| = |\Psi\rangle^\dagger$ is created by transposing a column into a row AND complex-conjugating every entry. That is $\begin{pmatrix}1\\ i\end{pmatrix}^\dagger = \begin{pmatrix}1 & -i\end{pmatrix}$.

Exercise 9: Write down the following vectors (as columns and rows), $|+i\rangle$, $\langle +i|$, $|-i\rangle$, $\langle -i|$.

Exercise 10: Find the matrix $\hat{Y}$ the eigenvalues of which are $+1$ and -1 corresponding to the eigenvectors $|+i\rangle$ and $|-i\rangle$.

Exercise 11: Show that the pair of states $|+i\rangle$ and $|-i\rangle$ form a basis in the vector space of our qubit. Basis means any other states can be expressed as a linear superposition of the basis states.

Exercise 12: Find the analog of the Hadamard matrix which would convert basis states $|+i\rangle$ and $|-i\rangle$ into $|0\rangle$ and $|1\rangle$ and back.

Exercise 13: Same question as above but this time let's convert between the basis $|+\rangle$, $|-\rangle$ into $|+i\rangle$, $|-i\rangle$

You now know the three pairs of most common qubit basis states, which are the eigenstates (eigenvectors) of the three matrices (operators) $\hat{X}, \hat{Y}, \hat{Z}$. They are called Pauli matrices (operators). We will use the terms operators and matrices interchangeably, although it's important to keep in mind that the matrix is just a representation of the operator in a given basis. Unless specified, our default basis is the computational one.

B. Quantum measurement rules

As you have surely noticed, there is an infinite number of possible qubit states, given by the continuous choice of the two angles θ and ϕ in Eq. (1). So where's discretness (quantumnes!) coming from? It comes from the act of measurement.

To formulate the rules for the measurement outcome we first must choose what is being measured. In quantum mechanics, this means choosing an operator $\hat{L}$ representing some observable related to the qubit (no reason for choosing L over other letters). The only theoretical constraint on $\hat{L}$ is that it must be a hermitian operator, $\hat{L}^\dagger = \hat{L}$. We'll get to see why it is so a bit later.

Exercise 14: Consider Pauli operators $\hat{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \hat{Y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \hat{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (with their matrices written in the computational basis) as well as their linear combinations $\hat{X} \pm \hat{Z}$, $\hat{X} \pm \hat{Y}$, $\hat{X} \pm i\hat{Y}$. Which one(s) cannot represent an observable?

Rules of quantum measurement. Let's imagine a qubit in an arbitrary state $|\Psi\rangle$ and a measurement apparatus (a device) which somehow reads the value of a qubit observable defined by a hermitian operator $\hat{L}$. Building such a device is a problem for quantum engineers, and we will touch on this topic later in the course. For now let's just assume that we have an instrument that can measure any hermitian operator $\hat{L}$. Let's also denote eigenstates and eigenvalues of our measurement operator $\hat{L}$ as $|\psi_j\rangle$ and λ_j ($j = 0, 1$). The reading will be either λ_1 or λ_2, at random, and nothing else! The probability of getting λ_j is given by $|\langle\psi_j|\Psi\rangle|^2$. Furthermore, immediately after the measurement, the qubit state changes (collapses) from $|\Psi\rangle$ to $|\psi_j\rangle$. How weird is that?

Consider a specific example of the measurement operator $\hat{Z}$, the eigenstates of which are $|0\rangle$ and $|1\rangle$ (our computational basis) and the eigenvalues are $+1$ and -1, respectively. The qubit is prepared in the state $|\Psi\rangle$ given by Eq. 1. There are two options for the measurement outcome: i) with a probability $(\cos(\theta/2))^2$ the reading shows $+1$ and the qubit is prepared (initialized) in state $|0\rangle$ and ii) with a probability $(\sin(\theta/2))^2$ the reading is -1 and the qubit is prepared in state $|1\rangle$.

Note, repeating the measurement second time would give the same result as the first one. The randomness is gone. That's because if the first measurement collapses the qubit state into $|0\rangle$, the second measurement must yield $+1$ reading with probability 1 and the qubit state $|0\rangle$ remains unchanged. Likewise, if the first measurement already collapsed the qubit to state $|1\rangle$, the second one will output -1 with probability 1. Randomness happens only during the very first measurement. The observer can't notice this randomness unless someone provides a second qubit prepared in exactly the same initial state $|\Psi\rangle$. In this case the observer can notice that the outcome of the first measurement of the first qubit does not necessarily match with that of the first measurement of the second qubit. By getting $N \gg 1$ copies of the qubit in state $|\Psi\rangle$, one can measure the frequency of measurement outcomes $+1$ and -1 and evaluate the probabilities $(\cos(\theta/2))^2$ and $(\sin(\theta/2))^2$.

What's the meaning of measuring $\hat{Z}$ operator for a qubit? Imagine a qubit as some kind of an arrow with a unit length in the real space. We call such a quantum arrow a "spin-1/2 system" or just "spin". Measuring the $\hat{Z}$-operator is asking for the value of the projection of the arrow onto Z-axis. Measuring a classical arrow would give a continuum of projection values, from $+1$ to -1. But in the quantum case the Z-projection can only take discrete values $+1$ or -1. It cannot be zero! Before we make the measurement, not only we don't know what's the Z-projection of the arrow, the arrow itself does not know it. Each time the measurement of the Z-projection of our qubit (or spin) gives us the value of $+1$ we know the qubit is now prepared in state $|0\rangle$ (spin is pointing along the Z axis) and each time the measurement value is -1 we know the qubit is now prepared in state $|1\rangle$ (the spin is pointing against the Z-axis). We often say **"measure the qubit along the Z-axis"** or **"measure the Z-projection"** or **"measure the qubit in the computational basis"**, which are all equivalent to **choosing the measurement operator to be $\hat{Z}$** and applying the **rules of quantum measurement** formulated above.

Exercise 15: Consider a qubit prepared in state $|+\rangle$. Suppose we have a large number of copies of this qubit and we measure the $\hat{Z}$ for each qubit. What would be the mean value of the outcome? Same question for state $|-\rangle$.

Exercise 16: Consider the same experiment as in the exercise above but the qubit state is now $|\Psi\rangle$ given by Eq. (1). Plot the mean value of the measurement outcome as a function of θ. Does the answer make sense in the context of the Z-projection of a classical arrow?

Exercise 17: Show that the average measurement value of the Z-projection in the previous exercise can be compactly written as $\langle\Psi|\hat{Z}|\Psi\rangle$. Hint: just multiply the three corresponding matrices.

Now let's consider measurement of other observables of the qubit. We can perform a measurement of $\hat{Z}$ and check that the reading is $+1$, in which case the qubit is guaranteed to be in state $|1\rangle$. Otherwise we would ask for another qubit and repeat until we get the desired $|1\rangle$ state initialization. What would we get if, following the initialization to $|1\rangle$, we now measure operator $\hat{X}$? In the spin analogy, this is measuring the spin's X-projection. Following the quantum measurement rule, we recall that eigenstates of $\hat{X}$ are $|+\rangle$ and $|-\rangle$ and the eigenvalues are $+1$ and -1, respectively. Therefore, the reading on the measurement device would be $+1$ with probability $|\langle+|1\rangle|^2 = 1/2$ and -1 with probability $|\langle-|1\rangle|^2 = 1/2$. The mean value of the reading would be $|\langle+|1\rangle|^2 \times (+1) + |\langle+|1\rangle|^2 \times (-1) = \langle 1|\hat{X}|1\rangle = 0$.

Exercise 18: Plot $\langle\Psi|X|\Psi\rangle$ as a function of the angle θ. Compare it to the previously calculated $\langle\Psi|Z|\Psi\rangle$. Do these quantities behave like X- and Z-projection of a unit vector?

Exercise 19: Consider a qubit in state $|0\rangle$ and a measurement of $\hat{Z}$ and $\hat{X}$. We know that if we repeat each measurement many times (each time with a fresh qubit initialized to state $|0\rangle$), the mean value for $\hat{Z}$ would be $+1$ and the mean value for $\hat{X}$ would be 0, that is $\langle 0|\hat{Z}|\rangle = +1$, and $\langle 0|\hat{X}|0\rangle$ $=0$. Let's calculate the variance of the measurement outcome, that is $\langle 0|\hat{Z}^2|0\rangle - (\langle 0|\hat{Z}|0\rangle)^2$ and $\langle 0|\hat{X}^2|0\rangle$. How random is the measurement outcome in each case?

Exercise 20: Suppose we have a qubit in state $|+\rangle$ and measure operator $\hat{Y}$ (measure the spin's Y-projection). What reading would we get after one measurement, and what would be the mean value of the readings after many measurements (each time starting with a fresh qubit in state $|+\rangle$)?

Exercise 21: Is it too much to ask to measure $\hat{X}$ and $\hat{Z}$ at the same time, that is to learn both the Z-projection and the X-projection of our quantum arrow? Let's find out. Suppose you have a qubit prepared in state $|0\rangle$. Do a sequence of measurements $\hat{Z}$, $\hat{Z}$, $\hat{Z}$, ... You would get 1,1,1, etc. Now let's take a fresh qubit in state $|0\rangle$ and do a different measurement sequence $\hat{X}$, $\hat{X}$, $\hat{X}$. You will get either $1, 1, 1, ...$ or $-1, -1, -1, ...$ each sequence having a probability 50%. What do we get if instead we alternate the measurements $\hat{Z}$, $\hat{X}$, $\hat{Z}$, $\hat{X}$, etc?

Exercise 22: Now let's ask the same question about the mean values of the projections $\hat{X}$ and $\hat{Z}$. Let's take a qubit in the state $|\Psi\rangle$ given by Eq. 1. This time we change the measurement protocol. We take a fresh qubit in state $|\Psi\rangle$ each time we measure something. We first measure $\hat{Z}$, next time we measure $\hat{X}$, next time $\hat{Z}$, then $\hat{X}$, etc.. The average value of all $\hat{X}$ readings would be $\langle\Psi|\hat{X}|\Psi\rangle$ and similarly for $\hat{Z}$, we would get $\langle\Psi|\hat{Z}|\Psi\rangle$. What can we say about the simultaneous values of the mean projections X and Z of the spin? Do their values make more sense than the results of instantaneous measurements in the previous exercise?

C. Unitary and Hermitian operators

In quantum mechanics we usually deal with two types of operators: Hermitian and Unitary. A Hermitian operator $\hat{H}$ satisfies $\hat{H} = \hat{H}^\dagger$. A unitary operator U satisfies $\hat{U}^\dagger\hat{U} = \hat{I}$ (which is equivalent to $U^{-1} = U^\dagger$).

Eigenvectors of a Hermitian operator are orthogonal and form a basis in the vector space. This is why physical observables (the stuff that can be measured) must be represented by Hermitian operators. As a result of the measurement the quantum system finds itself in one of the eigenstates of the measurement operator. Any qubit state can be represented by a superposition of all possible measurement outcome states, which makes sense. Unitary operators preserve the vector's length, that is they keep the sum of all probabilities to a unity. Hence, any operator describing the evolution of a qubit's state (apart from the measurement collapse process) should be unitary.

Exercise 23: Check that any unitary operator $\hat{U}$ applied to a state $|\Psi\rangle$ in Eq. 1 creates a state $|\Psi'\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$ where $\alpha_0\alpha_0^* + \alpha_1\alpha_1^* = 1$ (and hence the new state can also be represented as a vector in a Bloch sphere).

Exercise 24: Check that the Pauli operators $\hat{X}$, $\hat{Y}$, $\hat{Z}$ are both Hermitian and unitary. Hence they can serve to represent physical observables (the projections of the spin onto the three orthogonal axis), and also as evolution operators. Illustrate your finding with vectors $|0\rangle$, $|1\rangle$, $|+\rangle$, $|-\rangle$, $|+i\rangle$, $|-i\rangle$

Because eigenvectors of any Hermitian operator $\hat{H}$ form a basis, we can write down this operator using Dirac notations in terms of the eigenvectors $|h_i\rangle$ and eigenvalues h_i, where $i = 0, 1$ for a qubit.

$\hat{H}|h_i\rangle = h_i|h_i\rangle$,
$\hat{H} = \sum_{\text{all eigenstates}} h_i|h_i\rangle\langle h_i|$

It's also useful to note a special case of this relation when $\hat{H} = \hat{I}$ (identity), in which case the eigenvector decomposition is called "completeness relation:

$\hat{I} = \sum_{\text{all eigenstates}} |h_i\rangle\langle h_i|$

We can use the above relations to find the matrices for the Pauli operators in the computational basis $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$. Indeed, $\hat{Z} = (+1)|0\rangle\langle 0| + (-1)|1\rangle\langle 1|) = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} - \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$. If eigenstates of the operator are not $|0\rangle$ and $|1\rangle$ we would have two more matrix elements to work out.

Exercise 25: Repeat the steps above for finding the matirx for $\hat{X}$-operator using its eigenvectors $|\pm\rangle$.

Exercise 26: Do the same as above but for $\hat{Y}$, using its eigenvectors $|\pm i\rangle$.

Exercise 27: Use the representation of a Hermitian operator above to prove that $\hat{H}^n = \sum_{\text{all eigenstates}} h_i^n |h_i\rangle\langle h_i|$.

Thus, to take a function f of a Hermitian matrix $\hat{H}$, assuming the function has a convergent power series, we simply need to find it's eigenvalues h_i and eigenvectors $|h_i\rangle$. Then $f(\hat{H}) = \sum_i f(h_i)|h_i\rangle\langle h_i|$.

Exercise 28: Show that any unitary operator $\hat{U}$ (represented by an $N \times N$ matrix) can be written as $\hat{U} = \exp(i\alpha\hat{H})$, where α is a real number and $\hat{H}$ is some hermitian operator (also represented by an $N \times N$) matrix.

D. Rotating the qubit state on the Bloch sphere

We have seen that operator $\hat{Z}$ is hermitian (and unitary), so operator $\exp(-i\alpha\hat{Z}/2)$ must also be a unitary (the factor 1/2 is there for some convenience later). What does this unitary do to the qubit state?

Using matrix exponentiation, we can show that $\exp(-i\alpha\hat{Z}/2) = \hat{I}\cos\alpha/2 - i\hat{Z}\sin\alpha/2 = \begin{pmatrix}\exp(-i\alpha/2) & 0\\0 & \exp(i\alpha/2)\end{pmatrix} = \exp(-i\alpha/2)\begin{pmatrix}1 & 0\\0 & \exp(i\alpha)\end{pmatrix}$. Applying this matrix to the state $|\Psi\rangle$ in Eq. 1 we get $\exp(-i\alpha\hat{Z}/2)|\Psi\rangle = \exp(-i\alpha/2)(\cos\theta/2|0\rangle + \sin\theta/2\exp i(\phi + \alpha)|1\rangle$. So, the operator $\exp(-i\alpha/2\hat{Z})$ rotates the qubit state in the XY-plane by an angle α. Perhaps an easier way to derive this would be use the relations:

$\exp(-i\alpha/2\hat{Z})|0\rangle = \exp(-i\alpha/2)|0\rangle$
$\exp(-i\alpha/2\hat{Z})|1\rangle = \exp(+i\alpha/2)|1\rangle$

Since only the phase difference between $|0\rangle$ and $|1\rangle$ matters, the effect of the operator is equivalent to doing nothing to $|0\rangle$ and multiplying $|1\rangle$ by $\exp(i\alpha)$.

Exercise 29: Show that $\exp(-i\alpha\hat{X}/2)$ is a rotation of the Bloch vector by an angle α around X-axis.

Exercise 30: Show that $\exp(-i\alpha\hat{Y}/2)$ is a rotation of the Bloch vector by an angle α around Y-axis.

Exercise 31: Show that a general qubit state $|\Psi\rangle$ given by Eq. 1 can be obtain by first rotating $|0\rangle$ by angle θ around Y axis and then rotating by angle ϕ around Z-axis: $|\Psi\rangle = \exp(-i\phi\hat{Z}/2)|0\rangle\exp(-i\theta\hat{Y}/2)$.

Exercise 32: Is the order of rotations important in the previous exercise?

Exercise 33: Now let's try a slightly more complicated rotation. Clearly, $\hat{M} = (\hat{X}+\hat{Z})/\sqrt{2}$ is a hermitian operator, so we can define a rotation $\exp(-i\alpha\hat{M}/2)$. Figure out what it does.
Hint: one way to approach this exercise is to figure out eigenvectors of $\hat{M}$ and find the matrix exponent this way.

Exercise 34: Based on the previous two exercises, is it true that $\exp(-i\alpha(\hat{X}+\hat{Z})) = \exp(-i\alpha\hat{X}) \times \exp(-i\alpha\hat{Z})$?

Exercise 35: How about a unitary operator $\exp(-i\alpha\hat{M}/2)$, where $\hat{M} = (\hat{X}+\hat{Y})/\sqrt{2}$? What kind of rotation on the Bloch sphere is this?

E. Qubit state tomography

Exercise 36: Suppose we have a qubit in a general state $|\Psi\rangle$ given by Eq. (1) and we want to measure the parameters θ and ϕ. How might we do this? Clearly, if we only have one copy of such a qubit we would only get 1's or -1's no matter which projection we measure. However, if we have many copies, we can measure mean values of $\langle\Psi|\hat{X}|\Psi\rangle$ and $\langle\Psi|\hat{Z}|\Psi\rangle$. Write down the values of θ and ϕ in terms of those mean values.

Exercise 37: Suppose now that we only have an instrument to measure $\hat{Z}$ and not $\hat{X}$. Can we still reconstruct the qubit state? All we need to do is to find a rotation which would turn $|+\rangle$ into $|0\rangle$ and $|-\rangle$ into $|1\rangle$ (or vice versa) and do this before measurement of $\hat{Z}$ to effectively obtain a measurement of $\hat{X}$. Come up with such a rotation and describe the protocol for measuring $\langle\Psi|\hat{X}|\Psi\rangle$ using an instrument that can only measures $\hat{Z}$.