Fall 2024: Final Project

COM-309: Quantum Information Processing

This homework should be done by teams of two students. The implementation should be done with PennyLane **AND** Qiskit. Only one of the team's students should upload on Moodle a PDF with the answers to the theory questions and the two notebooks. The name of the two students and their SCIPER should be written in the PDF.

1 Werner state

The goal of this mini-project is to study the entanglement properties of mixed states. A mixed state ρ can be seen as a statistical mixture of pure states,

$$\rho = \sum_i p_i \left|\phi_i\right\rangle \left\langle\phi_i\right|, \quad p_i \in [0,1], \quad \sum_i p_i = 1 \tag{1}$$

and can be represented by density matrices which are positive semi-definite, self-adjoint matrices of trace 1.

In this mini-project, we will focus on a specific class of states, called the Werner states. A Werner state ρ_W is a convex combination of the completely mixed state $\mathbf{I}$ and the maximally entangled Bell state $\rho_{11} = \left|B_{11}\right\rangle \left\langle B_{11}\right|$ where $\left|B_{11}\right\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$. This mixture will depend on a parameter $w \in [0,1]$ such that

$$\rho_W(w) = (1-w)\frac{\mathbf{I}}{4} + w\rho_{11}. \tag{2}$$

Question Theory 1: Give the matrix representation of $\rho_W(w)$, compute its eigenvalues and check that $\rho_W(w)$ is a valid state.

Werner state can be constructed with the following circuit

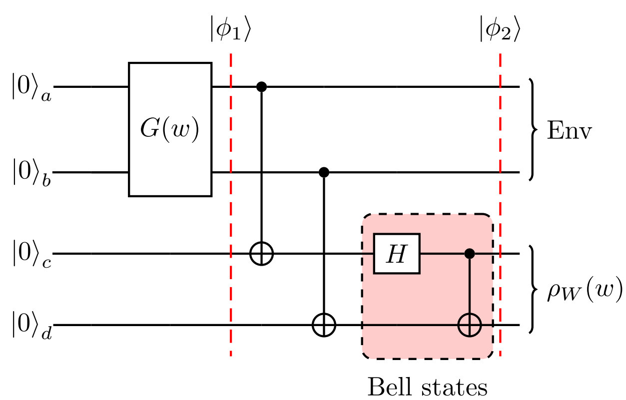

Figure 1: How to prepare Werner states

where $G(w)$ is the unitary such that $G(w)\left|00\right\rangle = \sqrt{\frac{1-w}{4}}\left(\left|00\right\rangle + \left|01\right\rangle + \left|10\right\rangle\right) + \sqrt{\frac{1+3w}{4}}\left|11\right\rangle$. The two qubits a and b can be considered as the environment acting on our system, the two qubits c and d. In the red box on Figure 1, you can recognize the Bell state preparation.

Question Theory 2: Compute the state $\left|\phi_2\right\rangle$ and check that $\rho_W(w) = \mathrm{Tr}_{a,b}(\left|\phi_2\right\rangle \left\langle\phi_2\right|)$.

Question Implementation 1: Implement the circuit to construct Werner states in Pennylane and Qiskit. How to implement $G(w)$ is given in the notebooks.

2 Separability and the Peres criterion

A pure state $|\psi\rangle_{c,d}$ is a product state if it can be decomposed as the tensor product of $|\psi_1\rangle_c$ and $|\psi_2\rangle_d$ such that $|\psi\rangle = |\psi_1\rangle_c \otimes |\psi_2\rangle_d$. The state is entangled otherwise. This notion can be generalized to mixed states.

A mixed state ρ is *separable* if it can be written in the form

$$\rho_{c,d} = \sum_i p_i \rho_c^{(i)} \otimes \rho_d^{(i)}, \quad p_i \in [0,1], \quad \sum_i p_i = 1. \tag{3}$$

You can easily check that a product state is separable.

One can identify if a state is separable or not thanks to the **Peres criterion**: A two-qubit state $\rho_{c,d}$ is entangled if and only if $(\mathbf{I}_c \otimes T_d)\rho_{c,d}$ has a negative eigenvalue. Here T_d is the operator that applies a transposition on the system d. In higher dimension, this criterion does not hold.

Question Theory 3: Prove that for a separable state $\rho_{c,d}$, $(\mathbf{I}_c \otimes T_d)\rho_{c,d}$ has only positive eigenvalues. We will admit the other direction of the proof. *Hint 1:* What can you say about the eigenvalues of the transpose matrix ? *Hint 2:* Remind that for a semi-positive definite matrix M, for any $|\phi\rangle$, we have $\langle\phi| M |\phi\rangle \geq 0$.

Question Implementation 2: Implement the Peres criterion to the Werner state for $w \in [0,1]$. For which w is $\rho_W(w)$ separable?

Question Theory 4: Check your implementation by studying the eigenvalues of $(\mathbf{I}_c \otimes T_d)\rho_W(w)$.

3 Bell (CHSH) inequality

In the lectures, we discussed the Bell inequalities. They are derived in relation to the experiments on quantum systems: if the results of an experiment can be explained by the local hidden-variable (LHV) model, such inequality should be satisfied. Its violation shows that in fact, LHV theory does not hold. Particularly, we looked at the CHSH inequality: in the corresponding experiment, Alice and Bob shared the two-qubit (pure) *Bell* state $|\Psi\rangle$ and estimated the correlation coefficient $\chi = \langle\Psi| \mathcal{B} |\Psi\rangle$ by measuring the observable

$$\mathcal{B} = A \otimes B + A \otimes B' - A' \otimes B + A' \otimes B',$$

where A, A', B, B' are observables

$$A = |\alpha\rangle\langle\alpha| - |\alpha_\perp\rangle\langle\alpha_\perp|, \quad A' = |\alpha'\rangle\langle\alpha'| - |\alpha'_\perp\rangle\langle\alpha'_\perp|,$$
$$B = |\beta\rangle\langle\beta| - |\beta_\perp\rangle\langle\beta_\perp|, \quad B' = |\beta'\rangle\langle\beta'| - |\beta'_\perp\rangle\langle\beta'_\perp|.$$

and $\{|\alpha\rangle, |\alpha_\perp\rangle\}$, $\{|\alpha'\rangle, |\alpha'_\perp\rangle\}$, $\{|\beta\rangle, |\beta_\perp\rangle\}$, $\{|\beta'\rangle, |\beta'_\perp\rangle\}$ are pairs of orthogonal states. The CHSH inequality states that $|\chi_{\mathrm{LHV}}| \leq 2$, but in practice, with certain choice of configuration $\alpha, \alpha', \beta, \beta'$, $|\chi_{\mathrm{QM}}|$ can reach up to $2\sqrt{2}$! In fact, one can prove that for any two-qubit pure *entangled* state $|\Psi\rangle$ there exists such configuration that the CHSH inequality is violated.

However, now we consider mixed states as well, so the correlation coefficient is now expressed as

$$\chi = \mathrm{Tr}\,(\rho\mathcal{B}).$$

Question Theory 5: In homework 6, you have shown that the inequality $\langle\Psi| \mathcal{B} |\Psi\rangle \leq 2\sqrt{2}$ holds for *any pure state* $|\Psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$ (Tsirelson's bound). i.e. $\chi(w) \leq 2\sqrt{2}$, holds. Using this, show that it holds for arbitrary two-qubit mixed state as well, i.e. $\chi(w) \leq 2\sqrt{2}$.

One can show that if the mixed state ρ is separable, the CHSH inequality is satisfied. This does not work in reverse: mixed non-separable states can satisfy CHSH inequality for any configuration. In this problem, we will see this with the example of Werner states, i.e. we will measure correlation coefficient

$$\chi(w) = \mathrm{Tr}\,(\rho_W(w)\ \mathcal{B}).$$

Question Theory 6: Show that the optimal configuration $\alpha, \alpha', \beta, \beta'$ does not depend on value of w.

One can prove that $\alpha = \frac{\pi}{4}, \alpha' = 0, \beta = -\frac{\pi}{8}, \beta' = -\frac{3\pi}{8}$ is an optimal configuration.

Question Theory 7: For this configuration, express A, A', B, B' as a linear combination of Pauli matrices. This will be used later for Pennylane and Qiskit implementations of observables.

Question Implementation 3: Run the circuits and measure $\mathcal{B}$ to evaluate $\chi(w)$ for $w \in [0, 1]$. Draw a plot of this dependence; compare it to 2 (validity of the CHSH inequality bound) and Tsirelson's bound. More details are provided in the corresponding notebooks.

Question Theory 8: From the implementation, what is the value of $\chi(0)$ and $\chi(1)$? What are the values of w s.t. the CHSH inequality is satisfied? Compare them with the values of w for those when $\rho_W(w)$ is separable.