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Exercise 1: Werner State

Question theory 1:

$$\rho_{11} = |B_{11}\rangle\langle B_{11}| = \frac{|01\rangle - |10\rangle}{\sqrt{2}} \cdot \frac{\langle 01| - \langle 10|}{\sqrt{2}} = \frac{1}{2}\left[|01\rangle\langle 01| - |01\rangle\langle 10| - |10\rangle\langle 01| + |10\rangle\langle 10|\right]$$

$$\rho_W(w) = (1-w)\mathbb{1} + w\rho_{11} = \frac{(1-w)}{4}\left(|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11|\right) + \frac{w}{2}\left[|01\rangle\langle 01| - |01\rangle\langle 10| - |10\rangle\langle 01| + |10\rangle\langle 10|\right]$$

$$\rho_W(w) = \frac{(1-w)}{4}\left[|00\rangle\langle 00| + |11\rangle\langle 11|\right] - \frac{w}{2}\left[|01\rangle\langle 10| + |10\rangle\langle 01|\right] + \frac{1+w}{4}\left[|01\rangle\langle 01| + |10\rangle\langle 10|\right]$$ "Dirac notation"

$$\rho_W(w) = \begin{bmatrix} \frac{1-w}{4} & 0 & 0 & 0 \\ 0 & \frac{1+w}{4} & -\frac{w}{2} & 0 \\ 0 & -\frac{w}{2} & \frac{1+w}{4} & 0 \\ 0 & 0 & 0 & \frac{1-w}{4} \end{bmatrix}$$

Thanks to Dirac notation we see that:

$$\rho_W(w)|B_{00}\rangle = \left[\left(\frac{1-w}{4}\right)\mathbb{1} + w|B_{11}\rangle\langle B_{11}|\right]|B_{00}\rangle = \frac{1-w}{4}|B_{00}\rangle + w|B_{11}\rangle\underbrace{\langle B_{11}|B_{00}\rangle}_{=0} = \frac{1-w}{4}|B_{00}\rangle$$

$$\rho_W(w)|B_{01}\rangle = \left[\left(\frac{1-w}{4}\right)\mathbb{1} + w|B_{11}\rangle\langle B_{11}|\right]|B_{01}\rangle = \frac{1-w}{4}|B_{01}\rangle + w|B_{11}\rangle\underbrace{\langle B_{11}|B_{01}\rangle}_{=0} = \frac{1-w}{4}|B_{01}\rangle$$

$$\rho_W(w)|B_{10}\rangle = \left[\left(\frac{1-w}{4}\right)\mathbb{1} + w|B_{11}\rangle\langle B_{11}|\right]|B_{10}\rangle = \frac{1-w}{4}|B_{10}\rangle + w|B_{11}\rangle\underbrace{\langle B_{11}|B_{10}\rangle}_{=0} = \frac{1-w}{4}|B_{10}\rangle$$

$$\rho_W(w)|B_{11}\rangle = \left[\left(\frac{1-w}{4}\right)\mathbb{1} + w|B_{11}\rangle\langle B_{11}|\right]|B_{11}\rangle = \frac{1-w}{4}|B_{11}\rangle + w|B_{11}\rangle\underbrace{\langle B_{11}|B_{11}\rangle}_{=1} = \frac{1+3w}{4}|B_{11}\rangle$$

eigenvectors and eigenvalues: $|B_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$ $\lambda_{00} = \frac{1-w}{4}$; $|B_{01}\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$ $\lambda_{01} = \frac{1-w}{4}$; $|B_{10}\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$

$\lambda_{10} = \frac{1-w}{4}$; $|B_{11}\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$ $\lambda_{11} = \frac{1+3w}{4}$.

We now check if $\rho_W(w)$ is a valid state, ie $\text{Tr}_{\mathcal{H}_A \otimes \mathcal{H}_B}\left[\rho_W(w)\right] = 1$.

Spectral decomposition: $\rho_W(w) = \frac{1-w}{4}|B_{00}\rangle\langle B_{00}| + \frac{1-w}{4}|B_{01}\rangle\langle B_{01}| + \frac{1-w}{4}|B_{10}\rangle\langle B_{10}| + \frac{1+3w}{4}|B_{11}\rangle\langle B_{11}|$ *

$$\text{Tr}_{\mathcal{H}_A \otimes \mathcal{H}_B}\left[\rho_W(w)\right] = \frac{1-w}{4} + \frac{1-w}{4} + \frac{1-w}{4} + \frac{1+3w}{4} = \frac{4+3w-3w}{4} = 1.$$

Therefore $\rho_W(w)$ is a valid state. Also $\rho_W(w) = \frac{1-w}{4}\mathbb{1}_4 + \frac{1-w}{4}|B_{01}\rangle\langle B_{01}| + \frac{1+3w}{4}|B_{11}\rangle\langle B_{11}|$

Question theory 2:

- $|\phi_1\rangle = G(w) \otimes \mathbb{1}_c \otimes \mathbb{1}_d \; |0\rangle_a \otimes |0\rangle_b \otimes |0\rangle_c \otimes |0\rangle_d$

$$|\phi_1\rangle = \left(\sqrt{\frac{1-w}{4}}\left(|00\rangle_{ab} + |01\rangle_{ab} + |10\rangle_{ab}\right) + \sqrt{\frac{1+3w}{4}}|11\rangle\right) \otimes |00\rangle_{cd}.$$

$$|\phi_1\rangle = \sqrt{\frac{1-w}{4}}\left(|0000\rangle_{abcd} + |0100\rangle_{abcd} + |1000\rangle_{abcd}\right) + \sqrt{\frac{1+3w}{4}}\,|1100\rangle_{abcd}.$$

- $|\phi_2\rangle$ we can track evolution $|\phi_1\rangle$ with the following unitary operations:

$$|\phi_2\rangle = (CNOT_{cd}\otimes \mathbb{1}_a \otimes \mathbb{1}_b)(\mathbb{1}_a\otimes\mathbb{1}_b\otimes H_c\otimes \mathbb{1}_d)(CNOT_{bd}\otimes \mathbb{1}_a\otimes\mathbb{1}_c)(CNOT_{ac}\otimes\mathbb{1}_b\otimes\mathbb{1}_d)|\phi_1\rangle$$

$$|\phi_2\rangle = (CNOT_{cd}\otimes \mathbb{1}_a \otimes \mathbb{1}_b)(\mathbb{1}_a\otimes\mathbb{1}_b\otimes H_c\otimes \mathbb{1}_d)(CNOT_{bd}\otimes \mathbb{1}_a\otimes\mathbb{1}_c)\left[\sqrt{\frac{1-w}{4}}\left(|0000\rangle_{abcd} + |0100\rangle_{abcd} + |1010\rangle_{abcd}\right) + \sqrt{\frac{1+3w}{4}}\,|1110\rangle_{abcd}\right]$$

$$|\phi_2\rangle = (CNOT_{cd}\otimes \mathbb{1}_a \otimes \mathbb{1}_b)(\mathbb{1}_a\otimes\mathbb{1}_b\otimes H_c\otimes \mathbb{1}_d)\left[\sqrt{\frac{1-w}{4}}\left(|0000\rangle_{abcd} + |0101\rangle_{abcd} + |1010\rangle_{abcd}\right) + \sqrt{\frac{1+3w}{4}}\,|1111\rangle_{abcd}\right]$$

$$|\phi_2\rangle = (CNOT_{cd}\otimes \mathbb{1}_a \otimes \mathbb{1}_b)\left[\sqrt{\frac{1-w}{4}}\left(\frac{|0000\rangle + |0010\rangle}{\sqrt{2}} + \frac{|0101\rangle + |0111\rangle}{\sqrt{2}} + \frac{|1000\rangle - |1010\rangle}{\sqrt{2}}\right) + \sqrt{\frac{1+3w}{4}}\left(\frac{|1101\rangle - |1111\rangle}{\sqrt{2}}\right)\right]$$

$$|\phi_2\rangle = \sqrt{\frac{1-w}{4}}\left(\frac{|0000\rangle + |0011\rangle}{\sqrt{2}} + \frac{|0101\rangle + |0110\rangle}{\sqrt{2}} + \frac{|1000\rangle - |1011\rangle}{\sqrt{2}}\right) + \sqrt{\frac{1+3w}{4}}\left(\frac{|1101\rangle - |1110\rangle}{\sqrt{2}}\right)$$

$$|\phi_2\rangle = \sqrt{\frac{1-w}{4}}\left(|00\rangle_{ab}\otimes|B_{00}\rangle_{cd} + |01\rangle_{ab}\otimes|B_{01}\rangle_{cd} + |10\rangle_{ab}\otimes|B_{10}\rangle_{cd}\right) + \sqrt{\frac{1+3w}{4}}\left(|11\rangle_{ab}\otimes|B_{11}\rangle_{cd}\right).$$

Density matrix:

$$|\phi_2\rangle\langle\phi_2| = \frac{1-w}{4}\left(|00\rangle_{ab}\otimes|B_{00}\rangle_{cd} + |01\rangle_{ab}\otimes|B_{01}\rangle_{cd} + |10\rangle_{ab}\otimes|B_{10}\rangle_{cd}\right)\left(\langle 00|_{ab}\otimes\langle B_{00}|_{cd} + \langle 01|_{ab}\otimes\langle B_{01}|_{cd} + \langle 10|_{ab}\otimes\langle B_{10}|_{cd}\right)$$

$$+ \sqrt{\frac{1-w}{4}}\cdot\sqrt{\frac{1+3w}{4}}\left(|00\rangle_{ab}\otimes|B_{00}\rangle_{cd} + |01\rangle_{ab}\otimes|B_{01}\rangle_{cd} + |10\rangle_{ab}\otimes|B_{10}\rangle_{cd}\right)\left(\langle 11|_{ab}\otimes\langle B_{11}|_{cd}\right)$$

$$+ \sqrt{\frac{1-w}{4}}\cdot\sqrt{\frac{1+3w}{4}}\left(|11\rangle_{ab}\otimes|B_{11}\rangle_{cd}\right)\cdot\left(\langle 00|_{ab}\otimes\langle B_{00}|_{cd} + \langle 01|_{ab}\otimes\langle B_{01}|_{cd} + \langle 10|_{ab}\otimes\langle B_{10}|_{cd}\right)$$

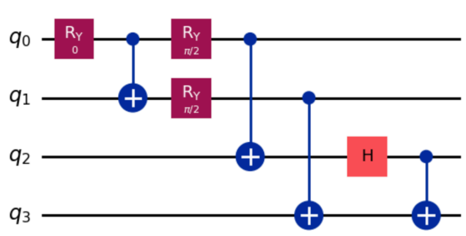
$$+ \frac{1+3w}{4}\left(|11\rangle_{ab}\langle 11|_{ab}\otimes|B_{11}\rangle_{cd}\langle B_{11}|_{cd}\right)$$

$$\mathrm{Tr}_{\mathcal{H}_a\otimes\mathcal{H}_B}\left(|\phi_2\rangle\langle\phi_2|\right) = \frac{1-w}{4}\left(|B_{00}\rangle_{cd}\langle B_{00}|_{cd} + |B_{01}\rangle_{cd}\langle B_{01}|_{cd} + |B_{10}\rangle_{cd}\langle B_{10}|_{cd}\right) + \frac{1+3w}{4}\left(|B_{11}\rangle_{cd}\langle B_{11}|_{cd}\right).$$

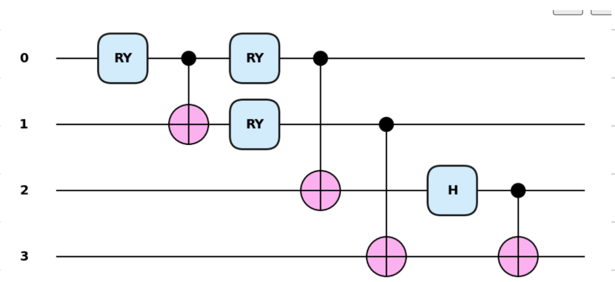
$$\mathrm{Tr}_{\mathcal{H}_a\otimes\mathcal{H}_B}\left(|\phi_2\rangle\langle\phi_2|\right) = \rho_W(w).\ \blacksquare$$ This is the spectral decomposition of $\rho_W(w)$ seen in question theory 1:*

$\boxed{\text{Question implementation 1:}}$

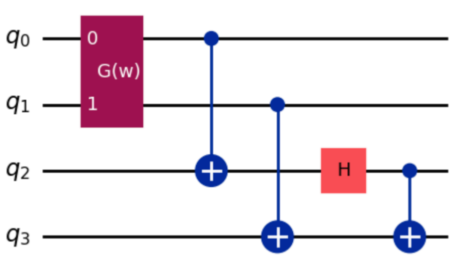
Qiskit:



Pennylane:



Equivalently:



Exercice 2: Separability and the Peres criterion

Question theory 3:

Proof: If ρ_{cd} is a separable state, then $(\mathbb{1}_c \otimes T_d)\rho_{cd}$ has only positive eigenvalues.

(1) Separable state: $\rho_{cd} = \sum_i p_i \, \rho_c^{(i)} \otimes \rho_d^{(i)}$.

$$(\mathbb{1}_c \otimes T_d)\,\rho_{cd} = \sum_i p_i \, \mathbb{1}_c \, \rho_c^{(i)} \otimes T_d \, \rho_d^{(i)}$$

$$(\mathbb{1}_c \otimes T_d)\,\rho_{cd} = \sum_i p_i \, \rho_c^{(i)} \otimes \rho_d^{(i)T}$$

(2) Positivity of each individual term of the sum:

↳ $\rho_c^{(i)}$ semi-positive definite (because it is a density matrix).

↳ $\rho_d^{(i)}$ is semi-positive definite: $\langle \phi | \rho_d^{(i)} | \phi \rangle \geqslant 0$.

↳ We will prove that: $\rho_d^{(i)T}$ is also semi-positive definite:

- $\left(\rho_d^{(i)}\right)_{kj} = \rho_{kj}$ & $\left(\rho_d^{(i)T}\right)_{kj} = \rho_{jk}$
- $\left(\rho_d^{(i)T} |\phi\rangle\right)_k = \sum_j \rho_{jk} \phi_j$
- $\langle \phi | \rho_d^{(i)T} | \phi \rangle = \sum_k \phi_k^* \left(\rho_d^{(i)T} |\phi\rangle\right)_k = \sum_k \phi_k^* \sum_j \rho_{jk} \phi_j = \sum_k \sum_j \rho_{jk} \phi_k^* \phi_j = \sum_j \phi_j \sum_k \rho_{jk} \phi_k^*$

$= \sum_j \phi_j \left(\rho_d^{(i)} |\phi^*\rangle\right)_j$ where $|\phi^*\rangle$ is the complexe conjugate of vector $|\phi\rangle$ (not its bra)

$\langle \phi | \rho_d^{(i)T} | \phi \rangle = \langle \phi^* | \rho_d^{(i)} | \phi^* \rangle$ but we said that $\rho_d^{(i)}$ is semi-positive definite because it is by definition a density matrix.

Therefore $\langle \phi | \rho_d^{(i)T} | \phi \rangle = \langle \phi^* | \rho_d^{(i)} | \phi^* \rangle \geqslant 0$ so $\langle \phi | \rho_d^{(i)T} | \phi \rangle \geqslant 0$ and $\rho_d^{(i)T}$ is therefore semi-positive definite and has $\geqslant 0$ eigenvalues eigenvalues ∎

- Also, we suppose $\rho_d^{(i)}$ admits a spectral decomposition:

$$\rho_d^{(i)} = \sum_j \lambda_j |\lambda_j\rangle\langle\lambda_j| \implies \rho_d^{(i)T} = \sum_j \lambda_j \left(|\lambda_j\rangle\langle\lambda_j|\right)^T = \sum \lambda_j |\lambda_j^*\rangle\langle\lambda_j^*|.$$

↳ $\rho_d^{(i)T}$ has the same eigenvalues as $\rho_d^{(i)}$ which are all $\geqslant 0$.

* ↳ $\rho_d^{(i)T}$ semi-positive definite, $\left[\left(\rho_d^{(i)}\right)^T\right]^\dagger = \left(\rho_d^{(i)}\right)^T$ and $\mathrm{Tr}\left(\rho_d^{(i)T}\right) = 1$ since it has the same eigenvalues as $\rho_d^{(i)}$ which sums to 1 and corresponds to the Trace.

$\implies \rho_d^{(i)T}$ is a valid state which makes sense to incorporate it in the convexe combination as a density matrix.

(4) Convexe combination: $(\mathbb{1}_c \otimes T_d)\rho_{cd} = \sum_i p_i \, \rho_c^{(i)} \otimes \rho_d^{(i)T}$.

↳ sum of semi-positive definite matrices (convexe combination) is also a semi-positive definite matrix.
Suppose $M = \sum_i p_i A_i$ with $\langle \psi | A_i | \psi \rangle \geq 0 \;\; \forall |\psi\rangle \in \mathcal{H}_A$. and $p_i \geq 0$, $\sum_i p_i = 1$.
$\langle \psi | M | \psi \rangle = \sum_i p_i \langle \psi | A_i | \psi \rangle \geq 0$ since $p_i \geq 0$.

↳ Since $\rho_d^{(i)T}$ is a valid state (see (*)) the sum stays a convex combination of valid states (density matrix) which are semi-positive definite. Thus, $(\mathbb{1}_c \otimes T_d)\rho_{cd}$ stays a seperable which is semi-positive by definition.

(5) Conclusion: Since $\rho_c^{(i)}$ and $\rho_d^{(i)}$ are density matrices, they are semi-positive definite. The sum of the tensor products of these matrices forms a density matrix ρ_{cd}, which represents a separable state. By definition, ρ_{cd} is semi-positive definite as a density matrix is.
In our case, we replace $\rho_d^{(i)}$ by $\rho_d^{(i)T}$, which remains a valid density matrix.
Therefore $(\mathbb{1}_c \otimes T_d)\rho_{cd}$ is also separable. By the previously established result, any separable is semi-positive definite, meaning that all its eigenvalues are greater than or equal to 0.
This ends the proof, as we have shown that if ρ_{cd} is a separable state, then $(\mathbb{1}_c \otimes T_d)\rho_{cd}$ is also semi-positive definite and has no negative eigenvalues. ☒

Rigourously, we should consider any eigenvector of $\rho_c^{(i)}$ and $\rho_d^{(i)T}$, let's say $\{|c_k^{(i)}\rangle\}$ for $\rho_c^{(i)}$ and $\{|d_l^{(i)}\rangle\}$ for $\rho_d^{(i)T}$. $\{|c_k^{(i)}\rangle\}$ and $\{|d_l^{(i)}\rangle\}$ form an orthonormal basis of $\mathbb{C}^2$ and $\{|c_k^{(i)}\rangle\} \otimes \{|d_l^{(i)}\rangle\}$ of $\mathbb{C}^2 \otimes \mathbb{C}^2$

Since $\rho_c^{(i)}$ and $\rho_d^{(i)T}$ are semi-positive definite, their eigenvalues are ≥ 0. Let's look at $\rho_c^{(i)} \otimes \rho_d^{(i)}$ whether it is semi-positive definite also or not:

$$\rho_c^{(i)} |c_k^{(i)}\rangle = \lambda_k^{c(i)} |c_k^{(i)}\rangle \quad \text{with } \lambda_k^{c(i)} \geq 0$$
$$\rho_d^{(i)T} |d_l^{(i)}\rangle = \lambda_l^{d(i)} |d_l^{(i)}\rangle \quad \text{with } \lambda_l^{d(i)} \geq 0.$$

Then $$\left(\rho_c^{(i)} \otimes \rho_d^{(i)T}\right) |c_k^{(i)}\rangle \otimes |d_l^{(i)}\rangle = \rho_c^{(i)} |c_k^{(i)}\rangle \otimes \rho_d^{(i)T} |d_l^{(i)}\rangle = \lambda_k^{c(i)} |c_k^{(i)}\rangle \otimes \lambda_l^{d(i)} |d_l^{(i)}\rangle$$

$$\left(\rho_c^{(i)} \otimes \rho_d^{(i)T}\right) |c_k^{(i)}\rangle \otimes |d_l^{(i)}\rangle = \lambda_k^{c(i)} \lambda_l^{d(i)} |c_k^{(i)}\rangle \otimes |d_l^{(i)}\rangle$$

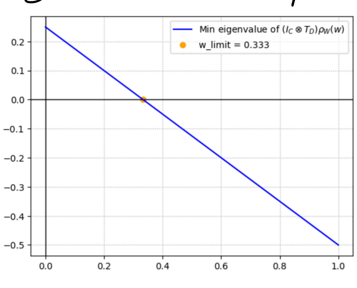
Therefore $\rho_c^{(i)} \otimes \rho_d^{(i)T}$ has eigenvectors $|c_k^{(i)}\rangle \otimes |d_l^{(i)}\rangle$ with its associated eigenvalue: $\lambda = \lambda_k^{c(i)} \lambda_l^{d(i)}$ and this quantity is greater than or equal to 0.

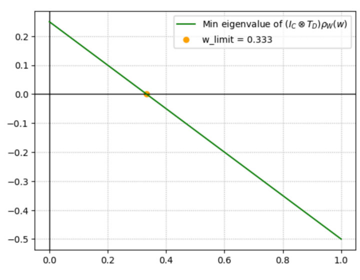
Finally, $\rho_c^{(i)} \otimes \rho_d^{(i)T}$ has positive eigenvalues and is hermitian, this is a sufficient condition to consider $\rho_c^{(i)} \otimes \rho_d^{(i)T}$ as a semi-positive definite matrix.

With this being said, we conclude again that $(\mathbb{1}_c \otimes T_d)\rho_{cd}$ is semi-positive definite and therefore has all its eigenvalues greater than or equal to 0. ☒

Question implementation 2:

Quiskit





Pennylane:

Question theory 4:

$$(\mathbb{1}_c \otimes T_d)\rho_w(w) = (\mathbb{1}_c \otimes T_d)\frac{1-w}{4}\mathbb{1} + (\mathbb{1}_c \otimes T_d)\, w.\, |B_{11}\rangle\langle B_{11}|$$

$$= \frac{1-w}{4}\mathbb{1} + w(\mathbb{1}\otimes T_d).\ \frac{|0\rangle\otimes|1\rangle - |1\rangle\otimes|0\rangle}{\sqrt{2}} \cdot \frac{\langle 0|\otimes\langle 1| - \langle 1|\otimes\langle 0|}{\sqrt{2}}$$

$$= \frac{1-w}{4}\mathbb{1} + \frac{w}{\sqrt{2}\cdot\sqrt{2}}(\mathbb{1}\otimes T_d)\left[|0\rangle\langle 0|\otimes|1\rangle\langle 1| - |0\rangle\langle 1|\otimes|1\rangle\langle 0| - |1\rangle\langle 0|\otimes|0\rangle\langle 1| + |1\rangle\langle 1|\otimes|0\rangle\langle 0|\right]$$

$$= \frac{1-w}{4}\mathbb{1} + \frac{w}{\sqrt{2}\cdot\sqrt{2}}\left[|0\rangle\langle 0|\otimes|1\rangle\langle 1| - |0\rangle\langle 1|\otimes|0\rangle\langle 1| - |1\rangle\langle 0|\otimes|1\rangle\langle 0| + |1\rangle\langle 1|\otimes|0\rangle\langle 0|\right]$$

$$= \frac{1-w}{4}\mathbb{1} + \frac{w}{2}\left[|01\rangle\langle 01| - |00\rangle\langle 11| - |11\rangle\langle 00| + |10\rangle\langle 10|\right]$$

$$|B_{00}\rangle\langle B_{00}| = \frac{|00\rangle + |11\rangle}{\sqrt{2}}\ \frac{\langle 00| + \langle 11|}{\sqrt{2}} = \frac{|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|}{\sqrt{2}\cdot\sqrt{2}}$$

$$|B_{01}\rangle\langle B_{01}| = \frac{|01\rangle + |10\rangle}{\sqrt{2}}\ \frac{\langle 01| + \langle 10|}{\sqrt{2}} = \frac{|01\rangle\langle 01| + |01\rangle\langle 10| + |10\rangle\langle 01| + |10\rangle\langle 10|}{\sqrt{2}\cdot\sqrt{2}}$$

$$|B_{10}\rangle\langle B_{10}| = \frac{|00\rangle - |11\rangle}{\sqrt{2}}\ \frac{\langle 00| - \langle 11|}{\sqrt{2}} = \frac{|00\rangle\langle 00| - |00\rangle\langle 11| - |11\rangle\langle 00| + |11\rangle\langle 11|}{\sqrt{2}\cdot\sqrt{2}}$$

$$|B_{11}\rangle\langle B_{11}| = \frac{|01\rangle - |10\rangle}{\sqrt{2}}\ \frac{\langle 01| - \langle 10|}{\sqrt{2}} = \frac{|01\rangle\langle 01| - |01\rangle\langle 10| - |10\rangle\langle 01| + |10\rangle\langle 10|}{\sqrt{2}\cdot\sqrt{2}}$$

$$|B_{00}\rangle\langle B_{00}| - |B_{10}\rangle\langle B_{10}| = \frac{1}{2}\left(2|00\rangle\langle 11| + 2|11\rangle\langle 00|\right). = |00\rangle\langle 11| + |11\rangle\langle 00|$$

$$|B_{01}\rangle\langle B_{01}| + |B_{11}\rangle\langle B_{11}| = \frac{1}{2}\left(2|01\rangle\langle 01| + 2|10\rangle\langle 10|\right) = |01\rangle\langle 01| + |10\rangle\langle 10|.$$

We see that: $|01\rangle\langle 01| - |00\rangle\langle 11| - |11\rangle\langle 00| + |10\rangle\langle 10| = |B_{10}\rangle\langle B_{10}| - |B_{00}\rangle\langle B_{00}| + |B_{01}\rangle\langle B_{01}| + |B_{11}\rangle\langle B_{11}|.$

We can rewrite: $(\mathbb{1}_c \otimes T_d)\rho_{cd} = \frac{1-w}{4}\mathbb{1} + \frac{w}{2}\left(|B_{10}\rangle\langle B_{10}| - |B_{00}\rangle\langle B_{00}| + |B_{01}\rangle\langle B_{01}| + |B_{11}\rangle\langle B_{11}|.\right)$

$(\mathbb{1}_c\otimes T_d)\rho_w(w)|B_{00}\rangle = \left(\frac{1-w}{4} - \frac{w}{2}\right)|B_{00}\rangle = \frac{1-3w}{4}|B_{00}\rangle$ eigenvalue: $\frac{1-3w}{4}$ eigenvector: $|B_{00}\rangle$

$(\mathbb{1}_c\otimes T_d)\rho_w(w)|B_{01}\rangle = \left(\frac{1-w}{4} + \frac{w}{2}\right)|B_{01}\rangle = \frac{1+w}{4}|B_{01}\rangle$ eigenvalue: $\frac{1+w}{4}$ eigenvector: $|B_{01}\rangle$

$(\mathbb{1}_c\otimes T_d)\rho_w(w)|B_{10}\rangle = \left(\frac{1-w}{4} + \frac{w}{2}\right)|B_{10}\rangle = \frac{1+w}{4}|B_{10}\rangle$ eigenvalue: $\frac{1+w}{4}$ eigenvector: $|B_{10}\rangle$

$(\mathbb{1}_c\otimes T_d)\rho_w(w)|B_{11}\rangle = \left(\frac{1-w}{4} + \frac{w}{2}\right)|B_{11}\rangle = \frac{1+w}{4}|B_{11}\rangle$ eigenvalue: $\frac{1+w}{4}$ eigenvector: $|B_{11}\rangle$

We need the eigenvalues of $(\mathbb{1}_c\otimes T_d)\rho_w(w)$ to be greater than or equal to 0:

$\frac{1-3w}{4} \geq 0$ & $\frac{1+w}{4} \geq 0$ $\boxed{w \leq \frac{1}{3}}$ $w \geq -1$

Exercise 3: Bell (CHSH inequality)

Question theory 5:

$$\rho_w(w)\,\mathcal{B} = \left[\left(\frac{1-w}{4}\right)\mathbb{1} + w\,|B_{11}\rangle\langle B_{11}|\right]\left[A\otimes B + A\otimes B' - A'\otimes B + A'\otimes B'\right]$$

$$\chi(w) = \mathrm{Tr}(\rho_w(w)\,\mathcal{B}) = \mathrm{Tr}\left(\left(\frac{1-w}{4}\right)\mathbb{1}\,\mathcal{B} + w\,|B_{11}\rangle\langle B_{11}|\,\mathcal{B}\right]$$

$$\chi(w) = \frac{1-w}{4}\,\mathrm{Tr}(\mathcal{B}) + w\,\mathrm{Tr}(|B_{11}\rangle\langle B_{11}|\,\mathcal{B}).$$

$|B_{11}\rangle$ is a pure state and we know from Homework 6 that it satisfies Tsirelson's bond such that $\langle B_{11}|\mathcal{B}|B_{11}\rangle \leq 2\sqrt{2}$

$$\Leftrightarrow\quad \chi(w) = \frac{1-w}{4}\,\mathrm{Tr}(A\otimes B + A\otimes B' - A'\otimes B + A'\otimes B) + w\,\mathrm{Tr}(\langle B_{11}|\mathcal{B}|B_{11}\rangle)$$

$$\Leftrightarrow \chi(w) = \frac{1-w}{4}\,\mathrm{Tr}(A\otimes B + A\otimes B' - A'\otimes B + A'\otimes B) + w\,\langle B_{11}|\mathcal{B}|B_{11}\rangle$$

$$\Leftrightarrow \chi(w) \leq \frac{1-w}{4}\,\mathrm{Tr}(A\otimes B + A\otimes B' - A'\otimes B + A'\otimes B) + 2\sqrt{2}\,w.$$

$$\mathrm{Tr}(A\otimes B) = \mathrm{Tr}_{\mathcal{H}_A\otimes\mathcal{H}_B}\left(\left(|\alpha\rangle\langle\alpha| - |\alpha_\perp\rangle\langle\alpha_\perp|\right)\otimes\left(|\beta\rangle\langle\beta| - |\beta_\perp\rangle\langle\beta_\perp|\right)\right).$$

$$\mathrm{Tr}(A\otimes B) = \mathrm{Tr}(|\alpha\rangle\langle\alpha| - |\alpha_\perp\rangle\langle\alpha_\perp|)\cdot\mathrm{Tr}(|\beta\rangle\langle\beta| - |\beta_\perp\rangle\langle\beta_\perp|)$$

$$\mathrm{Tr}(A\otimes B) = (1-1)(1-1) = 0.$$

By generalizing we have $\mathrm{Tr}(A\otimes B) = \mathrm{Tr}(A'\otimes B) = \mathrm{Tr}(A\otimes B') = \mathrm{Tr}(A'\otimes B') = 0$

$$\Leftrightarrow \chi(w) \leq \frac{1-w}{4}\cdot 0 + 2\sqrt{2}\,w$$

$\Leftrightarrow \chi(w) \leq 2\sqrt{2}\,w$ since $w\in[0;1]$ we can rewrite:

$\chi(w) \leq 2\sqrt{2}$ ☒ The Tirelson's bond holds also for abitrary two qubit mixed state as well i $\chi(w)\leq 2\sqrt{2}$ ☒

Question theory 6:

Recalling from last question:

$$\chi(w) = \mathrm{Tr}(\rho_w(w)\,\mathcal{B}) = \mathrm{Tr}\left[\left(\frac{1-w}{4}\mathbb{1} + w\,|B_{11}\rangle\langle B_{11}|\right)\mathcal{B}\right]$$

($\mathrm{Tr}(\mathbb{1}\mathcal{B}) = 0$ as seen previously).

$$= w\,\mathrm{Tr}(|B_{11}\rangle\langle B_{11}|\,\mathcal{B})$$

$$\chi(w) = w\,\langle B_{11}|\mathcal{B}|B_{11}\rangle \leq w\cdot 2\sqrt{2}$$

since $w \in [0;1]$, the quantity that maximizes $\chi(w)$ is the same quantity that maximizes $\langle B_u|\mathcal{B}|B_u\rangle$. Thus, $\langle B_u|\mathcal{B}|B_u\rangle$ being independant on w, the parameters maximizing $\chi(w)$ are independant on w. w is just affecting the amplitude of $\chi(w)$ but not the position of maxima.

Question theory 7:

$$\alpha = \frac{\pi}{4} \; ; \; \alpha' = 0 \; ; \; \beta = -\frac{\pi}{8} \; ; \; \beta' = -\frac{3\pi}{8}.$$

$$\hat{\Pi} = |\gamma\rangle\langle\gamma| - |\gamma\perp\rangle\langle\gamma\perp| = \begin{bmatrix}\cos(\gamma)\\ \sin(\gamma)\end{bmatrix}\begin{bmatrix}\cos(\gamma) & \sin(\gamma)\end{bmatrix} - \begin{bmatrix}-\sin(\gamma)\\ \cos(\gamma)\end{bmatrix}\begin{bmatrix}-\sin(\gamma) & \cos(\gamma)\end{bmatrix}$$

$$= \begin{bmatrix}\cos^2(\gamma) & \cos(\gamma)\sin(\gamma)\\ \sin(\gamma)\cos(\gamma) & \sin^2(\gamma)\end{bmatrix} - \begin{bmatrix}\sin^2(\gamma) & -\cos(\gamma)\sin(\gamma)\\ -\cos(\gamma)\sin(\gamma) & \cos^2(\gamma)\end{bmatrix}$$

$$= \begin{bmatrix}\cos^2(\gamma)-\sin^2(\gamma) & 2\sin(\gamma)\cos(\gamma)\\ 2\sin(\gamma)\cos(\gamma) & -(\cos^2(\gamma)-\sin^2(\gamma))\end{bmatrix}$$

$$\hat{\Pi} = \begin{bmatrix}\cos(2\gamma) & \sin(2\gamma)\\ \sin(2\gamma) & -\cos(2\gamma)\end{bmatrix} = \cos(2\gamma)\,\hat{\sigma}_z + \sin(2\gamma)\,\hat{\sigma}_x.$$

$$A = \begin{bmatrix}\cos(2\pi/4) & \sin(2\pi/4)\\ \sin(2\pi/4) & -\cos(2\pi/4)\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = \hat{\sigma}_x.$$

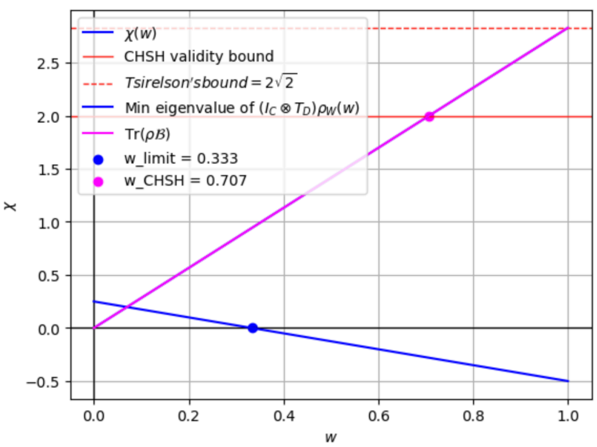
$$A' = \begin{bmatrix}\cos(2\cdot 0) & \sin(2\cdot 0)\\ \sin(2\cdot 0) & -\cos(2\cdot 0)\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = \hat{\sigma}_z$$

$$B = \begin{bmatrix}\cos(-2\cdot\pi/8) & \sin(-2\pi/8)\\ \sin(-2\pi/8) & -\cos(-2\pi/8)\end{bmatrix} = \begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ -1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix} = -\frac{1}{\sqrt{2}}\hat{\sigma}_x + \frac{1}{\sqrt{2}}\hat{\sigma}_z$$

$$B' = \begin{bmatrix}\cos(-2\cdot 3\pi/8) & \sin(-2\cdot 3\pi/8)\\ \sin(-2\cdot 3\pi/8) & -\cos(-2\cdot 3\pi/8)\end{bmatrix} = \begin{bmatrix}-1/\sqrt{2} & -1/\sqrt{2}\\ -1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix} = -\frac{1}{\sqrt{2}}\hat{\sigma}_x - \frac{1}{\sqrt{2}}\hat{\sigma}_z$$

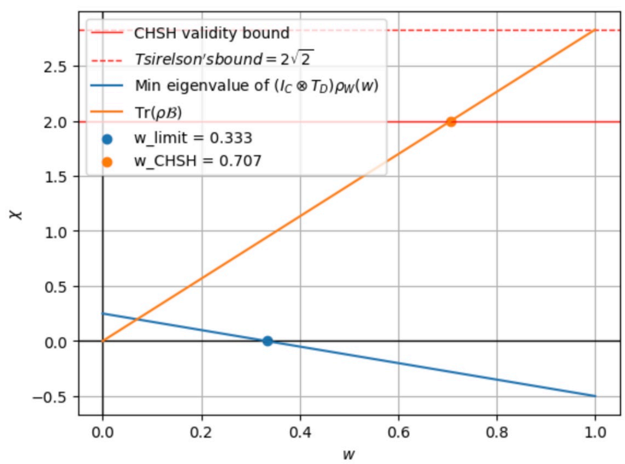
Question implementation 3:

Qiskit:




```
χ(w=0): -4.710277376051325e-16
χ(w=1): 2.8284271247461894
```

Pennylane:




```
χ(w=0): -4.440892098500626e-16
χ(w=1): 2.8284271247461885
```

Question theory 8:

We see that:

- $\chi(0) = 0$. This makes sense because $\rho_W(w) = \frac{1}{4}\mathbb{1}$ which is the most uncorrelated state since it is completly mixed. Therefore the simulation gives indeed 0 as expected.

- $\chi(1) = 2\sqrt{2}$. This also makes sense because here $\rho_W(w) = |B_{11}\rangle\langle B_{11}|$ which is a pure Bell state that is maximally entangled and violates CHSH inequality by reaching maximum of Tsirelson's bond with correlation equals to $2\sqrt{2}$. The simulation confirms this again by giving $2\sqrt{2}$

- $\chi(w) \in [0;2]$ for $w \in [0;\frac{1}{3}]$: For this configuration the state is separable by Peres criterion, because we saw that $(\mathbb{1}\otimes T)\rho_W(w)$ has positive eigenvalues for $w \leq \frac{1}{3}$. Therefore it should satisfy CHSH inequality since it is not entangled. This is confirmed by simulation where on the graph we have $\chi(w) \leq 2$,

- $\chi(w) \in [0;2]$ for $w \in [\frac{1}{3};\frac{1}{\sqrt{2}}]$: Here, the state $\rho_W(w)$ is not separable anymore, because we saw with Peres criterion, that $(\mathbb{1}\otimes T)\rho_W(w)$ has negative eigenvalues for w greater than $\frac{1}{3}$. But we see that it still satisfies CHSH inequality meaning that we have a weak entanglement for a state that is not fully entangled but not a separable state anymore.

- $\chi(w) \in [2;2\sqrt{2}]$ for $w \in [\frac{1}{\sqrt{2}};1]$: Here the state is also not separable for same reasons. But in this case we notice that CHSH is not satisfied anymore and we have a "stronger" entanglement reaching its maximum for $w = 1$ with $\chi(1) = 2\sqrt{2}$ since we now have $\rho_W(w) = |B_{11}\rangle\langle B_{11}|$, a pure state maximally entangled.