Exercise 2 : Sigmoid function

a/ Let $\sigma(z) = \frac{1}{1+e^{-z}}$. Find $\sigma'(z)$

- $\sigma'(z) = \frac{(1+e^{-z})'}{-(1+e^{-z})^2} = \frac{-1\cdot e^{-z}}{-(1+e^{-z})^2} = \frac{e^{-z}}{(1+e^{-z})^2}$

b/ Given: $\sigma'(z) = \frac{e^{-z}}{(1+e^{-z})^2}$; $\sigma(z) = \frac{1}{1+e^{-z}}$, show that $\sigma'(z) = \sigma(z)\cdot(1-\sigma(z))$:

- $\sigma'(z) = \frac{e^{-z}}{(1+e^{-z})^2} = \left[\frac{1}{1+e^{-z}}\right]^2 \cdot e^{-z} = \sigma^2(z)\cdot\left[\frac{1}{\sigma(z)} - 1\right] = \sigma(z)\cdot(1-\sigma(z))$

c/ Given $\zeta(z) = -\log(\sigma(-z))$, find $\zeta'(z)$ and $\zeta''(z)$.

- $\zeta'(z) = -\frac{\sigma'(-z)}{\sigma(-z)} = +\frac{\sigma(-z)(1-\sigma(-z))}{\sigma(-z)} = 1 - \frac{1}{1+e^{z}} = \frac{e^{z}}{1+e^{z}} = \frac{1}{1+e^{-z}} = \sigma(z)$
- $\zeta''(z) = \sigma'(z) = \sigma(z)(1-\sigma(z))$

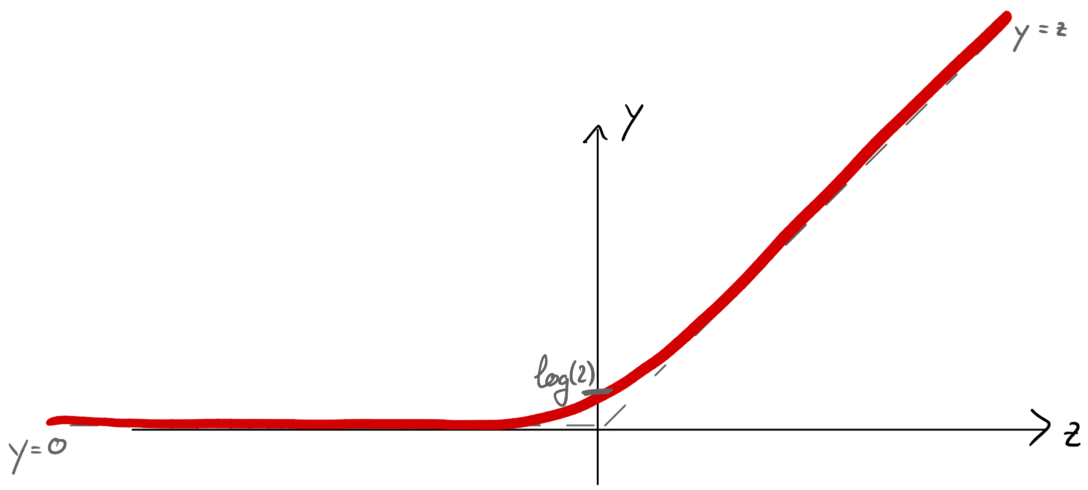
Find the asymptotes for $z \longrightarrow \pm\infty$

- $\lim\limits_{z\to-\infty} \zeta(z) = \lim\limits_{-\infty} -\log(\sigma(-z)) = -\log\left[\lim\limits_{-\infty}\sigma(-z)\right] = -\log\left[\underbrace{\lim\limits_{a\to+\infty}\frac{1}{1+e^{-a}}}_{1}\right] = 0$ $\to \lim\limits_{z\to-\infty}\zeta(z) = 0$
- $\lim\limits_{z\to\infty}\frac{\zeta(z)}{z} = \frac{\text{"}-\infty\text{"}}{\text{"}\infty\text{"}}$... Following l'Hospital rule we can sa :

$$\Longrightarrow \lim_{z\to\infty}\frac{\zeta(z)}{z} = \lim_{z\to\infty}\frac{\zeta'(z)}{z'} = \lim_{z\to\infty}\frac{\sigma(z)}{1} = 1$$

Thus, one may say : $\lim\limits_{z\to\infty}\zeta(z) = z$

- (for graph) $\zeta(0) = -\log(\sigma(-0)) = -\log\left(\frac{1}{1+e^{0}}\right) = -\log(1/2) = \log(2)$



f/ Show that $c_1(x) = (\sigma(x) - 1)^2$ is non-convexe

- $c_1'(x) = 2 \cdot \sigma'(x) \cdot (\sigma(x) - 1) = 2 \cdot \sigma(x) . (1 - \sigma(x))(\sigma(x) - 1) = -2 \cdot \sigma(x) \cdot (\sigma(x) - 1)^2 = -2 \cdot \sigma(x) \cdot c_1(x)$

- $c_1''(x) = -2 \cdot \left[\sigma'(x) \cdot c_1(x) + \sigma(x) \cdot c_1'(x) \right]$

$$= -2 \cdot \left[\sigma'(x) \cdot c_1(x) - 2\sigma^2(x) \cdot c_1(x) \right]$$

$$= -2c_1(x) \cdot \left[\sigma(x) \cdot (1 - \sigma(x)) - 2\sigma^2(x) \right]$$

$$= -2c_1(x) \cdot \left[\sigma(x) - 3\sigma^2(x) \right]$$

$$= -2 \cdot (\sigma(x) - 1)^2 \cdot \sigma(x) \cdot \left[1 - 2\sigma^2(x) \right]$$

$\Rightarrow c''(x) = 0 \iff \sigma(x) - 1 = 0 \Rightarrow \sigma(x) = 1 \Rightarrow \emptyset$

or $\sigma(x) = 0 \Rightarrow \emptyset$

or $1 - 2\sigma^2(x) = 0 \Rightarrow \sigma(x) = \sqrt{2}/2$

- Analysis when $x \to -\infty$, $\sigma(x) \to 0^+$

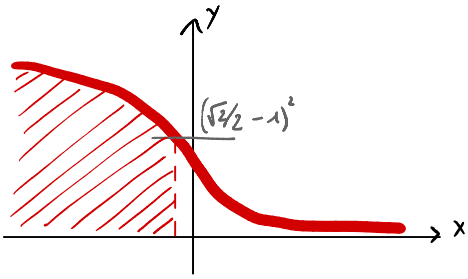
$\lim_{-\infty} c''(x) = -2(0^+ - 1)^2 \cdot 0^+ \cdot \left[1 - 2 . 0^{+2} \right] = 0^- \longrightarrow \underset{x \to -\infty}{c(x)}$ concave

- Analysis when $x \to \infty$, $\sigma(x) \to 1^-$

$\lim_{\infty} c''(x) = -2(1^- - 1)^2 . 1^- . \left[1 - 2 . 1^{-2} \right] = 0^+ \longrightarrow \underset{x \to \infty}{c(x)}$ convex

On the overall $\mathbb{R}$ domain, $c(x)$ is non-convex, as it's neither concave nor convex.

$c_1(x)$ may not converge when the initial value is in concave domain of $c(x)$, that is



g/ Given $c_2(x) = -\left(y \log(\sigma(\omega \cdot x)) + (1-y)\log(1-\sigma(\omega \cdot x))\right)$, find $c_2'(x)$ and $c_2''(x)$ with respect to ω.

$$c_2'(x) = -\left[y \cdot \frac{\sigma'(\omega \cdot x)}{\sigma(\omega \cdot x)} + (1-y) \cdot \frac{-\sigma'(\omega \cdot x)}{1-\sigma(\omega \cdot x)} \right]$$

- $\sigma'(\omega . x) = x \cdot \sigma(\omega . x)(1-\sigma(\omega . x))$

$$\Rightarrow c_2'(x) = -\left[y \cdot x(1-\sigma(\omega . x)) - (1-y) . x \cdot \sigma(\omega \cdot x) \right]$$

$$= \sigma(\omega . x) . xy - xy + \sigma(\omega . x) . x(1-y)$$

$$= \sigma(\omega . x) . x - xy$$

$$\Rightarrow c_2''(x) = \sigma'(\omega . x) . x$$

$$= x^2 . \sigma(\omega . x)(1-\sigma(\omega . x))$$

$\forall x \in \mathbb{R}$, $x^2 \geqslant 0$, $\sigma(\omega . x) . (1-\sigma(\omega . x)) \geqslant 0 \Rightarrow c_2(x) \geqslant 0 \Rightarrow c_2(x)$ is convex.