Applied Time-Series 2023

Computing Exercises for the 2023 Final Assignment

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2023-2024

Contents

Exercise 1 (2 points): The $AR(1) - ARCH(1)$ model.

Consider the model $Y_t = \beta_0 + \beta_1 Y_{t-1} + \epsilon_t$, $h_t = \mathbb{V}_{t-1}(\epsilon_t) = \alpha_0 + \alpha_1 \epsilon_{t-1}^2$ with α_0, $\alpha_1 > 0$ and $\alpha_1 < 1$.

1. Find the conditional and unconditional mean of Y_t.

2. Find the conditional and unconditional variance of Y_t.

1. Conditional mean of Y_t:

$$\begin{aligned}
\mathbb{E}\left[Y_t \mid F_{t-1}\right] &= \mathbb{E}\left[\beta_0 + \beta_1 * Y_{t-1} + \epsilon_t \mid F_{t-1}\right] \\
&= \beta_0 + \beta_1 * \mathbb{E}\left[Y_{t-1} \mid F_{t-1}\right] + \underbrace{\mathbb{E}\left[\epsilon_t \mid F_{t-1}\right]}_{=0} \\
&= \beta_0 + \beta_1 * Y_{t-1}
\end{aligned}$$

Unconditional mean of Y_t:

$$\begin{aligned}
\mathbb{E}\left[Y_t\right] &= \mathbb{E}\left[\beta_0 + \beta_1 * Y_{t-1} + \epsilon_t\right] \\
&= \beta_0 + \beta_1 * \mathbb{E}\left[Y_{t-1}\right] + \underbrace{\mathbb{E}\left[\epsilon_t\right]}_{=0} \\
&= \beta_0 + \beta_1 * \underbrace{\mathbb{E}\left[Y_t\right]}_{\mathbb{E}[Y_t] = \mathbb{E}[Y_{t-1}]^1} \\
\mathbb{E}\left[Y_t\right] &= \frac{\beta_0}{1 - \beta_1}
\end{aligned}$$

2. Conditional variance of Y_t:

$$\begin{aligned}
\mathbb{V}\left[Y_t \mid F_{t-1}\right] &= \mathbb{E}\left[\left(Y_t - \mathbb{E}\left[Y_t \mid F_{t-1}\right]\right)^2\right] \\
&= \mathbb{E}\left[\left(\beta_0 + \beta_1 Y_{t-1} + \epsilon_t - \beta_0 - \beta_1 * Y_{t-1}\right)^2 \mid F_{t-1}\right] \\
&= \mathbb{E}\left[\epsilon_t^2 \mid F_{t-1}\right] \\
&= \mathbb{V}_{t-1}(\epsilon_t) \\
\mathbb{V}\left[Y_t \mid F_{t-1}\right] &= \alpha_0 + \alpha_1 \epsilon_{t-1}^2
\end{aligned}$$

Unconditional variance of Y_t:

$$
\begin{aligned}
\mathbb{V}[Y_t] &= \mathbb{E}\left[(Y_t - \mathbb{E}[Y_t])^2\right] \\
&= \mathbb{E}\left[(\beta_0 + \beta_1 * Y_{t-1} + \epsilon_t - \beta_0 - \beta_1 * \mathbb{E}[Y_{t-1}])^2\right] \\
&= \mathbb{E}\left[(\epsilon_t + \beta_1 * (Y_{t-1} - \mathbb{E}[Y_{t-1}]))^2\right] \\
&= \mathbb{E}\left[\epsilon_t^2\right] + \beta_1^2 * \mathbb{V}[Y_{t-1}] + \underbrace{\mathbb{E}[\text{cross terms}]}_{=0} \\
&= \sigma_\epsilon^2 + \beta_1^2 * \mathbb{V}[Y_t] \;^2 \\
\mathbb{V}[Y_t] &= \frac{\sigma_\epsilon^2}{1 - \beta_1^2}
\end{aligned}
$$

As $\{\epsilon_t\}$ is a white noise process, we have that: $\mathbb{E}[\epsilon_{t-j}\epsilon_{t-k}] = 0 \; \forall j \neq k$ and $\mathbb{E}[\epsilon_t Y_{t-k}] = 0 \; \forall k$. Thus, $\mathbb{E}[\text{cross-terms}] = 0$.

[1]Condition of stationarity if $|\beta_1| < 1$: $\mathbb{E}[Y_t] = \mathbb{E}[Y_{t-1}]$

[2]Condition of stationarity: $\mathbb{V}[Y_t] = \mathbb{V}[Y_{t-1}]$

Exercise 2 (3 points): The seasonal $MA(1) \times MA(1)_{12}$ process.

Suppose that Y_t is a seasonal $MA(1) \times MA(1)_{12}$ process, that is:

$$Y_t = (1 - \theta L)(1 - \Theta L^{12})\epsilon_t$$

Where ϵ_t is a zero mean white noise process with variance $\mathbb{V}(\epsilon_t) = \sigma_\epsilon^2$.

1. Derive the expressions for $\mu = \mathbb{E}(Y_t)$ and $\gamma_0 = \mathbb{V}(Y_t)$.

2. Derive the auto-covariance function, that is, calculate $\gamma_k = \text{Cov}\left(Y_t, Y_{t-k}\right)$ for $k = 1, 2, 3..., 15$.

3. Derive the s-steps ahead forecasts of Y_t for $s = 1, 2, 3..., 15$.

1. Let's derive an expression for $\mu = \mathbb{E}\left[Y_t\right]$:

$$\begin{aligned}
\mu &= \mathbb{E}\left[Y_t\right] \\
&= \mathbb{E}\left[(1-\theta L)(1-\Theta L^{12})\epsilon_t\right] \\
&= \mathbb{E}\left[\left(1 - \Theta L^{12} - \theta L + \Theta\theta L L^{12}\right)\epsilon_t\right] \\
&= \underbrace{\mathbb{E}\left[\epsilon_t\right]}_{=0} - \Theta \underbrace{\mathbb{E}\left[\epsilon_{t-12}\right]}_{=0} - \theta \underbrace{\mathbb{E}\left[\epsilon_{t-1}\right]}_{=0} + \Theta\theta \underbrace{\mathbb{E}\left[\epsilon_{t-13}\right]}_{=0} \\
&= 0
\end{aligned}$$

As $\forall t \in \mathbb{N}$, $\mathbb{E}\left[\epsilon_t\right] = 0$ (White noise process). We have that: $\mu = \mathbb{E}\left[Y_t\right] = 0$.

For the variance of the process, $\gamma_0 = \mathbb{V}\left(Y_t\right)$:

$$\begin{aligned}
\gamma_0 &= \text{Cov}\left(Y_t, Y_t\right) \\
&= \mathbb{E}\left[\left(Y_t - \underbrace{\mu}_{=0}\right)\left(Y_t - \underbrace{\mu}_{=0}\right)\right] \\
&= \mathbb{E}\left[Y_t^2\right] \\
&= \mathbb{E}\left[\left(1 - \Theta L^{12} - \theta L + \Theta\theta L L^{12}\right)\epsilon_t \left(1 - \Theta L^{12} - \theta L + \Theta\theta L L^{12}\right)\epsilon_t\right] \\
&= \mathbb{E}\left[\left(\epsilon_t - \Theta\epsilon_{t-12} - \theta\epsilon_{t-1} + \Theta\theta\epsilon_{t-13}\right)^2\right] \\
&= \mathbb{E}[\left(\epsilon_t^2 - \theta\epsilon_t\epsilon_{t-1} - \Theta\epsilon_t\epsilon_{t-12} + \theta\Theta\epsilon_t\epsilon_{t-13}\right) \\
&\quad + \left(-\theta\epsilon_t\epsilon_{t-1} + \theta^2\epsilon_{t-1}^2 + \theta\Theta\epsilon_{t-1}\epsilon_{t-12} - \theta^2\Theta\epsilon_{t-1}\epsilon_{t-13}\right) \\
&\quad + \left(-\Theta\epsilon_t\epsilon_{t-12} + \theta\Theta\epsilon_{t-1}\epsilon_{t-12} + \Theta^2\epsilon_{t-12}^2 - \theta\Theta^2\epsilon_{t-12}\epsilon_{t-13}\right) \\
&\quad + \left(\theta\Theta\epsilon_t\epsilon_{t-13} - \theta^2\Theta\epsilon_{t-1}\epsilon_{t-13} - \theta\Theta^2\epsilon_{t-12}\epsilon_{t-13} + \theta^2\Theta^2\epsilon_{t-13}^2\right)] \\
&= \mathbb{E}\left[\epsilon_t^2\right] + \theta^2\mathbb{E}\left[\epsilon_{t-1}^2\right] + \Theta^2\mathbb{E}\left[\epsilon_{t-12}^2\right] + +\theta^2\Theta^2\mathbb{E}\left[\epsilon_{t-13}^2\right] + \mathbb{E}\left[\text{cross-terms}\right]
\end{aligned}$$

As $\{\epsilon_t\}$ is a white noise process, we have that: $\mathbb{E}\left[\epsilon_{t-j}\epsilon_{t-k}\right] = 0$, $\forall j \neq k$. Thus, $\mathbb{E}\left[\text{cross-terms}\right] = 0$. Hence,

$$\begin{aligned}
\gamma_0 &= \mathbb{E}\left[\epsilon_t^2\right] + \theta^2\mathbb{E}\left[\epsilon_{t-1}^2\right] + \Theta^2\mathbb{E}\left[\epsilon_{t-12}^2\right] + +\theta^2\Theta^2\mathbb{E}\left[\epsilon_{t-13}^2\right] + \underbrace{\mathbb{E}\left[\text{cross-terms}\right]}_{=0} \\
&= \mathbb{E}\left[\epsilon_t^2\right] + \theta^2\mathbb{E}\left[\epsilon_{t-1}^2\right] + \Theta^2\mathbb{E}\left[\epsilon_{t-12}^2\right] + +\theta^2\Theta^2\mathbb{E}\left[\epsilon_{t-13}^2\right]
\end{aligned}$$

Using the fact that, $\{\epsilon_t\}$ is a zero mean white noise process, we have: $\forall k$, $\mathbb{E}\left[\epsilon_{t-k}^2\right] = \mathbb{V}\left(\epsilon_t\right) = \sigma_\epsilon^2$. Thus, we obtain a value for γ_0:

$$\begin{aligned}
\gamma_0 &= \mathbb{E}\left[\epsilon_t^2\right] + \theta^2\mathbb{E}\left[\epsilon_{t-1}^2\right] + \Theta^2\mathbb{E}\left[\epsilon_{t-12}^2\right] + +\theta^2\Theta^2\mathbb{E}\left[\epsilon_{t-13}^2\right] \\
&= \sigma_\epsilon^2 + \theta^2\sigma_\epsilon^2 + \Theta^2\sigma_\epsilon^2 + \theta^2\Theta^2\sigma_\epsilon^2 \\
&= \sigma_\epsilon^2\left(1 + \theta^2 + \Theta^2 + \theta^2\Theta^2\right)
\end{aligned}$$

2. Let's now derive an expression of the auto-covariance function $\gamma_k = \text{Cov}\left(Y_t, Y_{t-k}\right)$ for $k = 1, 2, 3, ..., 15$.

$$\begin{aligned}
\gamma_k &= \text{Cov}\left(Y_t, Y_{t-k}\right) \\
&= \mathbb{E}\left[Y_{t-k}Y_t\right] \\
&= \mathbb{E}\left[\left(1 - \Theta L^{12} - \theta L + \Theta\theta LL^{12}\right)\epsilon_{t-k}\left(1 - \Theta L^{12} - \theta L + \Theta\theta LL^{12}\right)\epsilon_t\right] \\
&= \mathbb{E}\left[\left(\epsilon_{t-k} - \Theta\epsilon_{t-k-12} - \theta\epsilon_{t-k-1} + \Theta\theta\epsilon_{t-k-13}\right)\left(\epsilon_t - \Theta\epsilon_{t-12} - \theta\epsilon_{t-1} + \Theta\theta\epsilon_{t-13}\right)\right]
\end{aligned}$$

$$= \begin{cases}
-\theta\sigma_\epsilon^2(1+\Theta^2) & \text{if } k = 1 \\
\sigma_\epsilon^2(\theta\Theta) & \text{if } k = 11 \\
-\Theta\sigma_\epsilon^2(1+\theta^2) & \text{if } k = 12 \\
\sigma_\epsilon^2(\theta\Theta) & \text{if } k = 13 \\
0 & \text{otherwise}
\end{cases}$$

Exercise 3 (1 points): The $AR(1)$ process.

Consider the time series model:

$$Y_t = \mu + \phi^2 Y_{t-1} + \epsilon_t$$

Where μ and ϕ are fixed parameters and ϵ_t is a white noise process with mean zero and variance σ_ϵ^2.

1. Derive the $s = 1$ step ahead forecast of Y_t , $f_{t,1} = \mathbb{E}_t[Y_{t+s}]$.
2. The forecast error is the difference between the forecast and the true value, compute the forecast error, $e_{t,1} = f_{t,1} - y_{t+1}$ and it's variance.
3. When is this process stationary? Let's assume that stationarity holds. What is the forecast value $f_{t,s}$ when the lead time s is very large?

1. First step forecast

$$\begin{aligned}
f_{t,1} &= \mathbb{E}\left[Y_{t+1} \mid F_t\right] \\
&= \mathbb{E}\left[\mu + \phi^2 Y_t + \epsilon_{t+1} \mid F_t\right] \\
&= \mu + \phi^2 \underbrace{\mathbb{E}\left[Y_t \mid F_t\right]}_{=Y_t} + \underbrace{\mathbb{E}\left[\epsilon_{t+1} \mid F_t\right]}_{=0} \\
f_{t,1} &= \mu + \phi^2 Y_t
\end{aligned}$$

2. The forecast error

$$\begin{aligned}
e_{t,1} &= f_{t,1} - y_{t+1} \\
&= \mu + \phi^2 Y_t - \left(\mu + \phi^2 Y_t + \epsilon_{t+1}\right) \\
e_{t,1} &= -\epsilon_{t+1}
\end{aligned}$$

The variance of the forecast error

$$\begin{aligned}
\mathbb{V}\left[e_{t,1}\right] &= \mathbb{V}\left[-\epsilon_t\right] \\
&= \mathbb{V}\left[\epsilon_t\right] \\
&= \sigma_\epsilon^2
\end{aligned}$$

3. This process is stationary when $|\phi^2| < 1$

$$f_{t,1} = \mu + \phi^2 Y_t$$

$$
\begin{aligned}
f_{t,2} &= \mathbb{E}\left[Y_{t+2} \mid F_t\right] \\
&= \mu + \phi^2 \mathbb{E}\left[Y_{t+1} \mid F_t\right] \\
&= \mu + \phi^2 \left(\mu + \phi^2 Y_t\right) \\
f_{t,2} &= \mu \left(1 + \phi^2\right) + \phi^4 Y_t
\end{aligned}
$$

$$
\begin{aligned}
f_{t,3} &= \mathbb{E}\left[Y_{t+3} \mid F_t\right] \\
&= \mu + \phi^2 \mathbb{E}\left[Y_{t+2} \mid F_t\right] \\
&= \mu + \phi^2 \left(\mu \left(1 + \phi^2\right) + \phi^4 Y_t\right) \\
f_{t,3} &= \mu \left(1 + \phi^2 + \phi^4\right) + \phi^6 Y_t
\end{aligned}
$$

We can see that $\forall s > 1$

$$f_{t,s} = \frac{\mu}{1 - \phi^2} + \phi^{2s} Y_t$$

When time s tends to be very large

$$\lim_{s \to \infty} f_{t,s} = \frac{\mu}{1 - \phi^2}$$

as we assume stationarity, i.e. $|\phi^2| < 1$.

Exercise 4 (4 points): The $ARMA(2,2)$ process

Consider the following $ARMA(2,2)$ model:

$$Y_t = 1.3Y_{t-1} - 0.4Y_{t-2} + \epsilon_t - 1.2\epsilon_{t-1} + 0.2\epsilon_{t-2}$$

Where $\{\epsilon_t\}$ are independently and identically distributed by a normal distribution with zero mean and variance $\sigma_\epsilon^2 = 1$.

1. Is Y_t weakly stationary? If so compute the auto-covariance function. In this $ARMA(2,2)$, the general solution to the second order differential equation of γ_h is given by:

$$\gamma_h = c_1 0.5^h - c_2 0.8^h$$

Where c_1 and c_2 are constants to be determined by the three initial conditions (γ_0, γ_1 and γ_2).

2. Is Y_t invertible? If so, find the infinite order MA representation (coefficient of $\Psi_\infty(L)$). Note that we can write the equation of Y_t as:

$$Y_t = \Phi_2^{-1}(L)\Theta_2(L)\epsilon_t = \Psi_\infty(L)\epsilon_t$$

Where $\Psi_\infty(L) = \Psi_0 + \Psi_1 L + \Psi_2 L^2 + ...$ Note that the above equation implies that:

$$\Psi_\infty(L)\Phi_2(L) = \Theta_2(L)$$

Do the multiplication of the two polynomials from the left hand side to match terms (for each L operator) within the polynomial from the right hand side.

1. First, we can rewrite the Y_t process using lag operators:

$$\Phi_2(L)Y_t = \Theta_2(L)\epsilon_t$$

Where $\Phi_2(L) = 1 - 1.3L + 0.4L^2$ and $\Theta_2(L) = 1 - 1.2L + 0.2L^2$. This process is weakly stationary, if the roots of the AR component's characteristic equation: $1 - 1.3z + 0.4z^2 = 0$ lie outside of the unit circle (i.e. $\notin [0, 1]$). If we solve for z in the previous quadratic equation, we get $z_1 = 1.25$ and $z_2 = 2$. Thus, $|z_1| > 1$ and $|z_2| > 1$, the process is weakly stationary. Let's now solve the auto-covariance function using the fact that the process Y_t is stationary and that $\{\epsilon_t\}$ are i.i.d standard normal random variables.

$$\mathbb{E}[Y_t] = \mathbb{E}[1.3Y_{t-1} - 0.4Y_{t-2} + \epsilon_t - 1.2\epsilon_{t-1} + 0.2\epsilon_{t-2}] = 0.9\mathbb{E}[Y_t]$$

This implies that we have: $\mathbb{E}[Y_t] = 0$.

Hence,

$$
\begin{aligned}
\gamma_0 &= \mathrm{Cov}\left(Y_t, Y_t\right) \\
&= \mathbb{E}\left[Y_t^2\right] \\
&= \mathbb{E}\left[Y_t\left(1.3Y_{t-1} - 0.4Y_{t-2} + \epsilon_t - 1.2\epsilon_{t-1} + 0.2\epsilon_{t-2}\right)\right] \\
&= 1.3\mathrm{Cov}\left(Y_t, Y_{t-1}\right) - 0.4\mathrm{Cov}\left(Y_t, Y_{t-2}\right) + \mathbb{E}\left[Y_t\epsilon_t\right] - 1.2\mathbb{E}\left[Y_t\epsilon_{t-1}\right] + 0.2\mathbb{E}\left[Y_t\epsilon_{t-2}\right]
\end{aligned}
$$

We have that:

$$
\begin{aligned}
\gamma_1 &= \mathrm{Cov}\left(Y_t, Y_{t-1}\right) \\
\gamma_2 &= \mathrm{Cov}\left(Y_t, Y_{t-2}\right) \\
\mathbb{E}\left[Y_t\epsilon_t\right] &= \mathbb{V}\left(\epsilon_t\right) = 1 \\
\mathbb{E}\left[Y_t\epsilon_{t-1}\right] &= 1.3\mathbb{E}\left[Y_{t-1}\epsilon_{t-1}\right] - 1.2\mathbb{V}\left(\epsilon_{t-1}\right) = 1.3 - 1.2 = 0.1 \\
\mathbb{E}\left[Y_t\epsilon_{t-2}\right] &= 1.3\mathbb{E}\left[Y_{t-1}\epsilon_{t-2}\right] - 0.4\mathbb{E}\left[Y_{t-2}\epsilon_{t-2}\right] + 0.2\mathbb{V}\left(\epsilon_{t-2}\right) = 0.13 - 0.4 + 0.2 = -0.07
\end{aligned}
$$

Thus, we obtain:

$$
\begin{aligned}
\gamma_0 &= 1.3\gamma_1 - 0.4\gamma_2 + 1 - 0.12 - 0.014 \\
&= 1.3\gamma_1 - 0.4\gamma_2 + 0.866
\end{aligned}
$$

Similarly, we can give an expression of γ_h for $h \geq 1$:

$$
\begin{aligned}
\gamma_1 &= \mathrm{Cov}\left(Y_t, Y_{t-1}\right) = \mathbb{E}\left[Y_tY_{t-1}\right] = \mathbb{E}\left[Y_{t-1}\left(1.3Y_{t-1} - 0.4Y_{t-2} + \epsilon_t - 1.2\epsilon_{t-1} + 0.2\epsilon_{t-2}\right)\right] \\
&= 1.3\gamma_0 - 0.4\gamma_1 - 1.2 + 0.02 = 1.3\gamma_0 - 0.4\gamma_1 - 1.18 \\
\gamma_2 &= \mathrm{Cov}\left(Y_t, Y_{t-2}\right) = \mathbb{E}\left[Y_tY_{t-1}\right] = \mathbb{E}\left[Y_{t-2}\left(1.3Y_{t-1} - 0.4Y_{t-2} + \epsilon_t - 1.2\epsilon_{t-1} + 0.2\epsilon_{t-2}\right)\right] \\
&= 1.3\gamma_1 - 0.4\gamma_0 + 0.2
\end{aligned}
$$

Hence,

$$
\gamma_h = \begin{cases}
1.3\gamma_1 - 0.4\gamma_2 + 0.866 & \text{if } h = 0 \\
1.3\gamma_0 - 0.4\gamma_1 - 1.18 & \text{if } h = 1 \\
1.3\gamma_1 - 0.4\gamma_0 + 0.2 & \text{if } h = 2 \\
1.3\gamma_{h-1} - 0.4\gamma_{h-2} & \text{if } h > 2
\end{cases}
$$

Thus, we can find explicit values for γ_0, γ_1 and γ_2 by solving the following equation system:

$$\begin{cases} -0.866 + \gamma_0 - 1.3\gamma_1 + 0.4\gamma_2 & = 0 \\ 1.18 - 1.3\gamma_0 + 0.4\gamma_1 & = 0 \\ -0.2 + 0.4\gamma_0 - 1.3\gamma_1 + \gamma_2 & = 0 \end{cases}$$
$$\iff \begin{cases} \gamma_0 = {}^{10}\!/_{9} \\ \gamma_1 = {}^{17}\!/_{90} \\ \gamma_2 = {}^{1}\!/_{900} \end{cases}$$

We know that in this $ARMA(2,2)$, the general solution to the second order differential equation of γ_h is given by:

$$\gamma_h = c_1 0.5^h - c_2 0.8^h$$

Using the values found for γ_1 and γ_2, we can retrieve the coefficients c_1 and c_2:

$$\begin{cases} \gamma_1 = 0.5c_1 - 0.8c_2 \\ \gamma_2 = 0.25c_1 - 0.64c_2 \end{cases}$$
$$\iff \begin{cases} c_1 = 1 \\ c_2 = {}^{7}\!/_{18} \end{cases}$$

Finally, the explicit formula for the auto-covariance function of this process is given by:

$$\gamma_h = \begin{cases} \frac{10}{9} & \text{if } h = 0 \\ \frac{17}{90} & \text{if } h = 1 \\ \frac{1}{900} & \text{if } h = 2 \\ 0.5^h - \left(\frac{7}{18}\right) 0.8^h & \text{if } h > 2 \end{cases}$$

2. We showed in the first question that the process Y_t was weakly stationary. In particular, we showed that the AR component of the Y_t process was invertible as the roots of its characteristic equation lie outside of the unit circle. Hence, the AR component can be rewritten as an infinite MA process. We know from the instructions given above, that the equation of Y_t can be rewritten as:

$$\begin{aligned} Y_t &= \Phi_2^{-1}(L)\Theta_2(L)\epsilon_t = \Psi_\infty(L)\epsilon_t \\ \Psi_\infty(L) &= \Psi_0 + \Psi_1 L + \Psi_2 L^2 + ... \end{aligned}$$

And that the above equation implies that:

$$\begin{aligned} \Psi_\infty(L)\Phi_2(L) &= \Theta_2(L) \\ \left(\Psi_0 + \Psi_1 L + \Psi_2 L^2 + ...\right)\left(1 - 1.3L + 0.4L^2\right) &= 1 - 1.2L + 0.2L^2 \end{aligned}$$

Hence, by multiplying the polynomials from the left hand side and matching terms with the right hand side, we obtain:

$$\Psi_0 + L\left(-1.3\Psi_0 + \Psi_1\right) + L^2\left(0.4\Psi_0 - 1.3\Psi_1 + \Psi_2\right) + L^3\left(0.4\Psi_1 - 1.3\Psi_2 + \Psi_3\right) + ... = 1 - 1.2L + 0.2L^2$$

Thus, by matching terms we obtain an explicit value for Ψ_0, Ψ_1, Ψ_2 and an equation for Ψ_h, $\forall h > 2$.

$$\begin{aligned} &\begin{cases} 1 = \Psi_0 \\ -1.2 = -1.3\Psi_0 + \Psi_1 \\ 0.2 = 0.4\Psi_0 - 1.3\Psi_1 + \Psi_2 \\ \Psi_h = 1.3\Psi_{h-1} - 0.4\Psi_{h-2} \end{cases} \\ \iff &\begin{cases} \Psi_0 = 1 \\ \Psi_1 = 0.1 \\ \Psi_2 = -0.07 \\ \Psi_h = 1.3\Psi_{h-1} - 0.4\Psi_{h-2} \quad \forall h > 2 \end{cases} \end{aligned}$$

We recognized that the last equation for Ψ_h, $h > 2$ is similar to the one we solved in the first question for γ_h but with different initial condition Ψ_0, Ψ_1 and Ψ_2. Hence, we know that the general solution for Ψ_h with $h > 2$ is given by:

$$\Psi_h = c_1 0.5^h - c_2 0.8^h$$

$$
\begin{cases}
\Psi_1 = 0.5c_1 - 0.8c_2 \\
\Psi_2 = 0.25c_1 - 0.64c_2
\end{cases}
$$

$$
\iff \begin{cases}
c_1 = 1 \\
c_2 = {}^1\!/_2
\end{cases}
$$

Thus, we have that:

$$
\begin{aligned}
Y_t &= \Psi_\infty(L)\epsilon_t \\
Y_t &= \left(\Psi_0 + \Psi_1 L + \Psi_2 L^2 + \Psi_3 L^3 + ...\right)\epsilon_t \\
\text{With: } \Psi_h &= \begin{cases} 1 & \text{if } h = 0 \\ 0.5^h - \left(\dfrac{1}{2}\right) 0.8^h & \forall h > 0 \end{cases}
\end{aligned}
$$

Which is the explicit infinite order MA representation of the process Y_t.