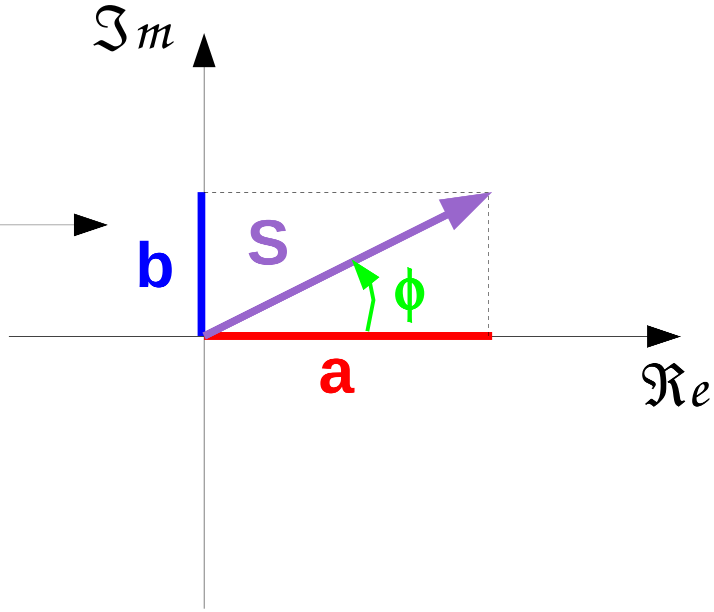
Monophase circuits

- Properties
- Power (instantaneous, active, reactive)
- Applications

Complex numbers

S = a + j b ⟷ associated vector

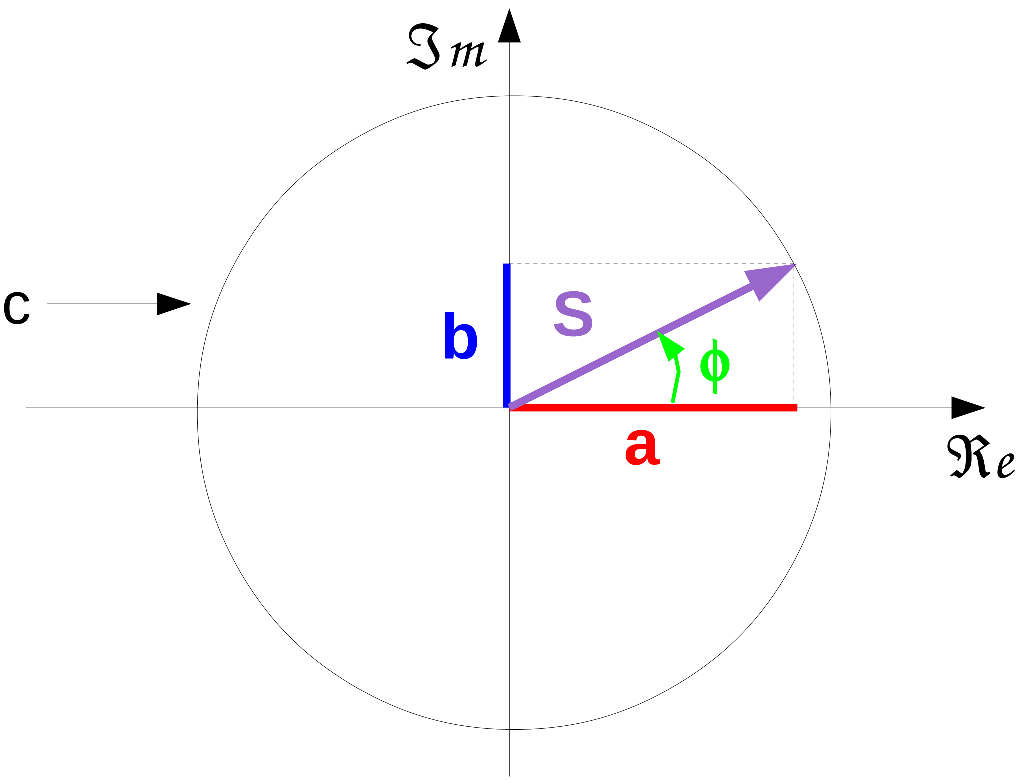


$$
\begin{aligned}
\Re e\{s\} &= a\,, \\
\Im m\{s\} &= b\,, \\
\tan\phi &= \frac{\Im m\{s\}}{\Re e\{s\}}\,, \\
&= \frac{b}{a}
\end{aligned}
$$

Complex numbers

S = a + j b

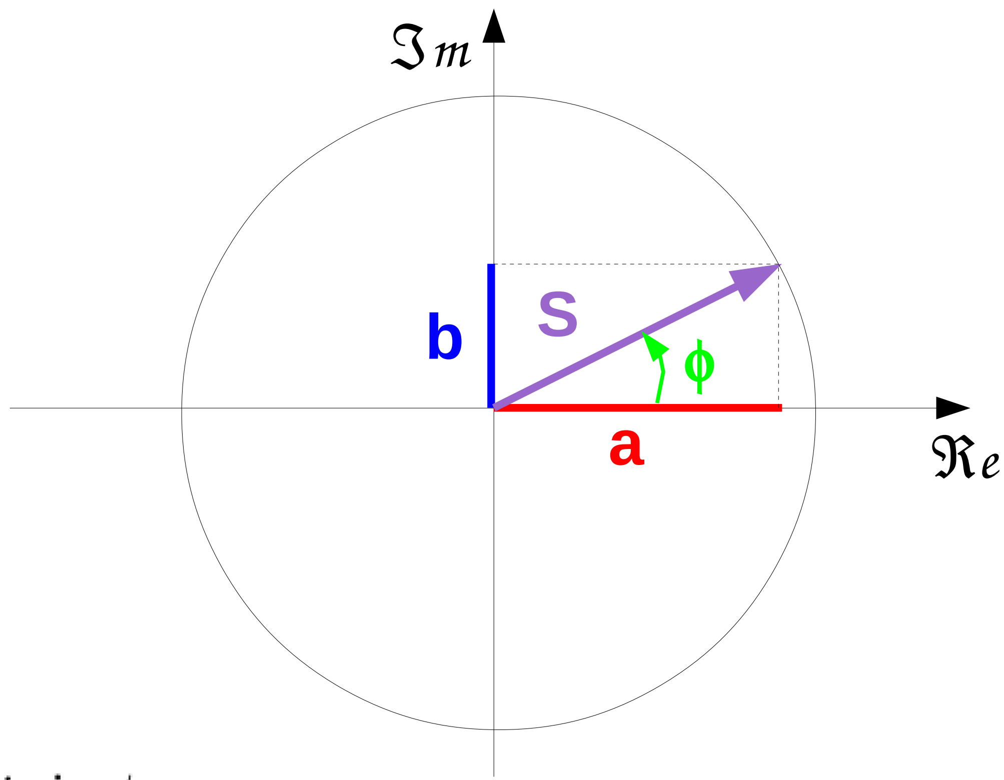
associated trigonometric circle



$$
\begin{aligned}
|s| &= \sqrt{\Re e^2\{s\} + \Im m^2\{s\}}\,, \\
&= \sqrt{a^2 + b^2}\,, \\
a &= |s| \cos\phi\,, \\
b &= |s| \sin\phi
\end{aligned}
$$

Complex numbers

S = a + j b



Euler's equations

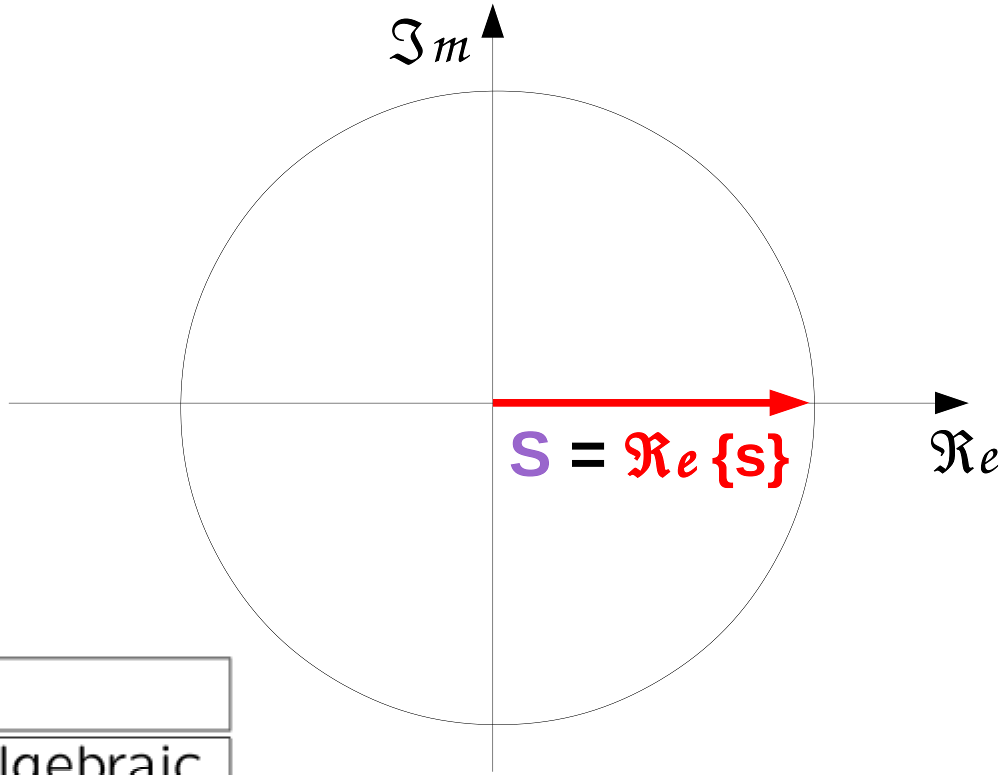
$$
\begin{aligned}
e^{\pm j\phi} &= \cos\phi \pm j \sin\phi \,, \\
\cos\phi &= \frac{e^{j\phi} + e^{-j\phi}}{2} \,, \\
\sin\phi &= \frac{e^{j\phi} - e^{-j\phi}}{2j}
\end{aligned}
$$

Complex numbers

$S = e^{j\phi}$

$S = \cos\phi + j\sin\phi$

$S = \Re e\{s\} + j\,\Im m\{s\}$



	Representation		
$\phi(rad)$	Complex	Trigonometric	Algebraic
0	e^{j0}	$\cos 0 + j\sin 0$	$\Re e\{s\}$

Complex numbers

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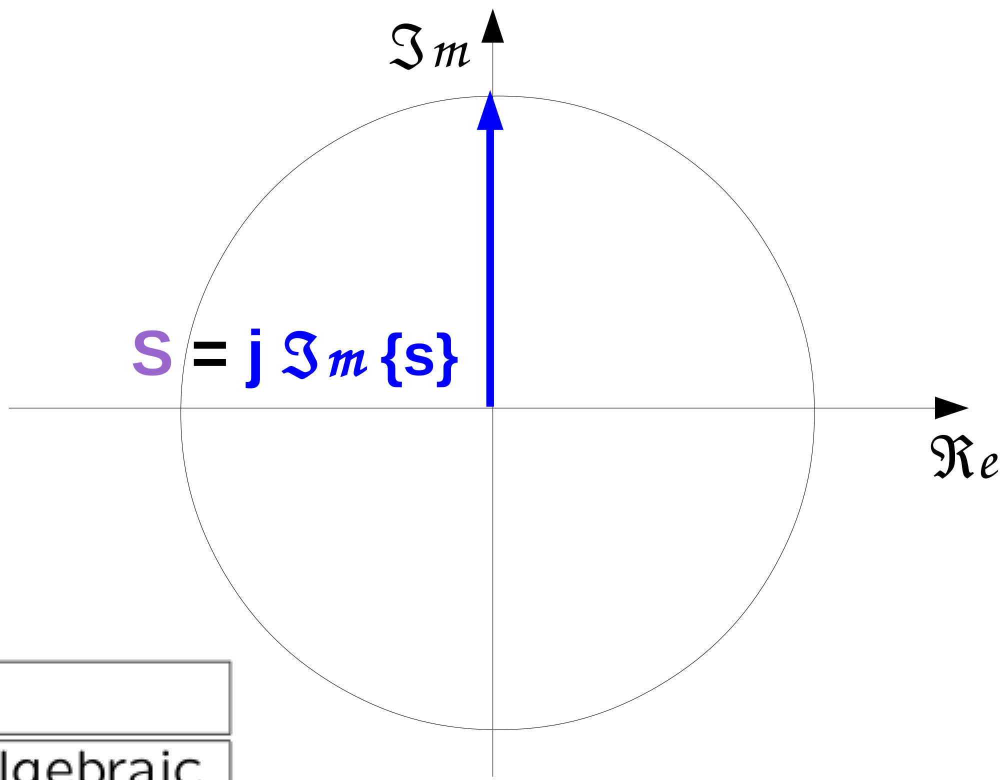

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$\phi(rad)$	Complex	Trigonometric	Algebraic
0	e^{j0}	$\cos 0 + j\sin 0$	$\Re e\{s\}$
$\frac{\pi}{2}$	$e^{j\frac{\pi}{2}}$	$\cos\frac{\pi}{2} + j\sin\frac{\pi}{2}$	$j\Im m\{s\}$

Complex numbers

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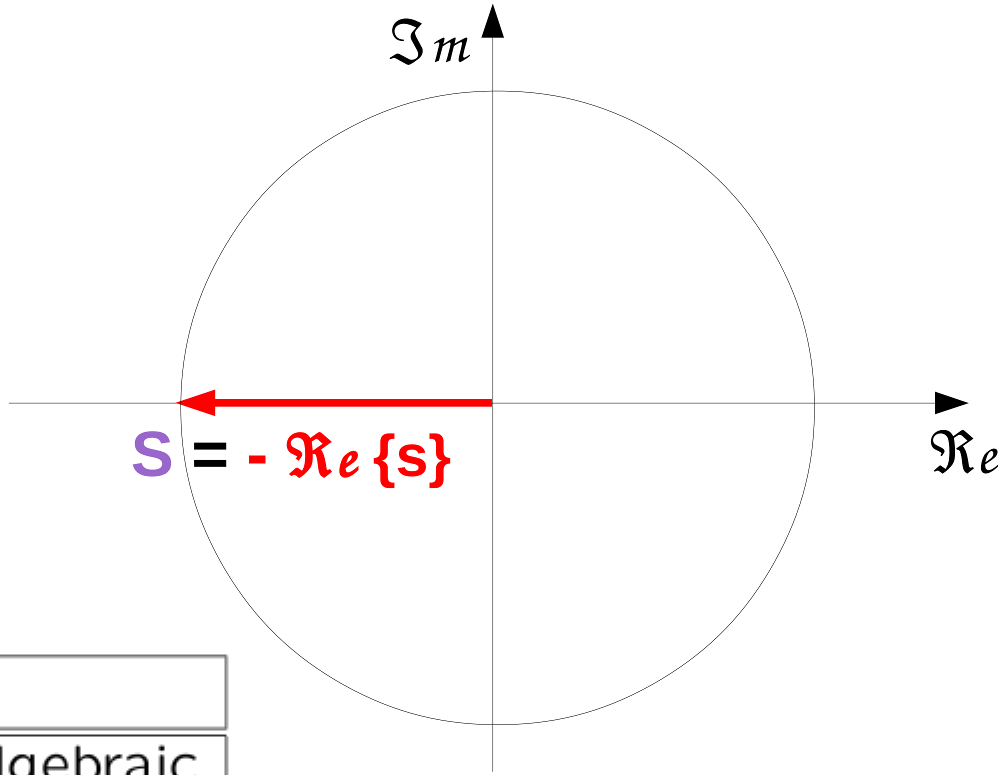

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$\phi(rad)$	Complex	Trigonometric	Algebraic
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π	$e^{j\pi}$	$\cos\pi + j\sin\pi$	$-\Re e\{s\}$

Complex numbers

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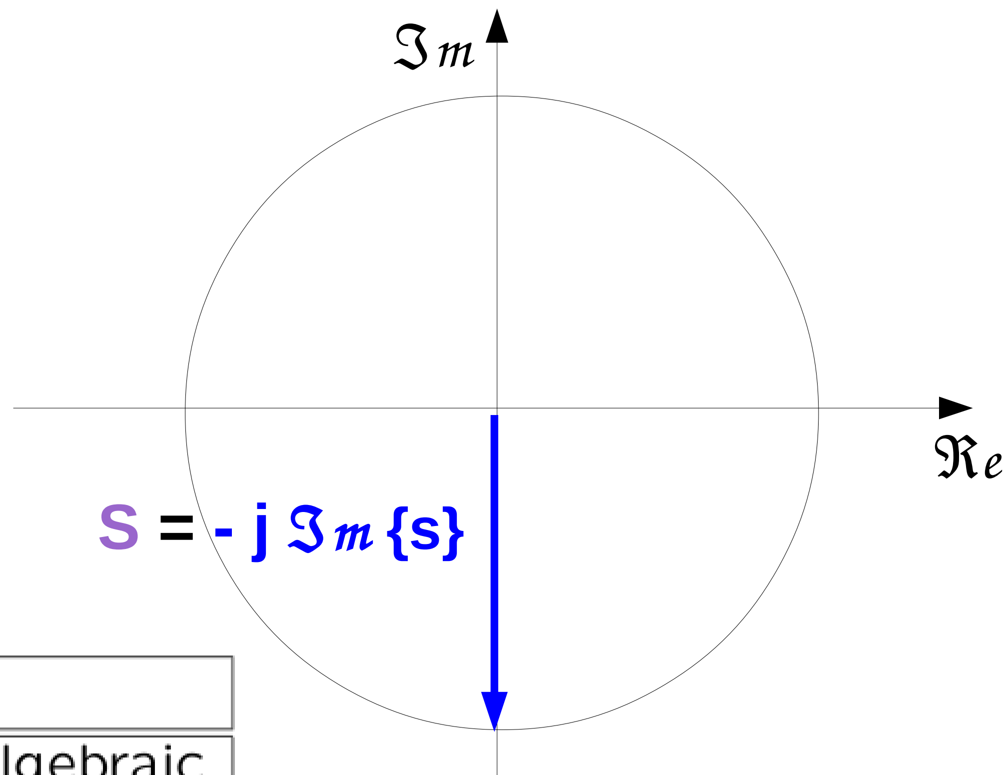
$S = \cos\phi + j \sin\phi$

$S = \Re e\{s\} + j\,\Im m\{s\}$



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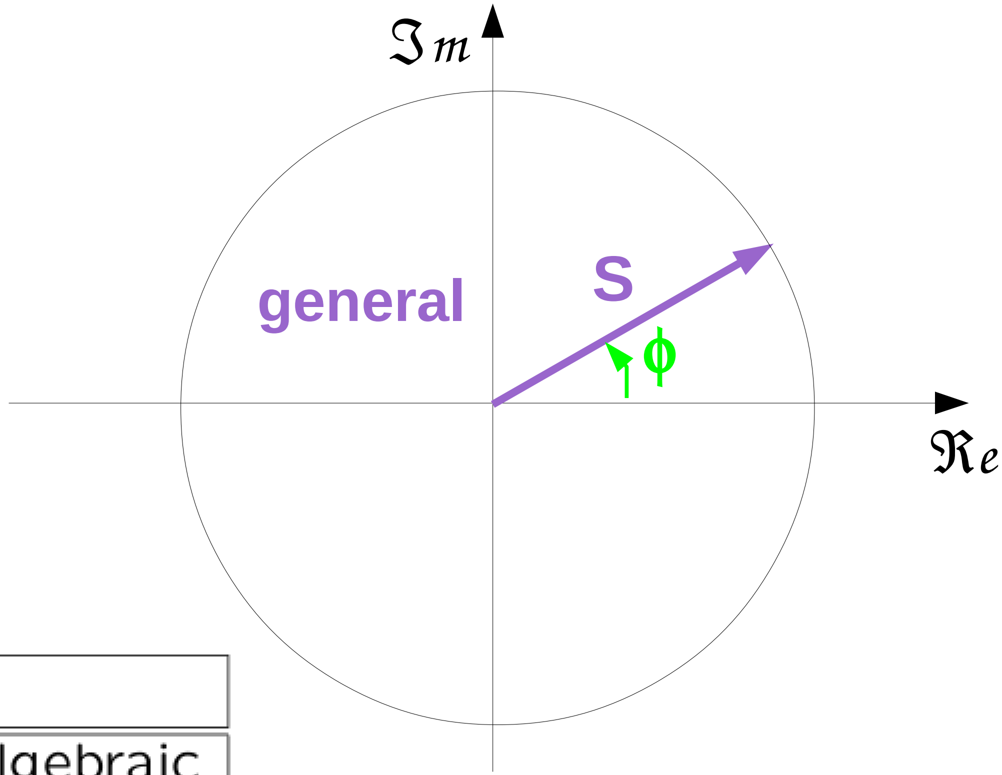
π	$e^{j\pi}$	$\cos\pi + j\sin\pi$	$-\Re e\{s\}$
$\frac{3\pi}{2}$	$e^{j\frac{3\pi}{2}}$	$\cos\frac{3\pi}{2} + j\sin\frac{3\pi}{2}$	$-j\Im m\{s\}$

Complex numbers

$S = e^{j\phi}$

$S = \cos\phi + j \sin\phi$

$S = \Re e\{s\} + j\, \Im m\{s\}$



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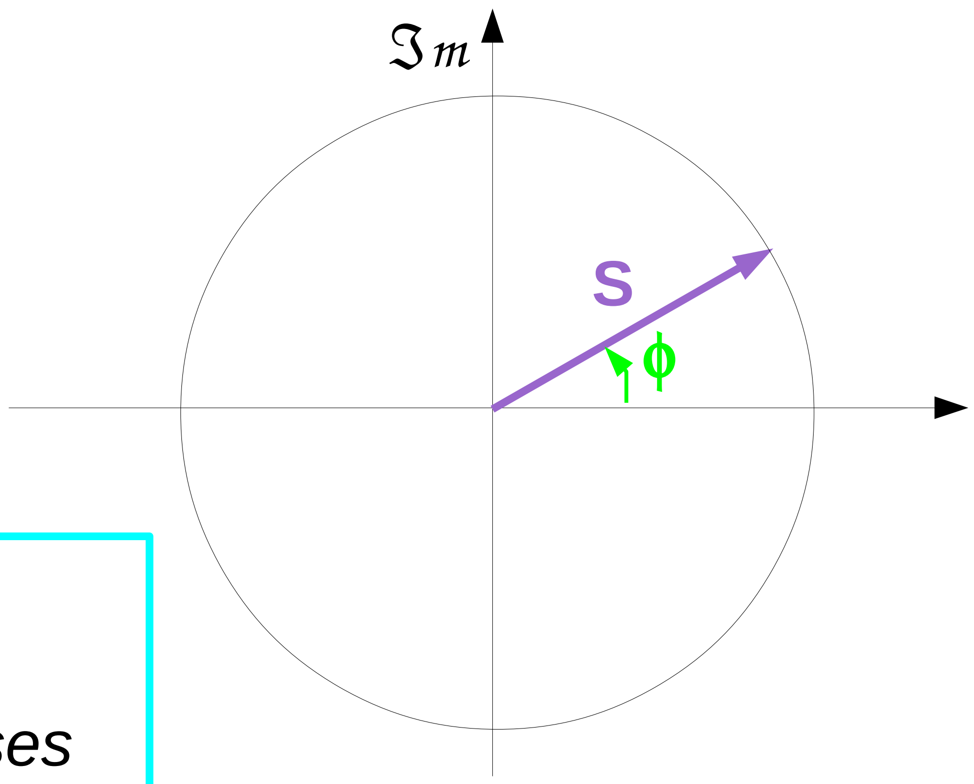
$\frac{3\pi}{2}$	$e^{j\frac{3\pi}{2}}$	$\cos\frac{3\pi}{2} + j\sin\frac{3\pi}{2}$	$-\Im m\{s\}$
2π	$e^{j2\pi}$	$\cos 2\pi + j\sin 2\pi$	$\Re e\{s\}$

Use the representation Most appropriate To the problem

Complex numbers



Practical points

- *Very useful to keep track of phases*
- *Physical quantities are taken by projecting onto the real axis*

Definitions

We define a signal s(t):

$$s(t) = \hat{A}\sin(\omega t - \phi) \qquad (1)$$

$\hat{A}$ amplitude

ω angular frequency ($rad \cdot s^{-1}$: $\omega = 2\pi f = 2\pi \frac{1}{T}$)

f frequency (Hz)

T period (s)

ϕ phase (rad)

Average value of the oscillating signal

Average value of s (i.e. rms value or s_{rms})

$$
\begin{aligned}
< s(t) >_T &= s_{rms} \equiv A\,, && (2)\\
A &= \sqrt{\frac{1}{T}\int_0^T s^2 dt}\,, && (3)\\
&= \sqrt{\frac{1}{T}\int_0^T \hat{A}^2 \sin^2(\omega t - \phi) dt}\,, && (4)\\
&= \sqrt{\frac{1}{T}\int_0^T \hat{A}^2 \frac{1-\cos(2\omega t - \phi)}{2} dt} && (5)\\
&= \sqrt{\frac{\hat{A}^2}{2T}\int_0^T dt}\,, && (6)\\
&= \frac{\hat{A}}{\sqrt{2}} && (7)
\end{aligned}
$$

Behaviour of different components

Resistance R

Capacitance C

Inductance L

Behaviour of different components

Resistance R

Capacitance C

Inductance L

Voltage signal (input)

$$V(t) = V_p \sin(\omega t)\,, \quad (8)$$
$$= V\sqrt{2}\sin(\omega t) \quad (9)$$

alternatively written as

$$V(t) = V_p e^{j\omega t}\,, \quad (10)$$
$$= V\sqrt{2}e^{j\omega t} \quad (11)$$

Resistor

Ohm's law

$$U = Ri\,, \tag{12}$$
$$R \in \mathbb{R} \tag{13}$$
$$i(t) = \frac{V(t)}{R}\,, \tag{14}$$
$$= \frac{V_p}{R}\sin(\omega t)\,, \tag{15}$$
$$= \frac{V\sqrt{2}}{R}\sin(\omega t) \tag{16}$$

alternatively

$$i(t) = \frac{V_p}{R}e^{j\omega t}\,, \tag{17}$$
$$= \frac{V\sqrt{2}}{R}e^{j\omega t} \tag{18}$$

Capacitor

From electromagnetism

$$Q = C \cdot V\,, \tag{19}$$

$$\begin{aligned} i &= \frac{dQ}{dt}\,, && (20) \\ &= C\frac{dV}{dt}\,, && (21) \\ &= C\omega V_p \cos(\omega t) && (22) \\ &= C\omega V_p \sin\left(\omega t + \frac{\pi}{2}\right) && (23) \end{aligned}$$

Not very practical!

Capacitor

Alternative form

$$i(t) = C\frac{d}{dt}V_p e^{j\omega t}, \quad (24)$$

$$= j\omega C V_p e^{j\omega t}, \quad (25)$$

$$= j\omega C V(t) \quad (26)$$

Generalizing Ohm's law

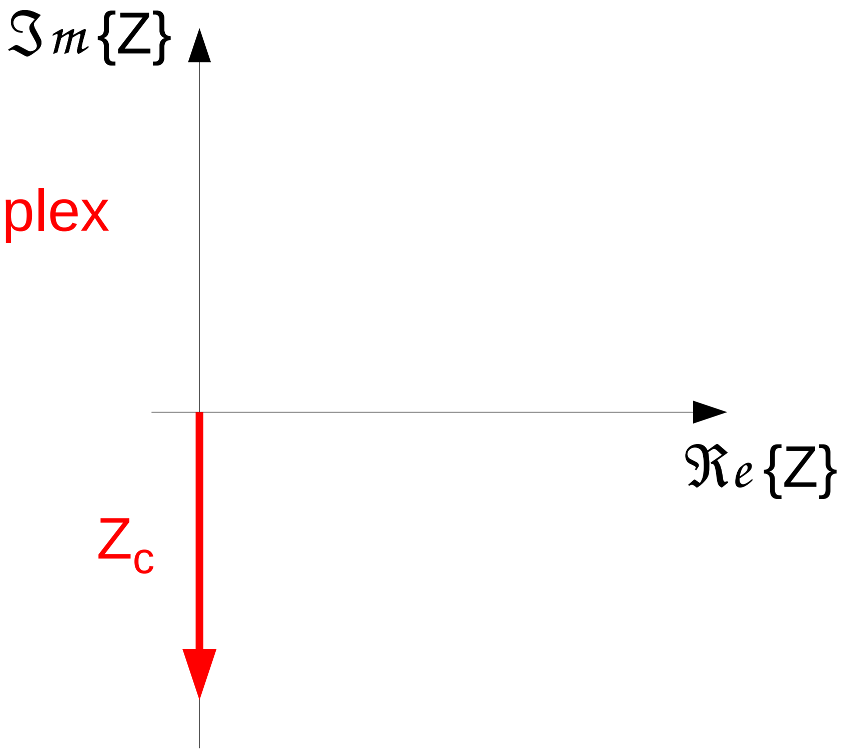
$$i(t) = \frac{V(t)}{Z}, \quad (27)$$

Capacitor's impedance →

$$Z_c = \frac{1}{j\omega C} \quad (28)$$

Capacitor

The capacitor's impedance is complex
with no real part!



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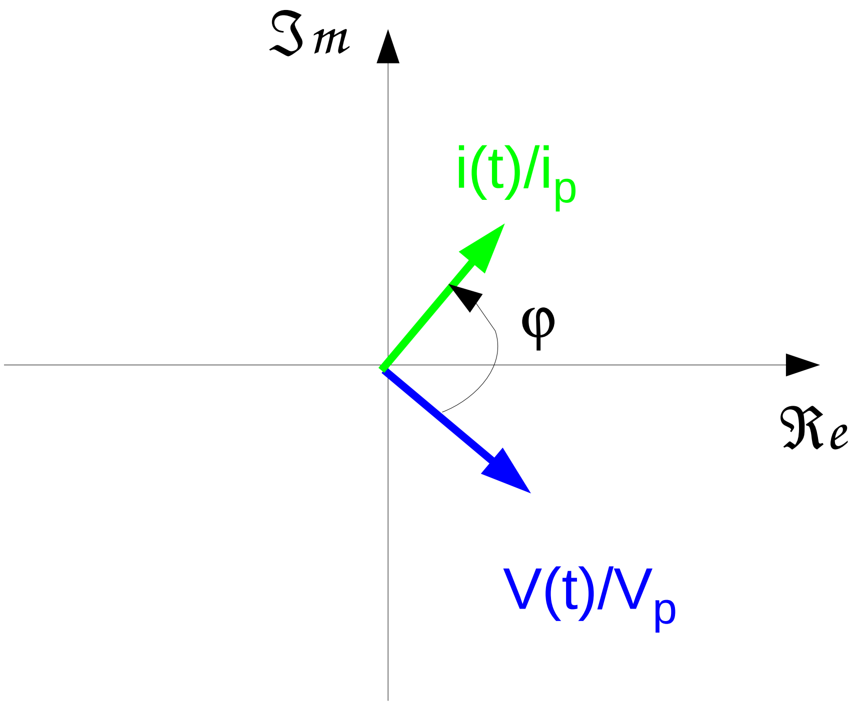
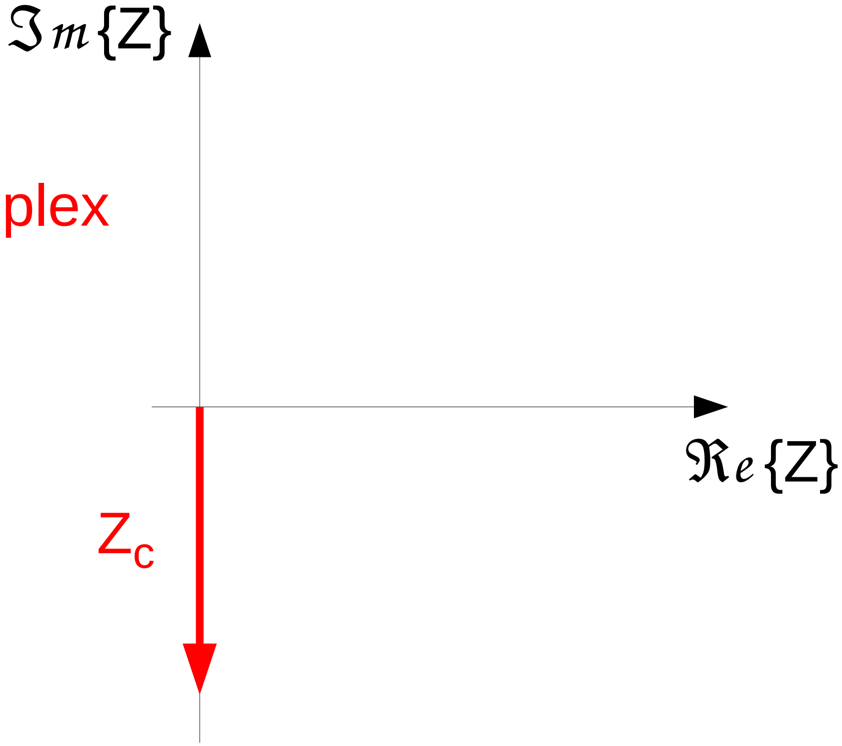




Voltage and current signals
are dephased (by $\varphi=\pi/2$)

Inductor

From electromagnetism

$$U = L\frac{di}{dt}, \tag{29}$$

$$i(t) = \frac{1}{L}\int u(t)dt, \tag{30}$$

$$= -\frac{1}{L\omega}V_p \cos(\omega t), \tag{31}$$

$$= \frac{V_p}{L\omega}\sin\left(\omega t - \frac{\pi}{2}\right) \tag{32}$$

Not very practical!

Inductor

Alternative form

$$i(t) = \frac{1}{L}\int V_p e^{j\omega t}, \quad (33)$$

$$= \frac{1}{j\omega L} V_p e^{j\omega t}, \quad (34)$$

$$= \frac{V(t)}{j\omega L} \quad (35)$$

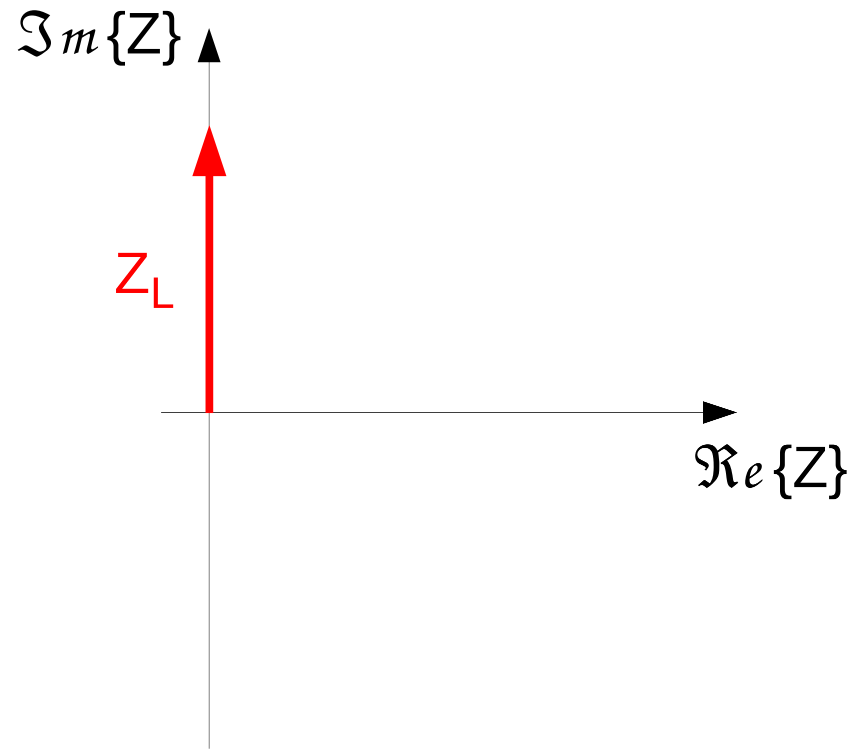
Generalizing Ohm's law

$$i(t) = \frac{V(t)}{Z}, \quad (36)$$

Inductor's impedance → $$Z_L = j\omega L \quad (37)$$

Inductor

The inductor's impedance is complex
with no real part!



Inductor

The inductor's impedance is complex
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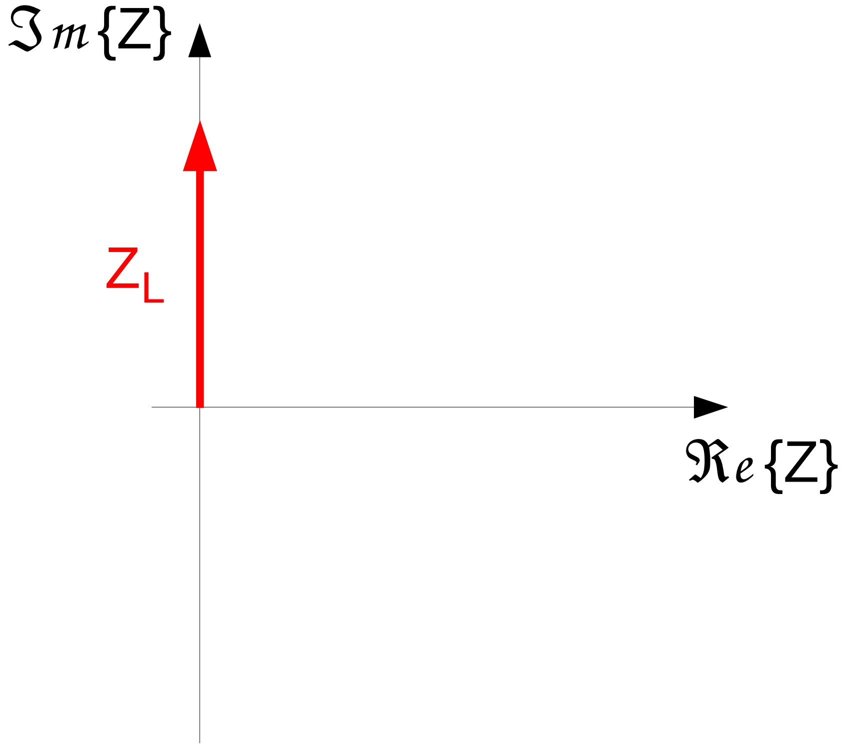

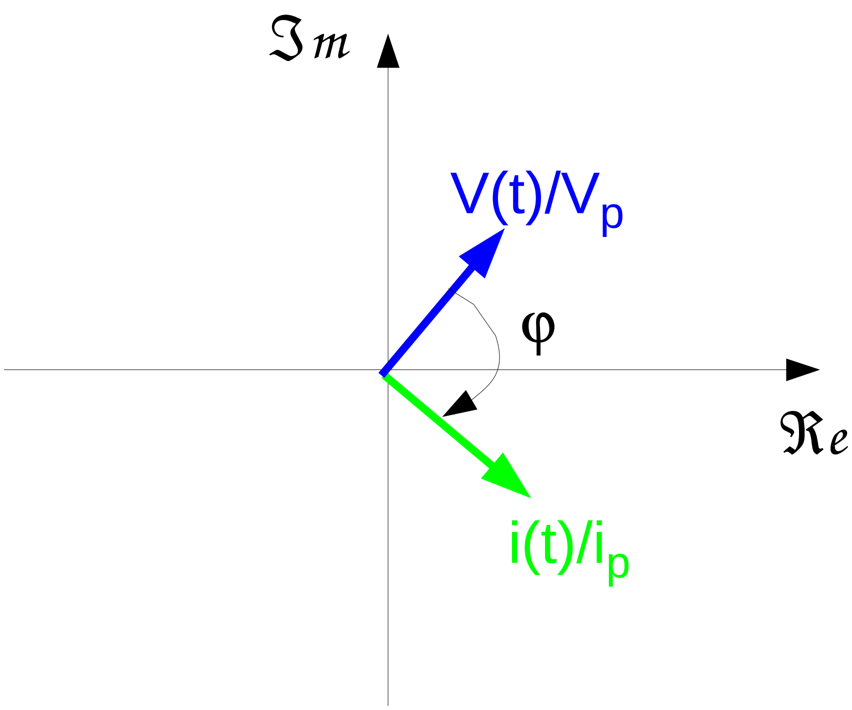

Voltage and current signals
are dephased (by $\varphi=-\pi/2$)

Definition

Correct
Not very
useful

The electrostatic (potential) energy is defined as the (opposite of the) work W done on a charge Q to displace it from an initial (reference) position (r_{ref}) to its actual position (r) in the presence of an electric field $\vec{E}$:

$$U_E(\vec{r}) \;=\; -W_{r_{ref}} = -\int_{r_{ref}}^{r} Q\vec{E}\cdot d\vec{s} \quad (38)$$

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Alternative definition

useful

A charge Q placed in an electrostatic potential $V(\vec{r})$ possesses an energy

$$U_E(\vec{r}) \;=\; QV(\vec{r}) \qquad (39)$$

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A charge Q placed in an electrostatic potential $V(\vec{r})$ possesses an energy

$$U_E(\vec{r}) = QV(\vec{r}) \qquad (39)$$

In practice, one works with electrical power ...

Definition

Power: Energy provided per unit time

$$P = \frac{dU_E}{dt}, \tag{40}$$

$$= \frac{dQ}{dt} V, \tag{41}$$

$$\Rightarrow \mathbf{P} = \mathbf{i} \cdot \mathbf{V} \tag{42}$$

where:

- we consider a charge flow $\left(\frac{dQ}{dt}\right)$ in a constant potential V $\left(\text{hence } \frac{dV}{dt} = 0\right)$
- we consider the potential position independent (no $\vec{r}$) in an electrical circuit around each terminal (i.e., we only consider potential differences across components)

Definition

Note: from now on we will use W for the electrical energy because of the so-called **passive sign convention**:
electric power consumed in a device is defined to have a positive sign, while power produced by a device is defined to have a negative sign

(opposite to the physical definition of U_E, whence the coincidence between the (negative) work W and the energy in this convention)

Application

Real components (resistor R)

Use Ohm's law:

$$\Rightarrow \mathbf{P} = \mathbf{i} \cdot \mathbf{V}\,, \tag{43}$$
$$= i^2 \cdot R\,, \tag{44}$$
$$= \frac{V^2}{R} \tag{45}$$

familiar expressions ...

General expression

For the sinusoidal (monophase) regime the expression generalizes to

$$
\begin{aligned}
P(t) &= i(t) \cdot V(t)\,, && (46)\\
&= i_P sin(\omega t - \beta) \cdot V_P \sin(\omega t - \alpha) && (47)\\
, &= i_P \left[\frac{e^{j(\omega t - \beta)} - e^{-j(\omega t - \beta)}}{2j} \right] \cdot \\
& \quad V_P \left[\frac{e^{j(\omega t - \alpha)} - e^{-j(\omega t - \alpha)}}{2j} \right]\,, && (48)\\
&= -\frac{i_P V_P}{4} \left[e^{j(2\omega t - \alpha - \beta)} + e^{-j(2\omega t - \alpha - \beta)} - \right. \\
& \quad \left. (e^{j(\beta - \alpha)} + e^{-j(\beta - \alpha)}) \right]\,, && (49)
\end{aligned}
$$

General expression

$$= \frac{i_P V_P}{2}\left[\cos(\beta-\alpha) - \cos(2\omega t - \alpha - \beta)\right] \qquad (50)$$

where α and β are two phase angles (to be determined by the problem).

Introducing

$$V_P = V_{rms}\sqrt{2}\,, \qquad (51)$$

$$i_P = i_{rms}\sqrt{2}\,, \qquad (52)$$

$$\phi \equiv \beta - \alpha\,, \qquad (53)$$

$$\gamma \equiv \alpha + \beta\,, \qquad (54)$$

General expression

we can rewrite

$$P(t) = \mathcal{P} + p(t)\,, \tag{55}$$

$$\mathcal{P} = V_{rms} i_{rms} \cos\phi\,, \tag{56}$$

$$p(t) = -V_{rms} i_{rms} \cos(2\omega t - \gamma)\,, \tag{57}$$

i.e., the power contains a **time-independent component** $\mathcal{P}$ and a **time dependent one** $p(t)$.

General expression

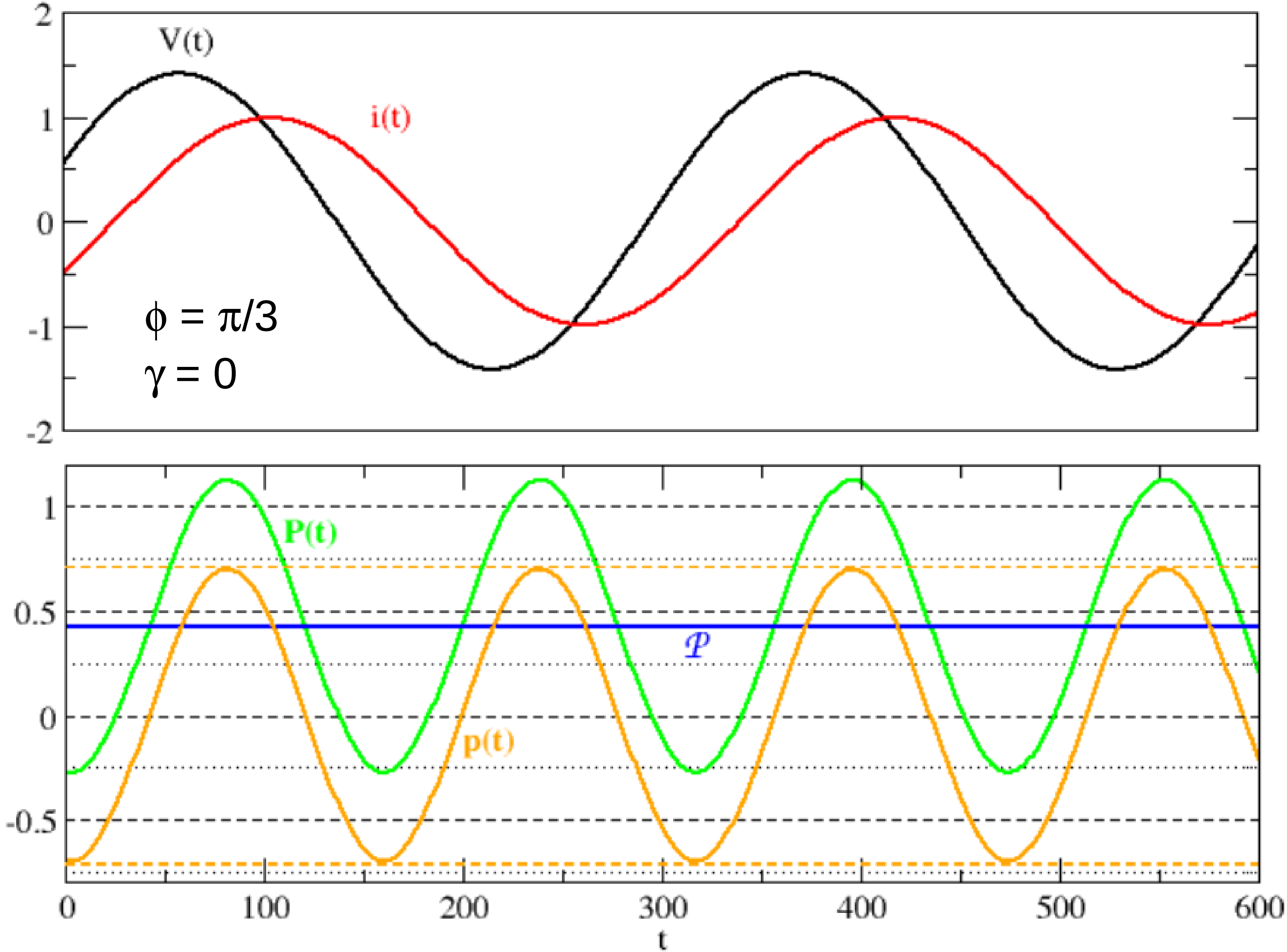
Average power

$$
\begin{aligned}
< P(t) >_T &= \frac{1}{T}\int_0^T P(t)dt\,, && (58)\\
&= \frac{1}{T}\int_0^T [\mathcal{P} + p(t)]dt\,, && (59)\\
&= \frac{\mathcal{P}}{T}\int_0^T dt\,, && (60)\\
&= \mathcal{P} && (61)
\end{aligned}
$$

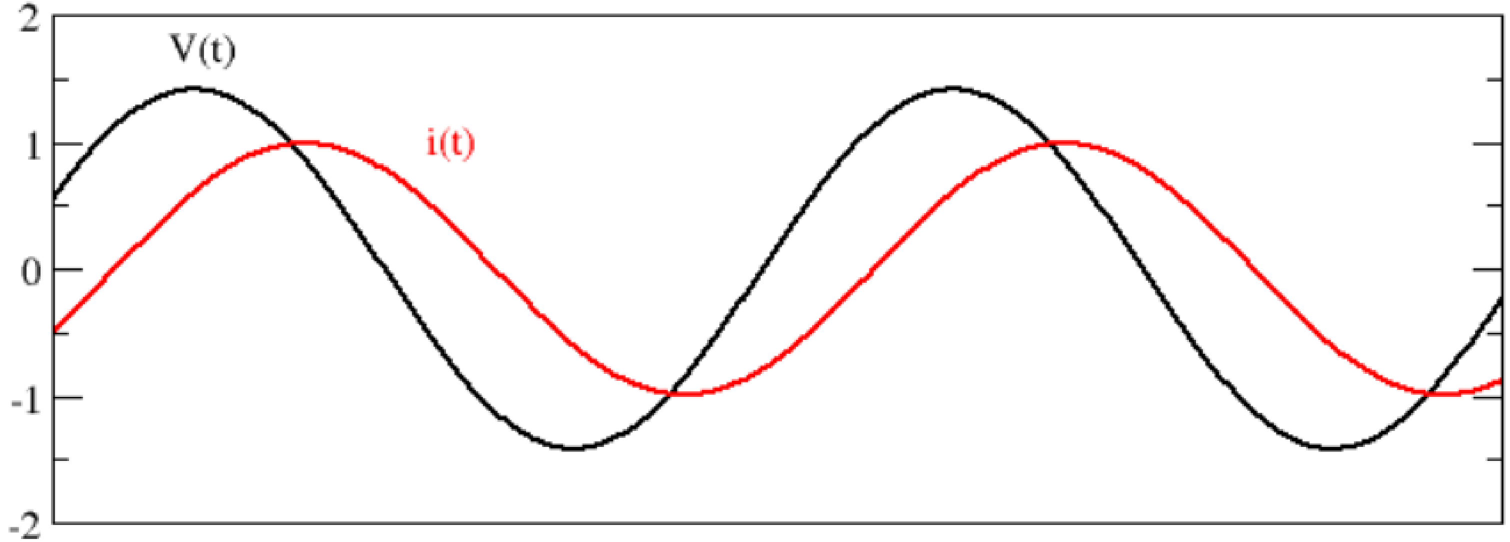
since

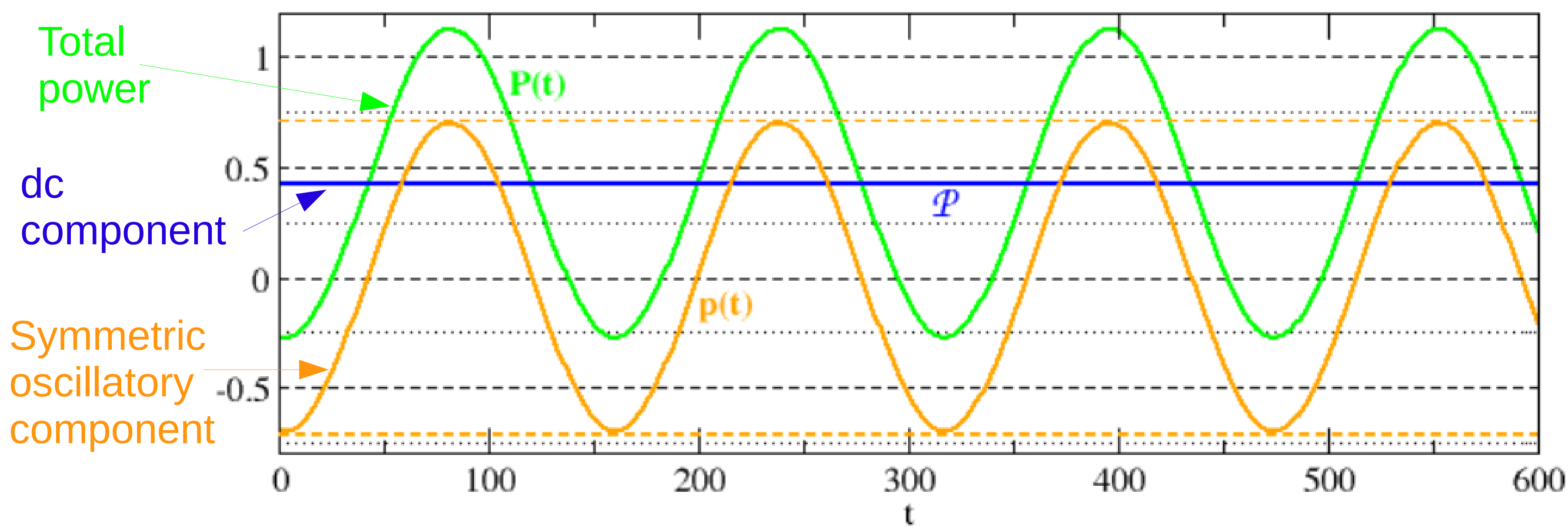
$$
\begin{aligned}
< p(t) >_T &= \frac{V_{rms} i_{rms}}{T}\int_0^T \cos(2\omega t - \gamma)dt\\
&= 0 && (62)
\end{aligned}
$$

Example: generic phase difference



Example

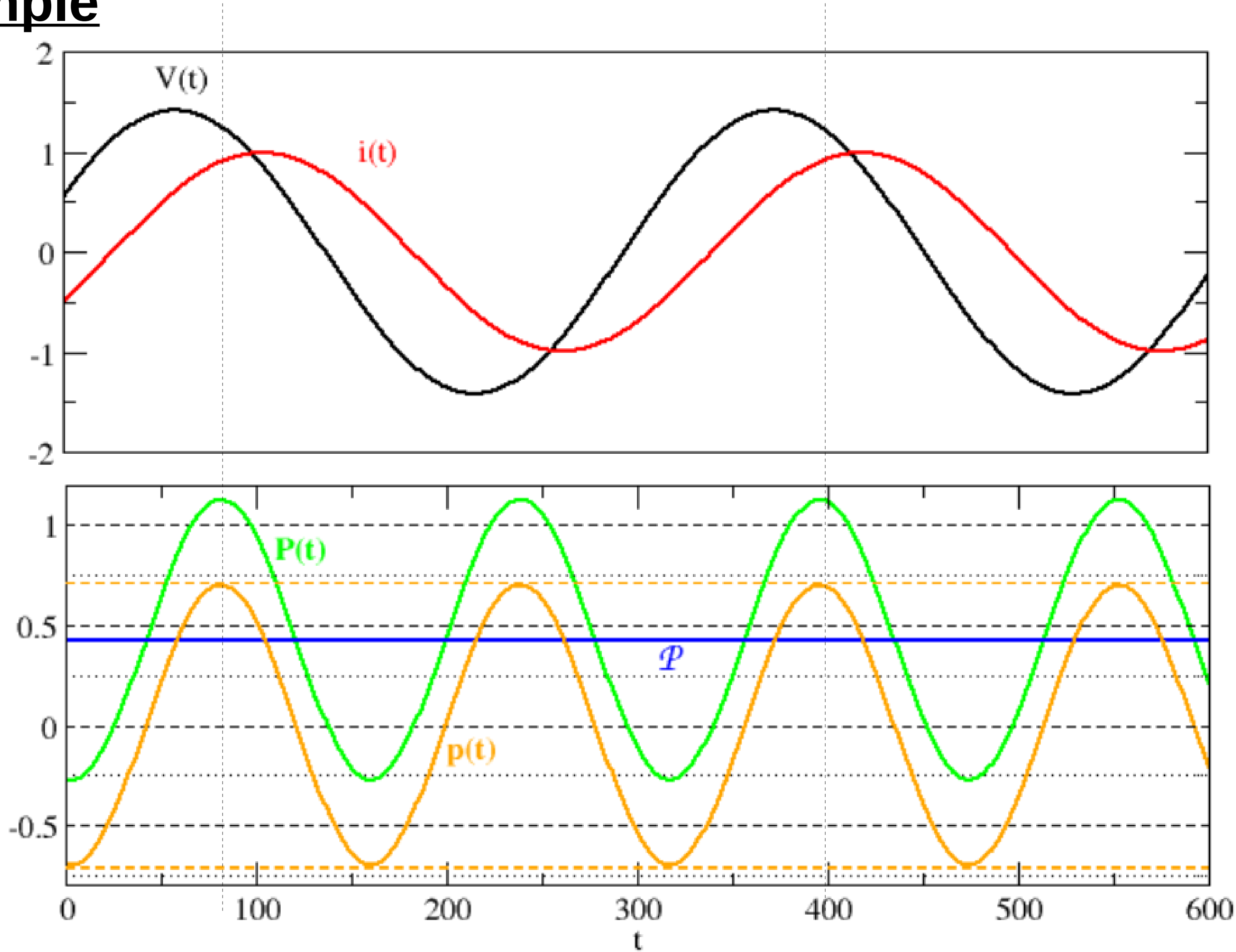

V(t)
i(t)
2
1
0
-1
-2


Total power
dc component
Symmetric oscillatory component
P(t)
$\mathcal{P}$
p(t)
1
0.5
0
-0.5
0
100
200
300
400
500
600
t

Electrical power

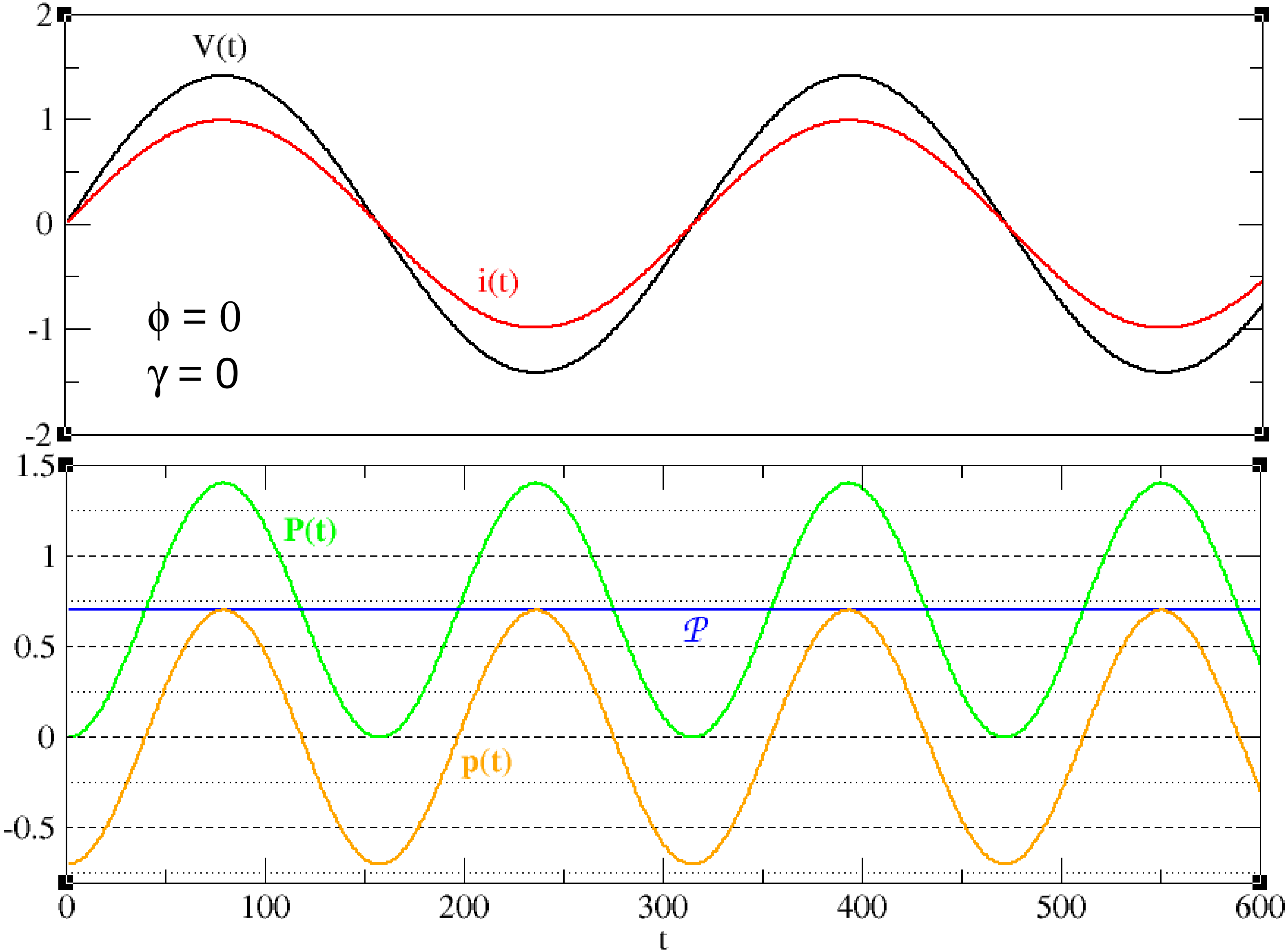
Example

P frequency: double of **V** and **i** frequency

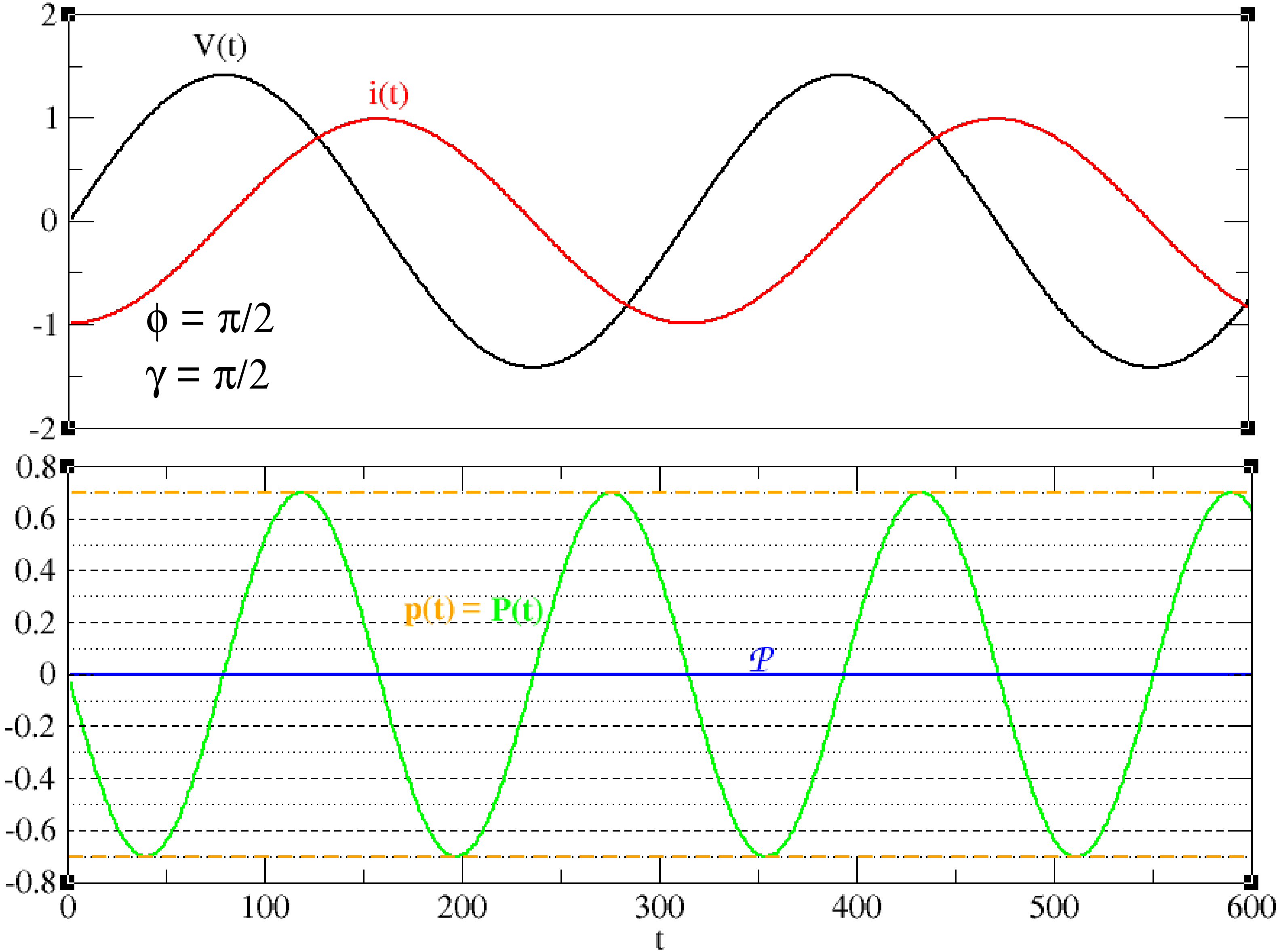


Example: voltage and current in phase

Maximum power



<u>Example</u>: voltage and current in quadrature



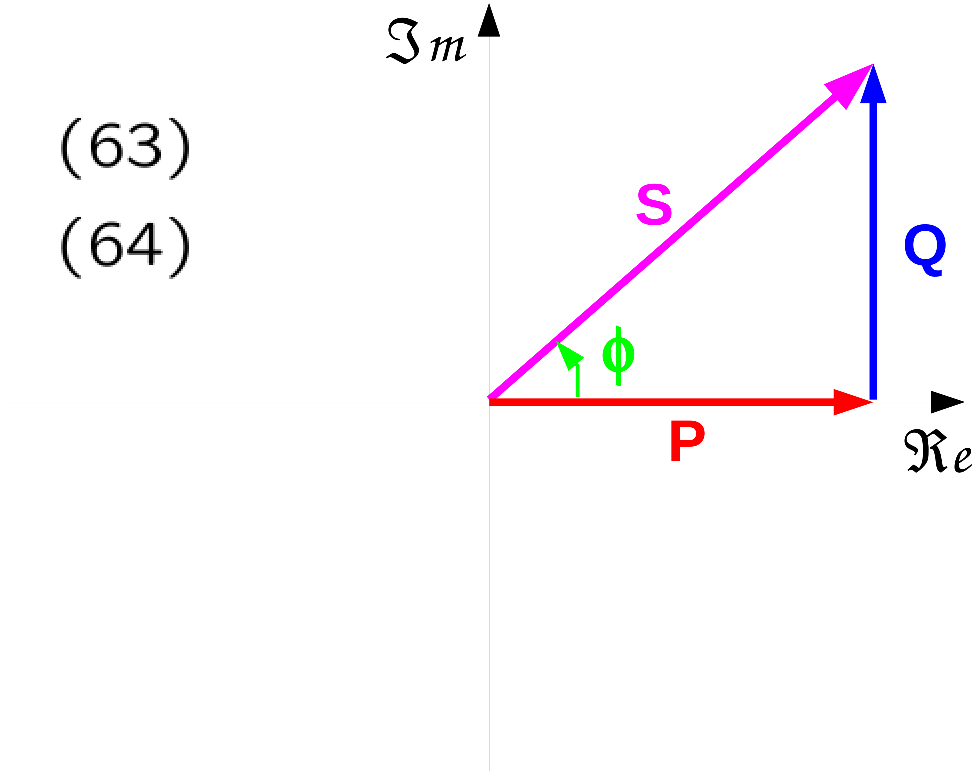
Time independent power

(no work provided by $p(t)$)

Power definitions (practical and useful)

$$P = |S| \cos \phi \tag{63}$$

$$Q = |S| \sin \phi \tag{64}$$



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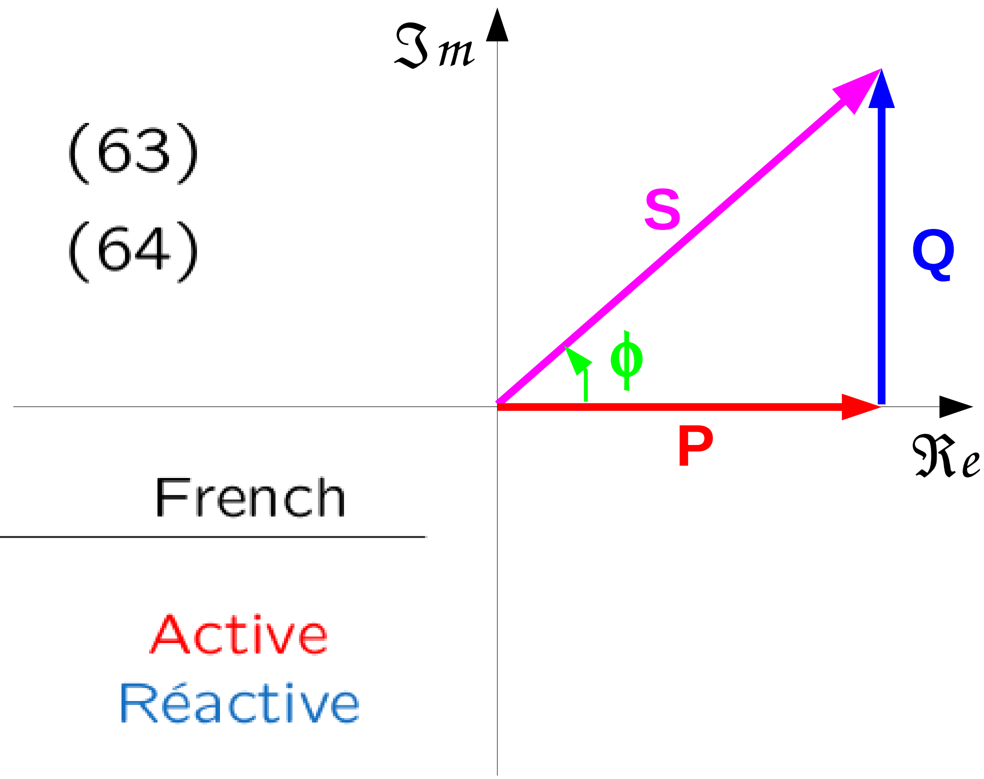
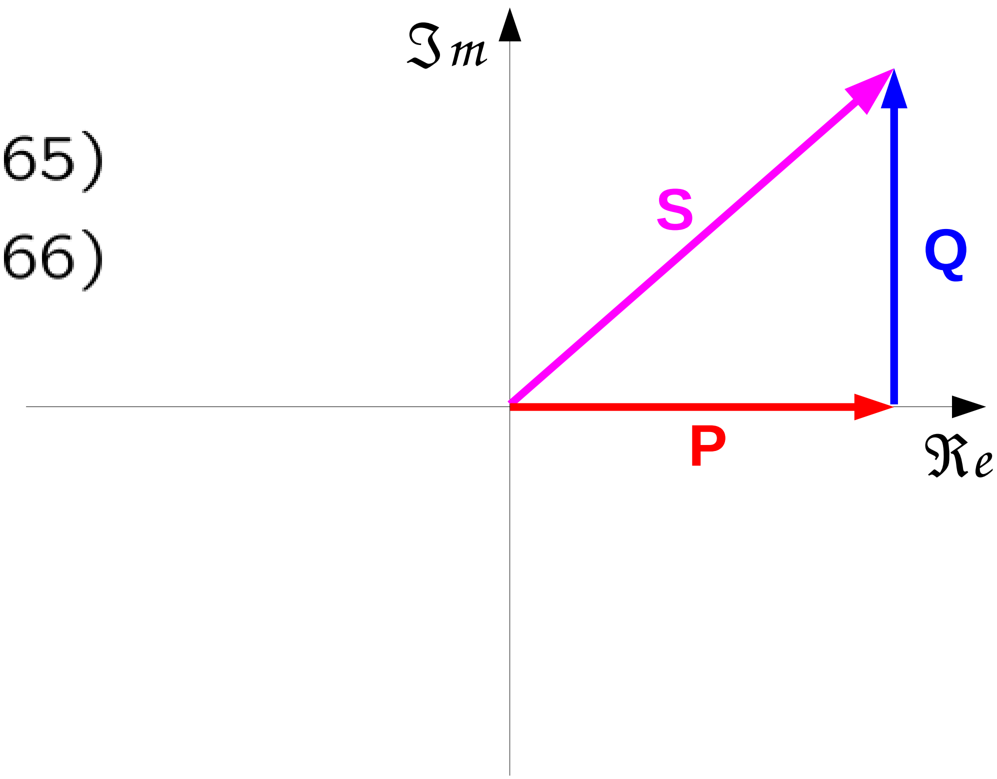


Quantity	Units	Definition	French
P	W	Real power	
		Active power	Active
Q	var	Reactive power	Réactive
S	VA	Complex power	
|S|	VA	Apparent power	Apparente
ϕ	deg	Voltage-to-current phase	

Active and reactive current

$$\vec{S} = \vec{V} \cdot \vec{i}^{\star} \quad (65)$$

$$= P + jQ \quad (66)$$

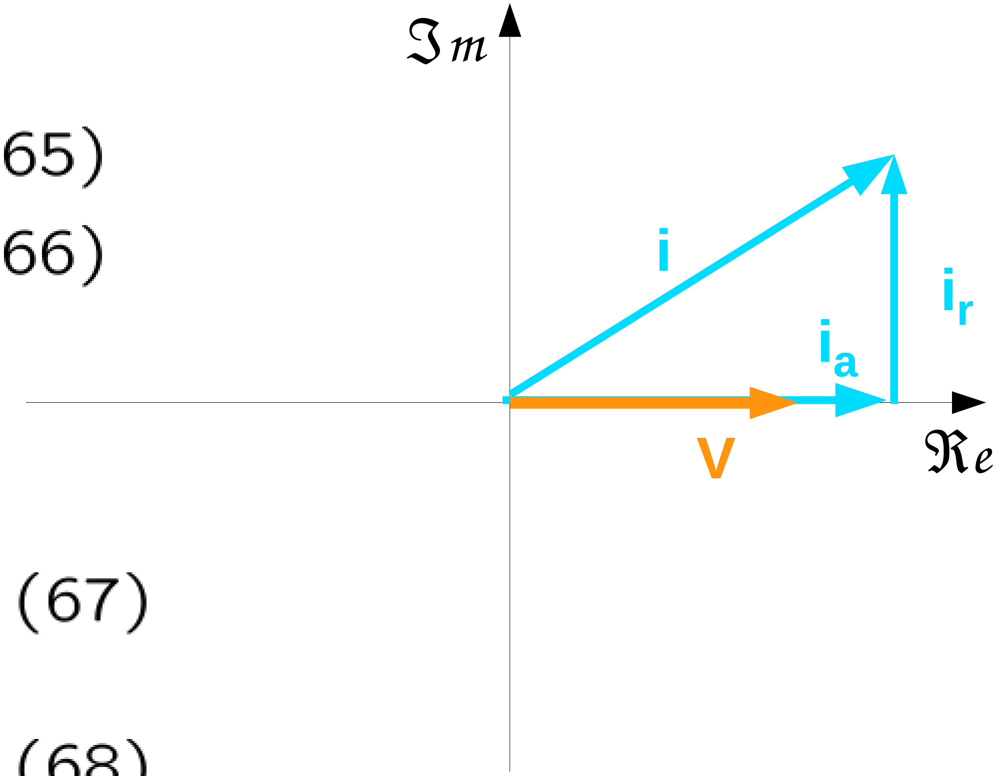


Active and reactive current

$$\vec{S} = \vec{V} \cdot \vec{i}^{\star} \tag{65}$$

$$= P + jQ \tag{66}$$

V aligned with real axis



$$\vec{i} = \frac{\vec{S}}{|V|} \tag{67}$$

$$= \frac{P}{|V|} + j\frac{Q}{|V|} \tag{68}$$

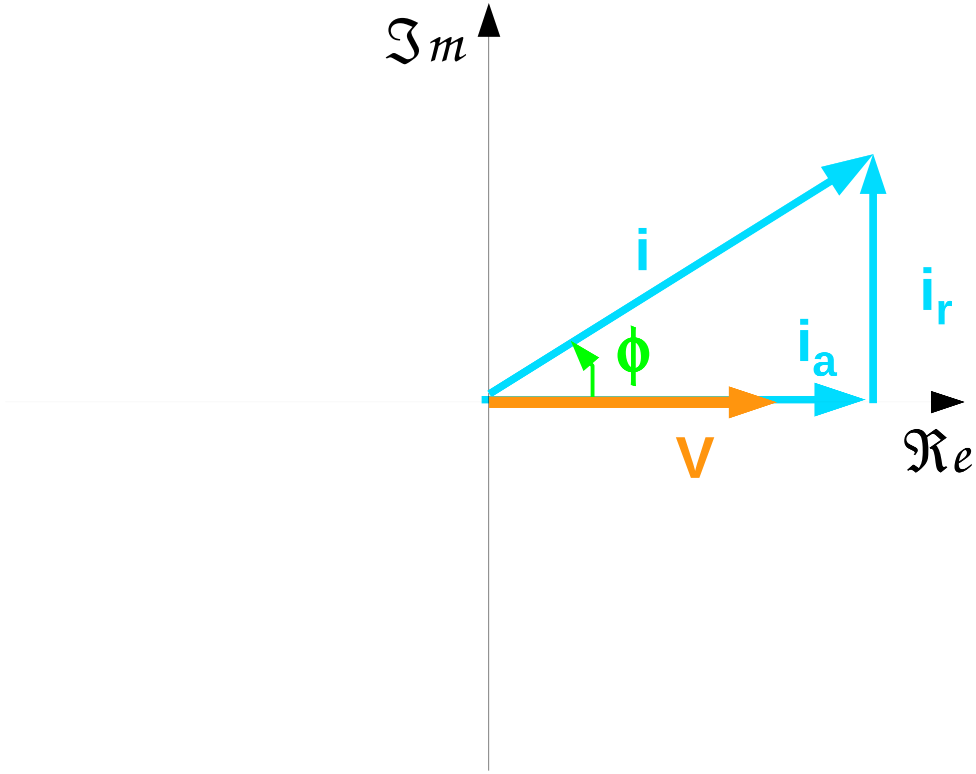
$$\equiv i_a + j i_r \tag{69}$$

Active and reactive current

$\phi > 0$

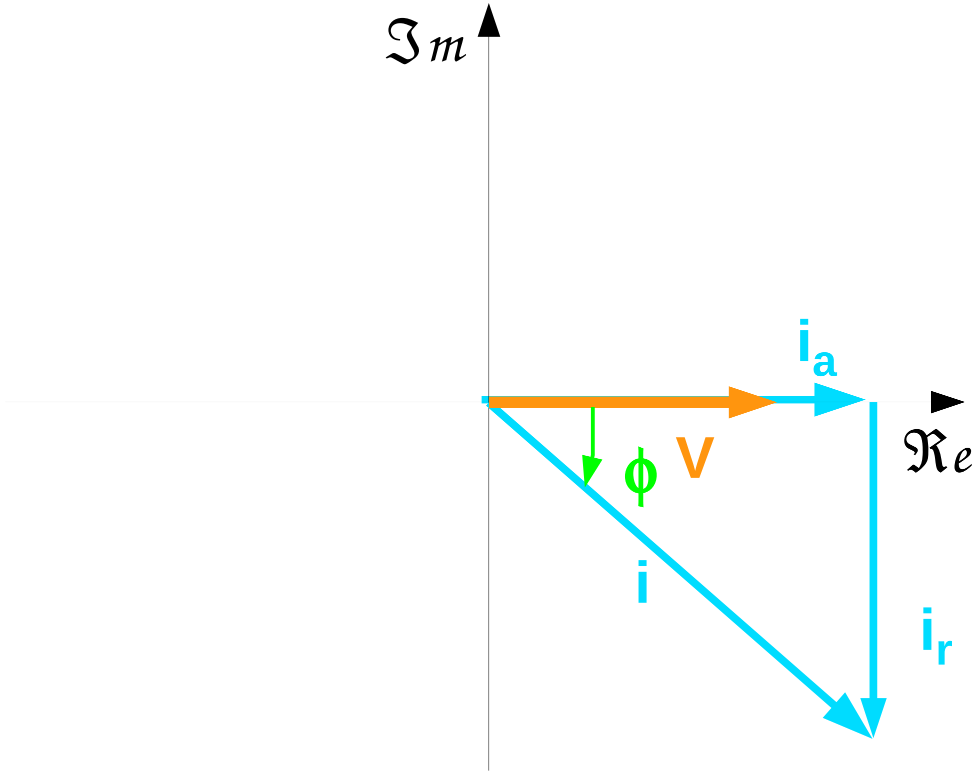
Predominantly capacitive load

Energy “stored” in the capacitance



Active and reactive current

$\phi < 0$
Predominantly inductive load
Energy “dissipated” in the inductance



Power definitions: comments

P: useful power

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Q: power transferred back and forth between reactive elements (ideal capacitors and ideal inductors)
Not **useful** for any purposes
BUT: a. dissipates energy
b. heats up wires and elements
=> needs to be accounted for!!!

Power definitions: comments

P: useful power

Q: power transferred back and forth between reactive elements (ideal capacitors and ideal inductors)
Not **useful** for any purposes
BUT: a. dissipates energy
b. heats up wires and elements
=> needs to be accounted for!!!

|S|: modulus of the complex vector

Power factor *(facteur de puissance)*

$$F_p = \frac{P}{|S|} \qquad (65)$$

Ratio between real and apparent power

Measures the efficiency of a power distribution system:
for same apparent power
the higher the power factor => the higher the real power
=> the lower the reactive power
=> the lower the fraction of circulating useless currents

Power factor *(facteur de puissance)*

For purely sinusoidal signals

$$F_p = \frac{P}{|S|} \quad \boxed{= \cos \phi} \qquad (65)$$

Ratio between real and apparent power

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for same apparent power
the higher the power factor => the higher the real power
=> the lower the reactive power
=> the lower the fraction of circulating useless currents

Power factor *(facteur de puissance)*

F_p	ϕ°	Comment
1	0	Voltage and current are in phase
0	- 90	Current leads voltage (pure inductive load)
0	+ 90	Current lags voltage (pure capacitive load)

Power factor *(facteur de puissance)*

F_p	ϕ°	Comment
1	0	Voltage and current are in phase
0	- 90	Current leads voltage (pure inductive load)
0	+ 90	Current lags voltage (pure capacitive load)

In equipment datasheets Fp denoted as "cos ϕ"

Application: an instrument has real power P = 1200 W with $\phi = 37^o$

Active current	$i_a = (P / 220\ V) \approx 5.45\ A$
Apparent power	$S = (1200/\cos\phi)\ VA \approx 1500\ VA$
Reactive power	$Q = (\lvert S \rvert \sin\phi)\ VA \approx 900\ var$
Reactive current	$i_r = (Q / 220\ V) \approx 4.09\ A$
Effective current	$\sqrt{(i_a^2 + i_r^2)} \approx 6.81\ A$

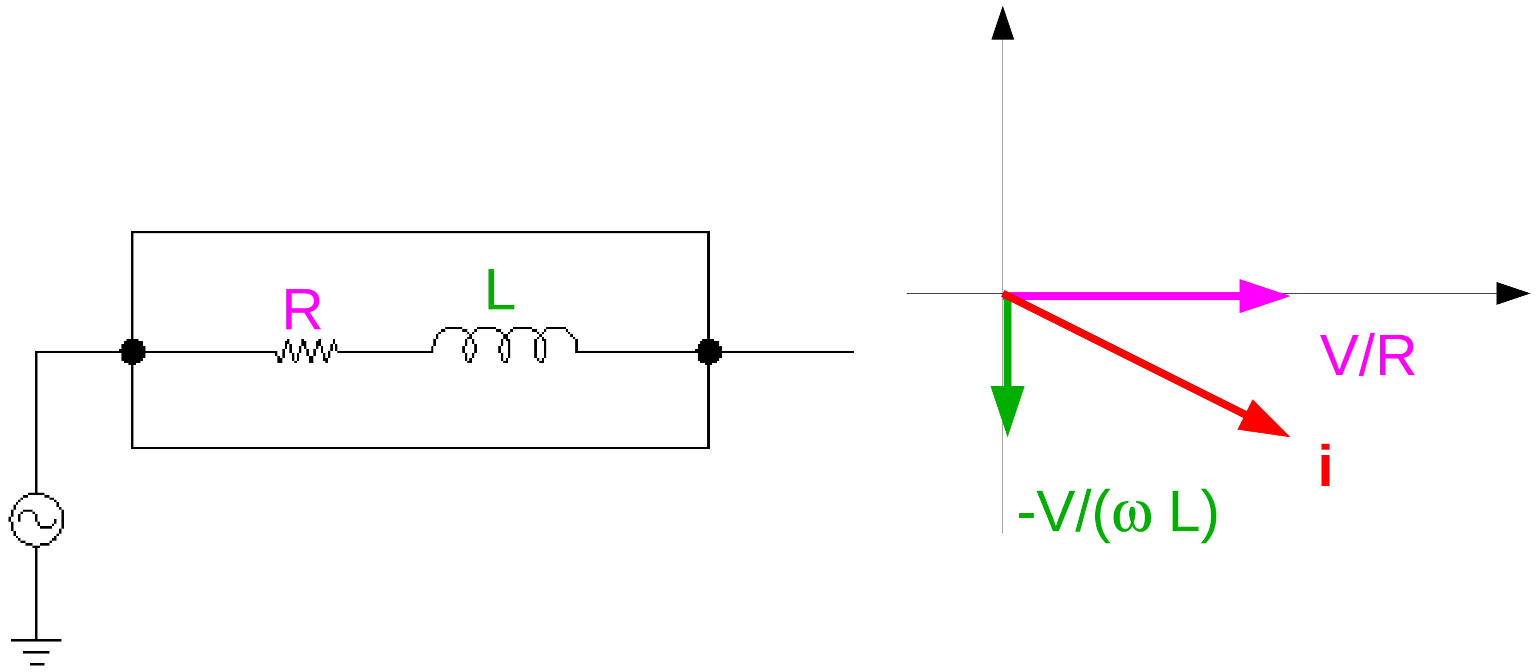
Bad news: reactive current/power wastes energy
(component and wire heating)

Reactive power control

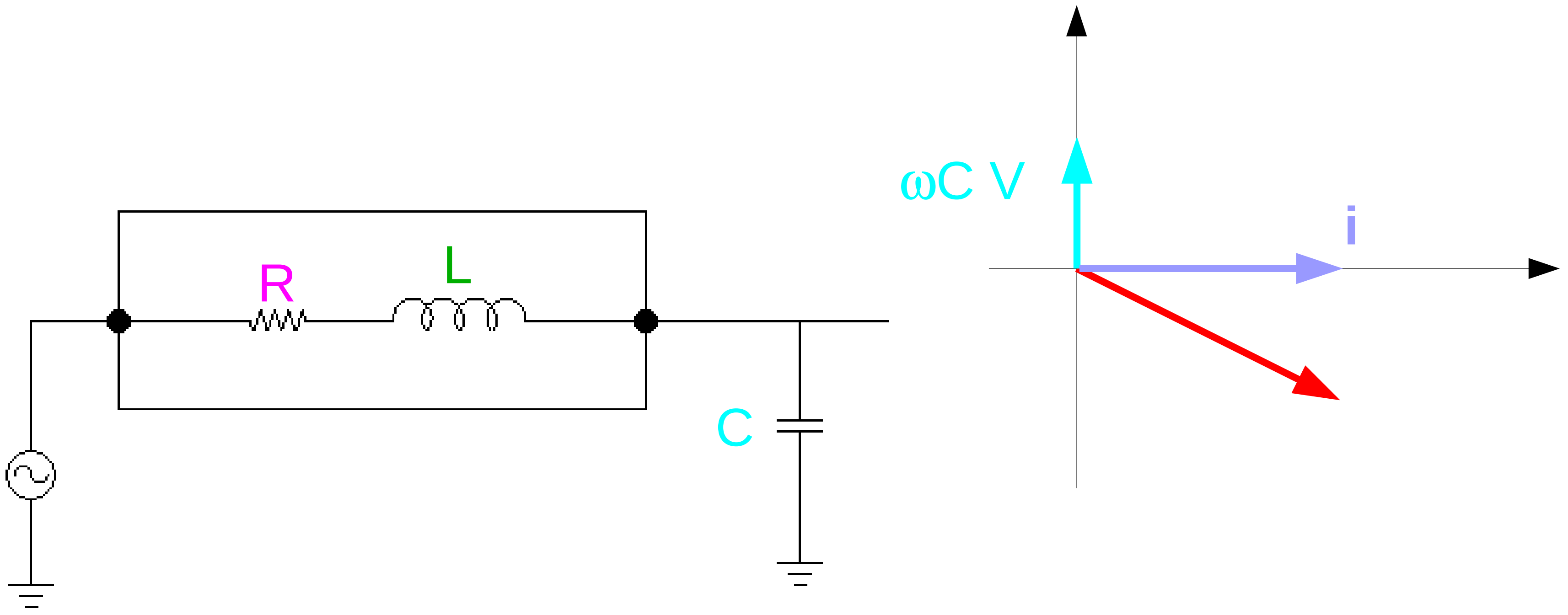
Bad news: reactive current/power wastes energy (component and wire heating)

Good news: reactive power can be controlled!

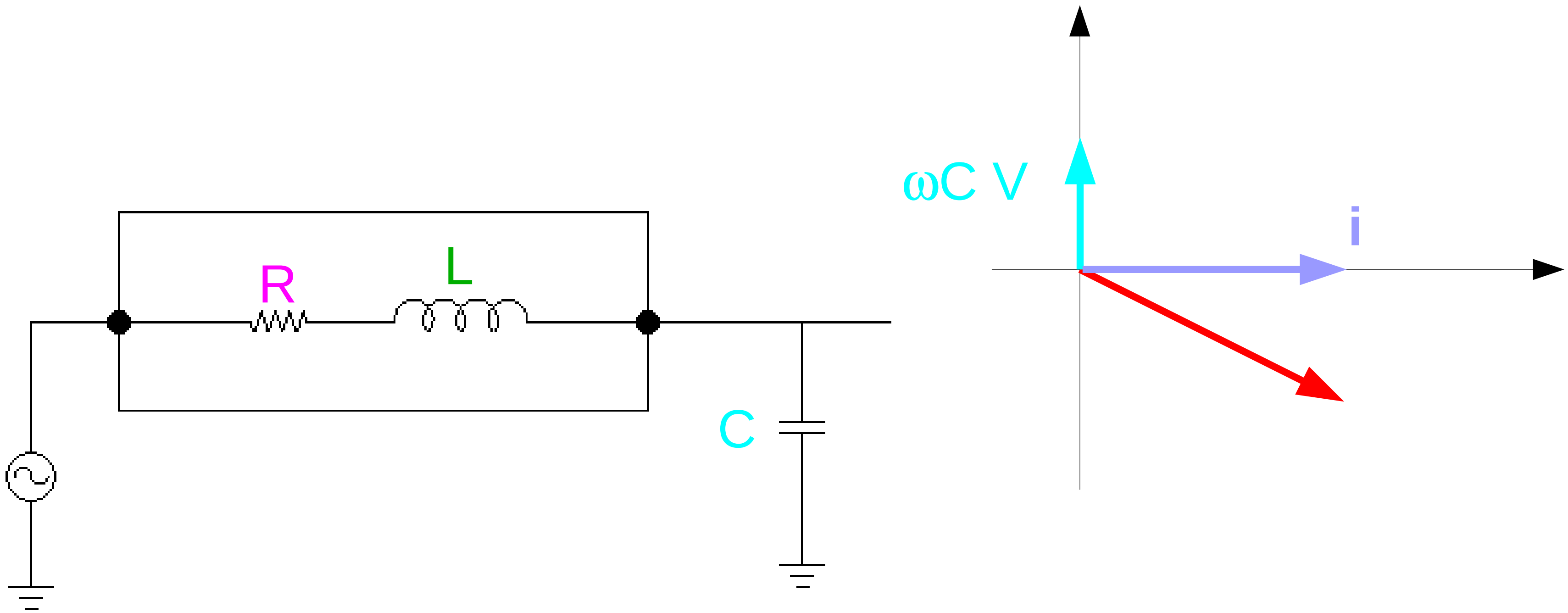
Controlling reactive power



Controlling reactive power



Controlling reactive power



And similarly for equipment with capacitive components ...

Boucherot's theorem

First formulation

The active (reactive) power absorbed by a ensemble of dipoles is equal to the sum of the active (reactive) power absorbed by each dipole.

For N dipoles whose power dissipation is given by

$$\vec{S}_k = P_k + jQ_k \tag{71}$$

the total active and reactive power absorbed by the ensemble of dipoles takes the form

$$P_t = \sum_{k=1}^{N} P_k \tag{72}$$

$$Q_t = \sum_{k=1}^{N} Q_k \tag{73}$$

Boucherot's theorem

Second formulation

The complex apparent power absorbed by an ensemble of dipoles is equal to the sum of the (complex) apparent power absorbed by each dipole

For N dipoles whose power dissipation is given by

$$\vec{S}_k = P_k + jQ_k \tag{74}$$

the total (complex) apparent power is

$$\vec{S}_t = \sum_{k=1}^{N} \vec{S}_k \tag{75}$$

Boucherot's theorem

Remarks

$$\vec{S}_t = \sum_{k=1}^{N} \vec{S}_k \tag{76}$$

$$= \sum_{k=1}^{N} P_k + j \sum_{k=1}^{N} Q_k \tag{77}$$

and

$$|S_t| = |\sum_{k=1}^{N} \vec{S}_k| \tag{78}$$

$$= \sqrt{\left(\sum_{k=1}^{N} P_k\right)^2 + \left(\sum_{k=1}^{N} Q_k\right)^2} \tag{79}$$

BUT

$$|S_t| \neq \sum_{k=1}^{N} |\vec{S}_k| \tag{80}$$