ELEC4 - EIEL821

Spectral Analysis — Solutions to exercises
Chapter 3: Parametric estimation

Vicente Zarzoso
vicente.zarzoso@univ-cotedazur.fr

Electrical Engineering Department
Polytech'Nice Sophia
Université Côte d'Azur

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3.1 Filtering random processes

Let $X(n)$ be a wide sense stationary random process with mean μ_X, ACS $r_X(k)$, and PSD $S_X(\omega)$ and $S_X(z)$. The process is filtered by a linear time-invariant system with transfer function $H(z)$. Find the expressions of the mean, ACS and PSD (in ω and z domains) of the filter output $Y(n)$.

Solution:

Recall that the output of a linear time-invariant system is given by the convolution

$$Y(n) = h(n) * X(n) \stackrel{\text{def}}{=} \sum_{k=-\infty}^{+\infty} h(k)X(n-k)$$

where $h(n)$ is the system impulse response, defined as

$$h(n) \stackrel{\text{def}}{=} \mathcal{Z}^{-1}\{H(z)\}.$$

Hence:

$$\mu_Y \stackrel{\text{def}}{=} \mathrm{E}\{Y(n)\} = \mathrm{E}\{h(n) * X(n)\} = \mathrm{E}\left\{\sum_{k=-\infty}^{+\infty} h(k)X(n-k)\right\}$$

$$= \sum_{k=-\infty}^{+\infty} h(k)\underbrace{\mathrm{E}\{X(n-k)\}}_{\mu_X} = \mu_X \sum_{k=-\infty}^{+\infty} h(k) = \mu_X \sum_{k=-\infty}^{+\infty} h(k)\mathrm{e}^{-\jmath 0\cdot k}$$

$$\mu_Y = \mu_X H(\mathrm{e}^{\jmath 0})$$

$$r_Y(k) \stackrel{\text{def}}{=} \mathrm{E}\{Y(n)Y^*(n-k)\} = \mathrm{E}\left\{\sum_{m=-\infty}^{+\infty} h(m)X(n-m) \sum_{p=-\infty}^{+\infty} h^*(p)X^*(n-k-p)\right\}$$

$$= \sum_{m=-\infty}^{+\infty} \sum_{p=-\infty}^{+\infty} h(m)h^*(p)\underbrace{\mathrm{E}\{X(n-m)X^*(n-k-p)}_{r_X(k+p-m)}$$

$$= \sum_{p=-\infty}^{+\infty} h^*(p) \sum_{m=-\infty}^{+\infty} h(m)r_X(k+p-m)$$

Now let us define

$$s(k) \stackrel{\text{def}}{=} h(k) * r_X(k) = \sum_{m=-\infty}^{+\infty} h(m)r_X(k-m) \quad \Rightarrow \quad \sum_{m=-\infty}^{+\infty} h(m)r_X(k+p-m) = s(k+p)$$

and therefore

$$r_Y(k) = \sum_{p=-\infty}^{+\infty} h^*(p)s(k+p) \underset{\substack{\uparrow \\ m=-p}}{=} \sum_{m=-\infty}^{+\infty} h^*(-m)s(k-m) = h^*(-k) * s(k).$$

So finally we have:

$$r_Y(k) = h^*(-k) * h(k) * r_X(k).$$

To compute $S_Y(\omega)$, we apply the definition (Wiener-Khinchin theorem):

$$S_Y(\omega) \stackrel{\text{def}}{=} \mathcal{F}\{r_Y(k)\} = \mathcal{F}\{h^*(-k) * h(k) * r_X(k)\} = \mathcal{F}\{h^*(-k)\}H(\omega)S_X(\omega)$$

$$\mathcal{F}\{h^*(-k)\} = \sum_{k=-\infty}^{+\infty} h^*(-k)e^{-\jmath\omega k} \underset{\substack{\uparrow \\ m=-k}}{=} \sum_{m=-\infty}^{+\infty} h^*(m)e^{\jmath\omega m} = \left(\sum_{m=-\infty}^{+\infty} h(m)e^{-\jmath\omega m}\right)^* = H^*(\omega)$$

Hence:

$$S_Y(\omega) = H^*(\omega)H(\omega)S_X(\omega) = |H(\omega)|^2 S_X(\omega).$$

Similarly:

$$S_Y(z) \stackrel{\text{def}}{=} \mathcal{Z}\{r_Y(k)\} = \mathcal{Z}\{h^*(-k) * h(k) * r_X(k)\} = \mathcal{Z}\{h^*(-k)\}H(z)S_X(z)$$

$$\mathcal{Z}\{h^*(-k)\} = \sum_{k=-\infty}^{+\infty} h^*(-k)z^{-k} \underset{\substack{\uparrow \\ m=-k}}{=} \sum_{m=-\infty}^{+\infty} h^*(m)z^{m} = \left(\sum_{m=-\infty}^{+\infty} h(m)(1/z^*)^{-m}\right)^* = H^*(1/z^*)$$

Hence:

$$S_Y(z) = H^*(1/z^*)H(z)S_X(z).$$

3.2 AR modeling

For the random process described in Problem 2.9:
a) Compute the AR spectral estimate of order 1, assuming that the true ACS values are available.

Solution:
We recall that the process is characterized by the true ACS:

$$r(k) = 2^{-|k|} + a\delta(k) \qquad a \in \mathbb{R}^+$$

The general form of an **AR(p) PSD estimate** is:

$$\hat{S}_{\mathrm{AR}(p)}(\omega) = \frac{\sigma_\varepsilon^2}{\left|1 + \sum_{k=1}^{p} a_k e^{-\jmath\omega k}\right|^2}.$$

To compute parameters $\boldsymbol{\theta}_p \stackrel{\text{def}}{=} [a_1, a_2, \dots, a_p]^\mathrm{T}$ and σ_ε^2, we resort to Yule-Walker equations.

For $p = 1$, the Yule-Walker equations become scalar:

$$\mathbf{R}_1\boldsymbol{\theta}_1 = -\mathbf{r}_1 \qquad \Rightarrow \qquad r(0)a_1 = -r(1) \qquad \Rightarrow \qquad a_1 = -\frac{r(1)}{r(0)}.$$

$$\sigma_\varepsilon^2 = r(0) + \mathbf{r}_1^\mathrm{H}\boldsymbol{\theta}_1 = r(0) + r(1)a_1 \qquad \Rightarrow \qquad \sigma_\varepsilon^2 = r(0) - \frac{r(1)^2}{r(0)} = \frac{r(0)^2 - r(1)^2}{r(0)}.$$

The AR(1) PSD estimate is then given by:

$$\hat{S}_{\mathrm{AR}(1)}(\omega) = \frac{\sigma_\varepsilon^2}{|1 + a_1 \mathrm{e}^{-\jmath\omega}|^2}.$$

But

$$\left|1 + a_1 \mathrm{e}^{-\jmath\omega}\right|^2 = \left(1 + a_1 \mathrm{e}^{-\jmath\omega}\right)\left(1 + a_1 \mathrm{e}^{-\jmath\omega}\right)^* = 1 + a_1^2 + 2a_1 \cos\omega$$

and then

$$\hat{S}_{\mathrm{AR}(1)}(\omega) = \frac{\sigma_\varepsilon^2}{1 + a_1^2 + 2a_1 \cos\omega} = \frac{r(0)[r(0)^2 - r(1)^2]}{r(0)^2 + r(1)^2 - 2r(0)r(1)\cos\omega}.$$

Replacing the true values of the ACS:

$$r(0) = 1 + a \qquad\qquad r(1) = 0.5$$

we finally obtain:

$$\hat{S}_{\mathrm{AR}(1)}(\omega) = \frac{(1+a)[4(1+a)^2 - 1]}{4(1+a)^2 + 1 - 4(1+a)\cos\omega}.$$

We remark that, just like the ACS, the PSD is parameterized by a.

b) Using MATLAB or Python, plot and compare the true PSD, Blackman-Tuckey's estimate obtained in Problem 2.9, and the AR(1) estimate for $a = 0$ and $a = 1$.

Solution:
The true PSD and the BT estimate based on a 5-point rectangular window were computed in [Problem 2.9]:

$$S(\omega) = \frac{(3+5a) - 4a\cos\omega}{5 - 4\cos\omega}$$

$$\hat{S}_{\mathrm{BT}}(\omega) = (1+a) + \cos\omega + \frac{1}{2}\cos(2\omega).$$

For the AR(1), we particularize the general expression found in the previous exercise:

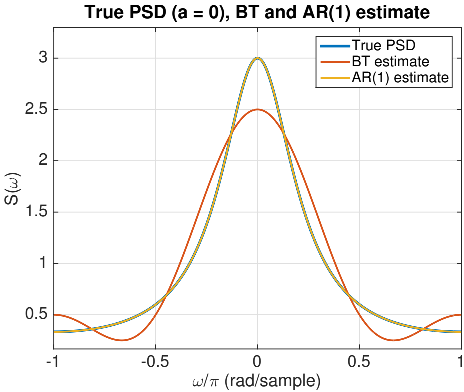
- $a = 0$:

$$\hat{S}_{\mathrm{AR}(1)}(\omega) = \frac{4-1}{4+1-4\cos\omega} = \frac{3}{5-4\cos\omega} = S(\omega).$$

- $a = 1$:

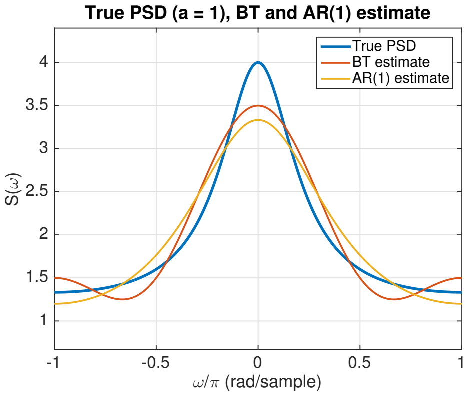
$$\hat{S}_{\mathrm{AR}(1)}(\omega) = \frac{2(16-1)}{16+1-8\cos\omega} = \frac{30}{17-8\cos\omega}.$$

For $a = 0$:



The AR(1) estimate coincides with the true PSD, as the observed signal can be modelled exactly as an AR(1) process.

For $a = 1$



Now AR(1) estimation does not look so accurate.

Q: How to quantify estimation performance?

One way to quantify estimation performance is to compute numerically the **mean squared error (MSE)** of the PSD estimate over the interval $\omega \in [-\pi, \pi]$:

$$\mathrm{MSE}_{\hat{S}(\omega)} = \int_{-\pi}^{\pi} [\hat{S}(\omega) - S(\omega)]^2 d\omega.$$

For $a = 0$, we obtain:

$$\mathrm{MSE}_{\hat{S}_{\mathrm{BT}}(\omega)} = 0.2617$$

$$\mathrm{MSE}_{\hat{S}_{\mathrm{AR}(1)}(\omega)} = 0.$$

For $a = 1$, we have:

$$\mathrm{MSE}_{\hat{S}_{\mathrm{BT}}(\omega)} = 0.2617$$

$$\mathrm{MSE}_{\hat{S}_{\mathrm{AR}(1)}(\omega)} = 0.3588.$$

Hence, while the AR(1) model is exact for $a = 0$, the BT estimate outperforms the AR(1) estimate for $a = 1$, since in that case the observed signal is no longer an AR(1) process.

3.3 AR modeling with varying order

The estimated ACS of a random process for lags $0 \leq k \leq 4$, is:

$$r(0) = 2 \qquad r(1) = 1 \qquad r(2) = 1 \qquad r(3) = 0.5 \qquad r(4) = 0.$$

Compute and plot the following power spectra:

a) True PSD (assuming $r(k) = 0,\ k \geq 4$).

Solution:
We apply the definition of the PSD of WSS random process (Wiener-Khinchin theorem):

$$S(\omega) = \mathcal{F}\{r(k)\} = \sum_{k=-\infty}^{+\infty} r(k)e^{-\jmath\omega k} = r(0) + r(1)\mathrm{e}^{-\jmath\omega} + r(2)\mathrm{e}^{-\jmath 2\omega} + r(3)\mathrm{e}^{-\jmath 3\omega}$$
$$= 2 + \mathrm{e}^{-\jmath\omega} + \mathrm{e}^{-\jmath 2\omega} + 0.5\mathrm{e}^{-\jmath 3\omega}.$$

Q: Do you agree?

No, I don't — we also need to take into account the negative values of k!

$$
\begin{aligned}
S(\omega) = \mathcal{F}\{r(k)\} &= \sum_{k=-\infty}^{+\infty} r(k)e^{-\jmath\omega k} \\
&= r(-3)e^{\jmath 3\omega} + r(-2)e^{\jmath 2\omega} + r(-1)e^{\jmath\omega} + r(0) + r(1)e^{-\jmath\omega} + r(2)e^{-\jmath 2\omega} + r(3)e^{-\jmath 3\omega}
\end{aligned}
$$

Since $r(k)$ has Hermitian symmetry and the process is real-valued: $r(-k) = r^*(k) = r(k)$, and then

$$
\begin{aligned}
S(\omega) &= r(0) + r(1)[e^{\jmath\omega} + e^{-\jmath\omega}] + r(2)[e^{\jmath 2\omega} + e^{-\jmath 2\omega}] + r(3)[e^{\jmath 3\omega} + e^{-\jmath 3\omega}] \\
&= r(0) + 2r(1)\cos\omega + 2r(2)\cos 2\omega + 2r(3)\cos 3\omega.
\end{aligned}
$$

Replacing the values of the ACS, we finally obtain:

$$S(\omega) = 2 + 2\cos\omega + 2\cos 2\omega + \cos 3\omega.$$

b) AR(p) models, with $p = 1, 2, 3$.

Solution:

AR(1) model: The general form of the AR(1) PSD estimate was computed in [Problem 3.2]:

$$\hat{S}_{\mathrm{AR}(1)}(\omega) = \frac{r(0)[r(0)^2 - r(1)^2]}{r(0)^2 + r(1)^2 - 2r(0)r(1)\cos\omega}.$$

In our particular case:

$$r(0) = 2 \qquad\qquad r(1) = 1.$$

Hence:

$$\hat{S}_{\mathrm{AR}(1)}(\omega) = \frac{2(4-1)}{4+1-4\cos\omega} = \frac{6}{5-4\cos\omega}.$$

Alternatively, we could solve the Yule-Walker equations:

$$\mathbf{R}_1\boldsymbol{\theta}_1 = -\mathbf{r}_1 \qquad \Rightarrow \qquad r(0)a_1 = -r(1) \qquad \Rightarrow \qquad a_1 = -\frac{r(1)}{r(0)} = -\frac{1}{2}$$

$$\sigma_\varepsilon^2 = r(0) + \mathbf{r}_1^{\mathrm{H}}\boldsymbol{\theta}_1 = r(0) + r(1)a_1 \qquad \Rightarrow \qquad \sigma_\varepsilon^2 = r(0) - \frac{r(1)^2}{r(0)} = 2 - \frac{1}{2} = \frac{3}{2}$$

yielding

$$\hat{S}_{\mathrm{AR}(1)}(\omega) = \frac{\sigma_\varepsilon^2}{|1 + a_1 \mathrm{e}^{-\jmath\omega}|^2} = \frac{\frac{3}{2}}{\left|1 - \frac{1}{2}\mathrm{e}^{-\jmath\omega}\right|^2}.$$

Also:

$$\left|1 - \frac{1}{2}\mathrm{e}^{-\jmath\omega}\right|^2 = \left(1 - \frac{1}{2}\mathrm{e}^{-\jmath\omega}\right)\left(1 - \frac{1}{2}\mathrm{e}^{-\jmath\omega}\right)^* = 1 + \frac{1}{4} - \cos\omega = \frac{5}{4} - \cos\omega$$

and therefore

$$\hat{S}_{\mathrm{AR}(1)}(\omega) = \frac{\frac{3}{2}}{\frac{5}{4} - \cos(\omega)} = \frac{6}{5 - 4\cos\omega}.$$

AR(2) model: The Yule-Walker equations are given by

$$\mathbf{R}_2\boldsymbol{\theta}_2 = -\mathbf{r}_2 \qquad \sigma_\varepsilon^2 = r(0) + \mathbf{r}_2^{\mathrm{H}}\boldsymbol{\theta}_2$$

with

$$\mathbf{R}_2 \stackrel{\text{def}}{=} \begin{bmatrix} r(0) & r(-1) \\ r(1) & r(0) \end{bmatrix} \qquad \mathbf{r}_2 \stackrel{\text{def}}{=} \begin{bmatrix} r(1) \\ r(2) \end{bmatrix} \qquad \boldsymbol{\theta}_2 \stackrel{\text{def}}{=} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}.$$

First, we need to solve the linear system:

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = -\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \Rightarrow \qquad a_1 = a_2 = -\frac{1}{3}.$$

And

$$\sigma_\varepsilon^2 = r(0) + r(1)a_1 + r(2)a_2 = r(0) + a_1[r(1) + r(2)] = 2 - \frac{2}{3} = \frac{4}{3}.$$

The PDS estimate is given by:

$$\hat{S}_{\mathrm{AR}(2)}(\omega) = \frac{\sigma_\varepsilon^2}{|1 + a_1 \mathrm{e}^{-\jmath\omega} + a_2 \mathrm{e}^{-\jmath 2\omega}|^2} = \frac{\frac{4}{3}}{|1 - \frac{1}{3}\mathrm{e}^{-\jmath\omega} - \frac{1}{3}\mathrm{e}^{-\jmath 2\omega}|^2}.$$

AR(3) model: The Yule-Walker equations are now

$$\mathbf{R}_3\boldsymbol{\theta}_3 = -\mathbf{r}_3 \qquad \sigma_\varepsilon^2 = r(0) + \mathbf{r}_3^{\mathrm{H}}\boldsymbol{\theta}_3$$

with

$$\mathbf{R}_3 \stackrel{\text{def}}{=} \begin{bmatrix} r(0) & r(-1) & r(-2) \\ r(1) & r(0) & r(-1) \\ r(2) & r(1) & r(0) \end{bmatrix} \qquad \mathbf{r}_3 \stackrel{\text{def}}{=} \begin{bmatrix} r(1) \\ r(2) \\ r(3) \end{bmatrix} \qquad \boldsymbol{\theta}_3 \stackrel{\text{def}}{=} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

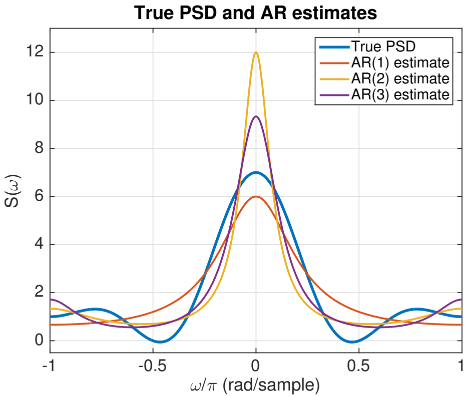
The linear system to be solved is:

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = - \begin{bmatrix} 1 \\ 1 \\ 0.5 \end{bmatrix} \qquad \Rightarrow \qquad a_1 = a_2 = -\frac{3}{8}, \quad a_3 = \frac{1}{8}.$$

$$\sigma_\varepsilon^2 = r(0) + r(1)a_1 + r(2)a_2 + r(3)a_3 = 2 - \frac{6}{8} + \frac{1}{16} = \frac{21}{16}.$$

And the PDS estimate is then:

$$\hat{S}_{\mathrm{AR}(3)}(\omega) = \frac{\sigma_\varepsilon^2}{\left|1 + a_1 \mathrm{e}^{-\jmath\omega} + a_2 \mathrm{e}^{-\jmath 2\omega} + a_3 \mathrm{e}^{-\jmath 3\omega}\right|^2} = \frac{\frac{21}{16}}{\left|1 - \frac{3}{8}\mathrm{e}^{-\jmath\omega} - \frac{3}{8}\mathrm{e}^{-\jmath 2\omega} + \frac{1}{8}\mathrm{e}^{-\jmath 3\omega}\right|^2}.$$



	AR(1)	AR(2)	AR(3)
σ_ε^2	1.5	1.3333	1.3125
$\text{MSE}_{\hat{S}(\omega)}$	4.1853	9.4161	3.7395

Q: Which performance criterion should be used in practice?

3.4 Model order selection

[HAY96, Problem 8.21, p. 483] Show that Akaike's final prediction error (FPE) criterion and Akaike's information criterion (AIC) are asymptotically equivalent, that is, for $N \gg 1$, estimating the order of an AR process by minimizing the FPE criterion is equivalent to minimizing AIC.

Hint: Show that for large N

$$N \log \mathrm{FPE}(p) \approx \mathrm{AIC}(p) + \text{constant}$$

using the fact that, if x is small, then $\log(1+x) \approx x$.

Solution: We recall the definitions of AIC and FPE:

$$\mathrm{AIC}(p) = N \log \sigma_\varepsilon^2(p) + 2p$$

$$\mathrm{FPE}(p) = \sigma_\varepsilon^2(p) \frac{N+p+1}{N-p-1}$$

Now

$$N \log \mathrm{FPE}(p) = N \log \left[\sigma_\varepsilon^2 \frac{N+p+1}{N-p-1}\right] = N \log \left[\sigma_\varepsilon^2 \left(1 + \frac{2p+2}{N-p-1}\right)\right]$$

since

$$\frac{N+p+1}{N-p-1} = 1 + \frac{2p+2}{N-p-1}.$$

Hence

$$N \log \left[\sigma_\varepsilon^2 \left(1 + \frac{2p+2}{N-p-1} \right) \right] = N \log \sigma_\varepsilon^2 + N \log \left(1 + \frac{2p+2}{N-p-1} \right).$$

Because $\log(1+x) \approx x$ when $x \approx 0$, the following approximation is valid when $N \to +\infty$:

$$N \log \left(1 + \frac{2p+2}{N-p-1} \right) \approx N \frac{2p+2}{N-p-1} \approx 2p+2.$$

Therefore

$$N \log \mathrm{FPE}(p) \approx N \log \sigma_\varepsilon^2 + 2p + 2 = \mathrm{AIC}(p) + \text{constant}.$$

Since $\log(\cdot)$ is a monotonically increasing function, it follows that

$$\arg\min_p \mathrm{FPE}(p) = \arg\min_p \mathrm{AIC}(p) \qquad \text{for } N \to \infty$$

and minimizing the FPE criterion is indeed asymptotically equivalent to minimizing the AIC.