ELEC4 - EIEL821

Spectral Analysis

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Introduction

Course organization

- 6 lectures
- 8 tutorial / lab sessions
- Course material sur Moodle: **Analyse spectrale - EIEL821**
- **Questions / answers** forum
- e-mail: vicente.zarzoso@univ-cotedazur.fr

Ressources

- Reference books
  - [HAY96] Hayes, *Statistical Digital Signal Processing and Modeling*, John Wiley, 1996.
  - [OPP89] Oppenheim, Schafer, *Discrete-time Signal Processing*, Prentice-Hall, 1989.
  - [STO05] Stoica, Moses, *Spectral Analysis of Signals*, Prentice-Hall, 2005.
- Course slides
- Tutorial + lab guide
- Solutions

Evaluation

- Written / multiple choice question exam(s): 50%
- Computer lab exam: 50%

Goals and organization

Course objectives

- Recognize the need for spectral estimation as an essential data analysis tool.
- Understand the main approaches to spectral estimation, including
  - motivation
  - performance
  - advantages and limitations.
- Implement, test and apply spectral estimation techniques using Python.

Syllabus

1. Introduction + Random processes (1 lecture)
2. Non-parametric spectral estimation (2 lectures)
3. Parametric spectral estimation (3 lectures)

Spectral analysis (or estimation)

Definition

From a finite record of a stationary data sequence, estimate how the signal power is distributed over frequency.

Useful in many application domains

- **Mechanics**: vibration monitoring, fault detection
- **Astronomy, finance**: hidden periodicity finding
- **Speech and audio processing**: speech recognition, audio compression, music recognition
- **Medicine**: physiological data analysis (electrocardiogram, electroencephalogram, ...)
- **Seismology**: earthquake analysis, focus localization, tremor prediction
- **Control systems**: dynamic behavior analysis, controller synthesis

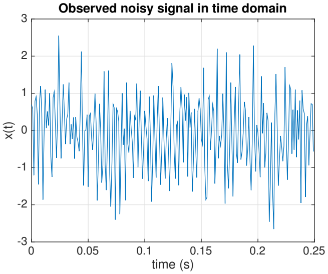
Fundamental tool

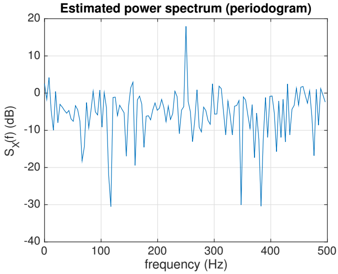
for the electrical engineer and the data scientist.

Spectral analysis — motivating examples (1/2)

Finding harmonic structure

Spectral analysis can often reveal repetitive or periodic components hidden in noisy data.





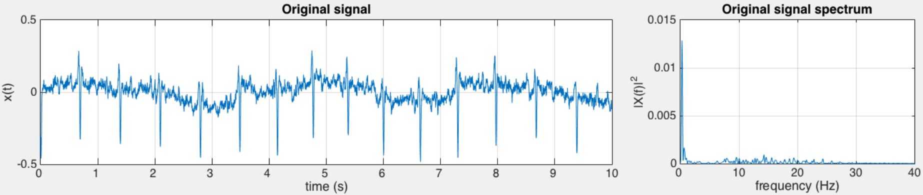
- [Left plot] Noisy data in time domain seem to lack ‘interesting’ components.
- [Right plot] Power spectral density (PSD) reveals periodic component at $f_0 = 250$ Hz.

Spectral analysis — motivating examples (2/2)

Artifact cancellation in biomedical data

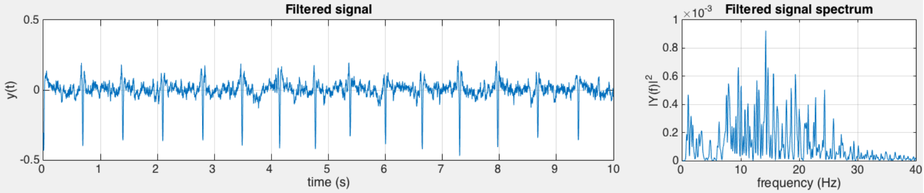
- Electrocardiogram (ECG) records are often corrupted by noise and artifacts.
- Spectral estimation allows the identification of corrupted frequency bands.
- Optimal frequency filters can be designed for artifact cancellation and signal enhancement.

Original signal: ECG record corrupted by baseline wandering and high-frequency noise

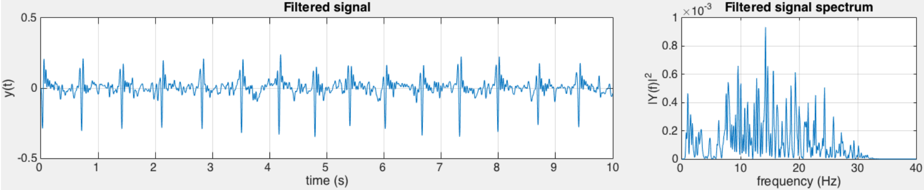


Spectral analysis — motivating examples (2/2, cont'd)

Highpass filtering ($f_c = 0.5$ Hz) $\rightarrow$ baseline wandering removal



Lowpass filtering ($f_c = 30$ Hz) $\rightarrow$ high-frequency noise suppression



Spectral analysis — approaches

Non-parametric approach

- Derived from the basic definitions of power spectral density (PSD).
- *Roughly speaking:* sweeping a narrowband filter over the data.

Parametric approach

- Assumes a parameterized functional form of the PSD.
- *Roughly speaking:* tune the parameters of a filter such that its output to white noise "ressembles" the data.

Can be applied to time and spatial series

- **Time series**
  - Observed data: variation of a physical magnitude as a function of time.
  - Basically what you have studied until now.
- **Spatial series**
  - Observed data: signal impinging on an antenna array.
  - Typical goal: locate direction of arrival of incoming signal(s).
  - Useful for source localization and interference suppression in radar, sonar (incl. underwater), communications, biomedical, seismology.

Chapter 1: Random processes

Definition

Random (or stochastic) processes

A fundamental tool for the electrical engineer and data scientist, including applications in

- **Communication systems**: channel equalization, interference cancellation
- **Computer networks**: trafic modeling and prediction, network optimization
- **Mechanics**: vibration monitoring, fault detection
- ...

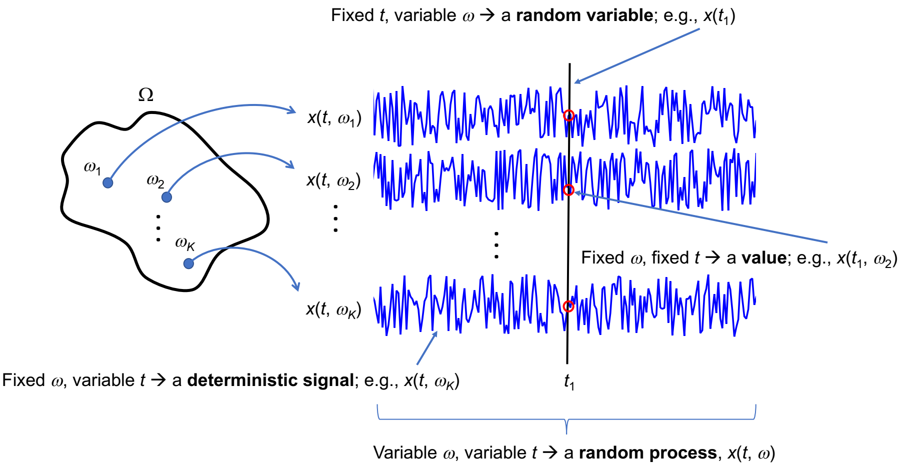
Definition

A **random process** is a function of the elements of a sample space Ω, as well as another independent variable, t, typically a time index. Given an experiment E, with sample space Ω, the random process $X(t)$ maps each possible outcome $\omega \in \Omega$ to a function of t, denoted $x(t, \omega)$.

Examples

Flipping coins, sinusoids, random telegraph processes, Gaussian noise, Gaussian increments, ...

Definition (cont'd)



Example: sinusoids with random initial phase

Let $x(t, \phi) = A\cos(wt + \phi)$ be a random process, where

- $A, w \in \mathbb{R}$ are real-valed constants
- $\phi \equiv U(-\pi, \pi)$ is a random variable uniformly distributed in the interval $[-\pi, \pi[$.

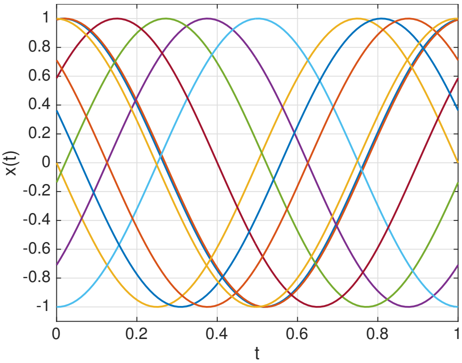


Figure: Ten realizations of the sinusoidal r.p. with uniform initial phase, for $A = 1$ and $w = 2\pi$. Each plot (color) represents $x(t, \phi)$ as a function of time index t for a particular realization of random variable ϕ.

Statistics

Let $X(t)$ be a random process.

- Symbol t can represent continuous ($t \in \mathbb{R}$) or discrete time ($t \in \mathbb{Z}$) index. This will be clear from the context.
- Its probability density function (pdf) is noted $f_X(x;t)$.

Mean, variance, power

$$\mu_X(t) = \mathrm{E}\{X(t)\} = \int_{-\infty}^{+\infty} x\, f_X(x;t)dx$$

$$\sigma_X^2(t) = \mathrm{E}\{[X(t) - \mu_X(t)]^2\} = \int_{-\infty}^{+\infty} [x - \mu_X(t)]^2 f_X(x;t)dx$$

$$P_X(t) = \mathrm{E}\{X(t)^2\} = \int_{-\infty}^{+\infty} x^2 f_X(x;t)dx = \mu_X^2(t) + \sigma_X^2(t)$$

Autocorrelation function

$$R_X(t_1, t_2) = \mathrm{E}\{X(t_1)X(t_2)\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1 x_2 f_X(x_1, x_2; t_1, t_2)dx_1 dx_2$$

Through the change of variable $t_1 = t$ and $t_2 = t + \tau$, we can also write

$$R_X(t, t+\tau) = \mathrm{E}\{X(t)X(t+\tau)\}$$

Statistics (cont'd)

Autocovariance function

$$C_X(t_1, t_2) = \mathrm{Cov}\{X(t_1)X(t_2)\} = \mathrm{E}\{[X(t_1) - \mu_X(t_1)][X(t_2) - \mu_X(t_2)]\}$$

The autocovariance function can be written as

$$C_X(t_1, t_2) = R_X(t_1, t_2) - \mu_X(t_1)\mu_X(t_2)$$

Crosscorrelation function

The *crosscorrelation function* of two random processes $X(t)$ and $Y(t)$ is defined as :

$$R_{XY}(t_1, t_2) = \mathrm{E}\{X(t_1)Y(t_2)\}$$

Crosscovariance function

$$C_{XY}(t_1, t_2) = \mathrm{Cov}\{X(t_1)Y(t_2)\} = \mathrm{E}\{[X(t_1) - \mu_X(t_1)][Y(t_2) - \mu_Y(t_2)]\}$$

It can also be written as

$$C_{XY}(t_1, t_2) = R_{XY}(t_1, t_2) - \mu_X(t_1)\mu_Y(t_2)$$

Stationary random processes

Stationarity: statistical properties of the r.p. are invariant to a shift in time origin.

Strict sense stationary random process (up to order n)

Random process $X(t)$ is strict sense stationary if and only if (iff) for any time shift τ

$$f_X(x_1, \dots, x_n; t_1, \dots, t_n) = f_X(x_1, \dots, x_n; t_1 + \tau, \dots, t_n + \tau)$$

By limiting stationarity to orders $n = 1$ and $n = 2$, we obtain a

Wide sense stationary random process (WSS)

Random process $X(t)$ is wide sense stationary if:

$$\mu_X(t) = \mu_X = \text{constant}$$

and

$$R_X(t, t + \tau) = R_X(\tau) \qquad \text{i.e., it only depends on time lag } \tau.$$

It can easily be checked that the variance and power of a WSS process are independent of t.

Example: harmonic process with random initial phase (cont'd)

Let $x(t) = A\cos(wt+\phi)$ be a continuous-time random process, where

- $A, w \in \mathbb{R}$ are real-valed constants
- $\phi \equiv U(-\pi,\pi)$ is a random variable uniformly distributed in the interval $[-\pi,\pi[$.

Mean

$$\mu_X(t) = \mathrm{E}\{x(t)\} = \int_{-\infty}^{\infty} x f_X(x;t)dx = \mathrm{E}\{A\cos(wt+\phi)\} = \int_{-\infty}^{\infty} A\cos(wt+\phi) f_\Phi(\phi)d\phi$$
$$= \frac{A}{2\pi}\int_{-\pi}^{\pi}\cos(wt+\phi)d\phi = \frac{A}{2\pi}[\sin(wt+\phi)]_{-\pi}^{\pi} = 0.$$

Variance

$$\sigma_X^2(t) = \mathrm{E}\{[x(t)-\mu_X(t)]^2\} \underset{\substack{\uparrow \\ \mu_X(t)=0}}{=} \mathrm{E}\{x(t)^2\} = \int_{-\infty}^{\infty} x^2 f_X(x;t)dx$$
$$= \mathrm{E}\{[A\cos(wt+\phi)]^2\} = \int_{-\infty}^{\infty} A^2\cos^2(wt+\phi) f_\Phi(\phi)d\phi = \frac{A^2}{2\pi}\int_{-\pi}^{\pi}\cos^2(wt+\phi)d\phi$$
$$= \frac{A^2}{4\pi}[\phi+\sin(2wt+2\phi)]_{-\pi}^{\pi} = \frac{A^2}{2}.$$

Example: harmonic process with random initial phase (cont'd)

Autocorrelation function

$$\begin{aligned}
R_X(t_1, t_2) &= \mathrm{E}\{X(t_1)X(t_2)\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1 x_2 f_X(x_1, x_2; t_1, t_2)dx_1 dx_2 \\
&= \mathrm{E}\{A^2\cos(wt_1+\phi)\cos(wt_2+\phi)\} = \int_{-\infty}^{\infty} A^2\cos(wt_1+\phi)\cos(wt_2+\phi)f_\Phi(\phi)d\phi \\
&= \frac{A^2}{4\pi}\int_{-\pi}^{\pi}[\cos(w(t_1-t_2)) + \cos(w(t_1+t_2)+2\phi)]d\phi = \frac{A^2}{2}\cos(w(t_1-t_2)).
\end{aligned}$$

In these calculations we have exploited a

Useful result

If $y = g(x)$ then

$$\mathrm{E}\{y\} = \int_{-\infty}^{\infty} y f_Y(y)dy = \mathrm{E}\{g(x)\} = \int_{-\infty}^{\infty} g(x)f_X(x)dx.$$

Example: harmonic process with random initial phase (cont'd)

Autocorrelation function (cont'd)

$$R_X(t_1, t_2) = \frac{A^2}{2}\cos(w(t_1 - t_2)) \underset{\tau \overset{\text{def}}{=} (t_2 - t_1)}{\underset{\uparrow}{=}} \frac{A^2}{2}\cos(w\tau) = R_X(\tau)$$

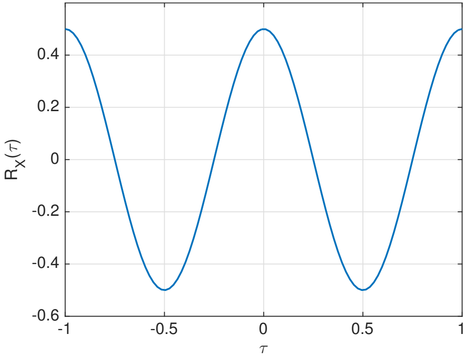


Figure: Autocorrelation function of the sinusoidal r.p. with uniform initial phase, for $A = 1$ and $w = 2\pi$.

Example: harmonic process with random initial phase (cont'd)

We deduce that **this is a WSS process** because:

- $\mu_X(t) = \text{constant}, \forall t.$
- $R_X(t_1, t_2) = \frac{A^2}{2}\cos(w(t_1 - t_2)) = \frac{A^2}{2}\cos(w\tau) = R_X(\tau)$

  Autocorrelation function only depends on the time lag $\tau \stackrel{\text{def}}{=} (t_2 - t_1)$ and not on the actual values of t_1 and t_2.

Other remarks

- $\mu_X(t) = 0 \Rightarrow \quad C_X(t_1, t_2) = R_X(t_1, t_2)$

  Autocorrelation and autocovariance functions coincide for zero-mean processes.
- For any WSS process:
$$C_X(0) = \sigma_X^2.$$
- In this example: $R_X(0) = \frac{A^2}{2} = P_X = \sigma_X^2$, because $\mu_X = 0$.

Ergodicity

In practice, only one realization $x(t)$ of r.p. $X(t)$ is observed or measured.

- Can *only one* realization $x(t)$ represent the random process $X(t)$?
- Are the *ensemble statistics* (over probability space Ω) equal to the *time statistics* (over t)?

If this is the case, random process $X(t)$ is said to be **ergodic**.

Strict sense ergodicity

$$\langle f(x(t))\rangle \stackrel{\text{def}}{=} \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{+T/2} f(x(t))dt = \mathrm{E}\{f(X(t))\}.$$

Necessary condition: $X(t)$ must be strict sense stationary.

Wide sense ergodicity

$$m_X \stackrel{\text{def}}{=} \langle x(t)\rangle = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{+T/2} x(t)dt = \mathrm{E}\{X(t)\} = \mu_X$$

$$\Gamma_X(\tau) \stackrel{\text{def}}{=} \langle x(t)x(t+\tau)\rangle = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{+T/2} x(t)x(t+\tau)dt = \mathrm{E}\{X(t)X(t+\tau)\} = R_X(\tau).$$

Necessary condition: $X(t)$ must be wide sense stationary.

Ergodicity (cont'd)

For a wide sense ergodic random process $X(t)$, we can establish the following links between ensemble and time averages:

- **Mean = DC component**

$$\mu_X = \mathrm{E}\{X(t)\} = \langle x(t) \rangle = m_X$$

- **Square mean = power of the DC component**

$$\mu_X^2 = \mathrm{E}\{X(t)\}^2 = \langle x(t) \rangle^2 = m_X^2 = P_{\mathrm{DC}}$$

- **Variance = power of the AC component**

$$\sigma_X^2 = \mathrm{E}\{[X(t) - \mu_X]^2\} = \mathrm{E}\{X(t)^2\} - \mu_X^2 = \left\langle x(t)^2 \right\rangle - m_X^2 = \left\langle [x(t) - m_X]^2 \right\rangle = P_{\mathrm{AC}}$$

- **Mean square value = total power (DC + AC components)**

$$R_X(0) = \mathrm{E}\{X(t)^2\} = \left\langle x(t)^2 \right\rangle = \Gamma_X(0) = P_X = P_{\mathrm{DC}} + P_{\mathrm{AC}}$$

- **Root mean square (RMS) value**

$$\sigma_X = \sqrt{P_{\mathrm{AC}}}$$

It can easily be checked that the harmonic process with random initial phase is wide sense ergodic.

Power spectral density of a continuous-time process

Let $X(t)$ be a **continuous-time wide sense stationary** random process. We can define:

Wiener-Khinchin Theorem (continuous time)

$$S_X(f) = \mathcal{F}\left[R_X(\tau)\right] = \int_{-\infty}^{+\infty} R_X(\tau) \mathrm{e}^{-\jmath 2\pi f \tau} d\tau \tag{1}$$

$$R_X(\tau) = \mathcal{F}^{-1}\left[S_X(f)\right] = \int_{-\infty}^{+\infty} S_X(f) \mathrm{e}^{\jmath 2\pi f \tau} df. \tag{2}$$

- According to eqn. (2), the power of $X(t)$ (assuming a finite energy signal) is given by

$$P_X = R_X(0) = \int_{-\infty}^{+\infty} S_X(f) df.$$

- $S_X(f)df$: infinitesimal signal power in the frequency band $\left[f - \frac{df}{2}, f + \frac{df}{2}\right]$ (Hz).
- Hence, $S_X(f)$ is the **power spectral density (PSD)** of $X(t)$.
- Units: Watt/Hz.

Power spectral density of a discrete-time process

Let $X(t)$ be a **discrete-time wide sense stationary** random process (random sequence). $R_X(k)$ denotes its **autocorrelation sequence**.

Wiener-Khinchin Theorem (discrete time)

$$S_X(f) = \mathcal{F}[R_X(k)] = \sum_{k=-\infty}^{+\infty} R_X(k) \mathrm{e}^{-\jmath 2\pi fk} \tag{3}$$

$$R_X(k) = \mathcal{F}^{-1}[S_X(f)] = \int_{-1/2}^{1/2} S_X(f) \mathrm{e}^{\jmath 2\pi f\tau} df. \tag{4}$$

- According to eqn. (4), the power of $X(t)$ (assuming a finite energy signal) is given by

$$P_X = R_X(0) = \int_{-1/2}^{1/2} S_X(f) df.$$

- $S_X(f)df$: infinitesimal signal power in the frequency band $\left[f - \frac{df}{2}, f + \frac{df}{2}\right]$ (sample^{-1}).
- Hence, $S_X(f)$ is the **power spectral density (PSD)** of $X(t)$.
- Units: Watt$\times$sample.

Power spectral density of a discrete-time process (cont'd)

The rest of the course will be devoted to solving the

Spectral estimation problem

From a finite length record of a wide sense stationary random sequence $y(t)$

$$\{y(t)\}_{t=0}^{N-1}$$

determine an estimate $\hat{S}_Y(f)$ of its power spectral density $S_Y(f)$.

Two main **approaches**:

- **Non-parametric**: derived from the definitions of PSD [Chapter 2].
- **Parametric**: assumes a parameterized functional form of the PSD [Chapter 3].