ELEC4 - EIEL821

Spectral Analysis — Solutions to exercises
Chapter 2: Non-parametric estimation
Part 2: Improved periodogram-based methods

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2.8 Time-bandwidth product

Prove that the equivalent time-bandwidth product is equal to one for any lag window $w(k)$.

Solution: We recall the definitions given in the lecture:

- **Equivalent time width**

$$N_e = \frac{1}{w(0)} \sum_{k=-(M-1)}^{M-1} w(k)$$

- **Equivalent bandwidth**

$$\beta_e = \frac{1}{2\pi W(0)} \int_{-\pi}^{\pi} W(\omega) d\omega$$

Because $w(k) = 0$ for $|k| \geq M$, we can express:

$$N_e = \frac{1}{w(0)} \sum_{k=-(M-1)}^{M-1} w(k) = \frac{1}{w(0)} \underbrace{\sum_{k=-\infty}^{+\infty} w(k) e^{-\jmath \cdot 0 \cdot k}}_{W(0)} = \frac{W(0)}{w(0)}.$$

$$\beta_e = \frac{1}{2\pi W(0)} \int_{-\pi}^{\pi} W(\omega) d\omega = \frac{1}{W(0)} \underbrace{\frac{1}{2\pi} \int_{-\pi}^{\pi} W(\omega) e^{\jmath \omega \cdot 0} d\omega}_{w(0)} = \frac{w(0)}{W(0)}.$$

And the result immediately follows.

2.9 Blackman-Tuckey's method

A real-valued stochastic process has an autocorrelation sequence given by

$$r(k) = 2^{-|k|} + a\delta(k) \qquad a \in \mathbb{R}^+$$

a) Determine the PSD of the process.

Solution:

By the Wiener-Khinchin theorem: $S(\omega) = \mathcal{F}\{r(k)\} = \mathcal{F}\left\{2^{-|k|} + a\delta(k)\right\}$.

$$\mathcal{F}\{a\delta(k)\} = a.$$

$$\mathcal{F}\left\{2^{-|k|}\right\} = \sum_{k=-\infty}^{+\infty} 2^{-|k|} e^{-\jmath\omega k} = 1 + \underbrace{\sum_{k=1}^{+\infty} 2^{-k} e^{-\jmath\omega k}}_{S_1(\omega)} + \underbrace{\sum_{k=-\infty}^{-1} 2^{k} e^{-\jmath\omega k}}_{S_2(\omega)} = 1 + S_1(\omega) + S_2(\omega).$$

Now, we realize that:

$$S_2(\omega) = \sum_{k=-\infty}^{-1} 2^{k} e^{-\jmath\omega k} \underset{m=-k}{\overset{\uparrow}{=}} \sum_{m=1}^{+\infty} 2^{-m} e^{\jmath\omega m} = S_1^*(\omega)$$

so we only need to compute $S_1(\omega)$.

We identify $S_1(\omega)$ as a geometric series with ratio $\rho = 2^{-1}\mathrm{e}^{-\jmath\omega}$, and notice that it is a convergent series since $|\rho| < 1$. Hence, its sum can readily be computed as:

$$S_1(\omega) = \frac{2^{-1}\mathrm{e}^{-\jmath\omega}}{1 - 2^{-1}\mathrm{e}^{-\jmath\omega}} = \frac{\mathrm{e}^{-\jmath\omega}}{2 - \mathrm{e}^{-\jmath\omega}}.$$

Therefore:

$$\begin{aligned} S_1(\omega) + S_2(\omega) = S_1(\omega) + S_1^*(\omega) &= \frac{\mathrm{e}^{-\jmath\omega}}{2 - \mathrm{e}^{-\jmath\omega}} + \frac{\mathrm{e}^{\jmath\omega}}{2 - \mathrm{e}^{\jmath\omega}} \\ &= \frac{(2\mathrm{e}^{-\jmath\omega} - 1) + (2\mathrm{e}^{\jmath\omega} - 1)}{(2 - \mathrm{e}^{-\jmath\omega})(2 - \mathrm{e}^{\jmath\omega})} = \frac{4\cos(\omega) - 2}{5 - 4\cos(\omega)} \end{aligned}$$

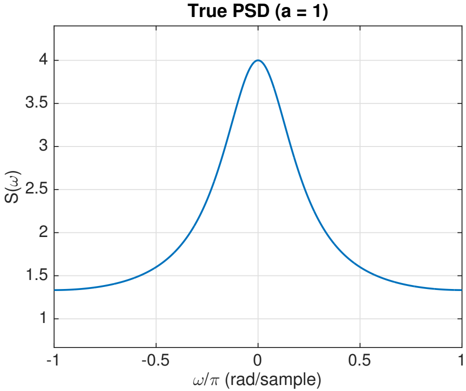
and then

$$\mathcal{F}\left\{2^{-|k|}\right\} = 1 + \frac{4\cos(\omega) - 2}{5 - 4\cos(\omega)} = \frac{3}{5 - 4\cos(\omega)}.$$

Putting it all together:

$$S(\omega) = a + \frac{3}{5 - 4\cos(\omega)} = \frac{(3 + 5a) - 4a\cos(\omega)}{5 - 4\cos(\omega)}.$$

Example: $a = 1$



b) Assuming that the ACS can be estimated exactly, determine the Blackman-Tuckey's spectral estimate with a 5-sample rectangular window.

Solution:

The **Blackman-Tuckey (BT) spectral estimate** is based on the windowed ACS:

$$\hat{S}_{\mathrm{BT}}(\omega) = \mathcal{F}\{w(k)\hat{r}(k)\}.$$

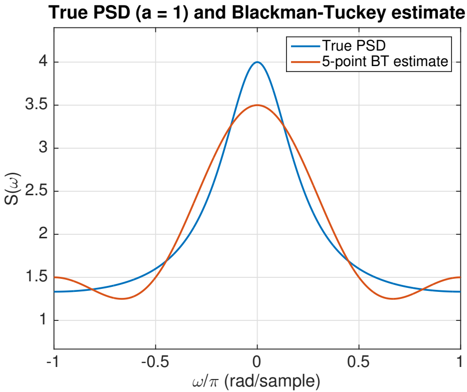
Because we assume that $\hat{r}(k) = r(k)$ and $w(k)$ is a 5-point rectangular window, we have:

$$\hat{S}_{\mathrm{BT}}(\omega) = \mathcal{F}\{w(k)r(k)\} = \sum_{k=-2}^{2} r(k)\mathrm{e}^{-\jmath\omega k} = r(-2)\mathrm{e}^{\jmath 2\omega} + r(-1)\mathrm{e}^{\jmath\omega} + r(0) + r(1)\mathrm{e}^{-\jmath\omega} + r(2)\mathrm{e}^{-\jmath 2\omega}.$$

Now we can exploit the Hermitian symmetry of $r(k)$, i.e., $r(-k) = r^*(k) = r(k)$, yielding:

$$\hat{S}_{\mathrm{BT}}(\omega) = r(0) + 2r(1)cos(\omega) + 2r(2)\cos(2\omega) = (1+a) + \cos(\omega) + \frac{1}{2}\cos(2\omega).$$

Example: $a = 1$


True PSD (a = 1) and Blackman-Tuckey estimate
True PSD
5-point BT estimate
S(ω)
ω/π (rad/sample)
4
3.5
3
2.5
2
1.5
1
-1
-0.5
0
0.5
1

c) What would be the expected value of Blackman-Tuckey's PSD if an unbiased estimate of the ACS was employed? And with the biased ACS estimate?

Solution: The expected value of BT estimate is given by

$$\mathrm{E}\{\hat{S}_{\mathrm{BT}}(\omega)\} = \mathrm{E}\big\{\mathcal{F}\{w(k)\hat{r}(k)\}\big\} = \mathcal{F}\big\{w(k)\mathrm{E}\{\hat{r}(k)\}\big\}.$$

Now, for the **unbiased ACS estimate**, we have $\mathrm{E}\{\hat{r}(k)\} = r(k)$, and therefore

$$\mathrm{E}\{\hat{S}_{\mathrm{BT}}(\omega)\} = \mathcal{F}\{w(k)r(k)\} = W(\omega) * S(\omega)$$

which coincides with the previous exercise (BT estimate computed from true ACS values).

For the **biased ACS estimate** computed from N samples, we have:

$$\mathrm{E}\{\hat{r}(k)\} = w_{\mathrm{B}}(k)r(k) = \left(1 - \frac{|k|}{N}\right) r(k)$$

and hence

$$\mathrm{E}\{\hat{S}_{\mathrm{BT}}(\omega)\} = \mathcal{F}\{w(k)w_{\mathrm{B}}(k)r(k)\} = W(\omega) * \underbrace{W_{\mathrm{B}}(\omega) * S(\omega)}_{\mathrm{E}\{\hat{S}_{\mathrm{P}}(\omega)\}} = W(\omega) * \mathrm{E}\{\hat{S}_{\mathrm{P}}(\omega)\}$$

This is the expected periodogram convolved with the Fourier transform of the rectangular window. This convolution will result in a loss of resolution over the periodogram.

2.10 Bartletts's method

The true PSD of a signal consists of a peak with 3-dB bandwidth of $\Delta f = 0.01$ sample^{-1}. The center frequency and the amplitude of this spectral component are both unknown. We wish to estimate the PSD of this random process using Bartlett's method.

a) Assuming that the number of available samples N is high, determine the window size L to guarantee a negligible bias in the PSD estimate.

Solution:
The expected Bartlett's estimate is given by the expected periodogram of any segment $1 \le j \le K$:

$$\mathrm{E}\{\hat{S}_{\mathrm{B}}(\omega)\} = \mathrm{E}\{\hat{S}_j(\omega)\} = W_{\mathrm{B}}(\omega) * S(\omega).$$

For segments composed of L samples, $W_{\mathrm{B}}(\omega)$ is the Bartlett (triangular) window of length $(2L-1)$ samples.

Its main lobe half-power bandwidth is $\Delta f_{3\mathrm{dB}} \approx 1/L$ sample^{-1}.

Hence, to guarantee sufficient resolution for the given signal, we should make sure that

$$\Delta f_{3\mathrm{dB}} \le \Delta f \quad \Rightarrow \quad \frac{1}{L} \le 0.01 \quad \Rightarrow \quad L \ge 100 \text{ samples.}$$

b) Explain why it would not be advantageous to increase L beyond the value found in a).

Solution:

By averaging K segments, the variance of Bartlett's estimate is reduced by a factor of K over that of the periodogram of each size-L segment:

$$\text{var}\{\hat{S}_\text{B}(\omega)\} \approx \frac{1}{K}\text{var}\{\text{E}\{\hat{S}_j(\omega)\}\}.$$

Since $K \cdot L \approx N$ and the number of available samples N is assumed to be constant, increasing the segment length L will result in a reduced number of segments K, and thus in an increased variance.

Therefore, the improved resolution may not compensate for the increased variance.

In Bartlett's method, the segment length L must balance a difficult tradeoff, as it should be

- high enough to ensure good resolution;
- low enough to ensure satisfactory variance reduction.

The segment overlap introduced in Welch's method can alleviate this tradeoff.

2.11 Welch's estimate

Using Welch's method, we wish to estimate the PSD of a continuous-time signal with bandwidth $B = 10$ kHz, sampled at $f_s = 20$ kHz during $T = 10$ s.

a) Compute the number of samples N available for spectral estimation.

Solution: Sampling at a rate of f_s samples per second over T seconds produces

$$N = f_s \cdot T = (2 \times 10^4) \cdot 10 = 2 \times 10^5 \text{ samples.}$$

b) We use the radix-2 fast Fourier transform (FFT) to approximate the DTFT. Compute the number of points of the FFT, $\tilde{N}$, to guarantee frequency samples spaced at most of 10 Hz.
Hint: The radix-2 FFT requires $\tilde{N} = 2^n$ points, with $n \in \mathbb{Z}^+$.

Solution: The FFT computes $\tilde{N}$ equally spaced points in the inverval $[0, 1]$ sample^{-1}, which corresponds to the interval $[0, f_s]$ Hz. Hence, two consecutive points will be spaced by:

$$\Delta f = \frac{f_s}{\tilde{N}} \leq 10 \text{ Hz} \qquad \Rightarrow \qquad \tilde{N} \geq \frac{f_s}{10} = 2 \times 10^3 \text{ points.}$$

The closest power of two fulfilling this constraint will be

$$\tilde{N}_{\min} = 2^{\lceil \log_2(2 \cdot 10^3) \rceil} = 2048 \text{ points.}$$

c) If the segment length L equals the number of FFT points $\tilde{N}$ computed above (i.e., no zero padding), how many segments K can be used for averaging without overlap?

Solution: We have $L = \tilde{N}_{\min} = 2048$, as computed in exercise b).

To have K nonoverlapping segments with signal samples, we must fulfil

$$K \cdot L \leq N \qquad \Rightarrow \qquad K \leq \frac{N}{L} \qquad \Rightarrow \qquad K_{\max} = \left\lfloor \frac{2 \times 10^5}{2048} \right\rfloor = 97 \text{ segments.}$$

d) We wish to reduce the estimator's variance by a factor of 10 while keeping the frequency sampling computed in exercise b). Propose two methods to do so. Explain their advantages and limitations.

Solution: We know that averaging achieves a variance reduction by a factor of K relative to the periodogram of any segment computed separately:

$$\mathrm{var}\{\hat{S}_{\mathrm{W}}(\omega)\} \approx \frac{1}{K}\mathrm{var}\{\mathrm{E}\{\hat{S}_j(\omega)\}.$$

Hence, to reduce the estimator's variance by a factor of 10 over the previous case, we must increase K by the same factor:

$$K' = 10 \cdot K_{\max} = 970 \text{ segments.}$$

There are mainly two methods to achieve this increase in the number of segments.

Method 1: nonoverlapping segments
The new segment length L' must decrease so as to ensure:

$$K' \cdot L' \leq N \qquad \Rightarrow \qquad L' = \left\lfloor \frac{N}{K'} \right\rfloor = \left\lfloor \frac{2 \times 10^5}{970} \right\rfloor = 206 \approx \frac{L}{10} \text{ samples/segment.}$$

- *Advantage:* variance reduction is optimal, as segments do not overlap.
- *Disadvantage:* significant loss in resolution, due to shorter segments.

Method 2: overlapping segments
To avoid losing resolution, we keep the segment length to $L = 2048$ samples but must allow certain overlap between consecutive segments to guarantee variance reduction.

We obtain each new segment by a shift of D samples, with $D < L$. To obtain K' segments:

$$(K' - 1) \cdot D + L \leq N \qquad \Rightarrow \qquad D \leq \frac{N - L}{K' - 1} = 204.28 \qquad \Rightarrow \qquad D = 204 \text{ samples.}$$

This yields an overlap of $\Delta = (L - D)/L = 0.9$, which is rather high.

- *Advantage:* no loss in resolution, as segments keep the original length.
- *Disadvantage:* variance reduction not actually by a factor of 10, due to the strong correlation between consecutive segments.

Q: How to improve variance reduction in the overlapping case?

Would there be any side effects?