ELEC4 - EIEL821

Spectral Analysis — Solutions to exercises
Chapter 2: Non-parametric methods
Part 1: Periodogram

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2.1 PSD definitions

Prove that the two definitions of PSD given at the beginning of Chapter 2 are equivalent.

Hint: Consider the following alternative definition for the autocorrelation sequence (ACS):

$$r(k) = \lim_{N\to\infty} \mathrm{E}\left\{ \frac{1}{N} \sum_{n=0}^{N-1} y(n)y^*(n-k) \right\}.$$

Replace this expression in the first definition of PSD and develop to reach the second definition. To simplify the development, use the windowed sequence defined as:

$$y_N(n) = \begin{cases} y(n) & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

and extend the summations from $-\infty$ to ∞.

Solution:

We recall the two definitions of the PSD given in [Chapter 2]:

- 1st definition — from the Wiener-Khinchin theorem:

$$S(\omega) = \mathcal{F}\{r(k)\} = \sum_{k=-\infty}^{+\infty} r(k) \mathrm{e}^{-\jmath\omega k}$$

- 2nd definition — based on ergodicity:

$$S(\omega) = \lim_{N\to\infty} \mathrm{E}\left\{ \frac{1}{N} \left| \sum_{n=0}^{N-1} y(n) e^{-\jmath\omega n} \right|^2 \right\}.$$

We replace $r(k)$ in the first definition by the equivalent expression given in the hint:

$$\begin{aligned} S(\omega) &= \sum_{k=-\infty}^{+\infty} \left[\lim_{N\to\infty} \mathrm{E}\left\{ \frac{1}{N} \sum_{n=0}^{N-1} y(n) y^*(n-k) \right\} \right] \mathrm{e}^{-\jmath\omega k} \\ &= \sum_{k=-\infty}^{+\infty} \left[\lim_{N\to\infty} \mathrm{E}\left\{ \frac{1}{N} \sum_{n=-\infty}^{\infty} y_N(n) y_N^*(n-k) \right\} \right] \mathrm{e}^{-\jmath\omega k}. \end{aligned}$$

The bounds of n can be extended because $y_N(n) = y(n)$ for $0 \leq n \leq (N-1)$ and zero elsewhere.

Now, we rearrange the order of the summation, limit and expectation operators:

$$\sum_{k=-\infty}^{+\infty}\left[\lim_{N\to\infty}\mathrm{E}\left\{\frac{1}{N}\sum_{n=-\infty}^{\infty}y_N(n)y_N^*(n-k)\right\}\right]\mathrm{e}^{-\jmath\omega k}$$

$$=\lim_{N\to\infty}\mathrm{E}\left\{\frac{1}{N}\sum_{n=-\infty}^{+\infty}\sum_{k=-\infty}^{\infty}y_N(n)y_N^*(n-k)\mathrm{e}^{-\jmath\omega k}\right\}$$

$$=\lim_{N\to\infty}\mathrm{E}\left\{\frac{1}{N}\sum_{n=-\infty}^{+\infty}y_N(n)\sum_{k=-\infty}^{\infty}y_N^*(n-k)\mathrm{e}^{-\jmath\omega k}\right\}.$$

We can compute the inner summation over k by making a change of variable:

$$\sum_{k=-\infty}^{\infty}y_N^*(n-k)\mathrm{e}^{-\jmath\omega k}\underset{m\overset{\text{def}}{=}(n-k)}{\underset{\uparrow}{=}}\sum_{m=-\infty}^{\infty}y_N^*(m)\mathrm{e}^{-\jmath\omega(n-m)}=\mathrm{e}^{-\jmath\omega n}\sum_{m=-\infty}^{\infty}y_N^*(m)\mathrm{e}^{\jmath\omega m}=\mathrm{e}^{-\jmath\omega n}Y_N^*(\omega).$$

Hence:

$$\sum_{n=-\infty}^{+\infty}y_N(n)\sum_{k=-\infty}^{\infty}y_N^*(n-k)\mathrm{e}^{-\jmath\omega k}=\sum_{n=-\infty}^{+\infty}y_N(n)\mathrm{e}^{-\jmath\omega n}Y_N^*(\omega)=Y_N(\omega)Y_N^*(\omega)=|Y_N(\omega)|^2.$$

In summary:

$$S(\omega) = \lim_{N\to\infty} \mathrm{E}\left\{\frac{1}{N}|Y_N(\omega)|^2\right\}.$$

The equivalence is finally established by noting that

$$Y_N(\omega) = \sum_{n=0}^{N-1} y(n)\mathrm{e}^{-\jmath\omega n}$$

and then

$$S(\omega) = \lim_{N\to\infty} \mathrm{E}\left\{\frac{1}{N}\left|\sum_{n=0}^{N-1} y(n)e^{-\jmath\omega n}\right|^2\right\}$$

which is indeed the second definition of the PSD.

2.2 ACS definitions

Prove that the two definitions given in Chapter 2 correspond indeed to a biased and an unbiased estimator of the ACS, respectively.

Solution:

The first ACS estimator is given by:

$$\hat{r}(k) = \frac{1}{N-k} \sum_{n=k}^{N-1} y(n) y^*(n-k), \qquad 0 \le k \le N-1.$$

To check for unbiasedness, we need to compute $\mathrm{E}\{\hat{r}(k)\}$. By linearity:

$$\mathrm{E}\{\hat{r}(k)\} = \frac{1}{N-k} \sum_{n=k}^{N-1} \mathrm{E}\{y(n) y^*(n-k)\} = \frac{1}{N-k} \sum_{n=k}^{N-1} r(k) = \frac{N-k}{N-k} r(k) = r(k).$$

Because $\mathrm{E}\{\hat{r}(k)\} = r(k)$, this definition of $\hat{r}(k)$ is indeed an **unbiased** estimator of the ACS.

The second ACS estimator is given by:

$$\hat{r}(k) = \frac{1}{N}\sum_{n=k}^{N-1} y(n)y^*(n-k), \qquad 0 \leq k \leq N-1.$$

Again, we compute $\mathrm{E}\{\hat{r}(k)\}$:

$$\mathrm{E}\{\hat{r}(k)\} = \frac{1}{N}\sum_{n=k}^{N-1}\mathrm{E}\{y(n)y^*(n-k)\} = \frac{1}{N}\sum_{n=k}^{N-1} r(k) = \frac{N-k}{N} r(k) = \left(1 - \frac{k}{N}\right) r(k).$$

Because $\mathrm{E}\{\hat{r}(k)\} \neq r(k)$, this definition of $\hat{r}(k)$ is a **biased** estimator of the ACS.

The biased estimator remains interesting because it presents lower variance than the unbiased one for $|k| > 0$. It gives rise to the periodogram, as proved in the next exercise.

2.3 Link between periodogram and correlogram

Prove that, if the biased estimate of the ACS is used, the correlogram and the periodogram are equivalent.

Hint: In the expression of the correlogram, express the biased ACS estimate in terms of the windowed sequence $y_N(n)$ defined above. Develop to reach the expression of the periodogram. The convolution theorem can also be exploited to simplify the derivations.

Solution: We start by recalling the correlogram:

$$\hat{S}_C(\omega) = \sum_{k=-(N-1)}^{N-1} \hat{r}(k) \mathrm{e}^{-\jmath\omega k}.$$

Let us consider the biased ACS estimate, which, for $0 \leq k \leq (N-1)$, can be expressed as:

$$\hat{r}(k) = \frac{1}{N} \sum_{n=k}^{N-1} y(n) y^*(n-k) = \frac{1}{N} \sum_{n=-\infty}^{+\infty} y_N(n) y_N^*(n-k).$$

Thanks to the definition of $y_N(n)$, this latter expression represents $\hat{r}(k)$ for any $k \in \mathbb{Z}$.

Hence:

$$\hat{S}_C(\omega) = \sum_{k=-(N-1)}^{N-1} \hat{r}(k) e^{-\jmath\omega k} = \sum_{k=-(N-1)}^{N-1} \left\{ \frac{1}{N} \sum_{n=-\infty}^{+\infty} y_N(n) y_N^*(n-k) \right\} e^{-\jmath\omega k}.$$

Because $\hat{r}(k) = 0$ for $|k| \geq N$, the summation over k can be extended to infinity. Also, the order of summations can be rearranged as before:

$$\sum_{k=-(N-1)}^{N-1} \left\{ \frac{1}{N} \sum_{n=-\infty}^{+\infty} y_N(n) y_N^*(n-k) \right\} e^{-\jmath\omega k} = \frac{1}{N} \sum_{n=-\infty}^{+\infty} y_N(n) \sum_{k=-\infty}^{+\infty} y_N^*(n-k) e^{-\jmath\omega k}.$$

This double sum was computed in [Problem 2.1], yielding

$$\hat{S}_C(\omega) = \frac{1}{N} |Y_N(\omega)|^2$$

which indeed corresponds to the periodogram, as detailed in the next exercise.

2.4 Periodogram

Prove that the periodogram of random process $Y(n)$ can simply be computed as

$$\hat{S}_{\mathrm{P}}(\omega) = \frac{1}{N} |Y_N(\omega)|^2$$

where $Y_N(\omega) = \mathcal{F}\{y_N(n)\}$ is the discrete-time Fourier transform (DTFT) of the sequence $y_N(n) = w_{\mathrm{R}}(n)y(n)$, with $w_{\mathrm{R}}(n)$ denoting the length-N rectangular window.

Solution: Recall the definition of the periodogram:

$$\hat{S}_{\mathrm{P}}(\omega) = \frac{1}{N} \left| \sum_{n=0}^{N-1} y(n) \mathrm{e}^{-\jmath \omega n} \right|^2 .$$

As proposed in the hint, we define the windowed sequence

$$y_N(n) = w_{\mathrm{R}}(n)y(n) = \begin{cases} y(n) & 0 \leq n \leq N-1 \\ 0 & \text{elsewhere.} \end{cases}$$

Then:

$$\sum_{n=0}^{N-1} y(n) \mathrm{e}^{-\jmath \omega n} = \sum_{n=-\infty}^{+\infty} y_N(n) \mathrm{e}^{-\jmath \omega n} = Y_N(\omega)$$

and the result immediately follows.

Remark

In general

$$Y(\omega) \neq Y_N(\omega)$$

but

$$Y(\omega) = \lim_{N\to\infty} Y_N(\omega)$$

Indeed, the computation of $Y(\omega)$ would require all samples of realization $y(n)$:

$$Y(\omega) = \mathcal{F}\{y(n)\} = \sum_{n=-\infty}^{+\infty} y(n)\mathrm{e}^{-\jmath\omega n}$$

but in practice only $\{y(n)\}_{n=0}^{N-1}$ are observed. These N samples generate $Y_N(\omega)$:

$$Y_N(\omega) = \mathcal{F}\{y_N(n)\} = \sum_{n=-\infty}^{+\infty} y_N(n)\mathrm{e}^{-\jmath\omega n} = \sum_{n=-\infty}^{+\infty} w_{\mathrm{R}}(n)y(n)\mathrm{e}^{-\jmath\omega n} = \sum_{n=0}^{N-1} y(n)\mathrm{e}^{-\jmath\omega n}.$$

Since $y_N(n) = w_{\mathrm{R}}(n)y(n)$ then by the convolution theorem:

$$Y_N(\omega) = Y(\omega) * W_{\mathrm{R}}(\omega).$$

2.5 Periodogram resolution

a) [HAY96, Example 8.2.3, p. 403] Let $X(n)$ be a discrete-time random process consisting of two equal amplitude sinusoids in zero-mean unit-variance white noise:

$$X(n) = A\sin(\omega_1 n + \phi_1) + A\sin(\omega_2 n + \phi_2) + \nu(n)$$

where $\omega_1 = 0.4\pi$ rad/sample, $\omega_2 = 0.45\pi$ rad/sample and $A = 5$. Determine the minimum required record length to resolve the two narrowband components using the periodogram.

Solution:
The periodogram cannot resolve two sinusoids that are separated less than

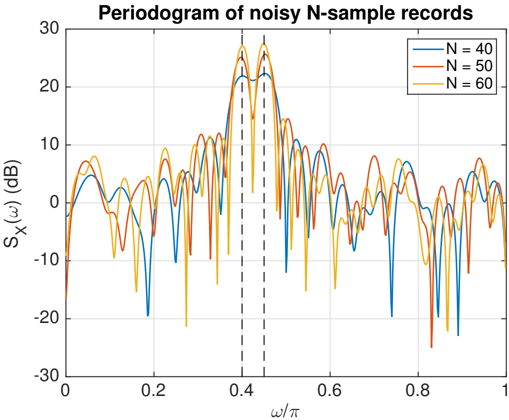
$$\Delta\omega_{\min} \approx \frac{2\pi}{N} \text{ rad/sample.}$$

This resolution limit is due to the **spectral smearing** or **smoothing** phenomenon caused by the main lobe of $|W_R(\omega)|^2$.

Hence, for our two sinusoids to be resolved, we must have

$$|\omega_1 - \omega_2| \geq \Delta\omega_{\min} \approx \frac{2\pi}{N} \text{ rad/sample} \qquad \Rightarrow \qquad N \geq \frac{2\pi}{|\omega_1 - \omega_2|} = \frac{2\pi}{0.05\pi} = 40 \text{ samples.}$$

Example



For particular realizations of the process, the spectral peaks can be distinguished with just $N = 40$ samples, but longer record lengths will generally be required to guarantee sufficient resolution.

This limitation is caused by the **spectral smoothing (or smearing)** phenomenon.

b) Assume that the two sinusoids are sufficiently separated in frequency relative to the available sample size. Determine the minimum amplitude of the weakest sinusoid to guarantee that it would be visible in the periodogram even in the absence of noise.

Solution:
If the two sinusoids are well separated in frequency, the worst case scenario occurs when the main spectral lobe of the weakest sinusoid is hidden by the principal sidelobe of the strongest sinusoid.

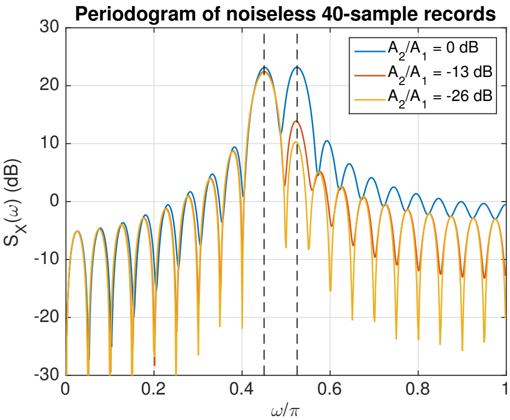
This overlap is due to the **power leakage** phenomenon caused by the sidelobes of $|W_\mathrm{R}(\omega)|^2$.

For the periodogram, the secondary lobe (principal sidelobe) of $|W_\mathrm{R}(\omega)|^2$ lies 26 dB below the main lobe level.

Hence, for the weakest sinusoid to be visible we must have:

$$P_2\ (\mathrm{dB}) \geq P_1\ (\mathrm{dB}) - 26\ \mathrm{dB} \quad \Rightarrow \quad A_2^2 \geq A_1^2 \times 10^{-26/10} \quad \Rightarrow \quad A_2 \geq 5 \times 10^{-26/20} = 0.25.$$

Example



If the weakest sinusoid falls below certain power level (23 dB for the periodogram), its spectral peak cannot be distinguished from the principal sidelobe of the strongest sinusoid.

This limitation is caused by the **power leakage** phenomenon.

Q: What is the influence of the sample size N on spectral smoothing and power leakage?

Can both phenomena be alleviated by increasing N?

2.6 Periodogram of white noise

[HAY96, Example 8.2.4, p. 405] Let $X(n)$ be zero-mean unit-variance white noise. Determine:
a) Its autocorrelation sequence and its power spectral density.

Solution: (see also [Problem 1.8])
Recall the definition of the ACS:

$$r(k) = \mathrm{E}\{x(n)x^*(n-k)\}.$$

For $k = 0$: $r(0) = \mathrm{E}\{|x(n)|^2\} = \sigma_X^2 + \mu_X^2 = 1$, due to the zero-mean unit-variance assumption.

For $k \neq 0$: $r(k) = \mathrm{E}\{x(n)\}\mathrm{E}\{x^*(n-k)\} = 0$, due to the whiteness and zero-mean assumptions.

Hence:

$$r(k) = \delta(k).$$

And then:

$$S(\omega) = \mathcal{F}\{r(k)\} = \sum_{k=-\infty}^{+\infty} r(k)\mathrm{e}^{-\jmath\omega k} = \sum_{k=-\infty}^{+\infty} \delta(k)\mathrm{e}^{-\jmath\omega k} = 1 \cdot \mathrm{e}^{-\jmath\omega \cdot 0} = 1.$$

b) The expected value of the periodogram.

Solution:

$$\mathrm{E}\{\hat{S}_{\mathrm{P}}(\omega)\} = S(\omega) * W_{\mathrm{B}}(\omega) \underset{\substack{\uparrow \\ * \text{ is commutative}}}{=} W_{\mathrm{B}}(\omega) * S(\omega) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} W_{\mathrm{B}}(\xi) S(\omega - \xi) d\xi$$

$$\underset{\substack{\uparrow \\ S(\omega)=1}}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} W_{\mathrm{B}}(\xi) d\xi = w_{\mathrm{B}}(0) = 1.$$

We remark that in this special case the periodogram is not only asymptotically unbiased, but also unbiased for any sample size N.

In general, the periodogram is a biased but asymptotically unbiased estimator of the PSD.

c) The variance of the periodogram. Can the variance be improved by increasing the record length?

Solution:
For N sufficiently large, we have:

$$\mathrm{var}\{\hat{S}_{\mathrm{P}}(\omega)\} \approx S^2(\omega).$$

In our case, since $S(\omega) = 1$:

$$\mathrm{var}\{\hat{S}_{\mathrm{P}}(\omega)\} \approx 1.$$

The variance coincides with the expected PSD, regardless of the sample size.

Hence, **the variance cannot be improved by increasing the record length**.

The periodogram is an inconsistent estimator of the PSD.