Digital Filters

Luc Deneire – deneire@unice.fr

Polytech Nice Sophia

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Introduction

a Digital Filter is :

- An algorithm ...
- who transforms a digital signal (discrete time - discrete values) to another signal
  - Noise suppression
  - Enhancement (music - graphic equalizer) (images ...)
  - Limit bandwith (telecoms)
  - ...

what we will do (and not do - This is a short course)

- Limit ourselves to Linear Time-Invariant (LTI) digital filters
- Learn to design Filters from Specifications (Templates)
- Learn to cope with different Sampling Frequencies (Multi-Rate)
- Learn to cope with Fixed Point filters

Course Schedule and Syllabus

In total, there are 6 lectures and 9 exercice sessions

Reminder
- LTI systems - Constant Coefficient Difference Equations - Z-transform - Pole-Zero and Frequency representation
- 1 lecture - 2 exercice sessions

FIR design
- Ideal Filters, Group and Phase delay - Ideal filter typology - windowing methods
- 1 lecture - 1 exercice session (with Python) (graded)

IIR design
- Specification and Design
- 1 lecture - 1 exercice session (with Python) (graded)

Multi-rate
- Sampling frequency changes - Multistage Filters - Polyphase Filters
- 2 lectures - 3 exercice sessions (Python) - 1 HW

Fixed-point
- Influence of Fixed-Point inputs and Fixed-Point Filter Taps (coefficients) - Basic design
- 1 lecture - 2 exercice sessions (with Python) - 1 HW

Informations

- Course to be found on Moodle (on your ent.unice.fr).
- Questions on Moodle (backup on deneire@unice.fr). Every student will have to ask at least one relevant question. Every student will have to answer at least one well documented answer !

Grades

HW1	HW2	Questions	Final Exam
1	1	1	3

Representation of Linear Time-Invariant (LTI) Filters

Linear Time Invariant Filters

- Linear : output is a linear combination of the input.
- Time Invariant : roughly – the output can be written as a CCDE : Constant Coefficients Difference Equation.

Rigorously : let's write $y[n] = \mathcal{H}\{x[n]\}$ the response of $x[n]$ to a LTI system :

- $y[n_1] = \mathcal{H}\{x[n_1]\}$ and $y[n_2] = \mathcal{H}\{x[n_2]\} \Rightarrow a.y[n_1] + b.y[n_2] = \mathcal{H}\{a.x[n_1] + b.x[n_2]\}$
- $y[n] = \mathcal{H}\{x[n]\} \Rightarrow y[n-k] = \mathcal{H}\{x[n-k]\}$

Impulse response

Let $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$, $y[n] = \mathcal{H}\{x[n]\} = \mathcal{H}\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\} =$
$\sum_{k=-\infty}^{\infty} \mathcal{H}\{x[k]\delta[n-k]\} = \sum_{k=-\infty}^{\infty} x[k]\mathcal{H}\{\delta[n-k]\} = \sum_{k=-\infty}^{\infty} x[k]h_k[n]$ where
$h_k[n] = \mathcal{H}\{\delta[n-k]\}$ is the impulse response at $n = k$.

Convolution and difference equation

Convolution

Let $h[n] = \mathcal{H}\{\delta[n]\}$ then $\mathcal{H}\{\delta[n-k]\} = h[n-k]$ and

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Which is the convolution product between $x[n]$ and $h[n]$.

Constant Coefficients Difference Equation

A system linking $y[n]$ to $x[n]$ can be described by it's difference equation :

$$\sum_{i=0}^{N} a_i y[n-i] = \sum_{l=0}^{M} b_l x[n-l]$$

.

The system is called recursive, unless if $a_i = 0$ for all $i \neq 0$.
Furthermore, given a_i and b_l, and considering causal signals, it is quite easy to determine the impulse response of the system (just feed it with an impulse).

Going to the frequency domain : The z-Transform

The z-Transform

The z-transform of $x[n]$ is $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$

where z is complex. In the following we assume $X(z)$ converges on the unit circle (for $r_1 < |z| < r_2$, with $r_1 < 1$ and $r_2 > 1$).
Moreover, $X(z) = 1$ if $x[n] = \delta[n]$, and $X(z) = z^{-1}$ if $x[n] = \delta(n-1)$.
Hence z^{-1} is a **Delay operator**. Hence if $Y(z) = X(z)z^{-k}$, $y[n] = x[n-k]$.

Filter in the z-domain

Filter in the z-domain

From $\sum_{i=0}^{N} a_i y[n-i] = \sum_{l=0}^{M} b_l x[n-l]$, we have

$$\sum_{i=0}^{N} a_i Y(z) z^{-i} = \sum_{l=0}^{M} b_l X(z) z^{-l} \Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{l=0}^{M} b_l z^{-l}}{\sum_{i=0}^{N} a_i z^{-i}}$$

$H(z)$ is called the *transfer function* of the LTI filter :

- If $H(z)$ is realizable filter the $H(z)$ is a *rational transfer function*
- $H(z)$ evaluated on the unit circle is the frequency response (see later)
- $H(z)$ is the z-transform of the filter's impulse response (indeed, $\mathcal{Z}\{\delta[n]\} = 1$)
- if $y[n] = x[n] * h[n] \Rightarrow \mathcal{Z}\{y[n]\} = Y(z) = H(z)X(z)$

Region of Convergence

The z-transform is a series that has to converge to exist !
Let's split the sum in two parts, the "anti-causal" and "causal" :

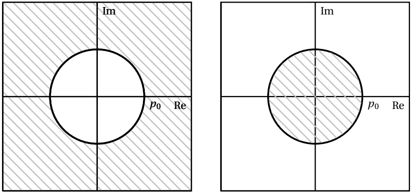
$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{-1} x[n]z^{-n} + \sum_{n=0}^{\infty} x[n]z^{-n} \\ &= \sum_{n=1}^{\infty} x[n]z^{n} + \sum_{n=0}^{\infty} \frac{x[n]}{z^n} \\ &= X_a(z) + X_c(z) \end{aligned}$$

Both $X_a(z), X_c(z)$ have to converge (and convergence for power series is equivalent to absolute convergence), hence :

$$z \in \text{ROC}\{X(z)\} \Leftrightarrow \sum_{n=-\infty}^{\infty} |x[n]z^{-n}| < \infty$$

Properties of the Region of Convergence (ROC)

- **The ROC has circular symmetry** (only depends on $|z|$)
- **If $X(z)$ of finite support** $\Rightarrow$ ROC = all complex plane
- **The ROC of a causal sequence extends to infinity**
  Indeed, suppose $X(z)$ converges for z_o, then, for z_1 such that' $|z_1| > |z_o|$, $|\frac{x[n]}{z_1^n}| \leq |\frac{x[n]}{z_o^n}|$.
- **The ROC of an anti-causal sequence extends to 0**
  Indeed, suppose $X(z)$ converges for z_o, then, for z_1 such that' $|z_1| < |z_o|$, $|x[n]z_1^n| \leq |x[n]z_o^n|$.



(a) (b)

ROC in hatched area for (a) causal sequence, (b) anticausal sequence

Stability and ROC

A system is said BIBO (bounded input, bounded output) stable if for any bounded input, the output is bounded.
In the time domain, if $x[n] < M$:

$$|y[n] \; = |h[n] * x[n]| = |\sum_k h[k]x[n-k]| \leq \sum_k |h[k]x[n-k]| \leq L \sum_k |h[k]|$$

Hence, BIBO stability is equivalent to have $h[n]$ to be absolutely summable,

$$\sum_k |h[k]| = \sum_k |h[k]z^{-k}|_{|z|=1}$$

A filter/system is stable if $|z| = 1$ is in the ROC

A system is BIBO stable if the ROC of $H(z)$ includes the unit circle .

In particular, for a causal system (which is usually the case, up to z^{-N} to account for some delay), the ROC must extend from $|z_o| < 1$ to infinity. We will come back to it when plotting poles and zeros.

Finding the frequency response of a filter

Replace z by $e^{j\omega}$

The frequency representation of a signal $x[n]$ is given by

$$\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n},$$

the frequency response of a filter is then given by

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} \text{ (can you prove this ?)}$$

Hence, as we have assumed that $H(z)$ converges on $z = 1$, we have :

$$H(e^{j\omega}) = H(z)|_{z=e^{j\omega}}$$

Frequency response to a sinusoid and Group delay

Let's take $x[n] = e^{j\omega n}$, a complex sinusoid ($X(z) = z^n$), then the output is

$$y[n] = H(e^{j\omega})e^{j\omega n} = |H(e^{j\omega})|e^{j\Theta(\omega)}e^{j\omega n} = |H(e^{j\omega})|e^{j(\omega n+\Theta(\omega))}$$

$H(e^{j\omega})$: frequency response

$$H(e^{j(\omega+2\pi k)}) = H(e^{j2\pi k}e^{j\omega}) = H(e^{j\omega}) \Rightarrow \text{ Periodic !}$$

$$\tau\omega = -\frac{d\Theta(\omega)}{d\omega} = \beta$$

If the group delay $\Theta(\omega)$ is linear in ω

$$y[n] = |H(e^{j\omega})|e^{j\omega(n+\beta)}$$

The influence on the phase is only a delay ! (desirable for a filter !!)

Example of frequency responses

- $y[n] = \dfrac{x[n] + x(n-1)}{2}$
- $y[n] = \dfrac{x[n] - x(n-1)}{2}$
- $h[n] = (1/2)^n u[n]$ and $x[n] = \sin(\omega_o n + \theta)$, find $y[n] = x[n] * h[n]$.

Poles and zeros plot - Frequency response

Let $H(z) = H_o z^{N-M} \frac{\prod_{l=1}^{M}(z - z_l)}{\prod_{i=1}^{M}(z - p_i)}$ where z_l and p_i are the Zeros and Poles of $H(z)$.

In the frequency domain this can be rewritten as :

$$H(e^{j\omega}) = H_o e^{j\omega(N-M)} \frac{\prod_{l=1}^{M}(e^{j\omega} - z_l)}{\prod_{i=1}^{N}(e^{j\omega} - p_i)}$$

$$|H(e^{j\omega})| = |H_o| \frac{\prod_{l=1}^{M}|e^{j\omega} - z_l|}{\prod_{i=1}^{M}|e^{j\omega} - p_i|}$$

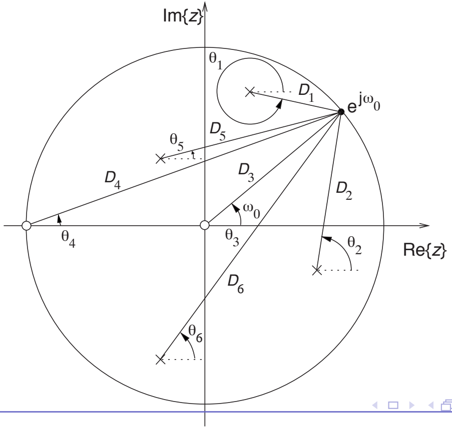
$$\text{and } \Theta(\omega) = \omega(N - M) + \sum_{l=1}^{M} \angle(e^{j\omega} - z_l) - \sum_{i=1}^{N} \angle(e^{j\omega} - p_i)$$

Poles and zeros : Graphical interpretation

Assume $H(z)$ has two zeros and four poles, then, for a given frequency ω_o

$$H(e^{j\omega_o}) = \frac{D_3 D_4}{D_1 D_2 D_5 D_6}$$

$$\Theta(\omega_o) = \theta_3 + \theta_4 - \theta_1 - \theta_2 - \theta_5 - \theta_6$$



FIR (Finite Impulse Response) and IIR (Inifinite Impulse Response) filters

FIR filter

$$y[n] = \sum_{i=0}^{M} h(i)x[n-i] \Rightarrow H(z) = \sum_{i=0}^{M} h(i)z^{-i}$$

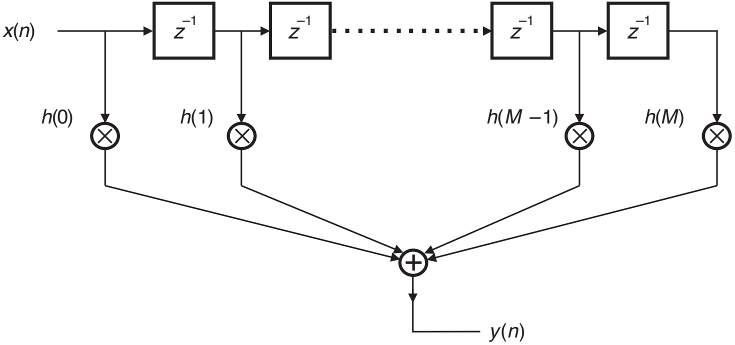
IIR filter

$$\sum_{i=0}^{N} a_i y[n-i] = \sum_{l=0}^{M} b_l x[n-l]$$

$$\sum_{i=0}^{N} a_i Y(z)z^{-i} = \sum_{l=0}^{M} b_l X(z)z^{-l} \Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{l=0}^{M} b_l z^{-l}}{\sum_{i=0}^{N} a_i z^{-i}}$$

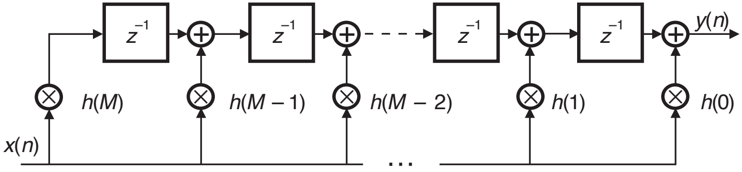
FIR filters

Direct form



FIR filters

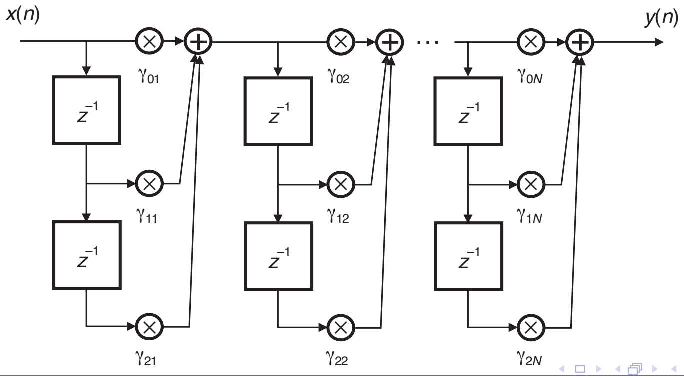
Alternative FIR direct form



FIR cascade form

FIR cascade form

$$H(z) = \prod_{k=1}^{n} (\gamma_{0k} + \gamma_{1k} z^{-1} + \gamma_{2k} z^{-2})$$



Linear Phase FIR filters

Linear Phase FIR filters

$$H(e^{j\omega}) = B(\omega)e^{-j(\omega\tau+\phi)}, B(\omega) \text{ real }, \tau, \phi \text{ constant.}$$

Implies : $h[n] = e^{2j\phi}h^*(2\tau - n)$.

- If $h[n]$ causal, $\tau = M/2$ and $h[n] = e^{2j\phi}h^*(M - n)$
- If $h[n]$ real : $h[n] = h^*[n] \Rightarrow e^{2j\phi} = \pm 1 \Rightarrow \phi = \frac{k\pi}{2}$ (k integer)
- $\Rightarrow H(e^{j\omega}) = B(\omega)e^{-j\omega\frac{M}{2}+j\frac{k\pi}{2}}$

Linear Phase FIR filter types

$$\text{Type I}: H(z) = \sum_{n=0}^{M/2-1} h[n]z^{-n} + h(M/2)z^{-M/2} + \sum_{n=M/2+1}^{M} h(z)z-n$$

$$H(z) = \sum_{n=0}^{M/2-1} h[n][z^{-n} + z^{-(M-n)}] + h(M/2)z^{-M/2}$$

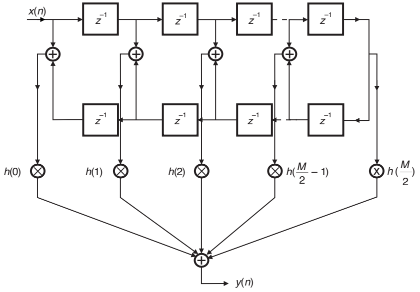
Linear phase FIR filter types

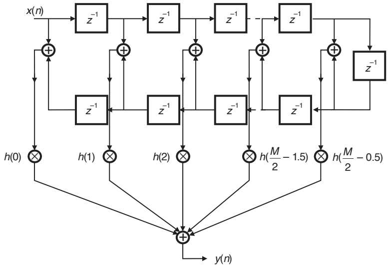
- Type I : k=0, M even : $H(e^{j\omega}) = e^{-j\omega M/2} \sum_{m=0}^{M/2} a(m)\cos(\omega m))$
- Type II : k=0, M odd :$H(e^{j\omega}) = e^{-j\omega M/2} \sum_{m=0}^{M/2} b(m)\cos(\omega(m-1/2))$
- Type III : k=1, M even : $H(e^{j\omega}) = e^{-j\omega M/2-\pi/2} \sum_{m=1}^{M/2} c(m)\sin(\omega m)$
- Type IV : k=1, M odd : $H(e^{j\omega}) = e^{-j\omega M/2-\pi/2} \sum_{m=1}^{M/2} d(m)\sin(\omega(m-1/2))$

FIR filter types

Type	M	$h(n)$	$H(e^{j\omega})$	$\Theta(\omega)$	τ
I	Even	Symmetric	$e^{-j\omega\frac{M}{2}}\sum_{m=0}^{\frac{M}{2}} a(m)\cos(\omega m)$ $a(0)=h\left(\frac{M}{2}\right); a(m)=2h\left(\frac{M}{2}-m\right)$	$-\omega\frac{M}{2}$	$\frac{M}{2}$
II	Odd	Symmetric	$e^{-j\omega\frac{M}{2}}\sum_{m=1}^{\frac{M+1}{2}} b(m)\cos\left[\omega\left(m-\frac{1}{2}\right)\right]$ $b(m)=2h\left(\frac{M+1}{2}-m\right)$	$-\omega\frac{M}{2}$	$\frac{M}{2}$
III	Even	Antisymmetric	$e^{-j(\omega\frac{M}{2}-\frac{\pi}{2})}\sum_{m=1}^{\frac{M}{2}} c(m)\sin(\omega m)$ $c(m)=2h\left(\frac{M}{2}-m\right)$	$-\omega\frac{M}{2}+\frac{\pi}{2}$	$\frac{M}{2}$
IV	Odd	Antisymmetric	$e^{-j(\omega\frac{M}{2}-\frac{\pi}{2})}\sum_{m=1}^{\frac{M+1}{2}} d(m)\sin\left[\omega\left(m-\frac{1}{2}\right)\right]$ $d(m)=2h\left(\frac{M+1}{2}-m\right)$	$-\omega\frac{M}{2}+\frac{\pi}{2}$	$\frac{M}{2}$

FIR linear phase Implementations



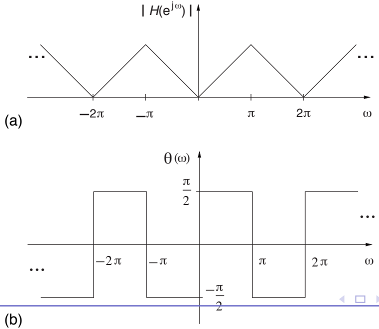


Special filters

Differentiator

$$y[n] = \left.\frac{dx_a(t)}{dt}\right|_{t=nT} \Rightarrow H(e^{j\omega}) = j\omega$$

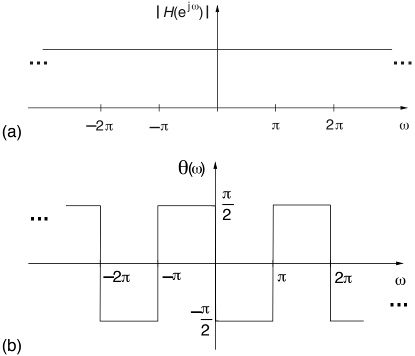
$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} j\omega e^{j\omega n} d\omega = \frac{(-1)^n}{n}(n \neq 0);\ h(0) = 0$$



Hilbert transform

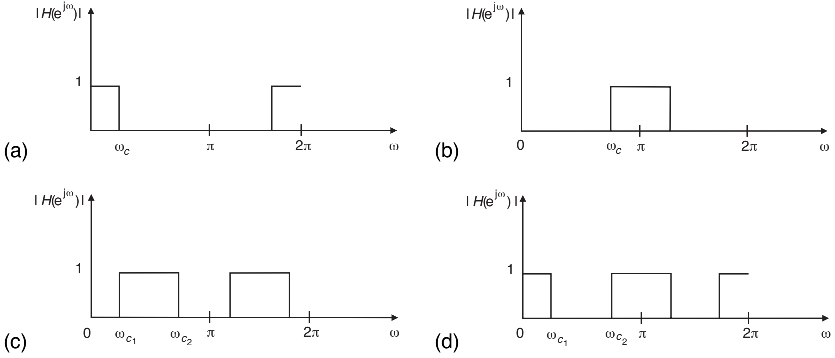
$$H(e^{j\omega}) = -j\text{sign}(\omega)$$

$$h[n] = \frac{1-(-1)^n}{\pi n}(n \neq 0);\, h(0) = 0$$



Ideal filter

Let's determine $h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega$ for ideal filters :



Low pass ideal filter

low pass ideal filter

$$|H(e^{j\omega})| = \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$

Which leads to (with a phase equal to zero)

$$h[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega = \begin{cases} \frac{\omega_c}{\pi}, & n = 0 \\ \frac{\sin(\omega_c n)}{\pi n}, & n \neq 0 \end{cases}$$

If M is odd, then the phase of a FIR must be $e^{-j\omega\frac{M}{2}}$, and

$$h[n] = \frac{\sin(\omega_c(n - \frac{M}{2}))}{\pi(n - \frac{M}{2})}$$

Filter type	Magnitude response $\lvert H(e^{j\omega})\rvert$	Impulse response $h(n)$
Lowpass	$\begin{cases} 1, & \text{for } 0 \le \lvert\omega\rvert \le \omega_c \\ 0, & \text{for } \omega_c < \lvert\omega\rvert \le \pi \end{cases}$	$\begin{cases} \dfrac{\omega_c}{\pi}, & \text{for } n = 0 \\ \dfrac{1}{\pi n}\sin(\omega_c n), & \text{for } n \neq 0 \end{cases}$
Highpass	$\begin{cases} 0, & \text{for } 0 \le \lvert\omega\rvert < \omega_c \\ 1, & \text{for } \omega_c \le \lvert\omega\rvert \le \pi \end{cases}$	$\begin{cases} 1 - \dfrac{\omega_c}{\pi}, & \text{for } n = 0 \\ -\dfrac{1}{\pi n}\sin(\omega_c n), & \text{for } n \neq 0 \end{cases}$

Filter type	Magnitude response $\lvert H(e^{j\omega})\rvert$	Impulse response $h(n)$
Bandpass	$\begin{cases} 0, & \text{for } 0 \le \lvert\omega\rvert < \omega_{c_1} \\ 1, & \text{for } \omega_{c_1} \le \lvert\omega\rvert \le \omega_{c_2} \\ 0, & \text{for } \omega_{c_2} < \lvert\omega\rvert \le \pi \end{cases}$	$\begin{cases} \dfrac{(\omega_{c_2} - \omega_{c_1})}{\pi}, & \text{for } n = 0 \\ \dfrac{1}{\pi n}[\sin(\omega_{c_2} n) - \sin(\omega_{c_1} n)], & \text{for } n \neq 0 \end{cases}$
Bandstop	$\begin{cases} 1, & \text{for } 0 \le \lvert\omega\rvert \le \omega_{c_1} \\ 0, & \text{for } \omega_{c_1} < \lvert\omega\rvert < \omega_{c_2} \\ 1, & \text{for } \omega_{c_2} \le \lvert\omega\rvert \le \pi \end{cases}$	$\begin{cases} 1 - \dfrac{(\omega_{c_2} - \omega_{c_1})}{\pi}, & \text{for } n = 0 \\ \dfrac{1}{\pi n}[\sin(\omega_{c_1} n) - \sin(\omega_{c_2} n)], & \text{for } n \neq 0 \end{cases}$

FIR approximation by frequency sampling

Frequency response of an FIR

$$H(e^{j\frac{2\pi k}{N}}) = \sum_{n=0}^{N-1} h[n]e^{-j\frac{2\pi k}{N}}, k = 0..N-1$$

... $h[n]$ inverse DFT of $H(e^{j\omega})$. Let

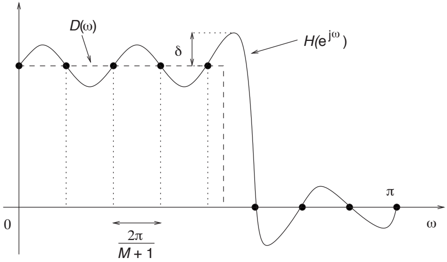
$$A[k]e^{j\theta[k]} = D(\omega_s k/N)$$

In Scilab `hst = fsfirlin(hd,1)` where `hd` is the desired response and "1" means "Type I" filter.

Filter type	Impulse response $h(n)$, for $n = 0, 1, \ldots, M$	Condition
Type I	$\frac{1}{M+1}\left[A(0) + 2\sum_{k=1}^{\frac{M}{2}} (-1)^k A(k) \cos \frac{\pi k(1+2n)}{M+1}\right]$	
Type II	$\frac{1}{M+1}\left[A(0) + 2\sum_{k=1}^{\frac{M-1}{2}} (-1)^k A(k) \cos \frac{\pi k(1+2n)}{M+1}\right]$	$A\left(\frac{M+1}{2}\right) = 0$
Type III	$\frac{2}{M+1}\sum_{k=1}^{\frac{M}{2}} (-1)^{k+1} A(k) \sin \frac{\pi k(1+2n)}{M+1}$	$A(0) = 0$
Type IV	$\frac{1}{M+1}\left[(-1)^{\frac{M+1}{2}+n} A\left(\frac{M+1}{2}\right) + 2\sum_{k=1}^{\frac{M-1}{2}} (-1)^k A(k) \sin \frac{\pi k(1+2n)}{M+1}\right]$	$A(0) = 0$

Problems of frequency sampling

- Does not control ripple
- Does not control ringing at edges



FIR approximation by windowing

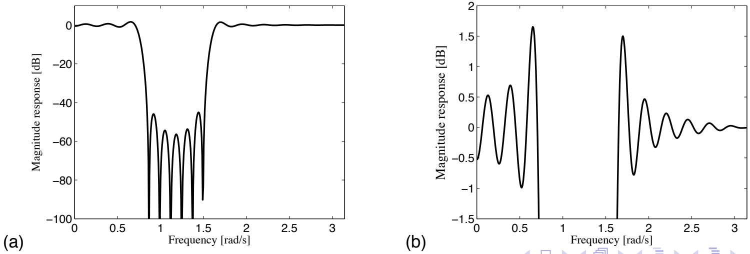
Rectangular FIR approximation

Assuming M even :

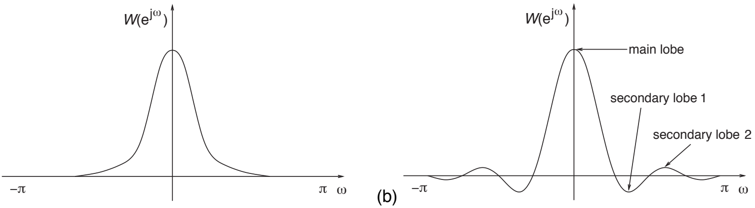
$$h'[n] = \begin{cases} h[n], & |n| \leq \frac{M}{2} \\ 0, & |n| > \frac{M}{2} \end{cases}$$

$$h'[n] = h[n].w[n]$$

$$H'(e^{j\omega}) = H(e^{j\omega}) \otimes W(e^{j\omega})$$



FIR approximation by windowing



FIR approximation by windowing

Rectangular window frequency response

let $w_r[n] = 1$ for $|n| \leq M/2$:

$$
\begin{aligned}
W_r(e^{j\omega}) &= \sum_{n=-M/2}^{M/2} e^{j\omega n} \\
&= \frac{e^{j\omega \frac{M}{2}} - e^{-j\omega \frac{M}{2}} e^{-j\omega}}{1 - e^{-j\omega}} \\
&= e^{-j\omega/2} \frac{e^{j\omega \frac{M+1}{2}} - e^{-j\omega \frac{M+1}{2}}}{1 - e^{-j\omega}} \\
&= \frac{\sin(\omega((M+1)/2))}{\sin(\omega/2)}
\end{aligned}
$$

Other windows

Triangular window

$$w_t(n) = \begin{cases} -\dfrac{2|n|}{M+2} + 1, & \text{for } |n| \leq \frac{M}{2} \\ 0, & \text{for } |n| > \frac{M}{2} \end{cases}$$

Hamming/Hanning window

$$w_H(n) = \begin{cases} \alpha + (1-\alpha)\cos\left(\dfrac{2\pi n}{M}\right), & \text{for } |n| \leq \frac{M}{2} \\ 0, & \text{for } |n| > \frac{M}{2} \end{cases}$$

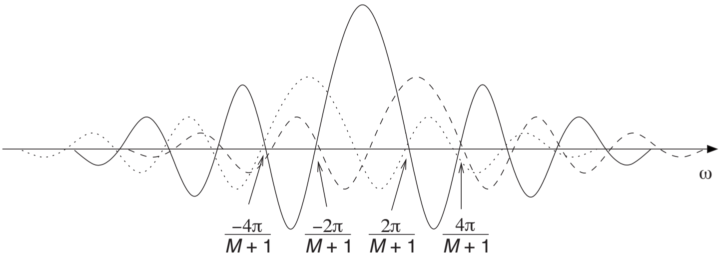
$\alpha = 0.54$: Hamming window ; $\alpha = 0.5$: Hanning window

Relation between Hamming and rectangular windows

$$w_H(n) = w_r(n)\left[\alpha + (1-\alpha)\cos\left(\frac{2\pi n}{M}\right)\right]$$

$$W_H\left(e^{j\omega}\right) = W_r\left(e^{j\omega}\right) * \left[\alpha\delta(\omega) + \left(\frac{1-\alpha}{2}\right)\delta\left(\omega - \frac{2\pi}{M}\right) + \left(\frac{1-\alpha}{2}\right)\delta\left(\omega + \frac{2\pi}{M}\right)\right]$$

$$W_H\left(e^{j\omega}\right) = \alpha W_r\left(e^{j\omega}\right) + \left(\frac{1-\alpha}{2}\right) W_r\left(e^{j(\omega - \frac{2\pi}{M})}\right) + \left(\frac{1-\alpha}{2}\right) W_r\left(e^{j(\omega + \frac{2\pi}{M})}\right)$$


ω
$\frac{-4\pi}{M+1}$
$\frac{-2\pi}{M+1}$
$\frac{2\pi}{M+1}$
$\frac{4\pi}{M+1}$

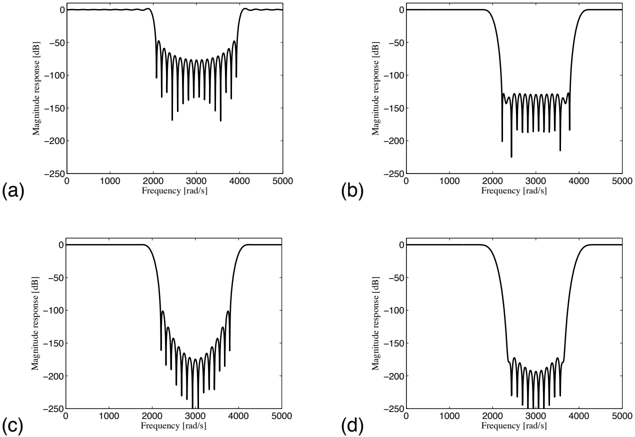
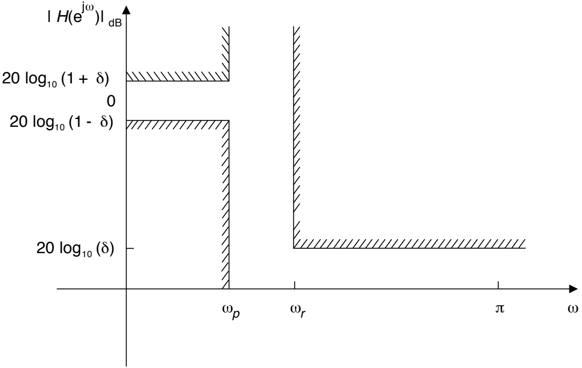
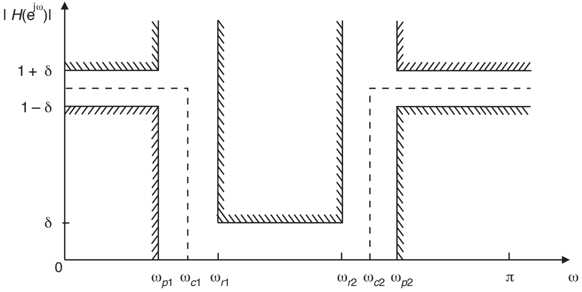


Figure 9: Magnitude responses when using: (a) rectangular; (b) Hamming; (c) Hann; (d) Blackman windows.

Kaiser and Chebyshev windows : specification



Kaiser and Chebyshev windows : specification



Kaiser window

The Kaiser window yields an optimal window in the sense that the side lobe ripple is minimized in the least squares sense for a certain main lobe width.

$$K_N[n] = \begin{cases} \frac{I_o(\beta\sqrt{1-[2n/(N-1)]^2})}{I_o(\beta)}, & -(N-1)/2 \leq n \leq (N-1)/2 \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

- $I_o(x)$ is the modified zeroth-order Bessel function
- β constant which controls the trade-off of the side-lobe heights and the width of the main lobe.

Chebyshev window

Minimizes the amount of ripple in the side lobes for a given main lobe width and filter length.
For δ_f (main lobe width/2) and δ_p (side lobe height) known, N (window length) is obtained from

$$N \geq 1 + \frac{\cosh^{-1}((1+\delta_p)/(\delta_p))}{\cosh^{-1}(1/(\cos(\pi\delta_f)))}. \tag{2}$$

For N and δ_p known, δ_f is obtained from

$$\delta_f = \frac{1}{\pi}\cos^{-1}(1/\cosh(\cosh^{-1}((1+\delta_p)/\delta_p)/(N-1))). \tag{3}$$

Finally, for N and δ_f known, δ_p is obtained from

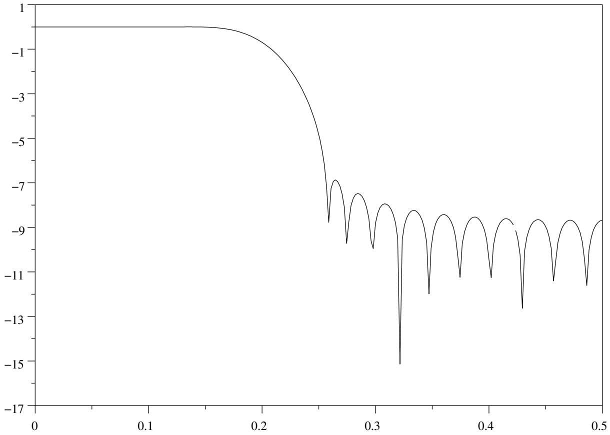
$$\delta_p = [\cosh((N-1)\cosh^{-1}(1/\cos(\pi\delta_f))) - 1]^{-1}. \tag{4}$$

`wfir` in Scilab

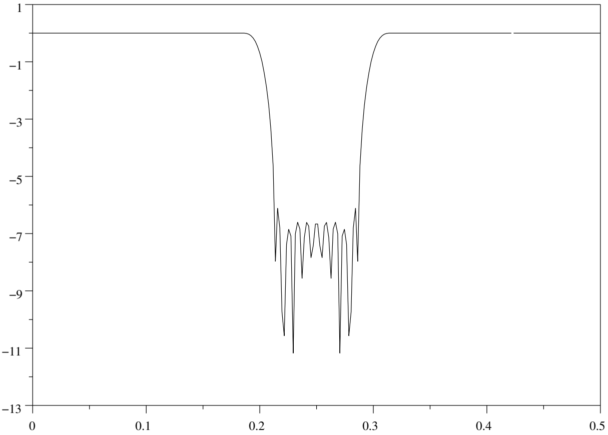
```
-> [wft,wfm,fr]=wfir()
```
Interactive tool

```
-> [wft,wfm,fr]=wfir(ftype,forder,cfreq,wtype,fpar)
```

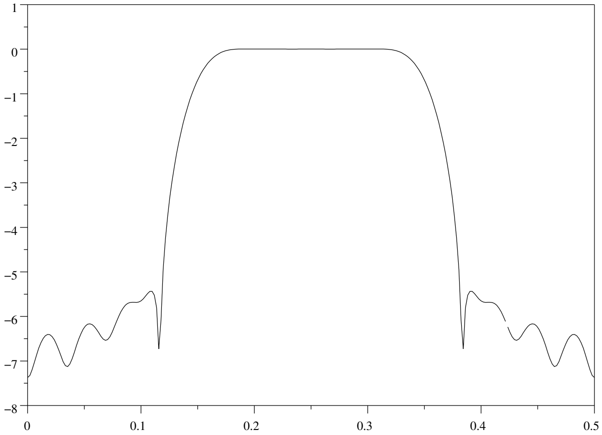
- `wft` : vector with the windowed filter coefficients of size *n*.
- `wfm` : vector with the frequency response of size 256.
- `fr` : vector with the frequency axis $0 \leq fr \leq .5$ associated to the values in wfm.
- `ftype` : `'lp','hp','bp','sb'`
- `forder` : Order of the filter
- `cfreq` : 2 elements vector with the cutoff frequencies (first element non zero for low-pass and high-pass filters)
- `wtype` : `'re','tr','hm','hn','kr','ch'`
- `fpar` : Parameters of the windows : `fpar(1)` sidelobe height, `fpar(2)` main-lobe width,



Low pass filter with Kaiser window, $n = 33$, $\beta = 5.6$



Stop band with Hamming window, $n = 127$, $\alpha = .54$



Band pass filter with Chebyshev window, $n = 55$, $dp = .001$, $df = .0446622$

FIR filter design by Optimization techniques

- Principle : Minimize the Maximum error between desired response and obtained response
- Uses “classical” optimisation algorithms (Matlab/Scilab/Scipy ...)
- For us : use of `eqfir` (see exercice session)

Infinite Impulse Response (IIR) Filter

IIR filter

$$H(z) = \frac{N(z)}{D(z)} = \frac{\sum_{i=0}^{M} b_i z^{-i}}{1 + \sum_{i=1}^{N} a_i z^{-i}}$$

IIR Design

IIR Design based on Transformation from Analog Filters :

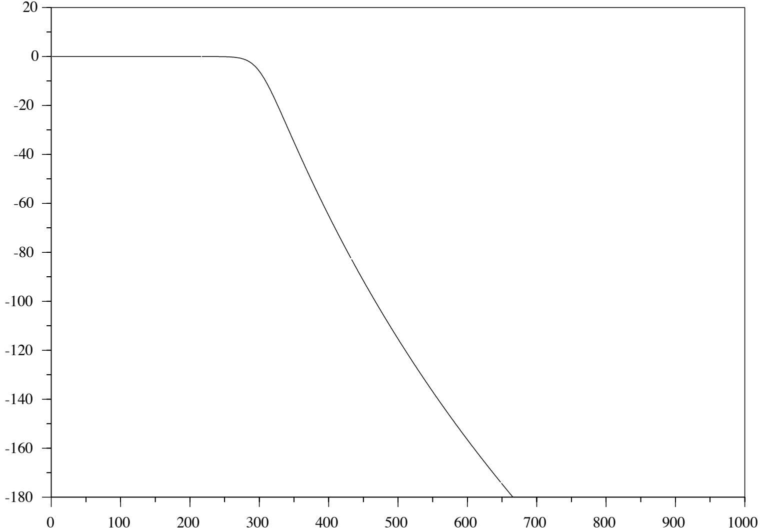
- Butterworth
- Elliptic
- Chebyshev I & II

Butterworth IIR filter characteristics

- Flat frequency response in the Pass Band and Stop Band
- Large transition band
- Decrease of $-6ndB$ / octave
- In the analog domain, for a LP filter with $2n$ pairs of conjugate poles, with a cutoff frequency of ω_c :

$$|H(\omega)|^2 = \frac{1}{1 + (\frac{\omega}{\omega_c})^{2n}}$$

Butterworth IIR filter characteristics



Magnitude in dB of Butterworth low-pass Filter with $n = 13$ and $\omega_c = 300Hz$.

Butterworth IIR filter characteristics

Order determination:

The filter order, n, completely specifies a Butterworth filter. In general n is determined by giving a desired attenuation $\frac{1}{A}$ at a specified "normalized" frequency $f = \frac{\omega_r}{\omega_c}$. The filter order, n, is given by the solution of $\frac{1}{A^2} = h_n^2(f)$. We obtain immediately:

$$n = \frac{\log_{10}(A^2 - 1)}{2 \log_{10}(f)} \tag{5}$$

Chebyshev I and II IIR filter characteristics

- Chebyshev I : specifies ripple in the Pass band
- Chebyshev II : specifies ripple in the Stop band (flat in Pass band)
- Narrow(er) transition band

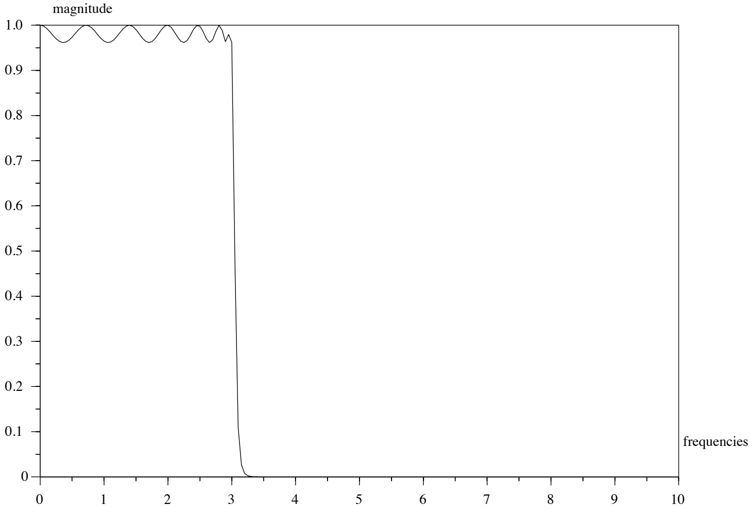
The n^{th} order Chebyshev polynomial $T_n(x)$:

$$T_n(x) = \begin{cases} \cos(n \cos^{-1}(x)) & \text{if } |x| < 1 \\ \cosh(n \cosh^{-1}(x)) & \text{otherwise} \end{cases}$$

$$|h_{1,n}(\omega \mid \omega_c, \epsilon)|^2 = \frac{1}{1 + \epsilon^2 T_n^2(\frac{\omega}{\omega_c})}$$

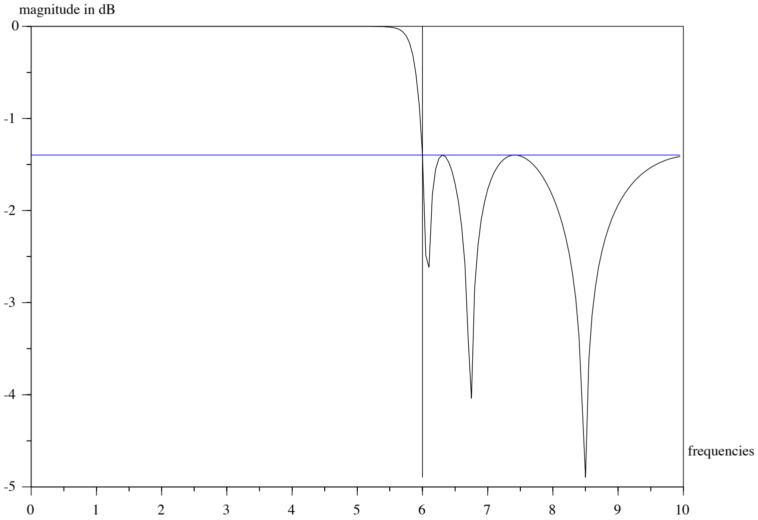
$$|h_{2,n}(\omega \mid \omega_r, A)|^2 = \frac{1}{1 + \frac{A^2 - 1}{T_n^2(\frac{\omega_r}{\omega})}}$$

Chebyshev I IIR filter characteristics



Magnitude in dB of Chebyshev I low-pass Filter with $n = 13, \epsilon = 0.2$ and $\omega_c = 3Hz$.

Chebyshev II IIR filter characteristics



Magnitude in dB of Chebyshev I low-pass Filter with $n = 10, \epsilon = 1/A = 0.2$ and $\omega_c = 6Hz$

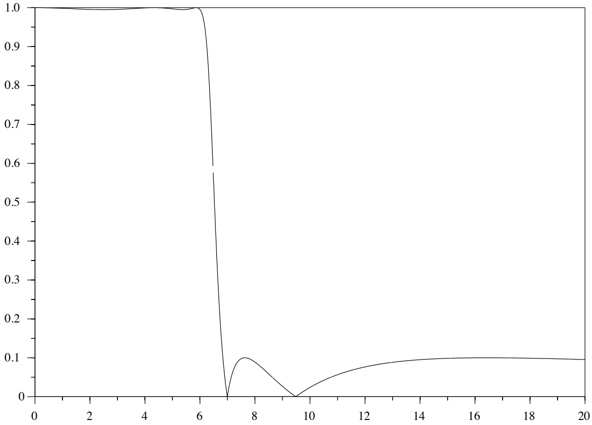
Chebyshev order determination

$$n = \frac{\cosh^{-1}\left(\frac{\sqrt{A^2-1}}{\epsilon}\right)}{\cosh^{-1}(f)} = \frac{\log(g + \sqrt{(g^2 - 1)})}{\log(f + \sqrt{(f^2 - 1)}}$$

where $g = \sqrt{\dfrac{A^2 - 1}{\epsilon^2}}$

Elliptic IIR filter characteristics

- Control both Passband and Stopband ripple
- 4 parameters (2 ripple parameters, Passband Cutoff frequency and Stopband Cutoff frequency (or n))



Elliptic frequency with $\omega_c = 6$, $\omega_r = 6.83 (n = 5)$, $A = 10$; $\epsilon = 0.1$

Design of IIR filters with Scilab

`-> [hz]=iir(n,ftype,fdesign,frq,delta)`

- `ftype` : `'lp','hp','bp','sb'`
- `fdesign` : `'butt','cheb1','cheb2','ellip'`
- `frq` : 2-vector with cut-off frequencies
- `delta` : 2-vector with ripples

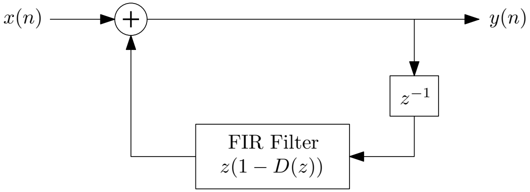
```
->
[cells,fact,zers,pols]=eqiir(ftype,fdesign,frq,delta1,delta2)
```

Product of second-order filters

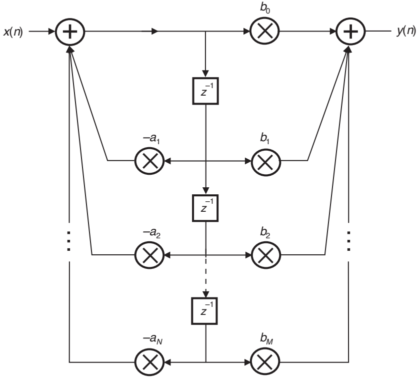
IIR : implementation

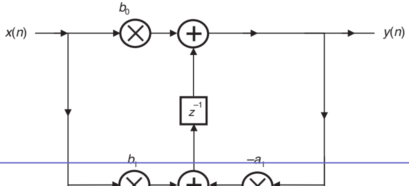
$$H(z) = \frac{N(z)}{D(z)} = \frac{\sum_{i=0}^{M} b_i z^{-i}}{1 + \sum_{i=1}^{N} a_i z^{-i}}$$

We can realize $1/D(z)$ as :



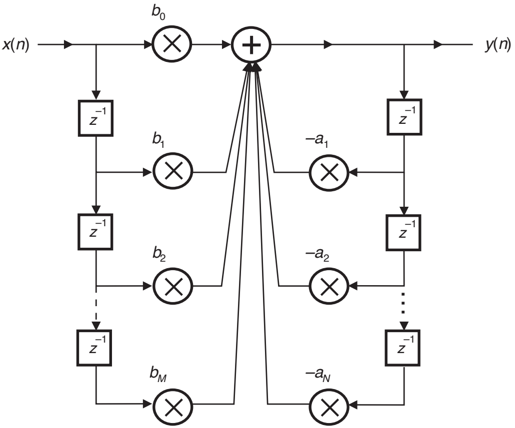
Canonic IIR direct-form realization





Noncanonic IIR direct-form realization

Non canonic realization because needs N+M delays



A note on allpass filters

A typical allpass filter :

- $$H(z) = \frac{m_2 z^2 + m_1 z + 1}{z^2 + m_1 z + m_2}$$

- $$H(z) = z^2 \frac{m_2 + m_1 z^{-1} + z^{-2}}{z^2 + m_1 z + m_2} = z^2 \frac{A(z^{-1})}{A(z)}$$

  $$|H(e^{j\omega})| = \frac{|A(e^{-j\omega})|}{|A(e^{j\omega})|} = 1$$

- Poles and zeroes related : $z_1^* = \frac{1}{p_1}; z_1 = \frac{1}{p_1^*};$

Can be used to correct non constant group delay

reminder : Ideal Continuous/Discrete (time) converter

$x_c(t)$ (continuous) and $x[n] = x_c(nT)$ discrete (and $x_s(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT)$ (sampled)). Then, the spectrum of $x[n]$ is (where $1/T$ is the sampling frequency) :

$$X(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(j(\frac{\omega}{T} - \frac{2\pi}{T}k\right)$$

$$\text{and } X_s(\imath\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(\imath(\Omega - \frac{2\pi}{T}k)\right)$$

where Ω is the frequency in the continuous time domain and ω is the frequency in the Discrete Time ($x[n]$) domain.
In the frequency domain, a C/D converter can be seen as a sequence of :

1. Repetition of the spectrum at $\frac{2\pi}{T}$ (in non-normalised frequency)
2. Scaling of the amplitude by $1/T$
3. Normalization of the frequency by $1/T$

You should be able, from this equation, to show the aliasing effect.

reminder : Ideal D/C converter

Given $x[n]$, and it's samples version $x_s(t)$, $x_s(t)$ has to be filtered $x[n]$ by an ideal filter to recover the "center of the spectrum" :

$$H_r(j\Omega) = \begin{cases} T & |\Omega| < \frac{\pi}{T} \\ 0 & |\Omega| \geq \frac{\pi}{T} \end{cases}$$

This corresponds to

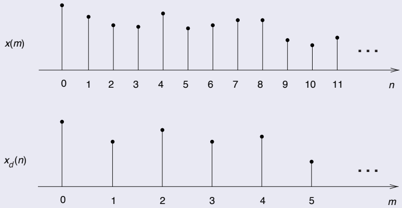
$$x_r(t) = \sum_n x[n] sinc\left(\frac{t - nT}{T}\right)$$

$$X_r(j\Omega) = \begin{cases} TX_s(j\Omega) & |\Omega| < \frac{\pi}{T} \\ 0 & |\Omega| \geq \frac{\pi}{T} \end{cases}$$

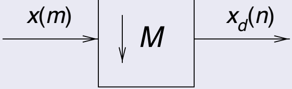
Downsampling - Decimation

Decimation

The decimation operation (by M) consists in taking one sample out of M samples in the input signal, ($x_d[n] = x(nM)$) as indicated in the following figure :



It is usually represented as :



Decimation in the frequency domain

Let $x'(m) = x(m)$ for $m = nM$, n integer and $x'(m) = 0$ otherwise. In other words, $x'(m) = x(m) \sum_{n=-\infty}^{\infty} \delta(m - nM)$, then :

$$
\begin{aligned}
X_d(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x_d[n] e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} x(nM) e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} x'(nM) e^{-j\omega n} \\
&= \sum_{l=-\infty}^{\infty} x'(l) e^{-j(\omega/M)l} = X'(e^{j\frac{\omega}{M}})
\end{aligned}
$$

Decimation in the frequency domain

and, from $x'(m) = x(m) \sum_{n=-\infty}^{\infty} \delta(m - nM)$,

$$\begin{aligned} X'(e^{j\omega}) &= X(e^{j\omega}) * TF\left\{ \sum_{n=-\infty}^{\infty} \delta(m - nM) \right\} \\ &= X(e^{j\omega}) * \frac{2\pi}{M} \sum_{k=0}^{M-1} \delta(\omega - 2\pi k/M) \\ &= \frac{1}{M} \sum_{k=0}^{M-1} X(e^{j(\omega - 2\pi k/M)}) \end{aligned}$$

Hence :

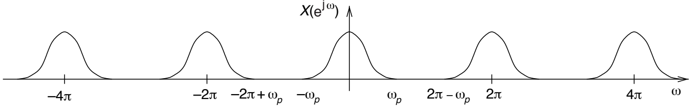
$$X_d(e^{j\omega}) = X'(e^{j\frac{\omega}{M}}) = \frac{1}{M} \sum_{k=0}^{M-1} X(e^{j\frac{\omega - 2\pi k}{M}})$$

Decimation and anti-aliasing

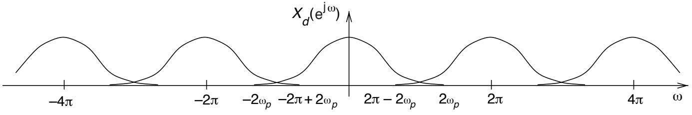
Usually, the decimation operation is preceded by a lowpass filter, which can be the ideal anti-aliasing filter :

$$H_d(e^{j\omega}) = \begin{cases} 1, & |\omega| \leq \frac{\pi}{M} \\ 0 & \text{otherwise} \end{cases}$$

In the normalized frequency domain , the decimation can be seen as :



(a)

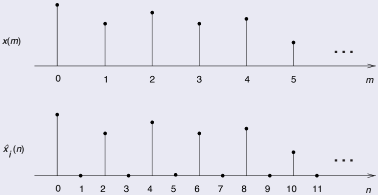


(b)

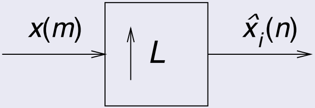
Upsampling

Upsampling

The upsamling operation (by L consists in inserting $L-1$ samples with value 0 between each sample of x[n], ($x_U[n] = x(n/L) for n = kL, k$ integer) as indicated in the following figure :



It is usually represented as :

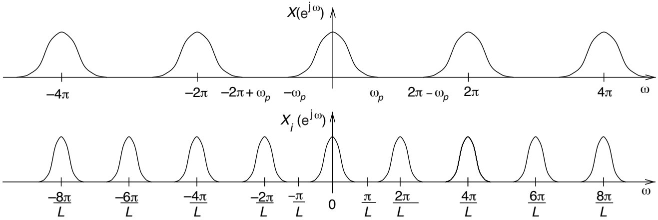


Upsampling in the frequency domain

It is straightforward to see that the spectrum is (go in the z domain)

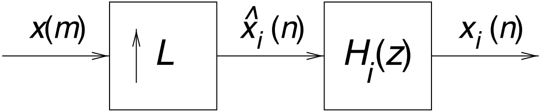
$$X_u(e^{j\omega}) = X(e^{j\omega L})$$

In the normalized frequency domain , the upsampling can be seen as :



Interpolation

To obtain a “smooth” version of $x(m)$, we will interpolate the upsampled signal by an interpolation filter like indicated below :

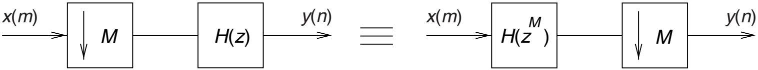


An ideal interpolator filter being :

$$H_i(e^{j\omega}) = \begin{cases} L, & |\omega| \leq \frac{\pi}{L} \\ 0 & \text{otherwise} \end{cases}$$

Noble Identities

The Noble Identities can be depicted as follows :



(a)

(b)

Let's look at the first identity (the second one can be proven in a similar way) : for the right term, we have :

$Y(z) = H(z).X_d(z)$, and $X_d(z) = \frac{1}{M}\sum_{k=0}^{M-1} X(z^{1/M}e^{-j\frac{2\pi k}{M}})$. Hence :

$$Y(z) = \frac{1}{M}H(z)\sum_{k=0}^{M-1} X(z^{1/M}e^{-j\frac{2\pi k}{M}})$$

Noble Identities

For the left term, denote $u[n]$ the output of the filter $H(z^M)$, we have

$$\begin{aligned} Y(z) &= \frac{1}{M}\sum_{k=0}^{M-1} U(z^{1/M}e^{-j\frac{2\pi k}{M}}) \\ &= \frac{1}{M}\sum_{k=0}^{M-1} X(z^{1/M}e^{-j\frac{2\pi k}{M}})H(ze^{-j\frac{2\pi Mk}{M}}) \\ &= \frac{1}{M}H(z)\sum_{k=0}^{M-1} X(z^{1/M}e^{-j\frac{2\pi k}{M}}) \end{aligned}$$

Let's write the z transform $H(z)$ of $h[n]$ as :

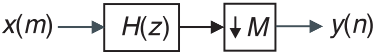
$$
\begin{aligned}
H(z) &= \sum_{k=-\infty}^{\infty} h[k] z^{-k} \\
&= \sum_{l=-\infty}^{\infty} \left[h(Ml) z^{-Ml} + h(Ml+1) z^{-(Ml+1)} + \ldots + h(Ml+M-1) z^{-(Ml+M-1)} \right] \\
&= \sum_{l=-\infty}^{\infty} \left[h(Ml) z^{-Ml} + z^{-1} h(Ml+1) z^{-Ml} + \ldots + z^{-(M-1)} h(Ml+M-1) z^{-M} \right. \\
&= \sum_{j=0}^{M-1} z^{-j} E_j(z^M)
\end{aligned}
$$

where

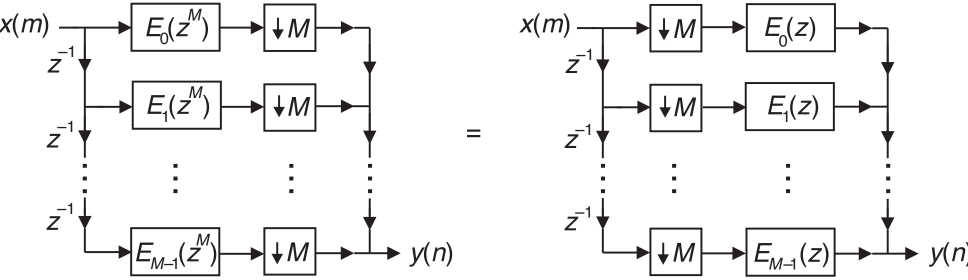
$$E_j(z) = \sum_{l=-\infty}^{\infty} h(Ml+j) z^{-l}$$

and $E_j(z)$ are called the **Polyphase components** of $H(z)$

Polyphase decomposition of $H(z)$ and decimation



(a)



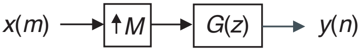
(b)

(c)

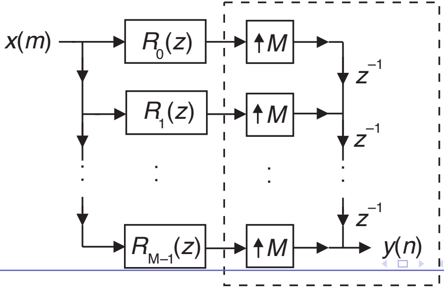
Polyphase decomposition of $H(z)$ and Interpolation

For the interpolation case, we define $R_j(z) = E_{M-1-j}(z)$, such that

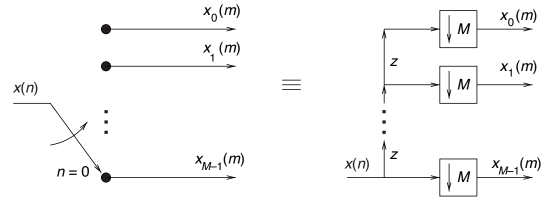
$$H(z) = \sum_{j=0}^{M-1} z^{-(M-1-j)} R_j(z^M)$$



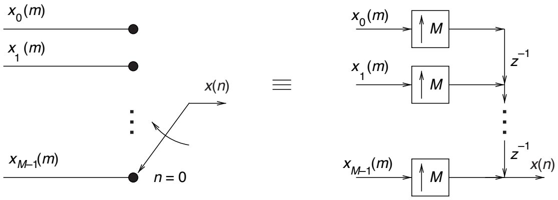
(a)



Polyphase decomposition and commutator model

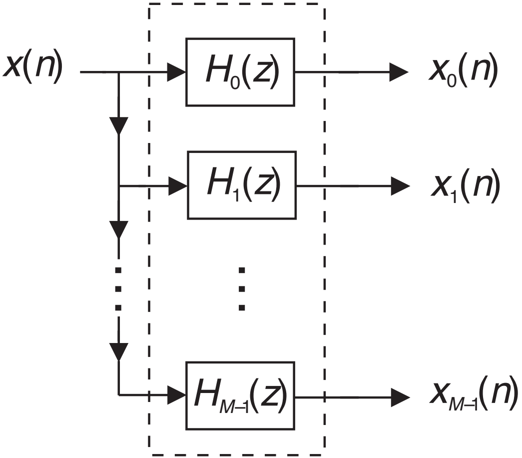


(a)



(b)

The Equalizer example : A filter bank



Introduction

- In a number of applications, it is necessary to split a digital signal into several frequency bands.
- After such decomposition, the signal is represented by more samples than in the original stage.
- Systems which decompose and reassemble the signals are generally called filter banks.
- In the following deal with filter banks, showing several ways in which a signal can be decomposed into critically decimated frequency bands, and recovered from them with minimum error.

Filter banks

- In some applications, such as signal analysis, signal transmission, and signal coding, a digital signal $x(n)$ is decomposed into several frequency bands.

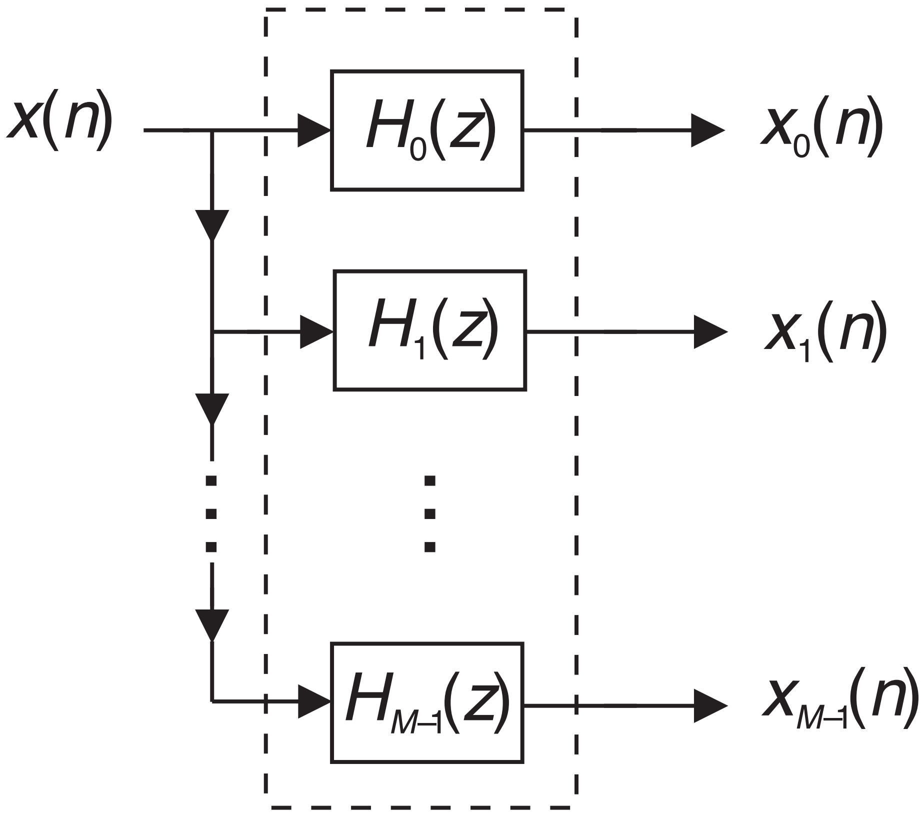


Figure 1: Decomposition of a digital signal into M frequency bands.

Filter banks

- The signal in each of the bands $x_k(n)$, for $k = 0, 1, \ldots, (M - 1)$, has at least the same number of samples as the original signal.
- This implies that after the M-band decomposition, the signal is represented with at least M times more samples than the original one.
- In the case where the signal is uniformly split in the frequency domain, each band has bandwidth M times smaller than the one of the original signal.
- The bands $x_k(n)$ can be decimated by a factor of M (critically decimated) without destroying the original.

Decimation of a bandpass signal

- If the input signal $x(m)$ is lowpass and band-limited to $[-\frac{\pi}{M}, \frac{\pi}{M}]$, the aliasing after decimation by a factor of M can be avoided.
- If before decimation the signal is split into M uniform real frequency bands, the kth band will be confined to $[-\frac{(k+1)\pi}{M}, -\frac{k\pi}{M}] \cup [\frac{k\pi}{M}, \frac{(k+1)\pi}{M}]$.
- This implies that band k, for $k \neq 0$, is necessarily not confined to $[-\frac{\pi}{M}, \frac{\pi}{M}]$.
- The spectrum contained in $[-\frac{(k+1)\pi}{M}, -\frac{k\pi}{M}]$ is mapped into $[0, \pi]$, if k is odd, or into $[-\pi, 0]$, if k is even.
- The spectrum contained in the interval $[\frac{k\pi}{M}, \frac{(k+1)\pi}{M}]$ is mapped into $[-\pi, 0]$, if k is odd, or into $[0, \pi]$, if k is even.
- Note that that both $\omega = \frac{2l\pi}{M}$ and $\omega = -\frac{2l\pi}{M}$ in the original signal are mapped to $\omega = 0$ in the decimated signal.
- In order to allow proper reconstruction of signals having components of the form $A_l \cos\left(\frac{2l\pi}{M}n\right)$, the ideal filters must have half the passband gain for $\omega = \pm\frac{2l\pi}{M}$.

Decimation of a bandpass signal

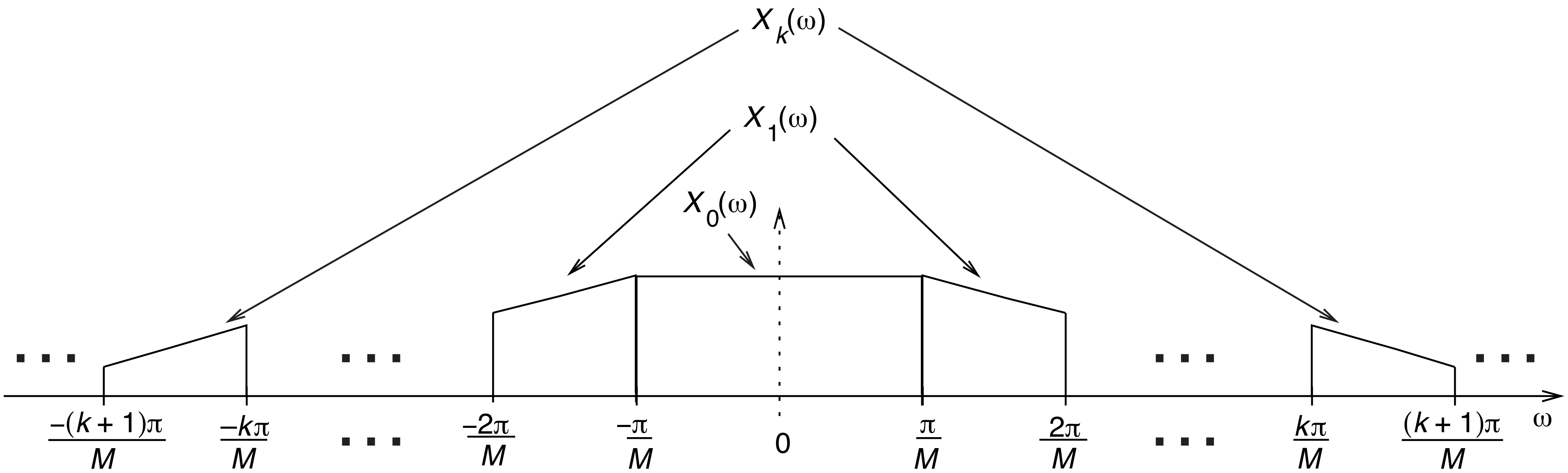


Figure 2: Uniform split of a signal into M real bands.

Decimation of a bandpass signal

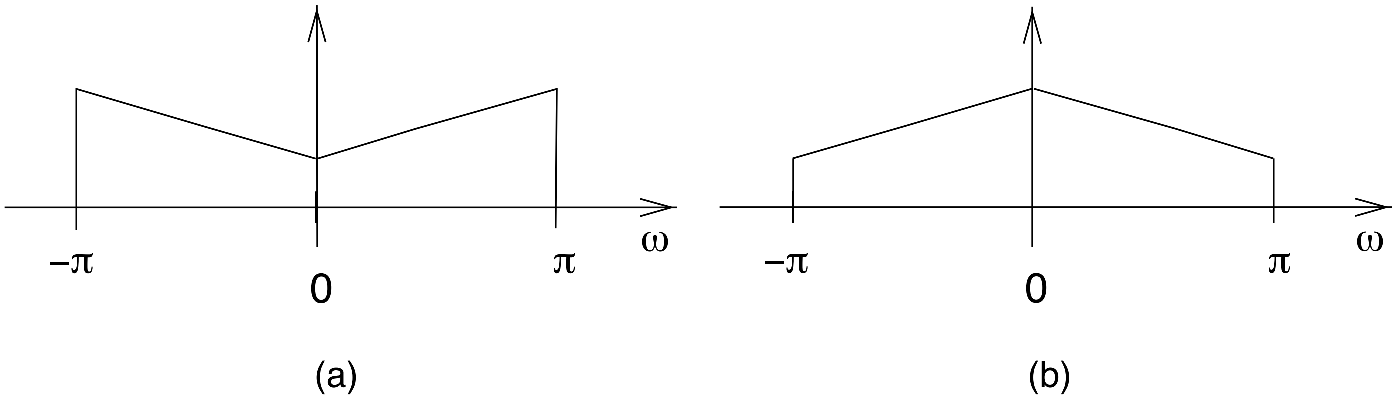


Figure 3: Spectrum of band k decimated by a factor of M: (a) k odd; (b) k even.

Inverse decimation of a bandpass signal

- A bandpass signal can be decimated by M without aliasing, provided that its spectrum is confined to $[-\frac{(k+1)\pi}{M}, -\frac{k\pi}{M}] \cup [\frac{k\pi}{M}, \frac{(k+1)\pi}{M}]$.
- The original bandpass signal be recovered from its decimated version by an interpolation operation in the bandpass case.

Inverse decimation of a bandpass signal

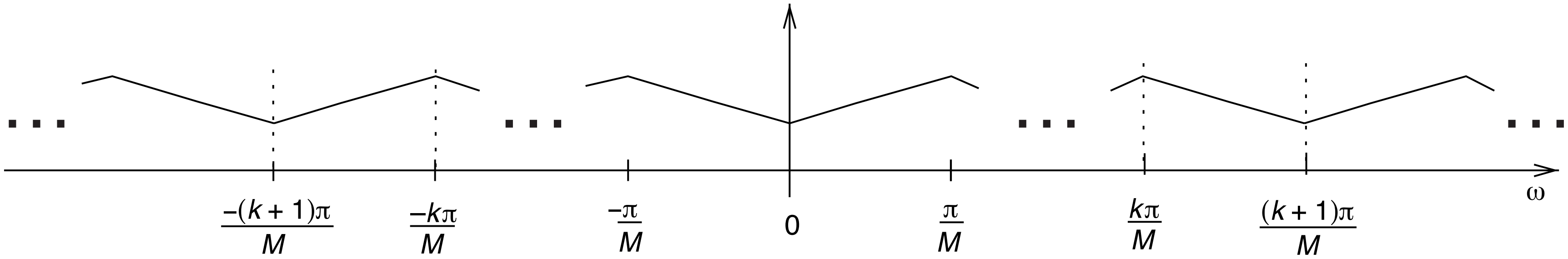


Figure 4: Spectrum of band k after decimation and interpolation by a factor of M for k odd.

Inverse decimation of a bandpass signal

- To recover band k, it suffices to keep the region of the spectrum in Figure 4 within $[-\frac{(k+1)\pi}{M}, -\frac{k\pi}{M}] \cup [\frac{k\pi}{M}, \frac{(k+1)\pi}{M}]$.
- For k even, the procedure is entirely analogous.
- The process of decimating and interpolating a bandpass signal is similar to the case of a lowpass signal, with the difference that for the bandpass case $H(z)$ must be a bandpass filter with bandwidth $[-\frac{(k+1)\pi}{M}, -\frac{k\pi}{M}] \cup [\frac{k\pi}{M}, \frac{(k+1)\pi}{M}]$.

Perfect Reconstruction: Critically decimated M-band filter banks

- If a signal $x(m)$ is decomposed into M non-overlapping bandpass channels B_k, with $k = 0, 1, \ldots, (M-1)$, such that $\bigcup_{k=0}^{M-1} B_k = [-\pi, \pi)$, then it can be recovered by just summing these M channels.

- Exact recovery of the original signal may not be possible if each channel is decimated by M.

- We examined a way to recover the bandpass signal from its decimated version. In fact, all that is needed are interpolations followed by filters with passband $[-\frac{(k+1)\pi}{M}, -\frac{k\pi}{M}] \cup [\frac{k\pi}{M}, \frac{(k+1)\pi}{M}]$.

Critically decimated M-band filter banks

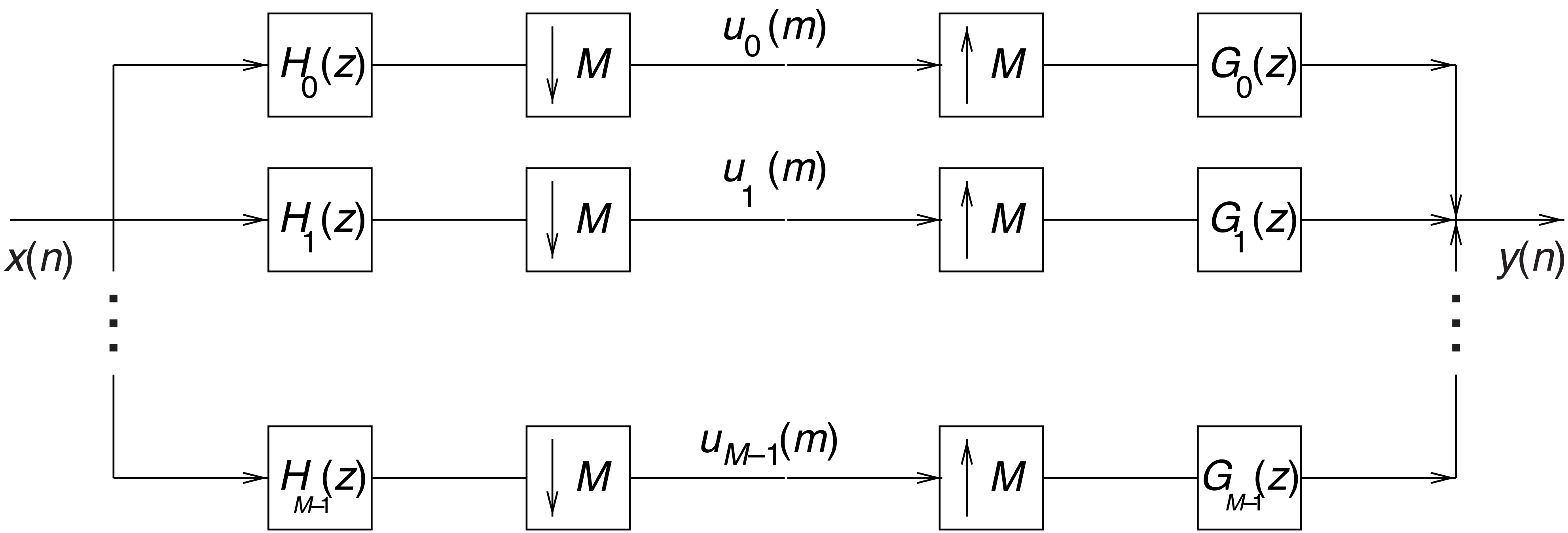


Figure 5: Block diagram of an M-band filter bank.

Critically decimated M-band filter banks

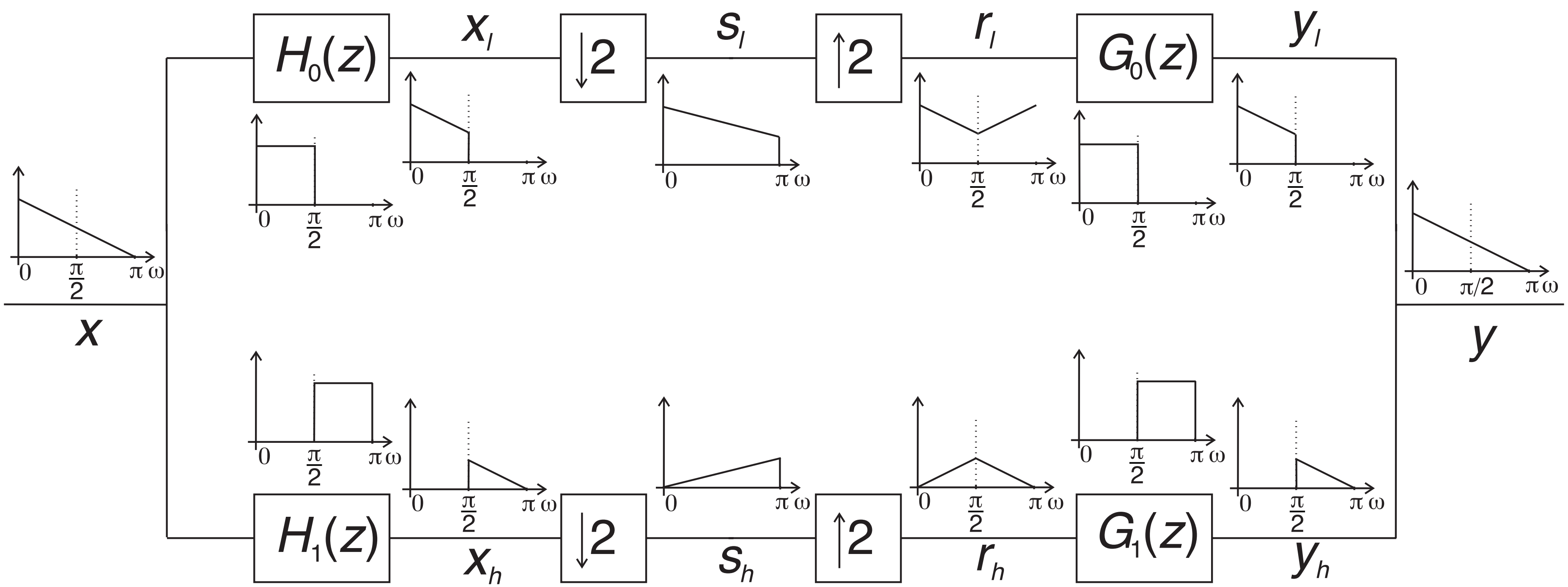


Figure 6: A 2-band perfect reconstruction filter bank using ideal filters.

Critically decimated M-band filter banks

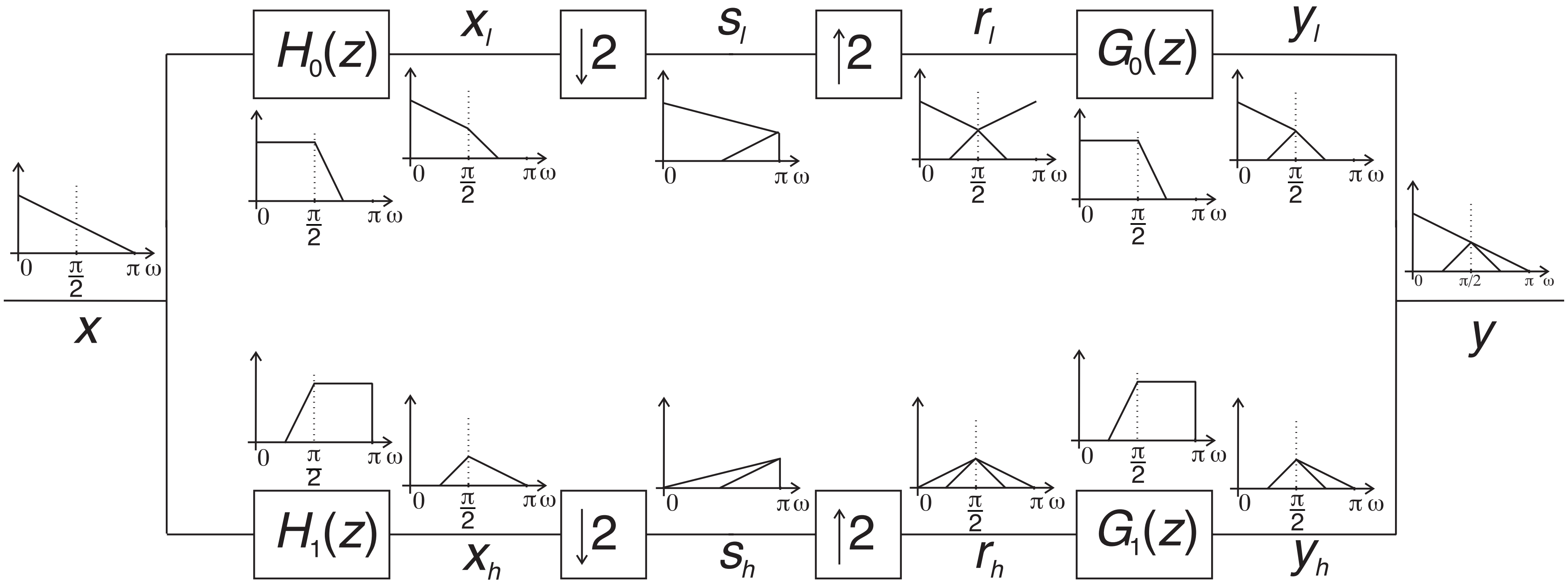


Figure 7: Two-band filter bank using realizable filters.

Critically decimated M-band filter banks

- The filters required for the M-band perfect reconstruction filter bank described above are not realizable.
- In a first analysis, the original signal would be only approximately recoverable from its decimated frequency bands.
- One can see that since the filters $H_0(z)$ and $H_1(z)$ are not ideal, the sub-bands $s_l(m)$ and $s_h(m)$ have aliasing.
- The signals $x_l(n)$ and $x_h(n)$ can not be correctly recovered from $s_l(m)$ and $s_h(m)$, respectively.

Critically decimated M-band filter banks

- Nevertheless, one can see that since $y_l(n)$ and $y_h(n)$ are added in order to obtain $y(n)$, the aliased components of $y_l(n)$ can be combined with the ones of $y_h(n)$.
- In principle, there is no reason why these aliased components could not be made to cancel each other, yielding $y(n)$ equal to $x(n)$. In such a case, the original signal could be recovered from its sub-band components.
- In an M-band filter bank as shown in Figure 5, the filters $H_k(z)$ and $G_k(z)$ are usually referred to as the analysis and synthesis filters of the filter bank.

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Perfect reconstruction: M-band filter banks in terms of polyphase components

- Representing $H_k(z)$ and $G_k(z)$ by their polyphase components,

$$H_k(z) = \sum_{j=0}^{M-1} z^{-j} E_{kj}(z^M) \tag{1}$$

$$G_k(z) = \sum_{j=0}^{M-1} z^{-(M-1-j)} R_{jk}(z^M) \tag{2}$$

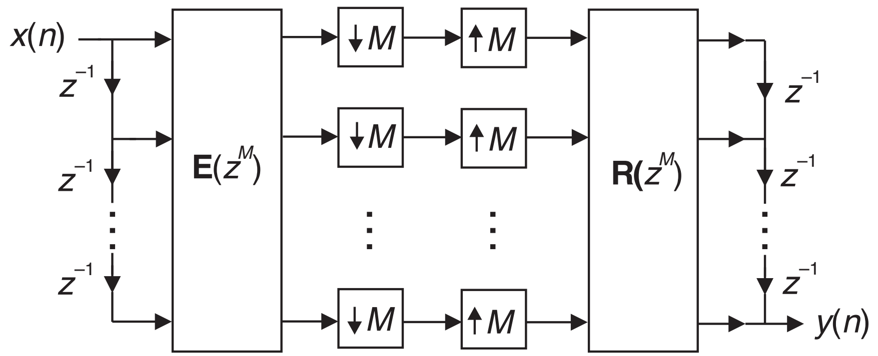
where $E_{kj}(z)$ is the jth polyphase component of $H_k(z)$, and $R_{jk}(z)$ is the jth polyphase component of $G_k(z)$.

Perfect reconstruction: M-band filter banks in terms of polyphase components

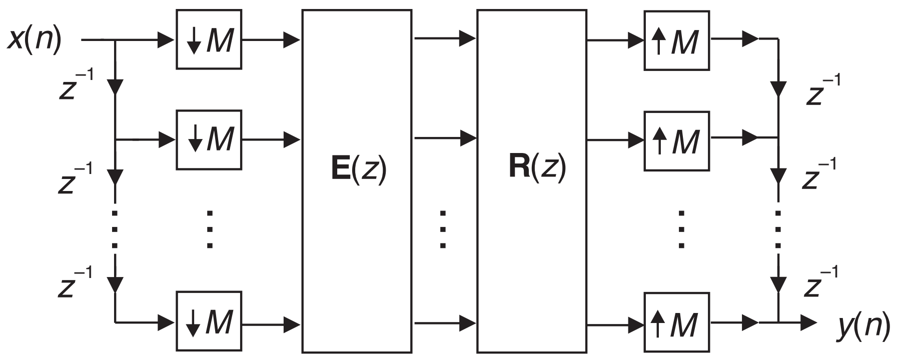
- Matrices $\mathbf{E}(z)$ and $\mathbf{R}(z)$ entries are $E_{ij}(z)$ and $R_{ij}(z)$, for $i, j = 0, 1, \ldots, (M-1)$

$$\begin{bmatrix} H_0(z) \\ H_1(z) \\ \vdots \\ H_{M-1}(z) \end{bmatrix} = \mathbf{E}(z^M) \begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-(M-1)} \end{bmatrix} \quad (3)$$

$$\begin{bmatrix} G_0(z) \\ G_1(z) \\ \vdots \\ G_{M-1}(z) \end{bmatrix} = \mathbf{R}^{\mathrm{T}}(z^M) \begin{bmatrix} z^{-(M-1)} \\ z^{-(M-2)} \\ \vdots \\ 1 \end{bmatrix} \quad (4)$$



(a)



(b)

Figure 8: M-band filter bank in terms of the polyphase components: (a) before application of the noble identities; (b) after application of the noble identities.

$\mathcal{M}$-band filter banks in terms of polyphase components

- In signal processing it is often advantageous to split a sequence $x(k)$ into several frequency bands prior to processing.

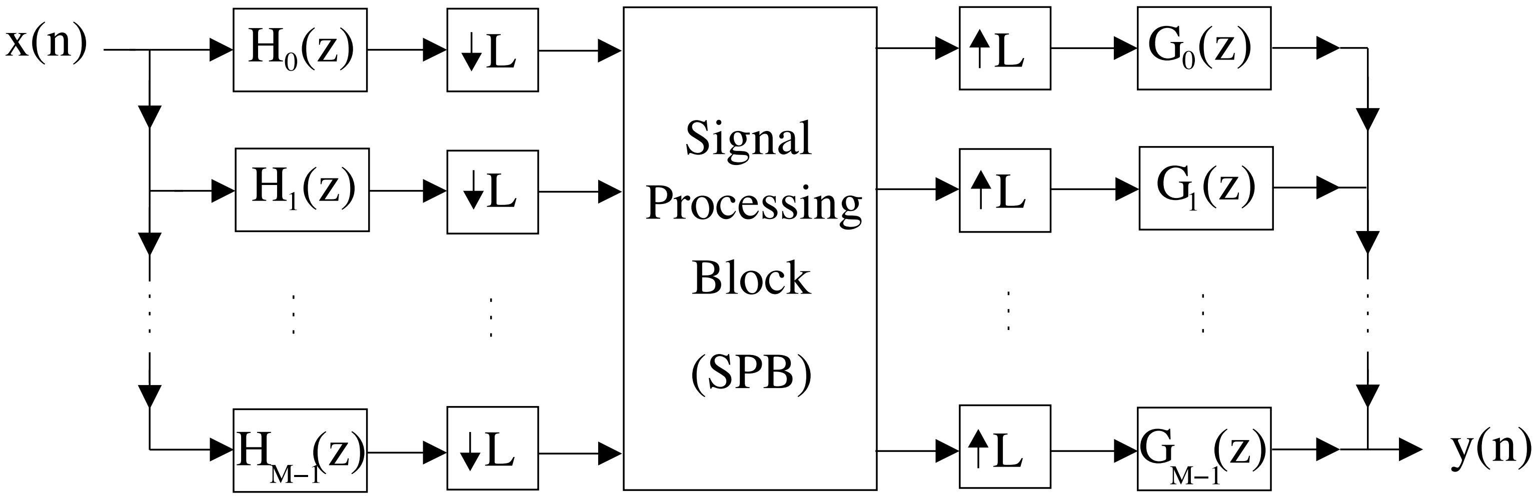


Figure 9: Signal processing in sub-bands.

M-band filter banks in terms of polyphase components

- Since the bandwidth of each analysis filter output is M times smaller than in the original signal, we can decimate each $x_i(k)$ by a factor of L smaller or equal to M and still avoid aliasing.

- For $L \leq M$, it is possible to retain all information contained in the input signal by properly designing the analysis filters in conjunction with the synthesis filters $G_i(z)$, for $i = 0, 1, \ldots, (M-1)$.

- If $L > M$ there is a loss of information due to aliasing which does not allow the recovery of the original signal.

- For $L = M$, we refer to the filter bank as maximally (or critically) decimated.

- For $L < M$, the filter bank is called oversampled (or noncritically sampled) since the set of sub-bands comprises more samples than the input signal.

Perfect reconstruction M-band filter banks

- If **R**(z)**E**$(z) =$ **I**, where **I** is the identity matrix, the M-band filter bank becomes that one shown in Figure 10.

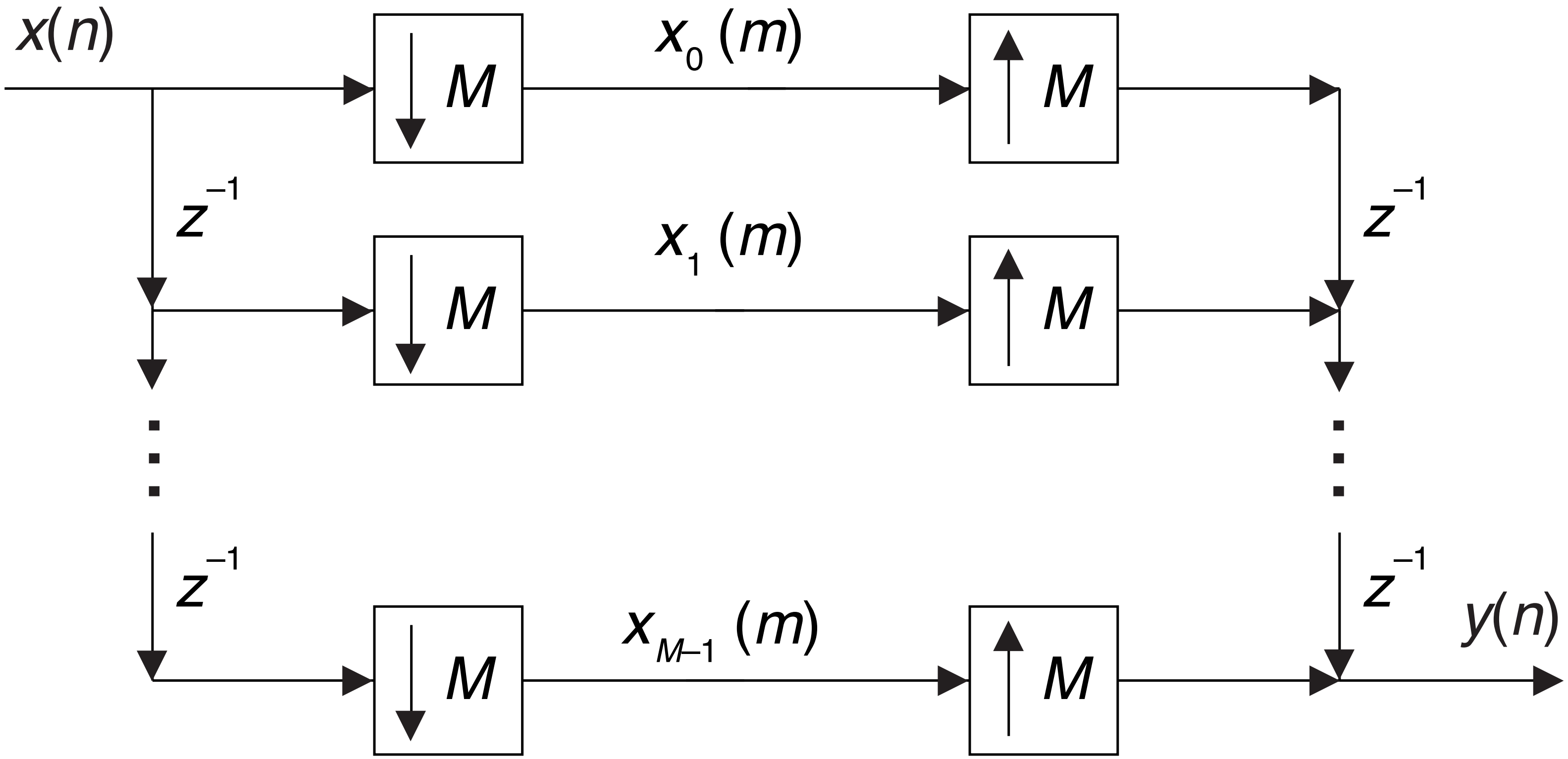


Figure 10: M-band filter bank when **R**(z)**E**$(z) =$ **I**.

Perfect reconstruction M-band filter banks

- By substituting the decimators and interpolators by the commutator models we arrive at the a scheme which is clearly equivalent to a pure delay.
- Therefore, the condition $\mathbf{R}(z)\mathbf{E}(z) = \mathbf{I}$ guarantees perfect reconstruction for the M-band filter bank.
- If $\mathbf{R}(z)\mathbf{E}(z)$ is equal to a pure delay one can still consider that perfect reconstruction holds.
- The weaker condition is sufficient for perfect reconstruction.

$$\mathbf{R}(z)\mathbf{E}(z) = z^{-\Delta}\mathbf{I} \tag{5}$$

- The total delay introduced by a perfect reconstruction filter bank is

$$\Delta_{\text{total}} = M\Delta + M - 1 \tag{6}$$

where $M\Delta$ is the delay originated from the polyphase matrices product and the term $(M - 1)$ accounts for the delay introduced by the commutator.

Perfect reconstruction M-band filter banks

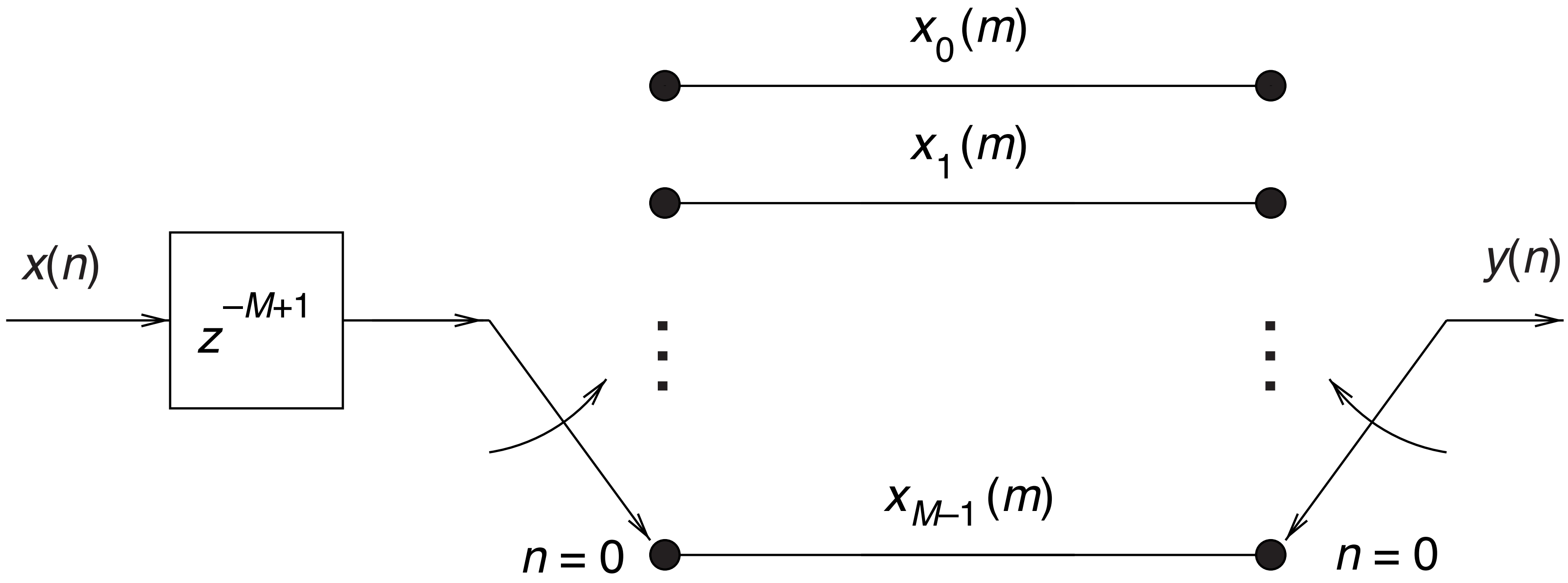


Figure 11: The commutator model of an M-band filter bank when $\mathbf{R}(z)\mathbf{E}(z) = \mathbf{I}$ is equivalent to a pure delay.

Perfect reconstruction M-band filter banks

How a simple perfect reconstruction filter bank can be built?

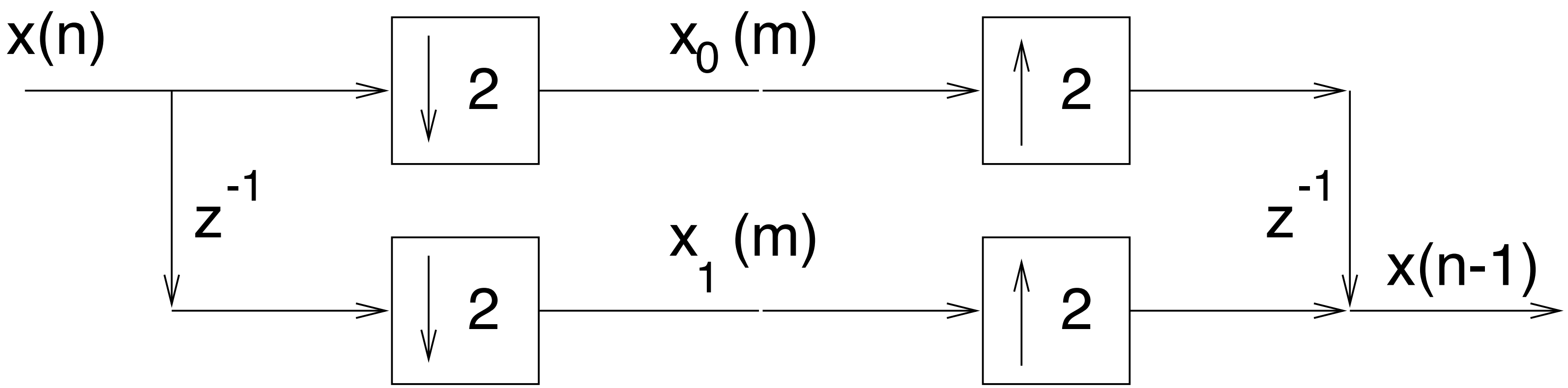


Figure 12: 2-band unit delay.

Perfect reconstruction M-band filter banks

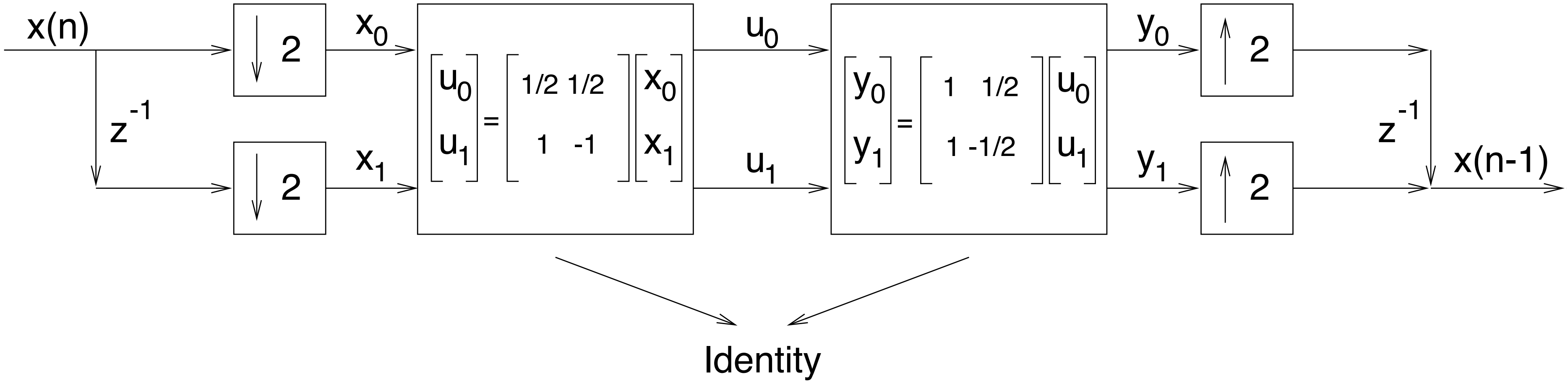


Figure 13: 2-band unit delay, including inverse matrices.

Perfect reconstruction M-band filter banks

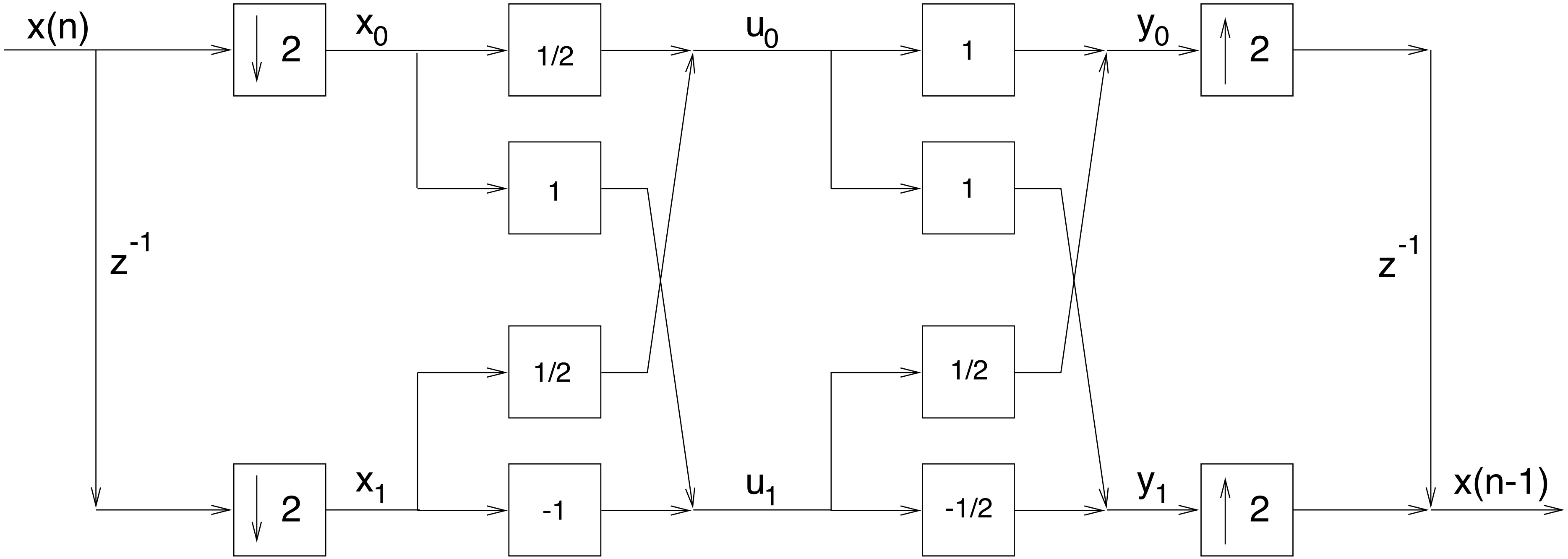


Figure 14: 2-band unit delay, with explicit realization of the matrix products.

Perfect reconstruction M-band filter banks

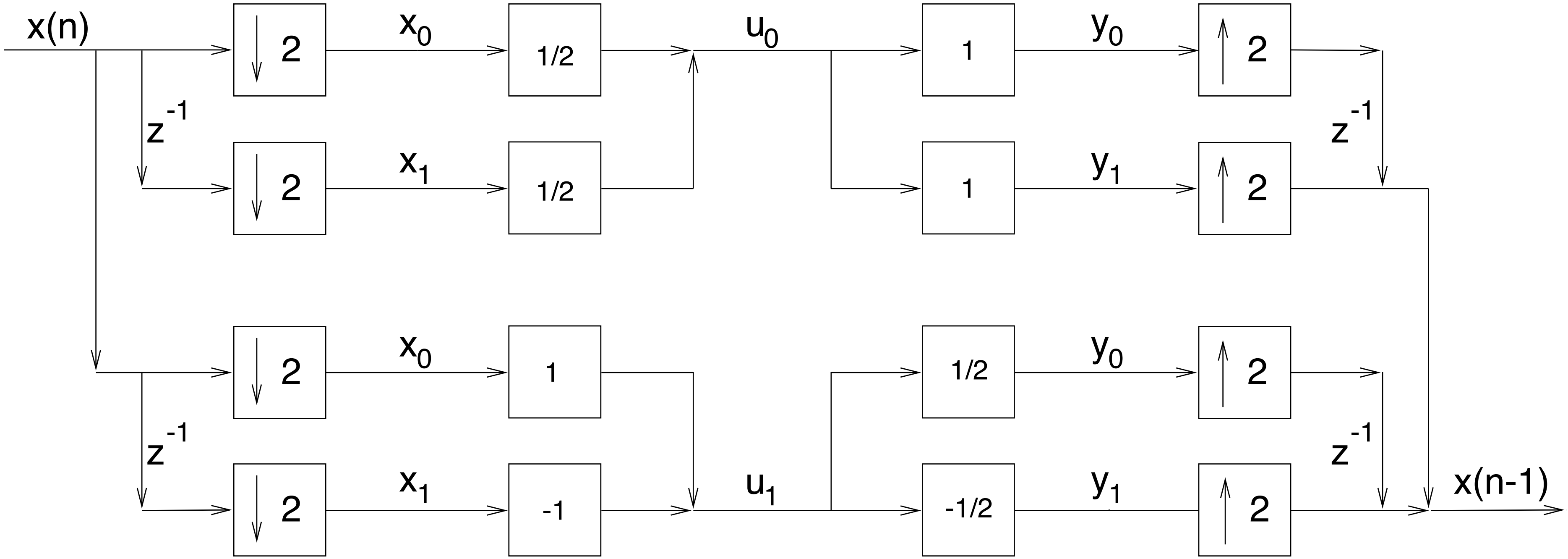


Figure 15: 2-band unit delay, splitting the decimators and interpolators.

Perfect reconstruction M-band filter banks

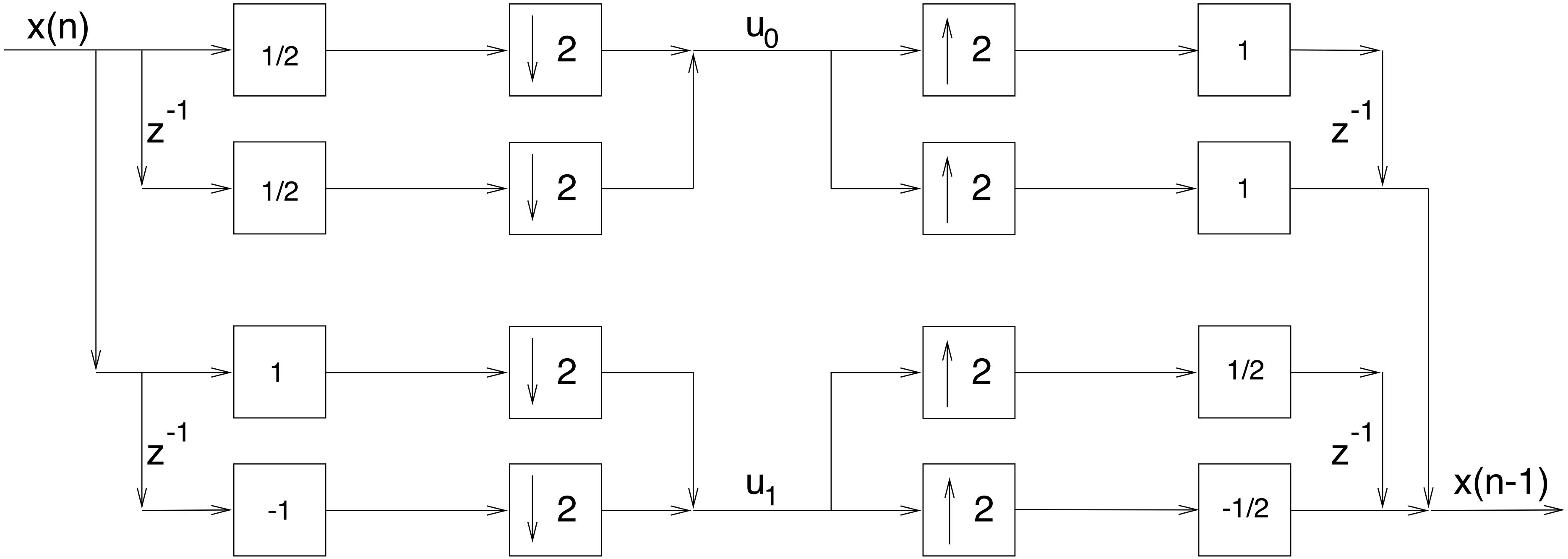


Figure 16: 2-band unit delay, moving the decimators and interpolators.

Perfect reconstruction M-band filter banks

By “merging” the decimators/interpolators we reach the realization of the unit delay as shown in Figure 17. Figure 17 is equivalent to the filter bank in Figure 18.

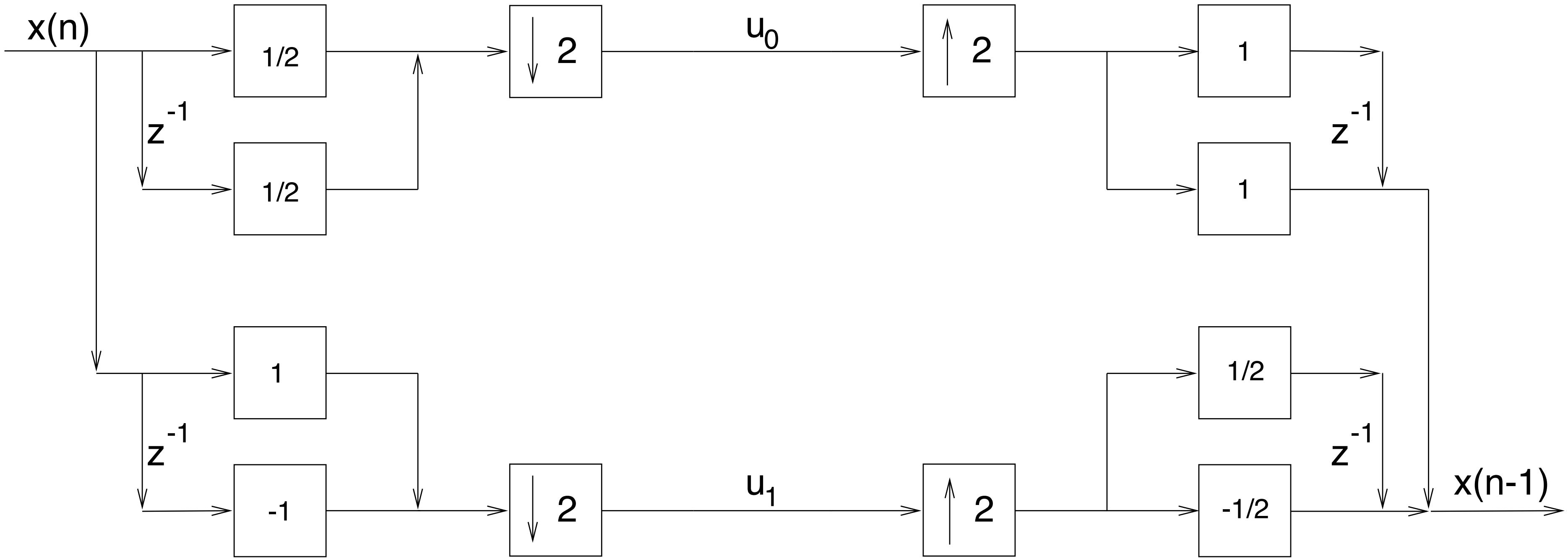


Figure 17: 2-band unit delay, merging the decimators and interpolators.

Perfect reconstruction M-band filter banks

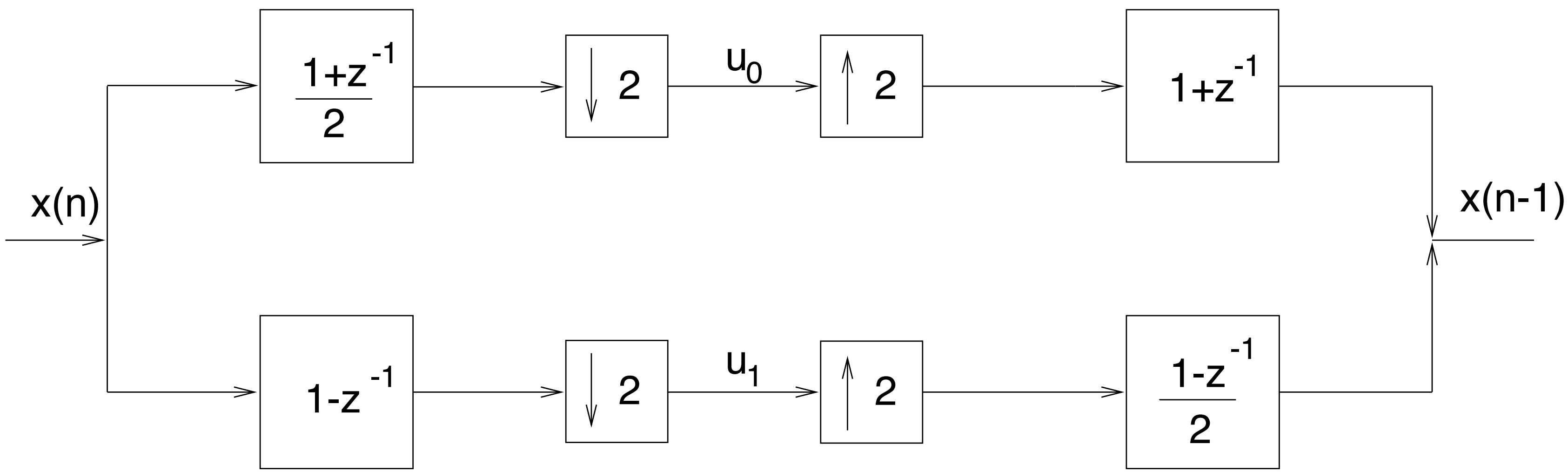


Figure 18: 2-band filter bank with perfect reconstruction.