ELEC4 - EIEL821

Spectral Analysis — Solutions to exercises
Chapter 1: Random processes

Vicente Zarzoso
vicente.zarzoso@univ-cotedazur.fr

Electrical Engineering Department
Polytech'Nice Sophia
Université Côte d'Azur

April–June 2020

1.1 Harmonic process with random phase

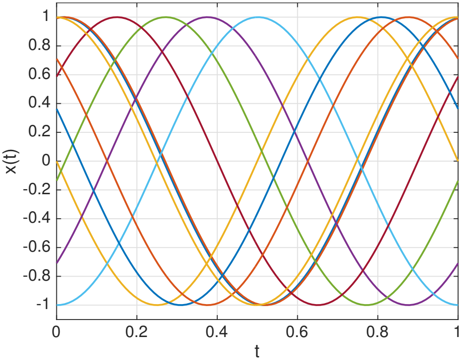
[HAY96, Example 3.3.1, p. 78] We define the stochastic process $X(t) = A\cos(wt+\theta)$, where θ is a random variable uniformly distributed in $[-\pi, \pi[$, and A, w are real-valued constants.
a) Plot several realizations $x(t)$ of process $X(t)$.

Solution: Each realization $x_k(t)$ is a sinusoid with the same frequency but random initial phase:

$$x_k(t) = A\cos(wt+\theta_k)$$

where θ_k is a realization of a random variable uniformly distributed in $[-\pi, \pi[$.

For instance, ten realizations $x_k(t)$, $k = 1, 2, \ldots, 10$, with $A = 1$ and $w = 2\pi$, could look like this:



b) Compute the ensemble mean of $X(t)$, $\mu_X(t)$.

Solution: By definition, $\mu_X(t) = \mathrm{E}\{X(t)\}$ and

$$\mathrm{E}\{X(t)\} = \int_{-\infty}^{\infty} x f_X(x;t)dx.$$

In theory, we would have to compute $f_X(x;t)$. But we do not need to because:

Useful result

If $y = g(x)$ then

$$\mathrm{E}\{y\} = \int_{-\infty}^{\infty} y f_Y(y)dy = \mathrm{E}\{g(x)\} = \int_{-\infty}^{\infty} g(x) f_X(x)dx.$$

Hence, we just need to express $X(t)$ as a function of random variable θ and compute the expectation in terms of θ:

$$\mathrm{E}\{X(t)\} = \mathrm{E}\{A\cos(wt+\theta)\} = \int_{-\infty}^{\infty} A\cos(wt+\theta) f_\Theta(\theta)d\theta.$$

$$E\{X(t)\} = E\{A\cos(wt+\theta)\} = \int_{-\infty}^{\infty} A\cos(wt+\theta) f_\Theta(\theta)\, d\theta.$$

Now, $\theta \equiv U(-\pi, \pi)$, and then:

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi} & |\theta| \leq \pi \\ 0 & |\theta| > \pi. \end{cases}$$

Therefore:

$$E\{x(t)\} = \frac{A}{2\pi}\int_{-\pi}^{\pi} \cos(wt+\theta)\, d\theta = \frac{A}{2\pi}\left[\sin(wt+\theta)\right]_{-\pi}^{\pi} = 0.$$

c) Compute the ensemble variance of $X(t)$, $\sigma_X^2(t)$.

Solution: By definition, $\sigma_X^2(t) = \mathrm{E}\{[x(t) - \mu_X(t)]^2\}$.

In the previous exercise, we computed $\mu_X(t) = 0$. Hence:

$$\mathrm{E}\{[X(t) - \mu_X(t)]^2\} = \mathrm{E}\{X(t)^2\} = \int_{-\infty}^{\infty} x^2 f_X(x;t)dx.$$

But, again, we need not compute $f_X(x;t)$ because:

$$\mathrm{E}\{X(t)^2\} = \mathrm{E}\{[A\cos(wt+\theta)]^2\} = \int_{-\infty}^{\infty} A^2\cos^2(wt+\theta)f_\Theta(\theta)d\theta = \frac{A^2}{2\pi}\int_{-\pi}^{\pi}\cos^2(wt+\theta)d\theta.$$

Useful trigonometric identity (linearization)

$$\cos(a)\cos(b) = \frac{1}{2}\left[\cos(a+b) + \cos(a-b)\right]$$

Using this identity, we have

$$\frac{A^2}{2\pi}\int_{-\pi}^{\pi}\cos^2(wt+\theta)d\theta = \frac{A^2}{4\pi}\left[\theta + \frac{1}{2}\sin(2wt+2\theta)\right]_{-\pi}^{\pi} = \frac{A^2}{2}.$$

Interestingly, this result coincides with the power of a deterministic sinusoid with amplitude A.

d) Compute the ensemble autocorrelation function of $X(t)$, $R_X(t_1, t_2)$.

Solution: We recall the definition of the autocorrelation function:

$$R_X(t_1, t_2) = \mathrm{E}\{X(t_1)X(t_2)\}.$$

Again, we can replace $X(t)$ by its definition in terms of r.v. θ:

$$\mathrm{E}\{X(t_1)X(t_2)\} = \mathrm{E}\{A^2 \cos(wt_1 + \theta)\cos(wt_2 + \theta)\}.$$

Exploiting again the trigonometric identity, we have:

$$E\{A^2 \cos(wt_1 + \theta)\cos(wt_2 + \theta)\} = \frac{A^2}{2}\mathrm{E}\Big\{\cos(w(t_1 + t_2) + 2\theta) + \cos(w(t_1 - t_2))\Big\}.$$

The mathematical expectation with respect to θ then yields:

$$R_X(t_1, t_2) = \frac{A^2}{2}\cos(w(t_1 - t_2)).$$

e) Study the stationarity of process $X(t)$.

Solution: To be wide sense stationary (WSS), random process $X(t)$ must fulfil two conditions:

C1) $\mu_X(t)$ is a constant independent of t.

C2) $R_X(t_1, t_2)$ only depends on the time lag $\tau \stackrel{\text{def}}{=} (t_2 - t_1)$.

Condition C1 is verified because $\mu_X(t) = 0$.

Condition C2 is also verified because

$$R_X(t_1, t_2) = \frac{A^2}{2} \cos(w(t_1 - t_2)) \underset{\tau \stackrel{\text{def}}{=} (t_2 - t_1)}{\underset{\uparrow}{=}} \frac{A^2}{2} \cos(w\tau) = R_X(\tau).$$

We can conclude that process $X(t)$ is indeed WSS.

We also realize that $R_X(0) = \frac{A^2}{2} = P_X = \sigma_X^2$, since $\mu_X = 0$.

1.2 Harmonic process with random amplitude

Repeat Problem 1.1 assuming A is a random variable uniformly distributed in $[0, 1]$ and w, θ are real constants.

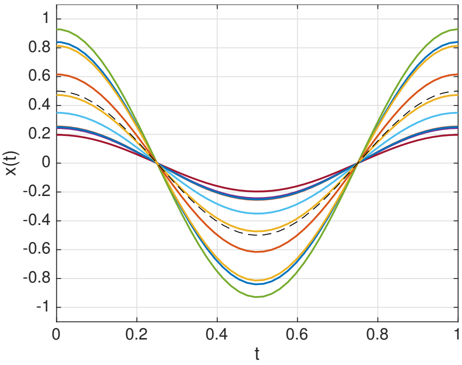
a) Plot several realizations $x(t)$ of process $X(t)$.

Solution: Each realization $x_k(t)$ is a sinusoid with the same frequency and initial phase, but random amplitude:

$$x_k(t) = A_k \cos(wt + \theta)$$

where A_k is a realization of a random variable uniformly distributed in $[0, 1]$.

For instance, ten realizations $x_k(t)$, $k = 1, 2, \ldots, 10$, with $w = 2\pi$ and $\theta = 0$, could look like this:



b) Compute the ensemble mean of $X(t)$, $\mu_X(t)$.

Solution: Now we have:

$$\mu_X(t) = \mathrm{E}\{X(t)\} = \mathrm{E}\{A\cos(wt+\theta)\}.$$

But only A is random, and then

$$\mathrm{E}\{A\cos(wt+\theta)\} = \mathrm{E}\{A\}\cos(wt+\theta) = \frac{1}{2}\cos(wt+\theta)$$

because

$$\mathrm{E}\{A\} = \int_{-\infty}^{\infty} a f_A(a)da = \int_0^1 a f_A(a)da = \left[\frac{a^2}{2}\right]_0^1 = \frac{1}{2}.$$

Function $\mu_X(t)$ is plotted in the dashed black line of the previous figure.

c) Compute the ensemble variance of $X(t)$, $\sigma_X^2(t)$.

Solution:

$$\sigma_X^2(t) = \mathrm{E}\{[X(t) - \mu_X(t)]^2\} = \mathrm{E}\{X(t)^2\} - \mu_X(t)^2.$$

Now

$$\mathrm{E}\{X(t)^2\} = \mathrm{E}\{A^2\cos^2(wt+\theta)\} = \mathrm{E}\{A^2\}\cos^2(wt+\theta) = \frac{1}{3}\cos^2(wt+\theta)$$

Q: Can you compute $\mathrm{E}\{A^2\}$?

Therefore:

$$\sigma_X^2(t) = \frac{1}{12}\cos^2(wt+\theta).$$

d) Compute the ensemble autocorrelation function of $X(t)$, $R_X(t_1, t_2)$.

Solution:

$$R_X(t_1, t_2) = \mathrm{E}\{X(t_1)X(t_2)\} = \mathrm{E}\{A^2 \cos(wt_1 + \theta)\cos(wt_2 + \theta)\}$$

$$= \mathrm{E}\{A^2\}\cos(wt_1 + \theta)\cos(wt_2 + \theta) = \frac{1}{3}\cos(wt_1 + \theta)\cos(wt_2 + \theta)$$

$$\underset{\substack{\uparrow \\ \cos(a)\cos(b)=\dots}}{=} \quad \frac{1}{6}\Big[\cos(w(t_1 + t_2) + 2\theta) + \cos(w(t_1 - t_2))\Big].$$

e) Study the stationarity of process $X(t)$.

Solution: The process is NOT WSS, because

- $\mu_X(t) \neq$ constant

$\Rightarrow$ condition C1 is violated.

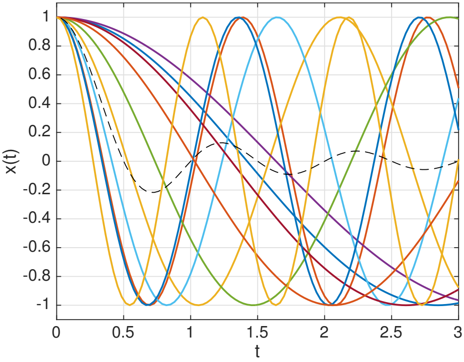
- $R_X(t_1, t_2) = \frac{1}{6}\left[\cos(w(t_1+t_2)+2\theta)+\cos(w(t_1-t_2)\right]$ is not a function of $\tau = (t_2 - t_1)$ only

$\Rightarrow$ condition C2 is violated.

1.3 Harmonic process with random frequency

Repeat Problem 1.1 assuming that w is a random variable uniformly distributed in $[0, 2\pi]$ and A, θ are real constants. To ease the calculations, you can assume that $\theta = 0$.
a) Realizations

Solution: Ten random realizations are shown in the figure below.



b) Ensemble mean

Solution:

$$\mu_X(t) = A\frac{\sin(2\pi t)}{2\pi t} = A\mathrm{sinc}(2\pi t).$$

Function $\mu_X(t)$ is shown in the dashed black line of the previous figure.

c) Ensemble variance

Solution:

$$\sigma_X^2(t) = \frac{A^2}{2}\left[1 + \mathrm{sinc}(4\pi t) - 2\mathrm{sinc}^2(2\pi t)\right].$$

d) Autocorrelation function

Solution:

$$R_X(t_1, t_2) = \frac{A^2}{2}\left[\mathrm{sinc}(2\pi(t_1 + t_2)) + \mathrm{sinc}(2\pi(t_1 - t_2))\right].$$

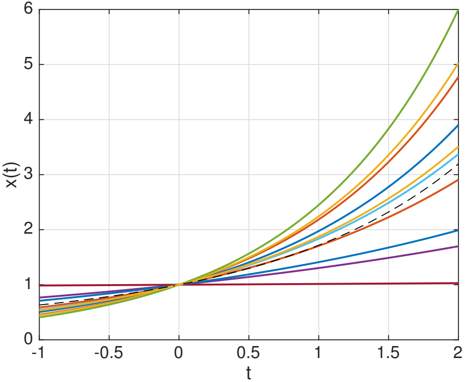
e) Stationarity

Solution: Clearly, this is not a WSS process.

1.4 Random exponential

We define the random process $X(t) = e^{at}$, consisting of a family of exponentials defined as a function of random variable a, with uniform probability density function (pdf) $f_A(a)$ in $[0, 1]$. Same questions as in Problem 1.1.
a) Realizations

Solution: Ten random realizations are shown below.



b) Ensemble mean

Solution:

$$\mu_X(t) = \frac{e^t - 1}{t}.$$

See dashed black line in previous plot.

c) Ensemble variance

Solution:

$$\sigma_X^2(t) = \frac{e^{2t} - 1}{2t} - \frac{(e^t - 1)^2}{t^2}.$$

d) Autocorrelation function

Solution:

$$R_X(t_1, t_2) = \frac{e^{t_1+t_2} - 1}{t_1 + t_2}.$$

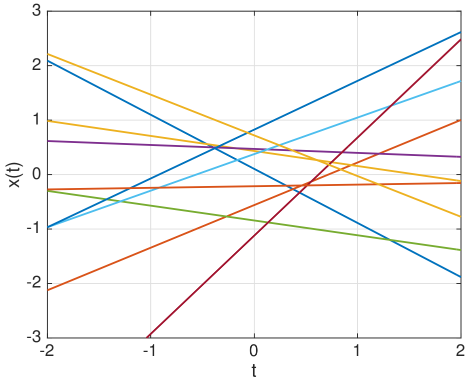
e) Stationarity

Solution: No WSS.

1.5 Two Gaussians

We define the stochastic process $X(t) = a + bt$ from two independent normalized (zero-mean, unit-variance) Gaussian random variables a and b.
Same questions as in Problem 1.1.
a) Realizations

Solution: The figure below plots ten realizations of this random process.



b) Ensemble mean

Solution:

$$\mu_X(t) = 0.$$

c) Ensemble variance

Solution:

$$\sigma_X^2(t) = 1 + t^2.$$

Q: Why $\mathrm{E}\{ab\} = 0$?

d) Autocorrelation function

Solution:

$$R_X(t_1, t_2) = 1 + t_1 t_2.$$

e) Stationarity

Solution: No WSS.

1.6 Ergodicity

Show that the harmonic process with random initial phase (Problem 1.1) is wide sense ergodic.

Solution:

We need to check whether the harmonic process with random initial phase fulfils

$$m_X \stackrel{\text{def}}{=} \langle x(t) \rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+T/2} x(t)dt = \mu_X$$

and

$$\Gamma_X(\tau) \stackrel{\text{def}}{=} \langle x(t)x(t+\tau) \rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+T/2} x(t)x(t+\tau)dt = R_X(\tau).$$

Concerning the time-averaged mean:

$$m_X = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+T/2} x(t)dt = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+T/2} A\cos(wt+\theta)dt.$$

Also

$$\int_{-T/2}^{+T/2} A\cos(wt+\theta)dt = \frac{A}{w}\left[\sin(wt+\theta)\right]_{-T/2}^{+T/2} \stackrel{\text{def}}{=} \xi(T), \quad \text{with } |\xi(T)| \leq \frac{2A}{w}.$$

Hence:

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+T/2} x(t)dt = \lim_{T\to\infty} \frac{\xi(T)}{T} = 0 = \mu_X.$$

Concerning the time-averaged autocorrelation:

$$\Gamma_X(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+T/2} x(t)x(t+\tau)dt = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+T/2} A^2 \cos(wt+\theta)\cos(w(t+\tau)+\theta)dt.$$

Using the trigonometric identity again, we note that

$$\cos(wt+\theta)\cos(w(t+\tau)+\theta) = \frac{1}{2}\left[\cos(2wt+2\theta)+\cos(w\tau)\right]$$

and then

$$\int_{-T/2}^{+T/2} A^2 \cos(wt+\theta)\cos(w(t+\tau)+\theta)dt = \frac{A^2}{2}\left[\frac{1}{2w}\sin(2wt+2\theta)\right]_{-T/2}^{+T/2} + \frac{A^2 T}{2}\cos(w\tau).$$

Dividing by T and taking limits:

$$\Gamma_X(\tau) = \frac{A^2}{2}\cos(w\tau) = R_X(\tau)$$

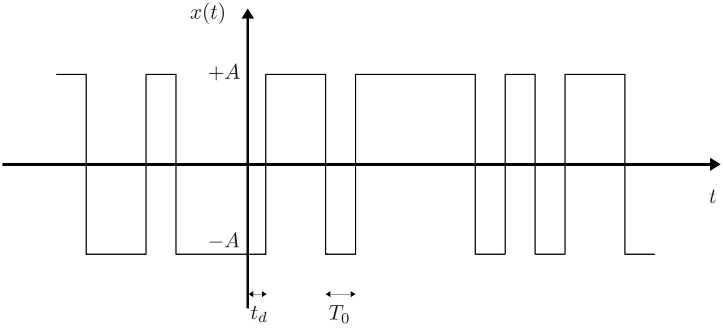
and the process is indeed wide sense ergodic.

1.7 Random binary sequence (Bernoulli process)

We consider the stochastic process $X(t)$ defined as a random sequence of binary symbols:

- Bits '0' and '1' are represented by an impulse of amplitude $-A$ and $+A$, respectively, and duration T_0 (constant value).
- Impulses are not synchronized, i.e., the delay t_d of the first impulse onset is a random variable, uniformly distributed in $[0, T_0[$.
- Bits '0' and '1' are equiprobable and independent.

A realization of $X(t)$ is illustrated in the following figure:



Compute:
a) The probability density function (pdf) of $X(t)$, $f_X(x;t)$.

Solution:

$$f_X(x;t) = \frac{1}{2}\delta(x-A) + \frac{1}{2}\delta(x+A).$$

b) The ensemble mean $\mu_X(t)$.

Solution:

$$\mu_X(t) = 0.$$

c) The autocorrelation function $R_X(t_1, t_2)$. Is the process stationary?

Solution:

$$R_X(t_1,t_2) = \begin{cases} A^2\left[1 - \frac{|t_1-t_2|}{T_0}\right], & |t_1-t_2| \le T_0 \\ 0, & |t_1-t_2| > T_0. \end{cases}$$

The process is WSS since $\mu_X(t)$ is constant and $R_X(t_1, t_2)$ depends only on $\tau = (t_2 - t_1)$.

d) The power spectral density (PSD) $S_X(f)$.

Solution:

$$S_X(f) = \mathcal{F}[R_X(\tau)] = A^2 T_0 \mathrm{sinc}^2(\pi f T_0).$$

e) The PSD of an impulse of amplitude A and duration T_0.

Solution:

$$S_X(f) = |\mathcal{F}[x(t)]|^2 = A^2 T_0^2 \mathrm{sinc}^2(\pi f T_0).$$

1.8 Discrete-time white noise

Let us consider the discrete-time random process $X(n)$, $n \in \mathbb{Z}$. For fixed n, $X(n)$ is a random variable with mean μ and variance σ^2. Random variables $X(n)$ and $X(m)$ are assumed to be statistically independent for $m \neq n$. Compute the autocorrelation sequence $R_X(k)$ and the power spectral density $S_X(\omega)$ of $X(n)$. If $\mu = 0$, this process is called *white noise*. Explain why.

Solution: The autocorrelation sequence is equivalent to the autocorrelation function for discrete-time processes. It is defined as:

$$R_X(n, n+k) = \mathrm{E}\{X(n)X(n+k)\} \qquad \text{with } k \in \mathbb{Z}.$$

According to the random sequence described in the exercise, we have:

$$\mathrm{E}\{X(n)X(n+k)\} = \begin{cases} \mathrm{E}\{X(n)^2\} = \sigma^2 + \mu^2, & k = 0 \\ \mathrm{E}\{X(n)\}\mathrm{E}\{X(n+k)\} = \mu^2, & k \neq 0. \end{cases}$$

This can be expressed compactly as

$$R_X(n, n+k) = \mu^2 + \sigma^2\delta(k).$$

We realize that $R_X(n, n+k)$ does not depend on n but only on the discrete-time lag k:

$$R_X(k) = \mu^2 + \sigma^2 \delta(k).$$

The process is hence WSS and its PSD is defined. It can be computed as

$$S_X(\omega) = \mathcal{F}[R_X(k)] = \sum_{k=-\infty}^{+\infty} R_X(k) e^{-\jmath \omega k}.$$

Recalling some discrete-time Fourier transform pairs will be useful at this point:

$x(k)$	$X(f), \quad \lvert f \rvert < 0.5$	$X(\omega), \quad \lvert \omega \rvert < \pi$
1	$\delta(f)$	$2\pi\delta(\omega)$
$e^{\jmath \omega_0 k}$	$\delta(f - f_0)$	$2\pi\delta(\omega - \omega_0)$
$\cos(\omega_0 k)$	$\frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$	$\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$
$\delta(k)$	1	1
$\delta(k - k_0)$	$e^{-\jmath 2\pi f k_0}$	$e^{-\jmath \omega k_0}$

Accordingly, we have:

$$S_X(\omega) = 2\pi\mu^2\delta(\omega) + \sigma^2.$$

If $\mu = 0$ (zero-mean sequence), Dirac's delta at the origin vanishes, and $S_X(\omega) = \sigma^2$: the PSD becomes flat and the signal power is distributed evenly across all frequencies.

This phenomenon is analogous to the spectrum of white light, to which all colors contribute equally. For this reason, the process is called *white noise*.