PNS ELEC4 2019–2020 — EIEL821 Spectral Analysis

Tutorials and Computer Labs

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1 $\boxed{\text{Stochastic processes}}$

1.1 Harmonic process with random phase

[HAY96, Example 3.3.1, p. 78] We define the stochastic process $X(t) = A\cos(wt + \theta)$, where θ is a random variable uniformly distributed in $[-\pi, \pi[$, and A, w are real-valued constants.

a) Plot several realizations $x(t)$ of process $X(t)$.

b) Compute the ensemble mean of $X(t)$, $\mu_X(t)$.

c) Compute the ensemble variance of $X(t)$, $\sigma_X^2(t)$.

d) Compute the ensemble autocorrelation function $X(t)$ of $R_X(t_1, t_2)$.

e) Study the stationarity of process $X(t)$.

1.2 Harmonic process with random amplitude

Repeat Problem 1.1 assuming that A is a random variable uniformly distributed in $[0, 1]$ and w, θ are real constants.

1.3 Harmonic process with random frequency

Repeat Problem 1.1 assuming that w is a random variable uniformly distributed in $[0, 2\pi]$ and A, θ are real constants. To ease the calculations, you can assume that $\theta = 0$.

1.4 Random exponential

We define the random process $X(t) = e^{at}$, consisting of a family of exponentials defined as a function of random variable a, with uniform probability density function (pdf) $f_A(a)$ in $[0, 1]$.

Same questions as in Problem 1.1.

1.5 Two Gaussians

We define the stochastic process $X(t) = a + bt$ from two independent normalized (zero-mean, unit-variance) Gaussian random variables a and b.

Same questions as in Problem 1.1.

1.6 Ergodicity

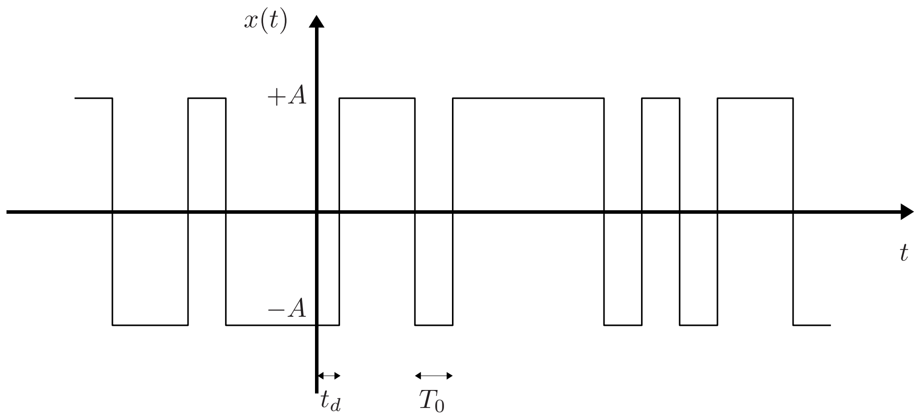
Show that the harmonic process with random initial phase (Problem 1.1) is wide sense ergodic.

1.7 Random binary sequence (Bernoulli process)

We consider the stochastic process $X(t)$ defined as a random sequence of binary symbols:

- Bits '0' and '1' are represented by an impulse of amplitude $-A$ and $+A$, respectively, and duration T_0 (constant value).
- Impulses are not synchronized, i.e., the delay t_d of the first impulse onset is a random variable, uniformly distributed in $[0, T_0[$.
- Bits '0' and '1' are equiprobable and independent.

A realization of $X(t)$ is illustrated in the following figure:



Compute:

a) The probability density function (pdf) of $X(t)$, $f_X(x;t)$.

b) The ensemble mean $\mu_X(t)$.

c) The autocorrelation function $R_X(t_1, t_2)$. Is the process stationary?

d) The power spectral density (PSD) $S_X(f)$.

e) The PSD of an impulse of amplitude A and duration T_0.

1.8 Discrete-time white noise

Let us consider the discrete-time random process $X(n)$, $n \in \mathbb{Z}$. For fixed n, $X(n)$ is a random variable with mean μ and variance σ^2. Random variables $X(n)$ and $X(m)$ are assumed to be statistically independent for $m \neq n$. Compute the autocorrelation sequence $R_X(k)$ and the power spectral density $S_X(\omega)$ of $X(n)$. If $\mu = 0$, this process is called *white noise*. Explain why.

1.9 Python lab

Download and complete **Jupyter notebook** for **Chapter 1: random processes**.

2 Non-parametric spectral estimation

2.1 PSD definitions

Prove that the two definitions of PSD given at the beginning of Chapter 2 are equivalent.

Hint: Consider the following alternative definition for the autocorrelation sequence (ACS):

$$r(k) = \lim_{N\to\infty} \mathrm{E}\left\{ \frac{1}{N} \sum_{n=0}^{N-1} y(n)y^*(n-k) \right\}. \tag{1}$$

Replace this expression in the first definition of PSD and develop to reach the second definition. To simplify the development, use the windowed sequence defined as:

$$y_N(n) = \begin{cases} y(n) & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

and extend the summations from $-\infty$ to ∞.

2.2 ACS definitions

Prove that the two definitions given in Chapter 2 correspond indeed to a biased and an unbiased estimator of the ACS, respectively.

2.3 Link between periodogram and correlogram

Prove that, if the biased estimate of the ACS is used, the correlogram and the periodogram are equivalent.

Hint: In the expression of the correlogram, express the biased ACS estimate in terms of the windowed sequence $y_N(n)$ defined above. Develop to reach the expression of the periodogram. The convolution theorem can also be exploited to simplify the derivations.

2.4 Periodogram

Prove that the periodogram of random process $Y(n)$ can simply be computed as

$$\hat{S}_{\mathrm{P}}(\omega) = \frac{1}{N}|Y_N(\omega)|^2$$

where $Y_N(\omega) = \mathcal{F}\{y_N(n)\}$ is the discrete-time Fourier transform (DTFT) of the sequence $y_N(n) = w_{\mathrm{R}}(n)y(n)$, with $w_{\mathrm{R}}(n)$ denoting the length-N rectangular window.

2.5 Periodogram resolution

a) [HAY96, Example 8.2.3, p. 403] Let $X(n)$ be a discrete-time random process consisting of two equal amplitude sinusoids in zero-mean unit-variance white noise:

$$X(n) = A\sin(\omega_1 n + \phi_1) + A\sin(\omega_2 n + \phi_2) + \nu(n)$$

where $\omega_1 = 0.4\pi$ rad/sample, $\omega_2 = 0.45\pi$ rad/sample and $A = 5$. Determine the minimum required record length to resolve the two narrowband components using the periodogram.

b) Assume that the two sinusoids are sufficiently separated in frequency relative to the available sample size. Determine the minimum amplitude of the weakest sinusoid to guarantee that it would be visible in the periodogram even in the absence of noise.

2.6 Periodogram of white noise

[HAY96, Example 8.2.4, p. 405] Let $X(n)$ be zero-mean unit-variance white noise. Determine:

a) Its autocorrelation sequence and its power spectral density.

b) The expected value of the periodogram.

c) The variance of the periodogram. Can the variance be improved by increasing the record length?

2.7 Python lab

Download and complete **Jupyter notebook** for **Chapter 2, part 1: periodogram**.

2.8 Time-bandwidth product

Prove that the equivalent time-bandwidth product is equal to one for any lag window $w(k)$.

2.9 Blackman-Tuckey's method

A real-valued stochastic process has an autocorrelation sequence given by

$$r(k) = 2^{-|k|} + a\delta(k) \qquad a \in \mathbb{R}^{+}$$

a) Determine the PSD of the process.

b) Assuming that the ACS can be estimated exactly, determine the Blackman-Tuckey's spectral estimate with a 5-sample rectangular window.

c) What would be the expected value of Blackman-Tuckey's PSD if an unbiased estimate of the ACS was employed? And with the biased ACS estimate?

2.10 Bartlett's method

The true PSD of a signal consists of a peak with 3-dB bandwidth of $\Delta f = 0.01$ sample^{-1}. The center frequency and the amplitude of this spectral component are both unknown. We wish to estimate the PSD of this random process using Bartlett's method.

a) Assuming that the number of available samples N is high, determine the window size L to guarantee a negligible bias in the PSD estimate.

b) Explain why it would not be advantageous to increase L beyond the value found in a).

2.11 Welch's estimate

Using Welch's method, we wish to estimate the PSD of a continuous-time signal with bandwidth $B = 10$ kHz, sampled at $f_s = 20$ kHz during $T = 10$ s.

a) Compute the number of samples N available for spectral estimation.

b) We use the radix-2 fast Fourier transform (FFT) to approximate the DTFT. Compute the number of points of the FFT, $\tilde{N}$, to guarantee frequency samples spaced at most of 10 Hz.

Hint: The radix-2 FFT requires $\tilde{N} = 2^n$ points, with $n \in \mathbb{Z}^{+}$.

c) If the segment length L equals the number of FFT points $\tilde{N}$ computed above (i.e., no zero padding), how many segments K can be used for averaging without overlap?

d) We wish to reduce the estimator's variance by a factor of 10 while keeping the frequency sampling computed in exercise b). Propose two methods to do so. Explain their advantages and limitations.

2.12 Python lab

Download and complete **Jupyter notebook** for **Chapter 2, part 2: improved periodogram-based methods**.

3 Parametric spectral estimation

3.1 Filtering random processes

Let $X(t)$ be a wide sense stationary random process with mean μ_X, ACS $r_X(k)$, and PSD $S_X(\omega)$ and $S_X(z)$. The process is filtered by a linear time-invariant system with transfer function $H(z)$. Find the expressions of the mean, ACS and PSD (in ω and z domains) of the filter output $Y(t)$.

3.2 AR modeling

For the random process described in Problem 2.9:

a) Compute the AR spectral estimate of order 1, assuming that the true ACS values are available.

b) Using MATLAB or Python, plot and compare the true PSD, Blackman-Tuckey's estimate obtained in 2.9, and the AR(1) estimate for $a = 0$ and $a = 1$.

3.3 AR modeling with varying order

The estimated ACS of a random process for lags $0 \leq k \leq 4$, is:

$$r(0) = 2 \qquad r(1) = 1 \qquad r(2) = 1 \qquad r(3) = 0.5 \qquad r(4) = 0.$$

Compute and plot the following power spectra:

a) True PSD (assuming $r(k) = 0,\ k \geq 4$).

b) AR(p) models, with $p = 1, 2, 3$.

3.4 Model order selection

[HAY96, Problem 8.21, p. 483] Show that Akaike's final prediction error (FPE) criterion and Akaike's information criterion (AIC) are asymptotically equivalent, that is, for $N \gg 1$, estimating the order of an AR process by minimizing the FPE criterion is equivalent to minimizing AIC.

Hint: Show that for large N

$$N \log \mathrm{FPE}(p) \approx \mathrm{AIC}(p) + \text{constant}$$

using the fact that, if x is small, then $\log(1 + x) \approx x$.

3.5 Python lab

Download and complete **Jupyter notebook** for **Chapter 3: parametric estimation**.