A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a tautology.A compound proposition that is always false is called a contradiction. A compound proposition that is neither a tautology nor a contradiction is called a contingency. Compound propositions that have the same truth values in all possible cases are called logically equivalent.

1. Show that each of these conditional statements is a tautology by using truth tables.
   (a) $[\neg p \wedge (p \vee q)] \rightarrow q$
   (b) $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$
   (c) $[p \wedge (p \rightarrow q)] \rightarrow q$
   (d) $[(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow r$
2. Show that the following proposition are logical equivalent. To do this, either show that both sides are true, or that both sides are false, for exactly the same combinations of truth values of the propositional variables in these expressions (whichever is easier).
   (a) Show that $p \leftrightarrow q$ and $(p \wedge q) \vee (\neg p \wedge \neg q)$ are logically equivalent.
   (b) Show that $\neg(p \leftrightarrow q)$ and $p \leftrightarrow \neg q$ are logically equivalent.
   (c) Show that $p \rightarrow q$ and $\neg q \rightarrow \neg p$ are logically equivalent.
   (d) Show that $\neg p \leftrightarrow a$ and $p \leftrightarrow \neg a$ are logicallv equivalent.
3. Determine is the following proposition p is true or false. p: This statement is false

Smullyan posed many puzzles about an island that has two kinds of inhabitants, knights, who always tell the truth, and their opposites, knaves, who always lie. You encounter two people A and B. What are A and B if A says " B is a knight" and B says "The two of us are opposite types?"

Solution: Let p and q be the statements that A is a knight and B is a knight, respectively, so that $\neg p$ and $\neg q$ are the statements that A is a knave and B is a knave, respectively.

We first consider the possibility that A is a knight; this is the statement that p is true. If A is a knight, then he is telling the truth when he says that B is a knight, so that q is true, and A and B are the same type. However, if B is a knight, then B 's statement that A and B are of opposite types, the statement $(p \wedge \neg q) \vee (\neg p \wedge q)$, would have to be true, which it is not, because A and B are both knights. Consequently, we can conclude that A is not a knight, that is, that p is false.

If A is a knave, then because everything a knave says is false, A's statement that B is a knight, that is, that q is true, is a lie. This means that q is false and B is also a knave. Furthermore, if B is a knave, then B 's statement that A and B are opposite types is a lie, which is consistent with both A and B being knaves. We can conclude that both A and B are knaves.

4. An explorer is captured by a group of cannibals. There are two types of cannibals - those who always tell the truth and those who always lie. The cannibals will barbecue the explorer unless he can determine whether a particular cannibal always lies or always tells the truth. He is allowed to ask the cannibal exactly one question.

(a) Explain why the question "Are you a liar?" does not work.

(b) Find a question that the explorer can use to determine whether the cannibal always lies or always tells the truth.

5. Express these system specifications using the propositions p "The user enters a valid password," q "Access is granted," and r "The user has paid the subscription fee" and logical connectives (including negations).

   (a) "The user has paid the subscription fee, but does not enter a valid password."

   (b) "Access is granted whenever the user has paid the subscription fee and enters a valid password."

   (c) "Access is denied if the user has not paid the subscription fee."

   (d) "If the user has not entered a valid password but has paid the subscription fee, then access is granted."

6. Explain, without using a truth table, why $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$ is true when $p, q,$ and r have the same truth value and it is false otherwise.

7. Explain, without using a truth table, why $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ is true when at least one of $p, q,$ and r is true and at least one is false, but is false when all three variables have the same truth value.

A compound proposition is satisfiable if there is an assignment of truth values to its variables that makes it true. When no such assignments exists, that is, when the compound proposition is false for all assignments of truth values to its variables, the compound proposition is unsatisfiable.

Note that a compound proposition is unsatisfiable if and only if its negation is true for all assignments of truth values to the variables, that is, if and only if its negation is a tautology.

When we find a particular assignment of truth values that makes a compound proposition true, we have shown that it is satisfiable; such an assignment is called a solution of this particular satisfiability problem. However, to show that a compound proposition is unsatisfiable, we need to show that every assignment of truth values to its variables makes it false. Although we can always use a truth table to determine whether a compound proposition is satisfiable, it is often more efficient not to do it.

EXAMPLE

Determine whether each of the compound propositions $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$, $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$, and $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p) \wedge (p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ is satisfiable.

SOLUTION:

Instead of using truth table to solve this problem, we will reason about truth values. Note that $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$ is true when the three variable $p, q,$ and r have the same truth value . Hence, it is satisfiable as there is at least one assignment of truth values for $p, q,$ and r that makes it true.

Similarly, note that $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ is true when at least one of $p, q,$ and r is true and at least one is false . Hence, $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ is satisfiable, as there is at least one assignment of truth values for $p, q,$ and r that makes it true.

Finally, note that for $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p) \wedge (p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ to be true, $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$ and $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ must both be true. For the first to be true, the three variables must have the same truth values, and for the second to be true, at least one of three variables must be true and at least one must be false. However, these conditions are contradictory. From these observations we conclude that no assignment of truth values to p, q, and r makes $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p) \wedge (p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ true. Hence, it is unsatisfiable.

8. Determine whether each of these compound propositions is satisfiable.
   (a) $(p \vee \neg q) \wedge (\neg p \vee q) \wedge (\neg p \vee \neg q)$
   (b) $(p \to q) \wedge (p \to \neg q) \wedge (\neg p \to q) \wedge (\neg p \to \neg q)$
   (c) $(p \leftrightarrow q) \wedge (\neg p \leftrightarrow q)$
9. Translate these specifications into English where $F(p)$ is "Printer p is out of service," $B(p)$ is "Printer p is busy" $L(j)$ is "Print job j is lost" and $Q(j\ )$ is "Print job j is queued."
   (a) $\exists p(F(p) \wedge B(p)) \to \exists j L(j)$
   (b) $\forall p B(p) \to \exists j Q(j)$
   (c) $\exists j(Q(j) \wedge L(j)) \to \exists p F(p)$
   (d) $(\forall p B(p) \wedge \forall j Q(j)) \to \exists j L(j)$
10. Express each of these system specifications using predicates, quantifiers, and logical connectives.
    (a) When there is less than 30 megabytes free on the hard disk, a warning message is sent to all users.
    (b) No directories in the file system can be opened and no files can be closed when system errors have been detected.
    (c) The file system cannot be backed up if there is a user currently logged on.
    (d) Video on demand can be delivered when there are at least 8 megabytes of memory available and the connection speed is at least 56 kilobits per second.
11. Let $P(x), Q(x)$, and $R(x)$ be the statements " x is a professor," " x is ignorant"' and " x is vain," respectively. Express each of these statements using quantifiers; logical connectives; and $P(x), Q(x)$, and $R(x)$, where the domain consists of all people.
    (a) No professors are ignorant.
    (b) All ignorant people are vain.
    (c) No professors are vain.
    (d) Does (c) follow from (a) and (b)?
12. Let $P(x), Q(x)$, and $R(x)$ be the statements " x is a clear explanation," " x is satisfactory," and " x is an excuse," respectively. Suppose that the domain for x consists of all English text. Express each of these statements using quantifiers, logical connectives, and $P(x), Q(x)$, and $R(x)$.
    (a) All clear explanations are satisfactory.

(b) Some excuses are unsatisfactory.

(c) Some excuses are not clear explanations.

(d) Does (c) follow from (a) and (b)?

13. Let $S(x)$ be the predicate " x is a student," $F(x)$ the predicate " x is a faculty member," and $A(x, y)$ the predicate " x has asked y a question," where the domain consists of all people associated with your school. Use quantifiers to express each of these statements.

    (a) Lois has asked Professor Michaels a question.

    (b) Every student has asked Professor Gross a question.

    (c) Every faculty member has either asked Professor Miller a question or been asked a question by Professor Miller.

    (d) Some student has not asked any faculty member a question.

    (e) There is a faculty member who has never been asked a question by a student.

    (f) Some student has asked every faculty member a question.

    (g) There is a faculty member who has asked every other faculty member a question.

    (h) Some student has never been asked a question by a faculty member.

14. Let $T(x, y)$ mean that student x likes cuisine y, where the domain for x consists of all students at your school and the domain for y consists of all cuisines. Express each of these statements by a simple English sentence.

    (a) $\neg T$ (Abdallah Hussein, Japanese)

    (b) $\exists x T(x,$ Korean $) \wedge \forall x T(x,$ Mexican $)$

    (c) $\exists y(T$ (Monique Arsenault, $y) \vee T($ Jay Johnson, $y)$)

    (d) $\forall x \forall z \exists y((x \neq z) \rightarrow \neg(T(x, y) \wedge T(z, y)))$

    (e) $\exists x \exists z \forall y(T(x, y) \leftrightarrow T(z, y))$

    (f) $\forall x \forall z \exists y(T(x, y) \leftrightarrow T(z, y))$

15. Express each of these system specifications using predicates, quantifiers, and logical connectives, if necessary.

    (a) Every user has access to exactly one mailbox.

    (b) There is a process that continues to run during all error conditions only if the kernel is working correctly.

    (c) All users on the campus network can access all websites whose url has a .edu extension.

    (d) * There are exactly two systems that monitor every remote server.

16. Express the negations of these propositions using quantifiers, and in English.

    (a) Every student in this class likes mathematics.

    (b) There is a student in this class who has never seen a computer.

    (c) There is a student in this class who has taken every mathematics course offered at this school.

(d) There is a student in this class who has been in at least one room of every building on campus.

17. Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all integers.

(a) $\forall x \forall y \left(x_0^2 = y^2 \rightarrow x = y\right)$

(b) $\forall x \exists y \left(y^2 = x\right)$

(c)) $\forall x \forall y (xy \geq x)$

Recall that the definition of limit

$$\lim_{x \to a} f(x) = L$$

For every real number $\epsilon > 0$ there exists a real number $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.

This definition of a limit can be phrased in terms of quantifiers by

$$\forall \epsilon \exists \delta \forall x (0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon)$$

where the domain for the variables δ and ϵ consists of all positive real numbers and for x consists of all real numbers.

This definition can also be expressed as

$$\forall \epsilon > 0 \exists \delta > 0 \forall x (0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon)$$

when the domain for the variables ϵ and δ consists of all real numbers, rather than just the positive real numbers. [Here, restricted quantifiers have been used. Recall that $\forall x > 0 P(x)$ means that for all x with $x > 0, P(x)$ is true.]

We can use quantifiers to express the fact that $\lim_{x \to a} f(x)$ does not exist where $f(x)$ is a real-valued function of a real variable x and a belongs to the domain of f

the statement $\lim_{x \to a} f(x) \neq L$ can be expressed as

$$\neg \forall \epsilon > 0 \exists \delta > 0 \forall x (0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon)$$

Successively applying the rules for negating quantified expressions, we construct this sequence of equivalent statements

$$\begin{aligned}
&\neg \forall \epsilon > 0 \exists \delta > 0 \forall x (0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon) \\
&\quad \equiv \exists \epsilon > 0 \neg \exists \delta > 0 \forall x (0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon) \\
&\quad \equiv \exists \epsilon > 0 \forall \delta > 0 \neg \forall x (0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon) \\
&\quad \equiv \exists \epsilon > 0 \forall \delta > 0 \exists x \neg (0 < |x - a| < \delta \rightarrow |f(x) - L| < \epsilon) \\
&\quad \equiv \exists \epsilon > 0 \forall \delta > 0 \exists x (0 < |x - a| < \delta \wedge |f(x) - L| \geq \epsilon)
\end{aligned}$$

In the last step we used the equivalence $\neg(p \rightarrow q) \equiv p \wedge \neg q$.

Because the statement " $\lim_{x \to a} f(x)$ does not exist" means for all real numbers L $\lim_{x \to a} f(x) \neq L$, this can be expressed as

$$\forall L \exists \epsilon > 0 \forall \delta > 0 \exists x (0 < |x - a| < \delta \wedge |f(x) - L| \geq \epsilon)$$

This last statement says that for every real number L there is a real number $\epsilon > 0$ such that for every real number $\delta > 0$, there exists a real number x such that $0 < |x - a| < \delta$ and $|f(x) - L| \geq \epsilon$

18. Using the definition show that $\lim_{x \to 0} sin(x) \neq 1/2$.

19. We call x the limit of the sequence (x_n) and we write $\lim_{n\to\infty} x_n = x$ if the following condition holds: - For each real number $\varepsilon > 0$, there exists a natural number N such that, for every natural number $n \geq N$, we have $|x_n - x| < \varepsilon$.
    (a) Use quantifiers to express the fact that $\lim_{n\to\infty} a_n = 5$
    (b) Use quantifiers to express the fact that $\lim_{n\to\infty} a_n \neq 2$

20. Suppose (f_n) is a sequence of functions sharing the same domain and codomain. The codomain is most commonly the reals.. The sequence (f_n) converges pointwise to the function f, often written as $\lim_{n\to\infty} f_n = f$ pointwise if and only if,

$$\lim_{n\to\infty} f_n(x) = f(x)$$

    for every x in the domain. The function f is said to be the pointwise limit function of f_n.
    (a) Use quantifiers to express the fact that $\lim_{n\to\infty} f_n = f$ pointwise
    (b) Use quantifiers to express the fact that $\lim_{n\to\infty} f_n \neq x^2$ pointwise

21. Suppose E is a set of the real line and $(f_n)_{n\in\mathbb{N}}$ is a sequence of real-valued functions on it. We say the sequence $(f_n)_{n\in\mathbb{N}}$ is uniformly convergent on E with limit f if for every $\epsilon > 0$, there exists a natural number N such that for all $n \geq N$ and $x \in E$

$$|f_n(x) - f(x)| < \epsilon$$

    We will use

$$f_n \rightrightarrows f,$$

    as the notation for uniform convergence of f_n to f .
    (a) Use quantifiers to express the fact that $f_n \rightrightarrows f$, pointwise
    (b) Use quantifiers to express the fact that $f_n \rightrightarrows 0$, pointwise
    (c) *Consider the sequence of function $f_n(x) = x^n$ on the interval $[0, 1]$, and the function

$$f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$$

    It can be prove (do it !!!) that $\lim_{n\to\infty} f_n = f$ pointwise. Show that, however $f_n \not\rightrightarrows f$, (Here $\not\rightrightarrows$ means that f_n does not converge uniformly to f in the interval $[0, 1]$) Hint: Study a neighborhood of the point 1. Show that for each $\epsilon > 0$ it is possible find a value of $N > 0$ and $\delta > 0$ such that if x is in the interval $[1 - \delta, 1)$ then $f_n(x) > \epsilon$.