1. Explain why $A \times B \times C$ and $(A \times B) \times C$ are not the same.
2. This exercise presents Russell's paradox. Let S be the set that contains a set x if the set x does not belong to itself, so that $S = \{x \mid x \notin x\}$.
   (a) Show the assumption that S is a member of S leads to a contradiction.
   (b) Show the assumption that S is not a member of S leads to a contradiction.

   By parts (a) and (b) it follows that the set S cannot be defined as it was. This paradox can be avoided by restricting the types of elements that sets can have.
3. Let A, B, and C be sets. Show that
   (a) $(A \cup B) \subseteq (A \cup B \cup C)$.
   (b) $(A \cap B \cap C) \subseteq (A \cap B)$
   (c) $(A - C) \cap (C - B) = \emptyset$
4. Let $A_i = \{1, 2, 3, \ldots, i\}$ for $i = 1, 2, 3, \ldots$ Find
   (a) $\bigcup_{i=1}^{n} A_i$
   (b) $\bigcap_{i=1}^{n} A_i$
5. Determine whether f is a function from $\mathbf{Z}$ to $\mathbf{R}$ if
   (a) $f(n) = \pm n$
   (b) $f(n) = \sqrt{n^2 + 1}$.
   (c) $f(n) = 1/\left(n^2 - 4\right)$.
6. If f and $f \circ g$ are one-to-one, does it follow that g is one-to-one? Justify your answer.
7. If f and $f \circ g$ are onto, does it follow that g is onto? Justify your answer.
8. Let f be a function from the set A to the set B. Let S and T be subsets of A. Show that
   (a) $f(S \cup T) = f(S) \cup f(T)$
   (b) $f(S \cap T) \subseteq f(S) \cap f(T)$.
9. Let S be a subset of a universal set U. The characteristic function f_S of S is the function from U to the set $\{0, 1\}$ such that $f_S(x) = 1$ if x belongs to S and $f_S(x) = 0$ if x does not belong to S. Let A and B be sets. Show that for all $x \in U$
   (a) $f_{A \cap B}(x) = f_A(x) \cdot f_B(x)$
   (b) $f_{A \cup B}(x) = f_A(x) + f_B(x) - f_A(x) \cdot f_B(x)$
   (c) $f_{\bar{A}}(x) = 1 - f_A(x)$
10. (a) Show that if a set S has cardinality m, where m is a positive integer, then there is a one-to-one correspondence between S and the set $\{1, 2, \ldots, m\}$
    (b) Show that if S and T are two sets each with m elements, where m is a positive integer, then there is a one-to-one correspondence between S and T. $*$**80**.

11. Show that a set S is infinite if and only if there is a proper subset A of S such that there is a one-to-one correspondence between A and S.
12. Show that if A and B are sets with the same cardinality, then $|A| \leq |B|$ and $|B| \leq |A|$
13. Show that if A and B are sets, A is uncountable, and $A \subseteq B$, then B is uncountable.
14. Show that a subset of a countable set is also countable.
15. If A is an uncountable set and B is a countable set, must $A - B$ be uncountable?
16. Suppose that a countably infinite number of buses, each containing a countably infinite number of guests, arrive at Hilbert's fully occupied Grand Hotel. Show that all the arriving guests can be accommodated without evicting any current guest.
17. Show that the set of real numbers that are solutions of quadratic equations $ax^2 + bx + c = 0$, where a, b, and c are integers, is countable.