MAD 2104 - HW03 - Romain Roux - PID: 6322237

Ex1. Prove that $1^2 + 3^2 + 5^2 + ... + (2n+ 1)^2 = (n+ 1)(2n+ 1)(2n+ 3) / 3$ whenever n is a nonnegative integer.

Base case: for n = 0

$(2 * 0 + 1)^2 = (0 + 1)(2 * 0 + 1)(2 * 0 + 3) / 3$

$1^2 = 1 * 1 * 3 / 3$

$1 = 1$

Assume that n = k

Inductive hypothesis :

$2 + 3^2 + 5^2 + ... + (2k + 1)^2 = (k + 1)(2k + 1)(2k + 3)/3$

Prove that :

$1^2 + 3^2 + 5^2 + ... + (2k + 1)^2 + (2k + 3)^2 = (k + 2)(2k + 3)(2k + 5)/3$

We have :

$$
\begin{aligned}
1^2 + 3^2 + 5^2 +...+ (2k+1)^2 + (2k+3)^2 &= (k + 1)(2k + 1)(2k + 3)/3 + (2k + 3)^2 \\
&= (2k + 3)[(k + 1)(2k + 1) / 3 + (2k + 3)] \\
&= (2k + 3)(k + 1)(2k + 1) + 3(2k + 3) / 3 \\
&= (2k + 3)(2k^2 + 3k + 1) + (6k + 9) / 3 \\
&= (2k + 3)(2k^2 + 9k + 10) / 3 \\
&= (2k + 3)(2k + 5)(k + 2) / 3 \\
&= (k + 2)(2k + 3)(2k + 5)/3
\end{aligned}
$$

Therefore, $1^2 + 3^2 + 5^2 + \cdot \cdot \cdot + (2n + 1)^2 = (n + 1)(2n + 1)(2n + 3) / 3$ is true for all $n \geq 0$.

Ex.3 Prove that 3^n < n! if n is an integer greater than 6

Base case : for n = 7

3^7 < 7!

2187 < 5040

Assume that n = k

Inductive hypothesis:

3^k < k!

Prove that :

3^(k + 1) < (k + 1)!

We have :

3^(k + 1) < (k + 1)!

3 * 3^k < (k + 1)k!

We know that 3k < k!, so we just need to show that 3 < (n + 1), which is always true since $k > 6$.

Therefore, $3^n < n!$ is true for all $n > 6$.

Ex.4 Prove that $2^n > n^2$ if n is an integer greater than 4

Base case : for $n = 5$

$2^5 > 5^2$
$32 > 25$

Assume that $n = k$
Inductive hypothesis :
$k! > 2^k$

Prove that :
$(k + 1)! > 2^{(k + 1)}$

We have :
$(k + 1)! > 2^{(k + 1)}$
$(k!) * (k + 1) > 2^k * 2$
We know that $k! > 2^k$, so we just need to show that $(k + 1) > 2$, which is always true since $k > 4$.

Therefore, $2^n > n^2$ is true for all $n > 4$.

Ex.7 Prove that 6 divides $n^3 - n$ whenever n is a nonnegative integer.

Base case: for $n = 0$
$0^3 - 0 = 0$ and 6 divides 0.

Asume than $n = k$
Inductive hypothesis:
6 divides $k^3 - k$

Prove that :
6 divides $(k + 1)^3 - (k + 1)$

We have :
$(k + 1)^3 - (k + 1)$
$= k^3 + 3k^2 + 3k + 1 - k - 1$
$= (k^3 - k) + (3k^2 + 3k)$
$= (k^3 - k) + 3k(k + 1)$

We know that 6 divide $k^3 - k$, so we just need to prove that 6 divide $3k(k + 1)$
One of the consecutive integers k and k + 1 is even, so 2 divides the product k(k +

1) and thus 2 divides $3k(k + 1)$.
3 divides $3k(k + 1)$.
So 6 divides $3k(k + 1)$.

Therefore, 6 divide $n^3 - n$ for any positive integer n.

Ex.8 Prove that $n^2 - 1$ is divisible by 8 whenever n is an odd positive integer.

If n is odd and positive, we define $n = 2k + 1$ where k is a non-negative integer.
We have :

$$\begin{aligned} n^2-1 &= (2k + 1)^2 - 1 \\ &= 4k^2 + 4k + 1 - 1 \\ &= 4k^2 + 4k \\ &= 4k(k + 1) \end{aligned}$$

Since k is a non-negative integer, we have two possible cases:
1. k is odd, in which case $k + 1 = 2q$
so we have :

$$\begin{aligned} n^2 - 1 &= 4k * (2q) \\ &=8kq \end{aligned}$$

therefore 8 divides $n^2 - 1$

2. k is even, then $k = 2q$
So we have :

$$\begin{aligned} n^2 - 1 &= 4(2q)(k + 1) \\ &= 8q(k + 1) \end{aligned}$$

therefore 8 divides n^2-1

In both cases, 8 divides n^2-1.

Therefore it is proved that 8 divides $n^2 - 1$ in for n is an odd positive integer.

Ex.9 What is wrong with this ”proof”? "Theorem" For every positive integer n,
$\Sigma(n,i=0)\ i = (n + 0.5)^2 / 2$

Base case : for $n = 0$
$\Sigma(n,i=0)\ i = 1$
$(1 + 0.5)^2 / 2 = 3.5^2 / 2 = 6.125$

The base case isn t true.
Therefore the proof cannot be valid.