Tile-Makers and Semi-Tile-Makers

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1. INTRODUCTION. A *development* of a convex polyhedron is a plane figure obtained by cutting the surface of the polyhedron. Cuts need not be confined to edges but can also be made through faces (Figure 1.1). Hence a convex polyhedron has infinitely many developments. It is shown in **[1]** and **[4]** that we can systematically get infinitely many convex developments of a regular tetrahedron using tilings. Similar results for a square dihedron are found in **[2]** and results for flat 2-foldings of convex polygons in **[3]**.

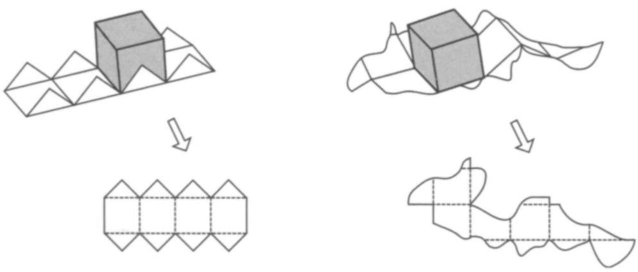


Figure 1.1. Two developments of a cube.

A plane figure is said to *tile* the plane $\mathbb{R}^2$ if copies of the figure cover the plane with no gaps nor overlaps when placed end to end. A convex polyhedron P is a *tile-maker* if every development of P tiles the plane.

An *edge-development* (Figure 1.2) of a convex polyhedron P is a development obtained by cutting the surface of P only along its edges. A convex polyhedron P is said to be a *semi-tile-maker* if every edge-development of P tiles the plane.

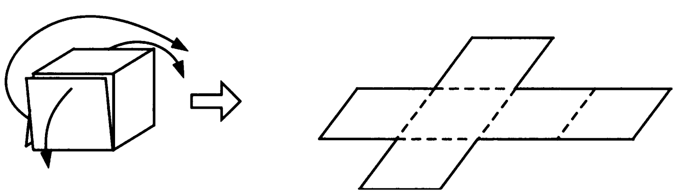
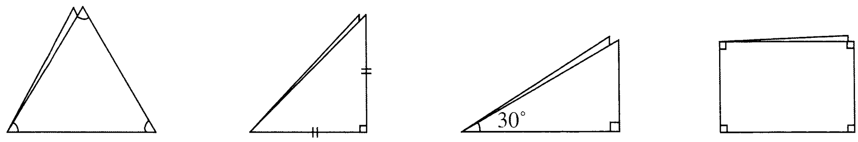


Figure 1.2. An edge-development of a cube.

In this paper, we determine all convex polyhedra that are tile-makers. We also present a conjecture about the convex polyhedra that could possibly be semi-tile-makers. We are concerned only with *monohedral* tilings (i.e., *tilings involving only one kind of tile*). Background sources on the topic of tilings are **[7]** and **[8]**, while **[6]** and **[9]** are references for the topic of developments.

The following are definitions of other geometric objects that will be encountered in this paper. A *dihedron* is a doubly-covered polygon. We denote a doubly-covered equilateral triangle, isosceles right triangle, half equilateral triangle, and rectangle by



(a) ETD (b) IRTD (c) HETD (d) RD

Figure 1.3. Although the figure looks like it has open sides, all of its sides are actually closed.

ETD, *IRTD*, *HETD*, and *RD*, respectively (Figure 1.3.). For a dihedron, one face is referred to as a *front face* and the other a *back face*.

A tetrahedron with four congruent triangular faces obtained by folding an acute triangle along its medians is called an *almost regular tetrahedron* and is denoted by *ART* (Figure 1.4). Note that there are infinitely many ARTs. A regular tetrahedron is, of course, an ART.

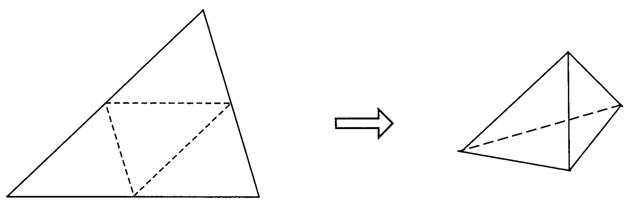


Figure 1.4. An ART.

2. TILING A PLANE WITH A REGULAR TETRAHEDRON. Take a regular tetrahedron R and carve distinct figures on each of its four faces, dip the faces in ink, and stamp the figures on a sheet of paper as follows: First, stamp the figure on one face, then arbitrarily choose one of the edges bounding the face and rotate the tetrahedron using the arbitrarily chosen edge as an axis and stamp the figure on the face adjacent to the axis. Continue this procedure to obtain a periodic tiling of the plane (Figure 2.1). The same pattern will result regardless of the direction or order in which the tetrahedron is rotated. We call this procedure *tiling by stamping* and refer to any polyhedron that accomplishes this as a *stamper*.

The regular tetrahedron is the only regular polyhedron that is a stamper. This is due to the fact that for any vertex V of a regular polyhedron there exists a positive number n such that, if T is the sum of the angles formed by the faces that meet at V, then $nT = 2\pi$. For a convex polyhedron to be a stamper, it is necessary that this number be an integer.

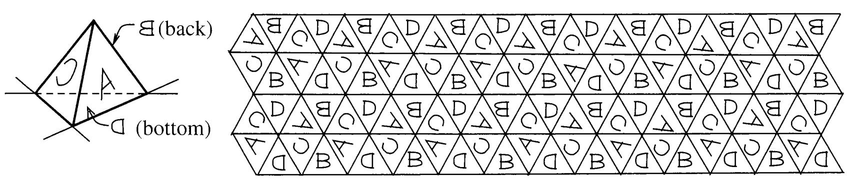


Figure 2.1. Carved regular tetrahedron R and the tiling by stamping with R.

We observe the positions of a point on the surface of R in the resulting tiling to determine which connected parts in the tiling can be a development of R. In order to state precisely how this is done, it is convenient to use oblique coordinates on the plane

$\gamma = \mathbb{R}^2$ and to consider a development of R as a figure on γ. A point $\mathrm{P}(x, y)$ of γ is called a *lattice point* if both x and y are integers.

Definition. Two points $\mathrm{P}(a, b)$ and $\mathrm{Q}(c, d)$ in the plane γ are *equivalent* if either the midpoint of the segment between them is a lattice point or both of the differences between corresponding coordinates are even integers. The former is called *symmetry type equivalence*, while the latter is called *parallel moving type equivalence* (Figure 2.2).

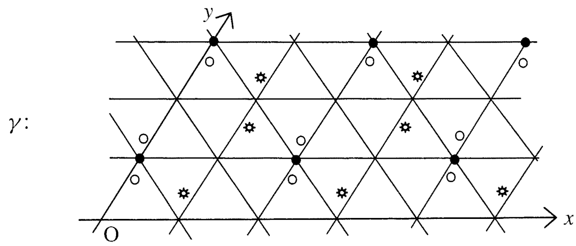


Figure 2.2. Points with the same marks are equivalent.

Note that equivalent points in the plane γ correspond to the same point on R (i. e., each point on R corresponds to a set of mutually equivalent points in the plane γ). Note also that if points $\mathrm{P}(a, b)$ and $\mathrm{Q}(c, d)$ are equivalent and if points $\mathrm{Q}(c, d)$ and $\mathrm{R}(e, f)$ are equivalent, then so are $\mathrm{P}(a, b)$ and $\mathrm{R}(e, f)$. Thus, the relation under discussion is an equivalence relation. The points in the plane γ are separated into equivalence classes with respect to this equivalence relation. This leads to the following result:

Theorem 1. *A development D of a regular tetrahedron R is a closed connected subset S of the plane γ containing representatives from all equivalence classes in γ and having the property that no two points in its interior are equivalent.*

Various examples of developments for a regular tetrahedron are shown in Figure 2.3.

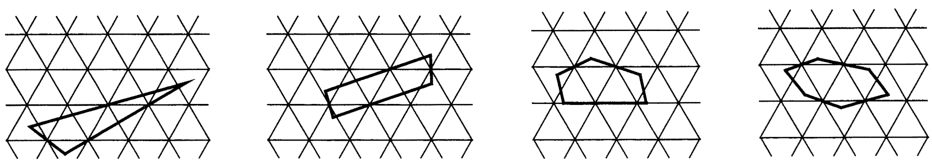


Figure 2.3. Examples of developments of a regular tetrahedron.

Observe that every vertex of the regular tetrahedron R corresponds to a lattice point in the plane γ.

3. A REGULAR TETRAHEDRON IS A TILE-MAKER. Recall that a convex polyhedron P is a *tile-maker* when every development of P tiles the plane. The following theorem identifies one such polyhedron:

Theorem 2. *A regular tetrahedron is a tile-maker.*

Proof. Let D be an arbitrary development of a regular tetrahedron R. It follows from Theorem 1 that D is a closed connected subset of the plane γ containing representatives from all equivalence classes in γ and with the property that no two points in its interior are equivalent. Take any lattice point $\mathrm{P}(k, l)$ from the boundary of D (Figure 3.1a). The existence of such a point is guaranteed, since at least four lattice points corresponding to the vertices of R are located on the boundary of D. Let E be the set obtained by rotating D by 180° with respect to the point $\mathrm{P}(k, l)$ (Figure 3.1b).

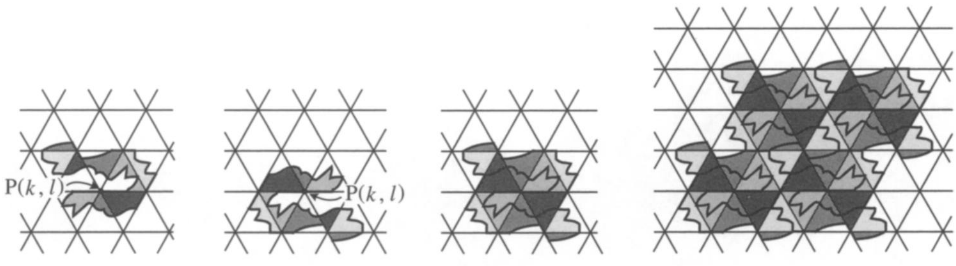


Figure 3.1.

Since any two points in the plane that are symmetric with respect to a lattice point are equivalent, the interiors of D and E are disjoint. Let U be the join of D and E in the plane (Figure 3.1c).

Let F be the set of all congruent copies of U obtained by shifting U parallel to each axis by even numbers of units (Figure 3.1d). We show that F covers the plane γ with neither overlaps nor gaps. Let $\mathrm{Q}_1(x_1, y_1)$ be an arbitrary point of γ. Since D is a set of all representatives of points in γ, there is a point $\mathrm{Q}_2(x_2, y_2)$ in D that is equivalent to Q_1, that is, for which either (a) both $x_1 - x_2$ and $y_1 - y_2$ are even numbers or (b) the midpoint $\mathrm{Q}_3((x_1 + x_2)/2, (y_1 + y_2)/2)$ of the segment between Q_1 and Q_2 is a lattice point.

If (a) holds, then the point Q_1 is a point in the set F by the definition of F. Suppose that (b) holds, and write $(x_1 + x_2)/2 = m$ and $(y_1 + y_2)/2 = n$, where m and n are integers. Then

$$x_1 = 2m - x_2 = (2k - x_2) + 2(m - k), \qquad y_1 = 2n - y_2 = (2l - y_2) + 2(n - l).$$

Since the point $\mathrm{Q}_4(2k - x_2, 2l - y_2)$ and the point Q_2 are symmetric with respect to the point $\mathrm{P}(k, l)$, the point Q_4 is in the set E. Because the point Q_1 is the point obtained by shifting Q_4 parallel to the x- and y-axes by the even numbers of units $2(m - k)$ and $2(n - l)$ units, respectively, the point Q_1 is in the set F, which establishes the theorem. ■

4. DETERMINATION OF ALL TILE-MAKERS. We exhibit a necessary condition for a convex polyhedron (including dihedrons) to be a tile-maker in the next lemma.

Lemma 1. *If a convex polyhedron or a dihedron is a tile-maker, then it is a stamper.*

Proof. Assume that a convex polyhedron P is a tile-maker. Let P have n vertices, and let D be an arbitrary edge-development of P. As D is an edge-development of P, we had to cut P along $n-1$ edges that induce a spanning tree having the vertices of P as its vertex set in order to obtain D. Denote these edges by e_i $(i = 1, 2, \dots, n-1)$.

Now construct a new development D' of P that differs from D as follows: for each i $(1 \le i \le n-1)$ when we cut along the edge e_i, we insert i zigzag cuts (Figure 4.1). Since we assumed that P is a tile-maker, the new development D' must tile the plane. Let T' be a tiling of the plane with D'. We note that for each i, a pair of adjacent faces of P sharing the edge e_i must touch each other in the tiling T', since otherwise the zigzags would not match. But then, removing the zigzags, we get a tiling T for the development D in which, for each k, a pair of adjacent faces of P sharing the edge e_k touch each other. This means that P is a stamper. ■

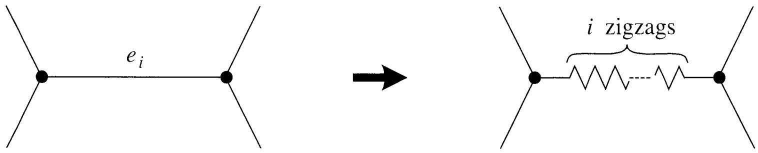


Figure 4.1.

It is proved in [**5**] that the only convex polyhedra, with the exception of dihedra, that can be stampers are the ARTs. The next theorem determines all dihedra that are stampers.

Theorem 3. *A dihedron is a stamper if and only if it is a member of one of the following four families:*

(a) ETD (doubly-covered equilateral triangles) (Figure 1.3*a),*
(b) IRTD (doubly-covered isosceles right triangles) (Figure 1.3*b),*
(c) HETD (doubly-covered half equilateral triangles) (Figure 1.3*c),*
(d) RD (doubly-covered rectangles) (Figure 1.3*d).*

Proof. It is easy to check that each of the dihedra in the families ETD, IRTD, HETD, and RD is stamper. We now turn to the converse. Let $Q = \mathrm{V}_1\mathrm{V}_2\cdots\mathrm{V}_k$ be a doubly-covered k-gon that is a stamper, and let V be any one of the vertices V_i $(i = 1, 2, \dots, k)$. Denote the angle at V by θ. The sum of the angles on the front and back faces of Q at V is 2θ, so 2π must be an integer multiple of 2θ. Hence $2n\theta = 2\pi$ (i.e., $\theta = \pi/n$). Since Q is convex, $n \ge 2$ and thus $\theta \le \pi/2$. On the other hand, the sum of the interior angles of a k-gon is $(k-2)\pi$. This implies that $(k-2)\pi \le k \cdot \pi/2$, which gives $k \le 4$. We consider the cases $k = 4$ and $k = 3$ separately.

Case 1: $k = 4$. The inequality holds only when $\theta = \pi/2$ for each vertex V_i, making Q an RD.

Case 2: $k = 3$. The k-gon is a triangle. Denote its three angles by x, y, and z $(x \ge y \ge z)$. Then

$$x + y + z = \pi \tag{$*$}$$

and, for positive integers l, m, and n, $x = \pi/l$, $y = \pi/m$, and $z = \pi/n$. Substituting these values into $(*)$, we obtain

$$\frac{1}{l} + \frac{1}{m} + \frac{1}{n} = 1.$$

Since $x \geq y \geq z$, $l = 2$ or 3. We have two subcases:

Case 2(a): $l = 2$. Since $\frac{1}{m} + \frac{1}{n} = \frac{1}{2}$ and $m \leq n$, we have $m = 3$ or 4, which leads to the following two possibilities: (i) $l = 2$, $m = 3$, and $n = 6$, in which case Q belongs to HETD and (ii) $l = 2$, $m = 4$, and $n = 4$, in which case Q is in IRTD.

Case 2(b): $l = 3$. This case gives $m = n = 3$, which implies that Q belongs to ETD.
■

All results on regular tetrahedra obtained in sections 2 and 3 carry over to ARTs and the four specified families of dihedra. Thus we have the following theorem:

Theorem 4. *A convex polyhedron or dihedron is a tile-maker if and only if it is a member of one of the following five families:* (a) ART *(almost regular tetrahedra),* (b) ETD, (c) IRTD, (d) HETD, (e) RD.

5. SEMI-TILE-MAKERS. We remind the reader that a *semi-tile-maker* is a convex polyhedron P such that every edge-development of P tiles the plane. We consider the following question: Which convex polyhedron are semi-tile-makers?

Proposition 1. *Cubes and regular octahedra are semi-tile-makers.*

Proof. It is well known that both a cube and a regular octahedron have exactly eleven distinct edge-developments, as shown in Figure 5.1 and Figure 5.2, respectively. Figure 5.3 and Figure 5.4 illustrate tilings corresponding to each of these edge-developments, proving that cubes and regular octahedra are semi-tile-makers. ■

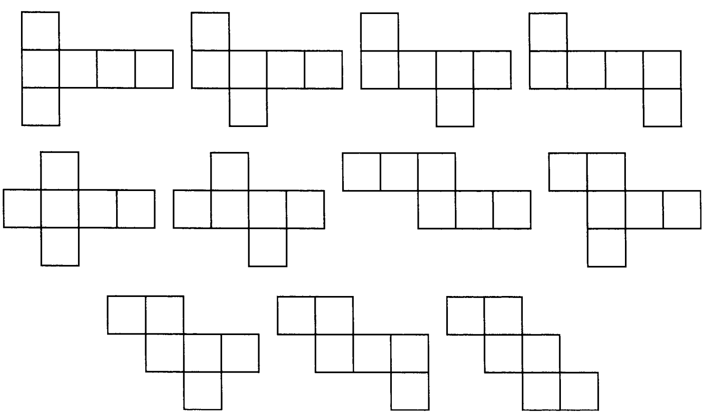


Figure 5.1. Edge-developments of a cube.

Suppose that we disregard the distinct figures carved on the faces of a polyhedron and consider the polyhedron with blank faces. If the faces tile the plane when they are rotated arbitrarily (like a stamper), then we call the polyhedron a *frame stamper*. Cubes, regular octahedra, and regular icosahedra are all frame stampers, and they seem to be the only ones. A polyhedron that is a semi-tile-maker is necessarily a frame stamper. However, a regular icosahedron is not a semi-tile-maker: at least one of its edge-developments does not tile the plane (Figure 5.5).

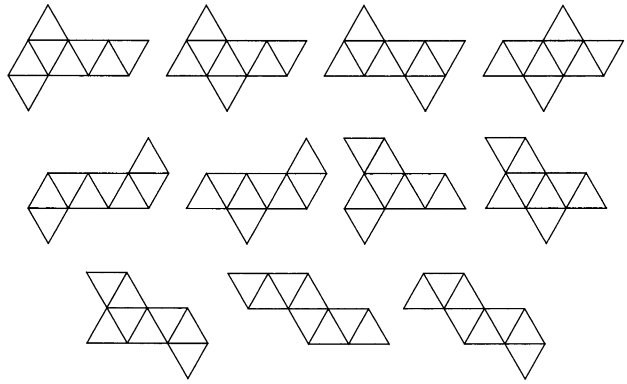


Figure 5.2. Edge-developments of a regular octahedron.

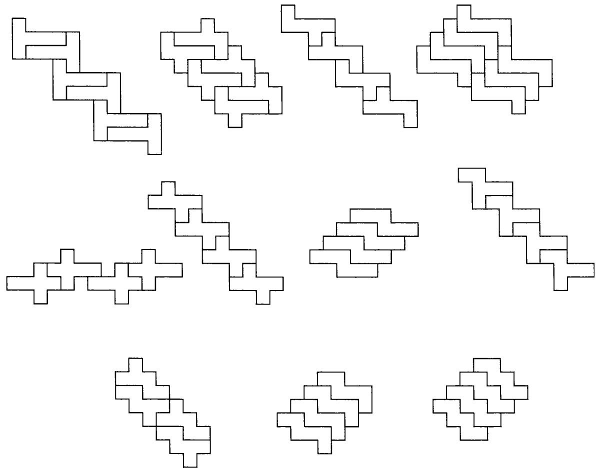


Figure 5.3. Tilings by each of the edge-developments of a cube.

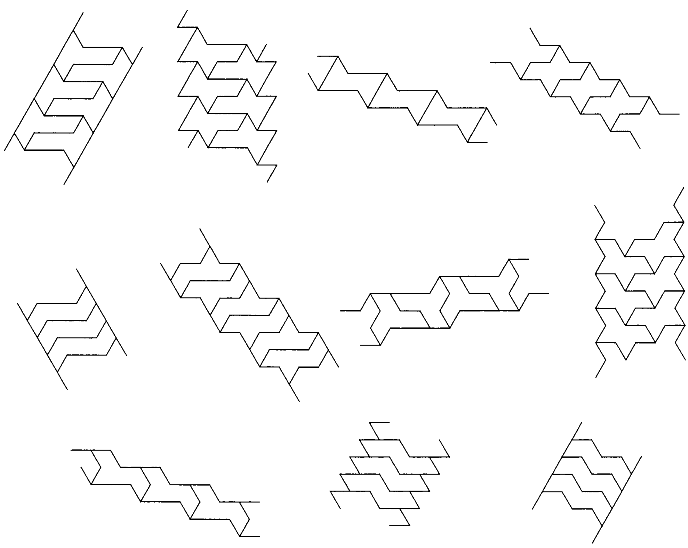


Figure 5.4. Tilings by each of the edge-developments of a regular octahedron.

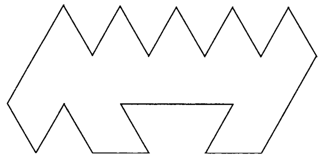


Figure 5.5. An edge-development of a regular icosahedron which does not tile the plane.

Note also that every tile-maker is a semi-tile-maker by definition. Based on these observations, we make the following conjecture:

Conjecture. A convex polyhedron or dihedron is a semi-tile-maker if and only if it is a member of one of the following seven families: (1) Cubes, (2) Regular octahedra, (3) ART, (4) ETD, (5) IRTD, (6) HETD, (7) RD.

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REFERENCES

1. J. Akiyama, K. Hirata, M. Kobayashi, and G. Nakamura, Convex developments of a regular tetrahedron, *Comput. Geom.* **34** (2006) 2–10.
2. ———, On convex developments of a doubly-covered square, in *Combinatorial Geometry and Graph Theory, Lecture Notes in Comput. Sci.*, no. 3330, Springer-Verlag, Berlin, 2004, 1–13.
3. J. Akiyama, K. Hirata, M. P. Ruiz, and J. Urrutia, Flat 2-foldings of convex polygons, in *Combinatorial Geometry and Graph Theory, Lecture Notes in Comput. Sci.*, no. 3330, Springer-Verlag, Berlin, 2004, 14–24.
4. J. Akiyama and C. Nara, Developments of polyhedra using oblique coordinates, *J. Indones. Math. Soc.* **13** (2007) 99–114.
5. J. Akiyama and G. Nakamura, Variations of the tiling problem, *Teaching Mathematics and Its Applications* **19** (2000) 8–12.
6. E. D. Demaine and J. O'Rourke, A survey of folding and unfolding in computational geometry, in *Combinatorial and Computational Geometry, Mathematical Sciences Research Institute Publications*, vol. 52, J. E. Goodman, J. Pach, and E. Welzl, eds., Cambridge University Press, Cambridge, 2005, 167–211.
7. A. W. M. Dress and D. H. Huson, On tilings of the plane, *Geom. Dedicata* **24** (1987) 295–310.
8. B. Grünbaum and G. C. Shephard, *Tilings and Patterns*, W.H. Freeman, New York, 1987.
9. J. O'Rourke, Folding and unfolding in computational geometry, in *Discrete and Computational Geometry, Lecture Notes in Comput. Sci.*, no. 1763, Springer-Verlag, Berlin, 2000, 258–266.

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