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Convex developments of a regular tetrahedron

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Abstract

The best-known developments of a regular tetrahedron are an equilateral triangle and a parallelogram. Are there any other convex developments of a regular tetrahedron? In this paper we will show that there are convex developments of a regular tetrahedron having the following shapes: an equilateral triangle, an isosceles triangle, a right-angled triangle, scalene triangles, rectangles, parallelograms, trapezoids, quadrilaterals which are not trapezoids, pentagons and hexagons, and furthermore these cases exhaust all the possibilities of convex developments with sides $n \leqslant 6$. And we will show that there are no convex n-gons which are developments of a regular tetrahedron when $n \geqslant 7$. Here, we mean by a development of a polyhedron a connected plane figure, from which one can construct the polyhedron by folding it without getting overlap or gap. In so folding we do not require that the sides of the development should end up as the edges of the polyhedron.

Keywords: Convexity; Development; Regular tetrahedron

1. Introduction

The best-known developments of a regular tetrahedron are an equilateral triangle and a parallelogram as shown in Figs. 1 and 2. There are other developments of a regular tetrahedron, however, such as rectangles shown in Figs. 3 and 4, and an isosceles triangle shown in Fig. 5. Are there any other convex developments of a regular tetrahedron?

In this paper we show the following theorems.

Theorem 1. *There are convex developments of a regular tetrahedron having the following shapes*: *an equilateral triangle, an isosceles triangle, a right-angled triangle, scalene triangles, rectangles, parallelograms, isosceles trapezoids, trapezoids with one of the sides being perpendicular to the bases* (*and hence with two right angles*)*, trapezoids which are neither isosceles nor with right angles, quadrilaterals which are not trapezoids, pentagons and hexagons, and furthermore these cases exhaust all the possibilities of convex developments with sides* $n \leqslant 6$.

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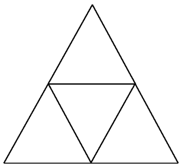

Fig. 1.

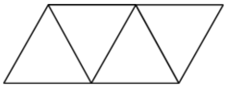

Fig. 2.

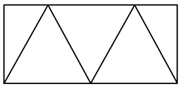

Fig. 3.

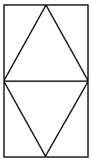

Fig. 4.

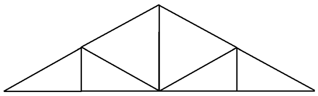

Fig. 5.

Theorem 2. *There are no convex n-gons which are developments of a regular tetrahedron when* $n \geqslant 7$.

Here, we mean by a *development* of a polyhedron a connected plane figure, from which one can construct the polyhedron by folding it without getting overlap or gap. In so folding we do not require that the sides of the development should end up as the edges of the polyhedron.

2. Equilateral triangular lattice

If we carve letters a, b, c, d on the four faces of a regular tetrahedron, stain the faces with ink and then role the tetrahedron over an equilateral triangular lattice, a pattern such as shown in Fig. 6 will emerge. It is important to note that we get the same pattern regardless of the direction or order of the rolling of the tetrahedron. This means that if we instead fold the plane figure of Fig. 6 along the lines (by cutting by scissors along some lines whenever necessary) to cover the faces of the tetrahedron many times, then the letters shown on the lattice will fit together exactly on each face of the tetrahedron. Furthermore, we notice that the pattern shown in Fig. 6 is symmetric about every lattice point (i.e., a point of intersection of lines) of the figure.

If we choose four contiguous triangles from Fig. 6 on which each of the letters a, b, c, d appears, then we obtain a development of the regular tetrahedron. For instance, the development of the shape of an equilateral triangle and that of the shape of a parallelogram can be obtained as indicated in Figs. 7 and 8, respectively. We will show in the next section that we can obtain, by modifying this process, developments, having various shapes, of a regular tetrahedron.

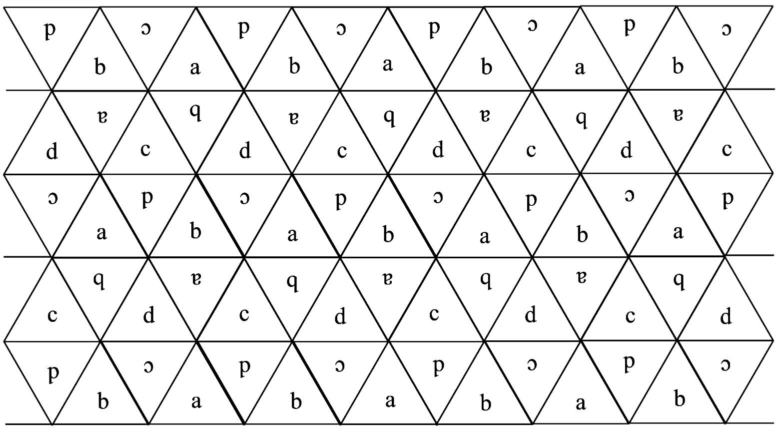

Fig. 6.

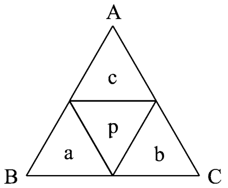

Fig. 7.

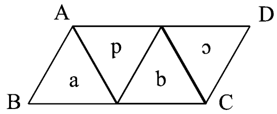

Fig. 8.

The statement that two figures lying on the triangular lattice of Fig. 6 are *congruent inclusive of labels* means that the two figures are congruent in the usual sense, and furthermore, the triangles or portions of triangles of the lattice contained in the two figures have labels (a, b, c, d) matching exactly.

3. A proof of Theorem 1

We will prove Theorem 1 in this section.

3.1. Triangles

An equilateral triangle ABC of Fig. 7 is a development of a regular tetrahedron. Shift the vertex A by 2 units to the right, and form a new triangle A′BC (Fig. 9). The triangles ABP and CA′P are congruent inclusive of labels. Therefore, the triangle A′BC gives a development of the regular tetrahedron, and this has the shape of an isosceles triangle.

If we shift the vertex A′ by 2 further units to the right and form the triangle A″BC as in Fig. 10, then the triangles A′BP and CA″P become congruent inclusive of labels. Therefore, the triangle A″BC also gives a development of the regular tetrahedron.

By repeating this process, we obtain triangles of the type shown in Fig. 11, etc. Thus we see that there are infinitely many distinct triangles which give developments of a regular tetrahedron. Here, we say that two figures in a plane are the same [distinct] if they are congruent [not congruent].

We can construct developments with the shape of a right-angled triangle as in Fig. 12.

Therefore we see that there are convex developments of a regular tetrahedron having the following shapes: an equilateral triangle, an isosceles triangle, a right-angled triangle, and infinitely many distinct scalene triangles.

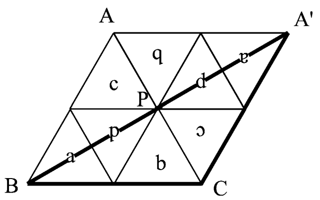

Fig. 9.

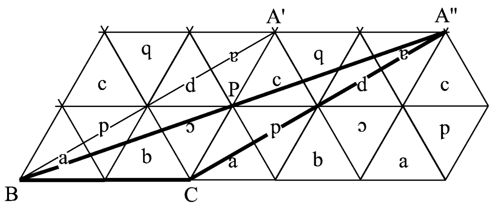

Fig. 10.

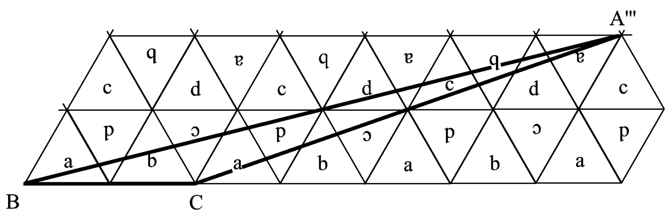

Fig. 11.

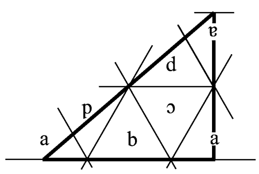

Fig. 12.

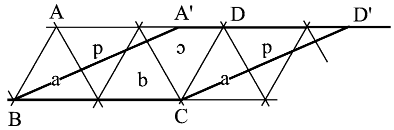

Fig. 13.

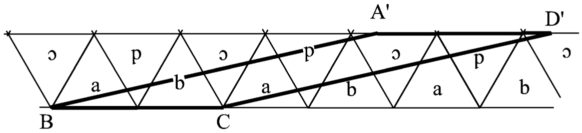

Fig. 14.

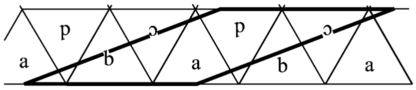

Fig. 15.

3.2. Parallelograms

We start with the parallelogram ABCD of Fig. 8. Shift the vertices A and D horizontally by the same distance and call the resulting points A′ and D′, respectively, as in Figs. 13 and 14. Then, the new parallelogram A′BCD′ formed gives a development of a regular tetrahedron.

Thus, there are infinitely many distinct parallelograms which give developments of a regular tetrahedron. Furthermore, figures obtained by translating the parallelograms obtained above horizontally, as in Fig. 15, give developments of the same tetrahedron (these figures are again parallelograms, but the lines of folding to construct the tetrahedron will be different).

3.3. Trapezoids

We start with the equilateral triangle of Fig. 7. If in Fig. 16 the equality AD = BE holds, then the triangles PAD and PBE become congruent inclusive of labels, and therefore, the trapezoid ADEC gives a development of a regular tetrahedron. If, in particular, AD = BE = AP/2, then we obtain a trapezoid with one of the sides being perpendicular to the bases. Starting with the triangles A′BC of Fig. 9, and A″BC of Fig. 10, and proceeding in the same way, we obtain the trapezoids ADEC of Figs. 17 and 18, respectively.

If we go through the same procedure on both sides of the vertex A of the equilateral triangle, as indicated in Fig. 19, we obtain an isosceles trapezoid DEFG, which is also a development of the regular tetrahedron. From a rectangle shown in Fig. 25 of the next subsection, we can also obtain by the same procedure a development of the regular tetrahedron with the shape of a trapezoid (see Fig. 20).

From the parallelogram A′BCD′ of Fig. 13, with vertices A′ and D′ lying on the lattice points, we can construct a trapezoid A*B*C*D* by going through the process indicated in Fig. 21, then the resulting trapezoid A*B*C*D* is a development of the regular tetrahedron, since in Fig. 21 the triangles PA′A* and PD′D* are congruent inclusive of labels and so are the triangles QBB* and QCC*. The trapezoid A*B*C*D* gives a development of the tetrahedron,

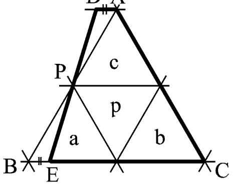

Fig. 16.

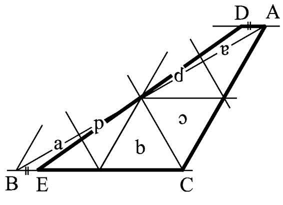

Fig. 17.

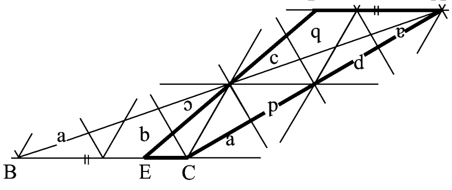

Fig. 18.

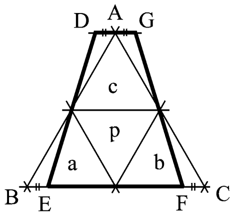

Fig. 19.

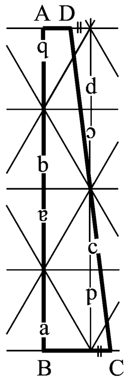

Fig. 20.

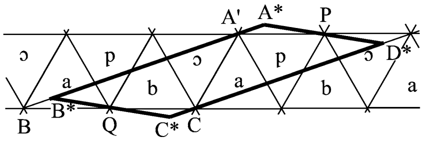

Fig. 21.

for which none of the sides of the trapezoid becomes an edge of the tetrahedron. We note also that there are infinitely many trapezoids of distinct shapes that are developments of the regular tetrahedron.

If we draw the line segment A*D* perpendicularly to the segment A′B, then the resulting figure A*B*C*D* becomes a trapezoid with one of the sides being perpendicular to the bases.

Therefore there are convex developments of a regular tetrahedron having the following shapes: isosceles trapezoids, trapezoids with one of the sides being perpendicular to the bases, trapezoids which are neither isosceles nor with right angles.

3.4. Rectangles

Fig. 22 shows a basic development of a regular tetrahedron of rectangular shape. By translating this rectangle horizontally, we obtain other type of developments of rectangular shape as in Fig. 23. We see further that the rectangles of Figs. 24 and 25 also give developments of a regular tetrahedron. If, in Fig. 21 of the preceding subsection, we draw the line segment A*D* perpendicularly to the segment A′B and the line segment B*C* perpendicularly to the segment BC, then the resulting figure A*B*C*D* becomes a rectangle (Fig. 26). From the parallelogram A″BCD″ of Fig. 27, we obtain in the same way a rectangle A*B*C*D*.

Thus we see that there are infinitely many distinct rectangles which are developments of the regular tetrahedron.

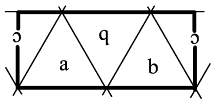

Fig. 22.

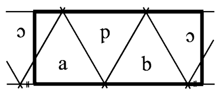

Fig. 23.

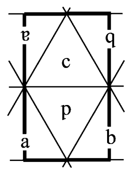

Fig. 24.

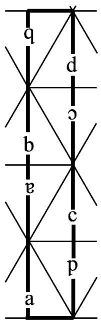

Fig. 25.

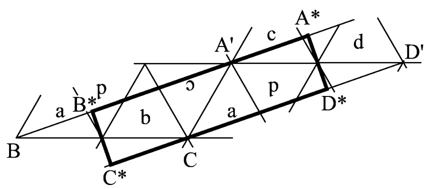

Fig. 26.

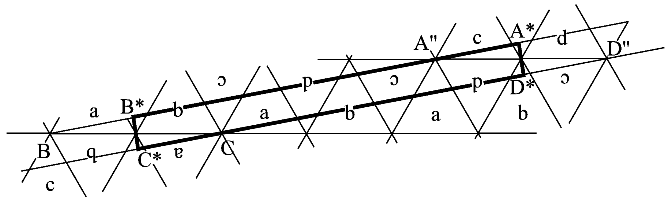

Fig. 27.

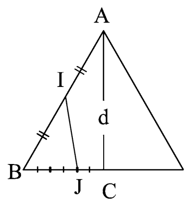

Fig. 28.

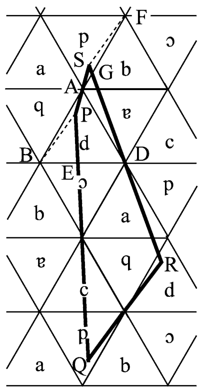

Fig. 29.

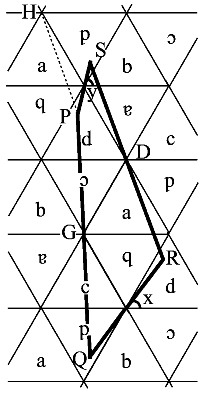

Fig. 30.

3.5. *Quadrilaterals which are not trapezoids*

Choose, as in Fig. 28, a triangle with the label “d” from the underlying triangular lattice, and call its left half the triangle ABC. Let I be the mid-point of the line segment AB, and J the point on the line segment BC, sub-dividing BC in 2:1 ratio. Let P be an arbitrary point lying in the interior of the triangle ABC but not on the line segment IJ. Next consider four triangles with the label d and situated as shown in Fig. 29, and call the four points P, Q, R, S, which correspond to the point P chosen in the triangle ABC above. The figure PQRS is a quadrilateral.

In order to prove that the quadrilateral PQRS gives a development of a regular tetrahedron, it suffices to show that PQRS contains all the parts of four triangles with labels a, b, c, d without overlaps. From Fig. 29, it is clear that this fact is true for triangles with labels a, b, c. So, we consider the case for the label d.

Let A, B, D, F be lattice points as indicated in Fig. 29. Let E be the intersection point of line segments PQ and BD, and G the intersection point of line segments SR and AF. The quadrilateral PQRS contains the quadrilateral APED, and also contains a triangle, which is congruent inclusive of labels to the triangle PBE. The triangle APB is congruent inclusive of labels to the triangle ASF, and PQRS contains the triangle ASG, and also contains a triangle, which is congruent inclusive of labels to the triangle FSG. Putting these facts together we conclude that the quadrilateral PQRS gives a development of a regular tetrahedron.

Next, we check that the opposite sides of the quadrilateral PQRS are not parallel. First, we note that the angle x of inclination of the line segment QR satisfies $0° < x < 60°$, while the angle y of inclination of the line segment PS satisfies $60° < y < 90°$, and therefore, sides QR and PS are not parallel (Fig. 30). Next, we note that if we pick a point H as indicated in Fig. 30, the line segments HP and SR become parallel. Consequently, the assertion that the line segments PQ and SR are parallel is equivalent to the assertion that the points H, P, G are collinear. Since, from our initial assumption, the point P does not lie on the line segment IJ in Fig. 28, the point P does not lie on the line segment HG. Thus, the line segments PQ and SR are not parallel, and we conclude that the quadrilateral PQRS is not a trapezoid.

There are infinitely many ways of choosing the point P to satisfy our requirement, and especially in such a way that the resulting quadrilaterals PQRS are all non-congruent. Thus there are infinitely many distinct quadrilaterals which give a development of a regular tetrahedron.

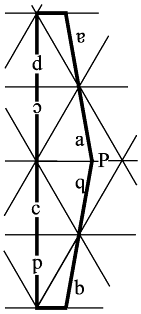

Fig. 31.

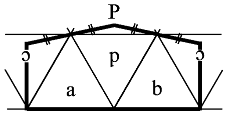

Fig. 32.

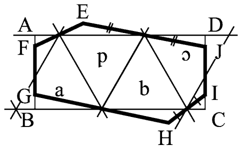

Fig. 33.

3.6. Pentagons

We can construct developments with the shape of a pentagon, as in Figs. 31 and 32, of a regular tetrahedron. Location of the point P in Figs. 31 and 32 can be chosen in infinitely many ways, so there are infinitely many distinct pentagons which give a development of a regular tetrahedron.

3.7. Hexagons

We can deform a rectangle ABCD to obtain a hexagon EFGHIJ, as indicated in Fig. 33, which gives a development of a regular tetrahedron. There are infinitely many distinct hexagons, which give a development of a regular tetrahedron.

Thus we complete the proof of Theorem 1.

4. A proof of Theorem 2

We will prove Theorem 2 in this section.

Let $n \geqslant 7$. Assume that a convex n-gon D is a development of a regular tetrahedron E. For a vertex A of D, we denote by $\theta(\mathrm{A})$ the interior angle of A. Let $\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3, \ldots, \mathrm{A}_n$ be the vertices of D. Then we have $\sum_{i=1}^{n} \theta(\mathrm{A}_i) = (n-2)\pi$. Since D is convex, we have $\theta(\mathrm{A}_i) < \pi$ $(1 \leqslant i \leqslant n)$.

Suppose that when D folds to a regular tetrahedron E, k vertices of D, say $\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3, \ldots, \mathrm{A}_k$ are joined together at a point P of E $(1 \leqslant k \leqslant n)$, and any other vertices are not joined together at the point P. Then the point P is either on a face of E, on an edge of E, or one of the vertices of E. When the point P is on a face of E or on an edge of E, it holds that $\sum_{i=1}^{k} \theta(\mathrm{A}_i) = \pi$ or 2π. When the point P is a vertex of E, it holds that $\sum_{i=1}^{k} \theta(\mathrm{A}_i) = \pi$ since E is a regular tetrahedron. In either case, we have $\sum_{i=1}^{k} \theta(\mathrm{A}_i) \leqslant 2\pi$.

Assume $k = n$. Then we have $(n-2)\pi = \sum_{i=1}^{k} \theta(\mathrm{A}_i) \leqslant 2\pi$, so $n \leqslant 4$, which is a contradiction. Hence we have $k < n$. So we obtain

$$\sum_{i=k+1}^{n} \theta(\mathrm{A}_i) \geqslant (n-2)\pi - 2\pi = (n-4)\pi. \tag{1}$$

Since $\theta(\mathrm{A}_i) < \pi$ for any i, we have

$$\sum_{i=k+1}^{n} \theta(\mathrm{A}_i) < (n-k)\pi. \tag{2}$$

From (1) and (2), we have $(n-4)\pi < (n-k)\pi$! Hence we have $k \leqslant 3$.

Assume $k = 1$. Then the point P is neither on a face of E, nor on an edge of E. If the point P is a vertex of E, then $\theta(\mathrm{A}_1) = \pi$, since E is a regular tetrahedron. This is a contradiction to $\theta(\mathrm{A}_1) < \pi$. So $k \neq 1$. Therefore we obtain $2 \leqslant k \leqslant 3$.

If $k = 2$, we have $\sum_{i=1}^{2} \theta(\mathrm{A}_i) = \pi$ since $\sum_{i=1}^{2} \theta(\mathrm{A}_i) = \pi$ or 2π and $\theta(\mathrm{A}_1)$, $\theta(\mathrm{A}_2) < \pi$. If $k = 3$, we have $\sum_{i=1}^{3} \theta(\mathrm{A}_i) \leqslant 2\pi$ since $\sum_{i=1}^{3} \theta(\mathrm{A}_i) = \pi$ or 2π.

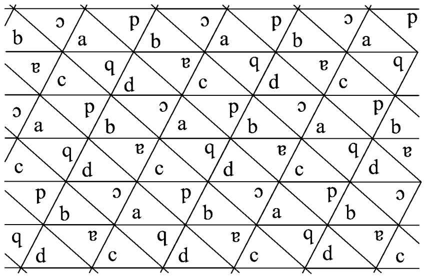

Fig. 34.

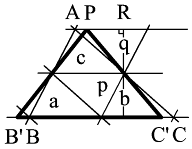

Fig. 35.

Suppose that there are x pairs of two vertices which are joined together at a point of E, and y sets of three vertices which are joined together at a point of E, where $x, y \geqslant 0$ and $2x + 3y = n$. Then we obtain $\sum_{i=1}^{n} \theta(\mathrm{A}_i) \leqslant x\pi + 2y\pi$, that is,

$$n - 2 \leqslant x + 2y. \tag{3}$$

From (3) and that $2x + 3y = n$, we obtain $x \leqslant 6 - n$. Since $n \geqslant 7$, we have $x \leqslant -1$, which is a contradiction. Hence there are no convex n-gons which are developments of a regular tetrahedron when $n \geqslant 7$. Thus we complete the proof of Theorem 2.

5. Concluding remarks

What we have explained above can be applied to obtain convex developments not only for a regular tetrahedron, but for any tetrahedron with four congruent faces. This is because, instead of taking equilateral triangular lattice as our underlying lattice, we can take the triangular lattice made up of contiguous congruent triangles of the faces of the tetrahedron, as indicated in Fig. 34. Topics related with such matters are discussed in [1].

Then we have the following result: there are convex developments of a tetrahedron with four congruent faces having the following shapes: an isosceles triangle, a right-angled triangle, scalene triangles, rectangles, parallelograms, isosceles trapezoids, trapezoids with one of the sides being perpendicular to the bases, trapezoids which are neither isosceles nor with right angles, quadrilaterals which are not trapezoids, pentagons and hexagons.

To prove this, we have only to construct an isosceles triangle which gives a development of a tetrahedron with four congruent faces, because the method of constructing the other shapes is similar to one in the case of a regular tetrahedron. In Fig. 35, we can choose a point P on a line segment AR so that $\mathrm{PB}' = \mathrm{PC}'$. Hence we see that there is an isosceles triangle which gives a development of a tetrahedron with four congruent faces.

Let P be a vertex of a tetrahedron with four congruent faces. Then the sum of three angles which meet at P is π (see Fig. 34). So we have the following result: there are no convex n-gons which are developments of a tetrahedron with four congruent faces when $n \geqslant 7$. The proof is similar to that of Theorem 2.

It was established in [2–5] that there are regular convex n-gons which are developments of convex polyhedrons for each n, where $n \geqslant 7$.

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