1 Extended fields

Let F be a totally ordered field. Let $F_\infty = F \cup \{\bot, \top\}$. We define operations $+$ and $\cdot$ on F_∞ so that (for all $a \in F_\infty$ and for all $p \in F, p > 0$) from top down:

$$
\begin{aligned}
\bot < a &< \top \\
\bot + a = \bot &= a + \bot \qquad \text{(includes } \bot + \top = \bot\text{)} \\
\top + a = \top &= a + \top \\
\bot \cdot p = \bot &= p \cdot \bot \\
\top \cdot p = \top &= p \cdot \top \\
\bot \cdot 0 = \bot &= 0 \cdot \bot \qquad \text{(the ``weird'' rule)} \\
\top \cdot 0 = 0 &= 0 \cdot \top
\end{aligned}
$$

All other cases of all operations and relations preserve their behavior from F. We keep the product between negative numbers and $\{\bot, \top\}$ undefined.

2 Farkas-like conjecture

Let $A \in F_\infty^{m \times n}$ and $b \in F^m$. Exactly one of the following is true:

- $\exists x \in F^n$ such that $0 \leqq x$ and $A\, x \leqq b$
- $\exists y \in F^m$ such that $0 \leqq y$ and $(-A^T)\, y \leqq 0$ and $b^T y < 0$

3 Counterexample

$$
A = \begin{pmatrix} \bot & \top \\ \top & \bot \end{pmatrix} \qquad\qquad b = \begin{pmatrix} -1 \\ -1 \end{pmatrix}
$$

Both are true.

4 Remark

For all other versions of Farkas lemma that I tried to generalize to F_∞ similar counterexample still applies. However, they might work if it was forbidden for a row of A to contain both $\bot$ and $\top$ and alike for a column.

5 New attempt

Let $A \in F_\infty^{m \times n}$ and $b \in F_\infty^m$. Assuming that no row and no column of A contains both $\bot$ and $\top$ elements and that A does not have $\bot$ on any row where b has $\bot$, exactly one of the following is true:

- $\exists x \in F^n$ such that $0 \leqq x$ and $A\, x \leqq b$
- $\exists y \in F^m$ such that $0 \leqq y$ and $(-A^T)\, y \leqq 0$ and $b^T y < 0$

6 Proof idea

We need to do the following steps in given order:

1. Delete all rows of $(A|b)$ where A has $\bot$ or b has $\top$ (they are tautologies).
2. Delete all columns of A that contain $\top$ (they force respective variables to be zero).
3. If b contains $\bot$ then the $(\exists x)$ part cannot be satisfied, but $y = 0$ satisfies the other part. Stop here.
4. Assume there is no $\bot$ in b. Use the normal Farkas. Whichever solution Farkas outputs, extend it with zeros on all deleted positions.

7 Proof sketch

- $A \in F_{\infty}^{I \times J}$
- $b \in F_{\infty}^{I}$
- hA: no row i is allowed to have both $A_{i,\star} \ni \bot$ and $A_{i,\star} \ni \top$
- hAb: no row i is allowed to have both $b_i = \top$ and $A_{i,\star} \ni \top$
- if $\bot \in b$ then easy; from now on assume $\bot \notin b$
- $I' := \{i \in I \mid b_i \neq \top \wedge \bot \notin A_{i,\star}\}$
- $J' := \{j \in J \mid \top \notin A_{\star,j}\}$
- $A' := A \restriction (I' \times J')$
- $A' \in F^{I' \times J'}$
- $b' := b \restriction I'$
- $b' \in F^{I'}$
- $(\exists x) \implies (\exists x')$ ... easy
- $(\exists x') \implies (\exists x)$ ... easy
- assume $\exists y \in F^{I}$ such that $0 \leqq y$ and $(-A^T)\, y \leqq 0$ and $b^T y < 0$
  - use $y' := y \restriction I'$
  - $y' \in F^{I'}$
  - $0 \leqq y'$ from $0 \leqq y$

- given $j \in J'$ show: $((-A'^T)\ y')_j = \sum_{i \in I'} y'_i \cdot (-A'^T)_{j,i} \le 0$
  using: $((-A^T)\ y)_j = \sum_{i \in I'} y'_i \cdot (-A^T)_{j,i} + \sum_{i \in I \setminus I'} y_i \cdot (-A^T)_{j,i} \le 0$
  suffices: $\sum_{i \in I \setminus I'} y_i \cdot (-A^T)_{j,i} = 0$
  fix $i \in I \setminus I'$ and show $y_i \cdot (-A)_{i,j} = 0$
  - from our I' we have either $b_i = \top$ or $\bot \in A_{i,\star}$
  - show $\bot \notin (-A^T)_{\star,i}$ that is $\top \notin A_{i,\star}$ by contradiction $\top \in A_{i,\star}$
    - if $b_i = \top$ then contradicts hAb
    - if $\bot \in A_{i,\star}$ then contradicts hA
  - show $y_i = 0$ by contradiction $y_i > 0$
    - if $b_i = \top$ then $b^T y = \top \ge 0$
      (we need $\bot \notin b$ here)
    - if $\bot \in A_{i,\star}$ hence $\top \in (-A^T)_{\star,i}$ then $((-A^T)\ y)_j = \top > 0$
      (we need $\bot \notin (-A^T)_{\star,i}$ here)
  - show $(-A)_{i,j} \neq \bot$
    - special case of what we already have
- we need to show $b'^T y' = \sum_{i \in I'} y'_i \cdot b'_i < 0$
  suffices… $\sum_{i \in I \setminus I'} y_i \cdot b_i = 0$
  show $y_i = 0$ by the same arguments as above
  finish using $\bot \notin b$

- assume $\exists y' \in F^{I'}$ such that $0 \leqq y'$ and $(-A'^T)\ y' \leqq 0$ and $b'^T y' < 0$
  - use $y := y'$ extended with 0 on $I \setminus I'$
  - $y \in F^I$
  - $0 \leqq y$ is trivial
  - assume $j \in J'$ and calculate… $((-A^T)\ y)_j =$
    $\sum_{i \in I'} y_i \cdot (-A^T)_{j,i} + \sum_{i \in I \setminus I'} y_i \cdot (-A^T)_{j,i} =$
    $\sum_{i \in I'} y'_i \cdot (-A'^T)_{j,i} + \sum_{i \in I \setminus I'} 0 \cdot (-A^T)_{j,i} =$
    $((-A'^T)\ y')_j + \sum_{i \in I \setminus I'} [0 \text{ or } \bot] \le ((-A'^T)\ y')_j \le 0$
    assume $j \notin J'$ hence $\top \in A_{\star,j}$ hence $\bot \in (-A^T)_{j,\star}$ and calculate...
    $((-A^T)\ y)_j = \sum_{i \in I} y_i \cdot (-A^T)_{j,i} = [\ge 0] \cdot \bot + (\dots) = \bot \le 0$
  - $b^T y = \sum_{i \in I'} y_i \cdot b_i + \sum_{i \in I \setminus I'} y_i \cdot b_i = \sum_{i \in I'} y'_i \cdot b'_i + \sum_{i \in I \setminus I'} 0 \cdot b_i = b'^T y' < 0$
    (note that we used $\bot \notin b$ for $0 \cdot b_i = 0$ in the last equality)