1 Extended fields

Let F be a totally ordered field. Let $F_\infty = F \cup \{\bot, \top\}$. We define operations $+$ and $\cdot$ on F_∞ so that (for all $a \in F_\infty$ and for all $p \in F, p > 0$) from top down:

$$
\begin{aligned}
\bot < a &< \top \\
\bot + a = \bot &= a + \bot \qquad \text{(includes } \bot + \top = \bot\text{)} \\
\top + a = \top &= a + \top \\
\bot \cdot p = \bot &= p \cdot \bot \\
\top \cdot p = \top &= p \cdot \top \\
\bot \cdot 0 = \bot &= 0 \cdot \bot \qquad \text{(the "weird" rule)} \\
\top \cdot 0 = 0 &= 0 \cdot \top
\end{aligned}
$$

All other cases of all operations and relations preserve their behavior from F. We keep the product between negative numbers and $\{\bot, \top\}$ undefined.

2 First attempt

Let $A \in F_\infty^{m \times n}$ and $b \in F^m$. Exactly one of the following is true:

- $\exists x \in F^n$ such that $0 \leqq x$ and $A\,x \leqq b$
- $\exists y \in F^m$ such that $0 \leqq y$ and $(-A^T)\,y \leqq 0$ and $b^T y < 0$

3 Counterexample

$$
A = \begin{pmatrix} \bot & \top \\ \top & \bot \end{pmatrix} \qquad\qquad b = \begin{pmatrix} -1 \\ -1 \end{pmatrix}
$$

Both are true.

4 Remark

For all other versions of Farkas lemma that I tried to generalize to F_∞ similar counterexample still applies. However, they might work if it was forbidden for a row of A to contain both $\bot$ and $\top$ and alike for a column.

The problem is when the conversion to the finite version requires us to delete both rows and columns, the rows must be deleted first both in the primar and in the dual, and so, if deleting a row stops deleting a column from triggering, it leads to a different result in the primar than in the dual because of the matrix transposition.

5 Extended Farkas

Let $A \in F_\infty^{m \times n}$ and $b \in F_\infty^m$. Assuming that no row and no column of A contains both $\bot$ and $\top$ elements and that A does not have $\bot$ on any row where b has $\bot$ and that A does not have $\top$ on any row where b has $\top$, exactly one of the following is true:

- $\exists x \in F^n$ such that $0 \leqq x$ and $A\,x \leqq b$
- $\exists y \in F^m$ such that $0 \leqq y$ and $(-A^T)\,y \leqq 0$ and $b^T y < 0$

6 Proof idea

We need to do the following steps in given order:

1. Delete all rows of $(A|b)$ where A has $\bot$ or b has $\top$ (they are tautologies).
2. Delete all columns of A that contain $\top$ (they force respective variables to be zero).
3. If b contains $\bot$ then the $(\exists x)$ part cannot be satisfied, but $y = 0$ satisfies the other part. Stop here.
4. Assume there is no $\bot$ in b. Use the normal Farkas. Whichever solution Farkas outputs, extend it with zeros on all deleted positions.

7 Proof sketch

- $A \in F_\infty^{I \times J}$
- $b \in F_\infty^I$
- hA: no row i is allowed to have both $A_i \ni \bot$ and $A_i \ni \top$
- hAT: no column j is allowed to have both $A_{\star,j} \ni \bot$ and $A_{\star,j} \ni \top$
- hAb: no row i is allowed to have both $b_i = \top$ and $A_i \ni \top$
- if $\bot \in b$ then easy; from now on assume $\bot \notin b$
- $I' := \{i \in I \mid b_i \neq \top \wedge \bot \notin A_i\}$
- $J' := \{j \in J \mid \top \notin A_{\star,j}\}$
- $A' := A \restriction (I' \times J')$
- $A' \in F^{I' \times J'}$
- $b' := b \restriction I'$
- $b' \in F^{I'}$

- $(\exists x) \implies (\exists x')$ ... easy
- $(\exists x') \implies (\exists x)$ ... easy
- assume $\exists y \in F^I$ such that $0 \leqq y$ and $(-A^T)\ y \leqq 0$ and $b^T y < 0$
    - use $y' := y \restriction I'$
    - $y' \in F^{I'}$
    - $0 \leqq y'$ from $0 \leqq y$
    - for every $i \in I \setminus I'$ show $\bot \notin (-A^T)_{\star,i}$
        - from our I' we have either $b_i = \top$ or $\bot \in A_i$
        - show $\top \notin A_i$ by contradiction $\top \in A_i$
            - if $b_i = \top$ then contradicts hAb
            - if $\bot \in A_i$ then contradicts hA
    - for every $i \in I \setminus I'$ show $y_i = 0$
        - from our I' we have either $b_i = \top$ or $\bot \in A_i$
        - show $y_i = 0$ by contradiction $y_i > 0$
            - if $b_i = \top$ then $b^T y = \top \geq 0$
              (we need $\bot \notin b$ here)
            - if $\bot \in A_i$ hence $\top \in (-A^T)_{\star,i}$ then $((-A^T)\ y)_j = \top > 0$
              (we use hAT here — TODO necessary?)
    - given $j \in J'$ show: $\quad ((-A'^T)\ y')_j = \sum_{i \in I'} y'_i \cdot (-A'^T)_{j,i} \leq 0$

      using: $\quad ((-A^T)\ \ y)_j \ = \ \sum_{i \in I'} y'_i \cdot (-A^T)_{j,i} + \sum_{i \in I \setminus I'} y_i \cdot (-A^T)_{j,i} \ \leq \ 0$

      suffices: $\quad \sum_{i \in I \setminus I'} y_i \cdot (-A^T)_{j,i} = 0$
        - $y_i = 0$
        - $(-A)_{i,j} \neq \bot$
    - we need to show $b'^T y' = \sum_{i \in I'} y'_i \cdot b'_i < 0$

      suffices: $\quad \sum_{i \in I \setminus I'} y_i \cdot b_i = 0$
        - $y_i = 0$
        - $\bot \notin b$
- assume $\exists y' \in F^{I'}$ such that $0 \leqq y'$ and $(-A'^T)\ y' \leqq 0$ and $b'^T y' < 0$
    - use $y := y'$ extended with 0 on $I \setminus I'$
    - $y \in F^I$
    - $0 \leqq y$ is trivial

- assume $j \in J'$ and calculate: $((-A^T)\ y)_j =$
$\sum_{i \in I'} y_i \cdot (-A^T)_{j,i} + \sum_{i \in I \setminus I'} y_i \cdot (-A^T)_{j,i} =$
$\sum_{i \in I'} y'_i \cdot (-A'^T)_{j,i} + \sum_{i \in I \setminus I'} 0 \cdot (-A^T)_{j,i} =$
$((-A'^T)\ y')_j + \sum_{i \in I \setminus I'} [0 \text{ or } \bot] \leq ((-A'^T)\ y')_j \leq 0$
assume $j \notin J'$ hence $\top \in A_{\star,j}$ hence $\bot \in (-A^T)_j$ and calculate: $((-A^T)\ y)_j = \sum_{i \in I} y_i \cdot (-A^T)_{j,i} = [\geq 0] \cdot \bot + (\dots) = \bot \leq 0$
- $b^T y = \sum_{i \in I'} y_i \cdot b_i + \sum_{i \in I \setminus I'} y_i \cdot b_i = \sum_{i \in I'} y'_i \cdot b'_i + \sum_{i \in I \setminus I'} 0 \cdot b_i = b'^T y' < 0$
(note that we used $\bot \notin b$ for $0 \cdot b_i = 0$ in the last equality)