Chapter 3

Abstraction

Mathematics in all of its manifestations and all of its vaguely defined layers is one enormous exercise in abstraction. A number is an abstraction. A geometric point is an abstraction. A circle is an abstraction. The Witch of Agnesi, a two-dimensional curve, is an abstraction. A probability of a certain event is an abstraction. A mathematical equation is an abstraction. An integral is an abstraction. So are a sphere and an elliptic paraboloid.

Both studying and doing mathematics requires a constant engagement with abstraction. In the **Prerequisites** chapter of this book we already saw one example of abstraction when we moved from concrete and specific numbers to their compact representation with symbols.

Such symbols can capture not only different numbers but different digitizable notions as well. For example, the symbol d can be an abstract representation of distance, while the symbol t can be an abstract representation of temperature. Both of these notions can be associated with numbers, but none of these notions have anything in common otherwise. The notion of distance has something to do with space, while the notion of temperature has something to do with kinetic energy.

Thus, it seems that the notion of a number and the consequent symbolization of numbers already does something very interesting to all digitizable things. As an abstraction, a number looks at all candidates and takes away their differences that it deems irrelevant, while preserving their commonality that it deems essential.

Abstraction, however, is not the solve purview of mathematics because the notion of abstraction finds its use in theoretical physics, in a form of a virtual particle, for example, and in theoretical computer science, in a form of an algorithm, for example.

Moreover, if the computational program of mechanical engineering amounts to a construction of an approximate solution of an exact problem, the computational program of theoretical physics amounts to a construction of an exact solution of an approximate problem.

The foundation of both of the above computational programs is mathematics and we now know that mathematics does the job of modeling physical phenomena unreasonably well, accomplishing that task with flying colors.

One of the fundamental reasons of why mathematics is so successful both on its own and in conjunction with other endeavors is because one of the corner stones of mathematics is abstraction.

For a specific example, one and the same mathematical equation:

$$\ddot{q} + \omega^2 q = 0$$

where the symbols $\ddot{q}$ represent the second derivative of the quantity q with respect to time, can describe very distinct processes:

- the process of oscillation of a material point attached to an end of a rigid rod (a pendulum)
- the process of oscillation of a point mass attached to an ideal spring
- the process of flow of an electrical current through a so-called LC-circuit

As such, how an abstraction can be defined or described?

In its first, ascetic, approximation, *abstraction* as a verb can be defined as a process of separation of what does not matter from what does matter.

In its second ascetic approximation, *abstraction* as a verb can be defined as a process of separation of what does not matter *at all* from *the only* things that matter.

In its third, less ascetic, approximation, *abstraction* as a verb can be defined is a process of separation of *everything* that does not matter at all from the only things that matter.

Finally, we may agree that *abstraction as a verb* means a process of separation of all the random and irrelevant details that do not matter in a particular context at all from the only details that matter in that context.

In such a process of elimination, the details that have been identified as the irrelevant minutiae are discarded from consideration completely and the details that have been identified as the only ones that matter are distilled and compressed into the *essence* of a given phenomenon or into the essence of the commonality of a collection of objects.

We, then, may agree that *abstraction as a noun* is simply the outcome of the above elimination rounds - a quintessential core that can be studied in its own right in total isolation from everything else and that can be applied in various contexts and animated with different specific material.

It is nothing short of a miracle that we live in a universe where such a lossy process, a process that actively discards things from consideration, actually helps us develop a deep understanding of a phenomenon or a collection of items of interest and then it allows us to re-purpose the established results elsewhere, often with an astounding success!

Intuitively speaking, abstraction is a peculiar process that allows us to gain via a loss.

In a modern society we deal with a multitude of abstractions of various scopes at all times.

For example, when we walk into an abstraction known as *a furniture store*, we know that the owners of such a store have already went through a massive elimination process and collected under the roof of their store only the convenient objects that we can bring into our homes to sit on, lie on, sit at or lie at.

Another example of an extremely useful abstraction in most developed societies is *a motor vehicle*.

In US, for example, every state government publishes a so-called *driver's manual* or a relatively small book that lists all the laws according to which all motorists in that state must behave on public roads.

However, there are many different makes, models and colors of passenger vehicles produced by multiple manufacturers and sold by multiple dealers.

Many of these same makes and models of passenger cars evolve over time - there are multiple copies of a blue 1967 Ford Mustang, and there are multiple copies of a purple 1983 Ford Mustang, and there are multiple copies of a yellow 2013 Ford Mustang and so on.

Let us now duplicate these examples over the entire bewildering array of different vehicles from different car makers and entertain the following thought experiment.

Imagine for a moment that we do not have this abstraction to which we refer as *a motor vehicle*.

It stands to reason that *then* every state government would have to publish a separate driver's manual for each make, model and color of a car produced by each manufacturer during each year that that manufacturer sells its vehicles to the US market!

In order to drive the point of the above thought experiment home, our state governments would have to publish:

- a separate driver's manual for the proverbial blue 1967 Ford Mustang and
- a separate driver's manual for the proverbial red 1967 Ford Mustang and
- a separate driver's manual for the proverbial yellow 1967 Ford Mustang

and so on for the years 1968, 1969, 1970 …

This set of driving manuals, however, would only cover the Ford Mustangs.

But what about the various Jeeps, Toyotas, Chryslers, Audis, Chevrolets, Hondas, Volkswagens and friends?

Come to think of it, this undertaking already sounds like a nightmare scenario to implement and maintain, doesn't it?

But wait, there is more.

Our police officers, patrolling the streets and the highways in a given state, would then have to know, remember and juggle all these presumably different rules in their mind in real time so that they can parse and validate the behavior of all these different cars according to such rules!

Moreover, we, as drivers, would have to learn and re-learn the different driving rules when we switched vehicles also. Selling our green Volkswagen and buying a blue Jeep would entail finding a driver's manual that explains all the rules of operating a blue Jeep on a road and re-learning all these rules from the ground up.

We hope that our readers now understand and appreciate the convenience and, by the same token, the power afforded by the abstraction known as *a motor vehicle*.

When the context of interest becomes *behavior on public roads*, then, so long that we find ourselves driving a motor vehicle on a public road, there is one and only one set of rules to learn, remember, use and apply.

That one set of driving rules inhales *all* the staggering visual and mechanical differences between the multitude of its subjects, crunches or trawls them through the above abstraction *process* and eliminates and throws all of them away mercilessly.

Such a set of driving rules is interested only in the fact that we are on a public road inside of a reasonably massive object that can propel itself at reasonably high speeds and maneuver in an acceptable manner in two-space at will.

The government of a given state, then, has to publish only one booklet that collects all of the above driving rules in one convenient place without any worry about the massive variance across all the vehicles, propelled by either an electric or an internal combustion engine.

The police officers of a given state have to learn only one set of driving-on-public-roads rules and so do we.

That state of driving affairs is a tangible benefit for a society that adopts the abstraction known as *a motor vehicle*.

Exercise 3.0.1: on your own, think about and identify at least one more useful everyday abstraction. Explain, which details such an abstraction eliminates and throws away and which details such an abstraction keeps as essential.

Solution: of this exercise is left to the reader. □

In mathematics, abstraction as a verb and as a noun operates in a way that does not deviate from the program explained above too much.

Mathematics, naturally, brings the intolerance to ambiguity, the specificity of the professional terms and the professional jargon to the table but at its core it is still preoccupied with the process of elimination of the irrelevant minutiae and the extraction of the bare essence.

Mathematics also capitalizes on the fact that one abstraction can absorb multiple other abstractions and, as a result, some abstractions are way more abstract than other abstractions.

Take the following suggestion with a grain of salt, because its verification is likely lost in the mist of time, but it is plausible to think that the very early humans or human-like creatures might have had the object-specific terms or names or sounds for the *numerocity* of things.

Crawling out of a proverbial cave, such an early human could have thought:

Hm, yesterday I saw *threebison* over there and today I see *fivebison* on the same spot!

Or:

Hm, yesterday I saw *twowooly* mammoths over there and now I see *sevenwooly* mammoths!

It is also likely that over time people started realizing that there is a certain *commonality* that runs through various everyday objects and that such a commonality can be attached to *all* of these objects, regardless of how drastically different they may be.

For example, what do stars in the sky, pebbles on the beach and bears in the forest have in common?

Ah, they all can be *counted* and, thus, give or take, a very abstract notion of *a number* is born.

Fast-forwarding by thousands and thousands of years and, as we already observed in the **Prerequisites** chapter, we study these counting numbers in lower school.

At some point, however, the descendants of the early humans have noticed that all these specific and concrete numbers, such as $1, 2$ and 3, have a particular *commonality* that can be captured and expressed without a direct reference to each and every such number specifically.

At first, in the works of the ancient Greek mathematician Euclid, we see how a geometric object, such as *a point*, for example, is adorned with a single-letter symbol, such as A, B or C, and later on we see how the specific numbers obtain their collective, symbolic, *representatives* via the lower case single letters, such as a, b and c, and the study of algebra is born.

That is, when we see three oranges on a table, we say that there are three *oranges* on a table, even though these oranges may be different in size and shape. No matter. The abstraction known as *an orange* swallows all these differences whole and ignores them blissfully.

When we see two *apples* on a table, we say that there are two *apples* on a table, even though these apples may be different in size, shape, color and type. No matter. The abstraction known as *an apple* swallows all these differences whole and ignores them all as well.

When we see three oranges and two apples on a table then what are we to do if we want to count them all?

We cannot add apples to oranges.

But we can introduce a new abstraction, *a fruit*, that will absorb these two existing abstractions and merge them into one unifying abstraction to which we refer as *a fruit*: an orange is a fruit and and an apple is a fruit. Thus, there are five *fruits* on a table.

Amazingly enough, later on we come to discover that there are *other fruits* out there and they all can be absorbed into the same abstraction …

Continuing the above marathon of abstractions, early on in the theory of groups we overcome the next big hurdle, as we do with operations, or calls to action, the same thing that our ancestors did with concrete numbers - we symbolize and abstract various mathematical calls to action into a new entity, *an* *abstract operation* or just *an operation* and, as we shall discover soon enough, we bring two abstract notions, the notion of *a collection of elements* and the notion of *a mathematical operation*, together under one roof of yet another abstraction notion of *a mathematical group*.