Chapter 7

A Difficult, Giant, Leap Forward

We hope that in the course of doing the exercises from and diligently working through the material presented in the early examples of a group our readers, sooner or later, started getting this feeling that the lengthy, wordy and awkward descriptions of various *operations*, seemingly mathematical in nature or not, are becoming more of a burden than a help.

We used these specific exercises in order to achieve three goals.

One. We needed some simple, tangible, visual, fun do-it-yourself introductory, consumable by a motivated upper middle school student, material with which to draw our readers in and involve them in doing the group-theoretic work because *doing* mathematics equates to *learning* mathematics.

Two. We wanted to demonstrate the mechanics of coming up with a certain abstraction.

Three. We wanted to at least try to present a plausible enough experiment that showcases how a symbolization of, not necessarily mathematical, operations comes about very naturally in or emerges from the course of such work.

During the switch from the lower school arithmetic of the naked numbers to the symbolic arithmetic of middle school algebra we had to overcome the proportional challenge of the said switch.

However, once the initial difficulties of the above transition were left behind, the advantages of the symbolic middle school algebra, which we already discussed earlier in the **Prerequisites** chapter, became apparent.

Likewise, it may be initially difficult to switch from the lengthy and wordy but warm and fuzzy, comfortable and easily comprehensible descriptions of various *operations* to their short and concise symbolic representation.

But once we jump *over* that hurdle, we let the theory of groups stretch its legs, flex its muscle and show its full potential, which, by the way, explains why the theory of groups permeates not only the modern grown-up mathematics but also leaks out into the modern:

- physics, see the so-called *Poincare Group*, for example
- chemistry, see *molecular symmetries* or *isomers enumeration* using the Polya Counting, for example
- crystallography
- theory of music, see its famous *circle of fifth* and *pitch classes enumeration*, for example
- preforming arts, see our square dancing group, for example
- visual arts, see the so-called *Wallpaper Group*, for example
- games, such as *Sudoku* or *Square-15*, for example
- architecture, see the *frieze patterns*, for example
- study of family relations

and so on.

One of the reasons that explains that awesome penetrating power of the theory of groups is its *abstractness*, which, in turn, is the source of grief for many students.

Historically, in real time, it is plausible to think that it took the brighter part of humanity about one hundred years to see that abstractness through.

In this text, however, we are compressing that enlightenment process into just one *week* or so.

To that end, not only the mathematical and the non-mathematical operations can by symbolized like numbers were symbolized but these operations can be symbolized to the various degrees of compactness.

That is, it is possible to saturate the symbolic representations of the operations with the necessary technical information:

$$R_{\lambda}^{\pi+2\pi k},\ k \in \mathbb{Z}$$

see below, but it is also possible, and in the group-theoretic context it is actually *advisable*, to compress such a symbolization even further into just r, a single-letter entity.

As an opener, let us symbolize the mathematical operation of *composition*.

Composition

The two *operations* that a composition glues together can be symbolized with just a and b and we understand that, by analogy with the symbolization of numbers, on the one hand, the symbols a and b *represent* certain operations that are not specified explicitly and that, on the other hand, the symbols a and b can be mechanically replaced with any suitable candidates on the moment's notice and such a mechanical replacement of the generic or abstract symbols with the suitable specific and concrete candidates will always yield a correct result regardless of whether the said result is obtained via a computation or via a line or reasoning.

The composition *itself* or the phrase *followed by* is traditionally symbolized with a small hollow same-level circle $\circ$, as in $b \circ a$, which is read as *bee of ay*.

In the Western civilization we read and write *left-to-right* and *top-to-bottom*.

In this particular mode of reading, within the current line of text, we discover the *leftmost* information first.

Thus, in the *verbal* or *wordy* description of a composition of the operations a and b:

the operation a followed by the operation b

not only we come across the demand to carry out the operation a first but the very logic of that description also dictates that we carry the operation a *first* and that we carry out the operation b *last*.

In mathematics, however, the modern or the more popular rule according to which the above verbal description of a composition is converted into its *symbolic representation* requires that in the symbolic equivalent the verbal order of the operations is swapped with respect to the $\circ$ symbol:

$$b \circ a$$

Conversely.

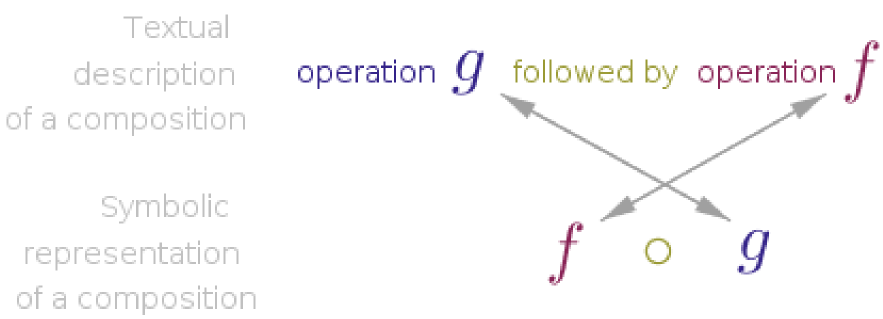
In order to convert a symbolic representation of a composition of two operations:

$$p \circ q$$

into its wordy description, swap the symbolic order of the operations with respect to the phrase *followed by*:

the operation q followed by the operation p

Pictorially (Figure 6.1):



Such a convention is consistent with the modern, see below, functional notation in which we write $f(x)$ and not $(x)f$, which was done in the earlier times but as recently as the 1980-ies (Figure 6.2):

$$(b)\omega = ((b)\varphi)\psi = (c)\psi = a.$$

Whether the result of the composition $b \circ a$ is the same as the result of the composition $a \circ b$ is a different story.

Speaking of the modern functional notation, we can now give an official definition of a composition of two mappings.

Definition 9.1: a composition of a mapping $f:\ X \to Y$ and a mapping $g:\ Y \to Z$ is a mapping $g \circ f:\ X \to Z$ whose result on the elements of X is given by the following formula:

$$(g \circ f)(x) = g(f(x))$$

From the above definition of a composition of two mappings it follows that even when both compositions $g \circ f$ and $f \circ g$ are defined then in general their results are distinct:

$$g \circ f \neq f \circ g$$

For a demonstration of the above observation consider the mapping $f: \mathbb{R} \to \mathbb{R}$ such that $f(x) = x + 1$ and the mapping $g: \mathbb{R} \to \mathbb{R}$ such that $g(x) = x^2$.

By **Definition 9.1** the composition $g \circ f$:

$$g \circ f = (x+1)^2$$

will be quite different from the composition $f \circ g$:

$$f \circ g = x^2 + 1$$

and the fact that there exists a unique real number $x_0 = 0$ at which the values of the above two compositions match is irrelevant because the required universal statement *for all* x does not hold - it is not the case that for all real numbers x the equality $(x+1)^2 = x^2 + 1$ holds.

As an informal exercise, in the light of the new knowledge, let us symbolize the duplex light switch group .

Inaction

The flipping action number 1 or no flips, which is the *do nothing* element in that group, becomes *the identity element* of the group.

We hope that by this time our readers have developed a reasonable intuitive grasp on the essence of the identity element of a group.

We remember that a group, once defined, has one and only one group operation, which becomes *the* group operation.

Very soon we will prove that a group has one and only one identity element, which becomes *the* identity element of a group.

Thus, the identity element of a group is the only element of that group that has the following property:

any element of a group paired up with the identity element of that group by the group operation corresponds to itself

In any given country across time and at any given time across different countries or even across different educational institutions of the same country the symbol used to designate the identity element of a group varies and in the literature on the subject we may see such identity element symbols as:

$$I, \ \mathrm{Id}, \ 1, \ e$$

to show a few.

In this text we will settle on the symbol e with which we will designate the identity element of a group and here comes our first symbolization lesson to learn.

The same mathematical entity may be symbolized in different ways. However. As long as we understand *the meaning* of the chosen symbol, we should have no problem with adjusting to and manipulating that symbol in the prescribed way.

In the older texts the identity element of a group was sometimes also called:

- *the unit element* of a group or
- *the unit* of a group or
- *the neutral element* of a group

Again, once we understand the essence of a mathematical entity, we should have no problem in selecting the appropriate terminology for it from a menu of choices.

Our choice in this text is *the identity element* of a group.

Other Group Elements

We symbolize the flipping action number 2 or flip the **l**eft lever with l.

It is that simple. Short and sweet.

But do notice the mighty effect of what just transpired: the above symbolization process compressed a large amount of technical descriptive information into a small but equivalent symbolic representation of it.

Thus, it can be said that by switching from the verbal gymnastics to a symbolic notation we increase *the efficiency* with which we communicate the mathematical ideas and socialize with ourselves, at different points in time, and with others.

In a certain vague way such an increase in the said efficiency tactically plays the role of *a time compressor* - by using a notation that is symbolic we tell more in a given amount of time and we spend less time conveying/comprehending a given unit of mathematical content.

We symbolize the flipping action number 3 or flip the **r**ight lever with r.

We symbolize the flipping action number 4 or flip **b**oth levers with b.

Thus, the elements of the duplex light switch group become:

$$\{e, l, r, b\}$$

which is *just four symbols** but stop and think for a moment about how much mathematical content is packed into the said four symbols.

Why?

Why did we replace the wordy descriptions of all the group elements with their compact symbolic equivalents?

By replacing the wordy descriptions of all the group elements with their compact symbolic equivalents, strategically, we *detached the specific underlying nature of these elements* from the domain-specific study of their properties and by so doing we gained the ability to manipulate these symbols in an abstract fashion or generically, without the reliance on the specific features of the said elements.

Group Operation

In the specific case of the duplex light switch group the group operation is *followed by* or *the composition* of the flipping actions.

Thus, the long and awkward phrase:

the flipping action number 1 followed by the flipping action number 1 is equivalent to the flipping action number 1

is compressed symbolically into:

$$e \circ e = e$$

The long and awkward phrase:

the flipping action number 2 followed by the flipping action number 2 is equivalent to the flipping action number 1

is compressed symbolically into:

$$l \circ l = e$$

The long and awkward phrase:

the flipping action number 2 followed by the flipping action number 3 is equivalent to the flipping action number 4 is compressed symbolically into:

$$r \circ l = b$$

and so on.

Conversely.

The neat little package of:

$$b \circ l = r$$

is ballooned into an incredible amount of the equivalent descriptive information as follows:

the flipping action number 2 followed by the flipping action number 4 is equivalent to the flipping action number 3

We, thus, now know how to jump back and forth between *neat-and-compact* and *unwieldy-and-large* effortlessly.

By doing so we can hide or pack an incredibly large amount of technical descriptive information into an incredibly small *symbolic* representation of that information, thus increasing the efficiency of conveying and communicating the relevant mathematical ideas.

Appending the identity element of a group to the corpus of our current knowledge, we have:

$$e \circ e = e$$

$$e \circ l = l \circ e = l$$

$$e \circ r = r \circ e = r$$

$$e \circ b = b \circ e = b$$

A note on the modern functional notation: in the duplex light switch group we apply the respective flipping actions to a light switch as a whole and such a light switch, as we have claimed earlier, is always in one of the four possible *states*: State 1, State 2, State 3 and State 4 in our nomenclature.

Thus, it makes sense to refer to such states with our favorite variables x, y, z and friends.

Moreover, we can now invent and explain the following symbolism quite naturally:

$$y = l(x)$$

Right?

The neat little package of the symbols above says that if to a duplex light switch in state x we apply the flipping action number 2, flip the **l**eft lever, then the said light switch will be taken into the state y for any state x.

How, then, would we record the composition $r \circ l$ in the new, functional, notation?

Well, by definition, the output of the $l(x)$ operation, which is y, becomes the input for the $r(y)$ operation:

$$z = r(y) = r(l(x))$$

and it so happens that in this particular group it will also be the case that $z = b(x)$:

$$r(l(x)) = z = b(x)$$

for any state x and we see that the modern functional notation $r(l(x))$ aligns nicely with the initially weird, but not anymore, notation $r \circ l$.

Borrowing A Symbol

We now borrow the composition symbol ∘ and in one sweeping decision, re-purposing that symbol, claim that we will use the small hollow same-level circle ∘ in order to designate *any* and *all* the candidate group operations.

In other words, from now on the symbol ∘ becomes *generic* or *abstract* and, officially, absorbs in itself all the incredible variety of the potential group operations.

In one specific case, such as the case of the duplex light switch group, the symbol ∘ may very well *stand for* or be *a representation of* the composition operation.

In another specific case, such as the the additive group of integers, the symbol ∘ may *stand for* or be *a representation of* the integer addition operation:

$$1 \circ 2 = 1 + 2 = 3$$

Yet, in the case of the numbers -1 and $+1$ the symbol ∘ may *stand for* or be *a representation of* the integer *multiplication of whole numbers* operation:

$$-1 \circ 1 = -1 \cdot 1 = -1$$

and so on.

Why?

Why did we introduce just one symbol for a group operation and what did it buy us?

By introducing just one symbol for a group operation we, strategically, *detached the specific underlying nature of the group operation* from the domain-specific study of the properties of these operations defined on the elements of the underlying sets.

The Net Effect

The symbolization of group elements and the uniform symbolization of a group operation gives us the ability to study the generic properties of groups without worrying or *knowing anything* about the specific underlying nature of the group's elements and of the group's operation.

The breathtaking scope and power of the above observation will sink in with our readers only after some time.

For now we shall remark, that the above ability means that as long as we can prove that our favorite candidate pair of a set and an operation that possesses certain restrictive properties is a group then we can already state many generic properties of this dynamic duo *without knowing anything about the specific and innate features of the elements of the underlying set and the satellite operation defined on that set*.

This is an awfully powerful ability and *now* it should be little wonder that the theory of groups, which, intuitively speaking, glues abstract elements with abstract operations, travels far and wide not only *in* but *out*side of mathematics as well.

Borrowing A Symbol, Again

Wait a second.

A small hollow circle $\circ$ looks like a blown up dot $\cdot$, which is a symbol that is traditionally used for *the multiplication of numbers*.

No matter.

We borrow that symbol also and from now on, let it be written and let it be said, in all of our tactical symbolic shuffling the $\cdot$ symbol will s*tand for* or will *represent* the group operation, which, due to the nature of the just borrowed symbol, now becomes *the group multiplication*:

$$r \circ l = r \cdot l = b$$

in the case of the duplex light switch group and:

$$1 \circ 2 = 1 \cdot 2 = 1 + 2 = 3$$

in the case of the additive group of integers and:

$$-1 \circ 1 = -1 \cdot 1 = -1 \cdot 1 = -1$$

in the case of a couple of groups which we shall study soon enough.

The formation $r \cdot l$ now becomes:

the group product

Naturally.

In other words, the phrase:

the group product

is a stand-in for the phrase:

the result of the group operation as it is applied to an ordered pair of the elements of the group

In the above case the group operation acts on the ordered pair of the elements l and r.

We can now restate the result of the **Exercise 5.1.1**: a group comprised of n elements has exactly n^2 *group products*.

Our readers should not be confused by the above middle group-theoretic transition:

$$1 \circ 2 = 1 \cdot 2 = 1 + 2 = 3$$

because now the symbol $\circ$ and the symbol $\cdot$ is abstract and now these symbols represents *any* candidate group operation, including, when called for, the good old multiplication of numbers.

In many texts, when a group is additive, the symbol $+$ is used directly as the group operation.

Vanishing A Symbol

When the symbols that we shuffle around mathematically are comprised of *a single letter only* then it becomes possible to vanish the symbol $\cdot$ from the juggling act completely and without ill effects.

However.

The two- and more-letter entities under such a convention become impossible, since we can no longer tell whether len is just a single and wholesome variable that stands for *the length* of something or it is a product of, say, three single-letter variables named l, e and n or is it a product of a single-letter variable named l and a two-letter variable named en and so on.

With such a common agreement in place, we can now write:

$$r \circ l = r \cdot l = rl = b$$

with all the previously agreed upon meanings preserved.

Borrowing Numeric Exponentiation

Remember the good old multiplication of like numbers:

$$a \cdot a = aa = a^2$$

that can be compressed into the respective integral power of the said numbers?

Well, group theorists borrow that notation from the algebra of numbers and bravely import it into the abstract algebra:

$$e \circ e = e \cdot e = ee = e^2$$

What does the above transition from a product of like symbols to the integral power of the group element e mean?

The above transition from a product of like symbols to the integral power of the group element e simply means that the group element named ee is now replaced with its equivalent shortcut version of e^2, which is still obtained by the application of the group operation, *whatever it happens to be*, to the same element, e, (of the same group).

In other words, e^2 is still an element of a given group - it is just expressed in this particular form.

How would we symbolize the application of the group operation to the same group element *two* times in a row?

The same way:

$$a \circ a \circ a = a \cdot a \cdot a = aaa = a^3$$

Three times in a row:

$$bbbb = b^4$$

and so on for $(n - 1)$ times in a row if n designates the number of the like elements:

$$\underbrace{cc \dots c}_{n \text{ of those}} = c^n$$

When $n = 0$ then we can simply *define* that:

$$c^0 = e$$

Thus, in the duplex light switch group:

$$l^2 = e, r^2 = e, b^2 = e$$

and so on.

Inverse Elements

Recall that in the duplex light switch group discussion we saw that every flipping action can be *undone* or *reversed* by a flipping action.

The official name for such a flipping action is *the inverse element* of a given element of a group.

Soon we will prove that any element of a group has one and only one inverse, which becomes *the* inverse of a given element.

The inverse element of a given element of a group has the following property:

if a given element of a group is paired up with its inverse by the group operation then the identity element of the group results

In order to symbolize the inverse elements, group theorists also borrow the notation from the algebra of numbers.

Namely, the inverse of a group element, say, l, is symbolized with *the negative integral power of the same element* l^{-1}:

$$l \circ l^{-1} = l \cdot l^{-1} = ll^{-1} = e$$

This approach to the symbolization of the inverses works because each given element of a group has not many but one and only one inverse, which is *the* inverse of that element.

It is also perfectly fine to designate an inverse with our favorite variable x:

$$x = l^{-1}$$

and, as a result of some symbolic manipulations, try to unmask it.

In other words, the following group-theoretic reasoning is completely normal. An inverse of an element l of the group this and that is an element x of that group such that:

$$lx = xl = e$$

We will use this approach in the **Group Axioms** chapter where we will also get into the delicate nuances of splitting the identity element into a left identity and a right identity and by splitting the inverses into a left inverse and and a right inverse.

From the above property of an inverse element it follows that the identity element of a group is its own inverse, always and in any group.

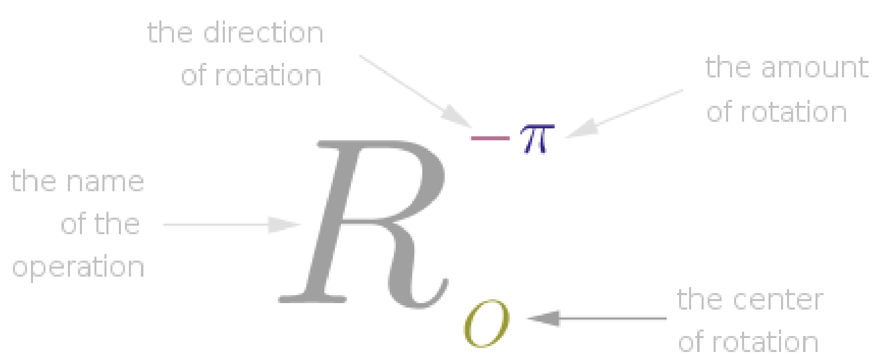
In the duplex light switch group it so happens that:

$$ee = e, \; ll = e, \; rr = e, \; bb = e$$

which means that in that group *every* element of the group is its own inverse - not just the group's identity element.

In the group of rigid symmetries of a non-square rectangle and the group of rigid symmetries of a square we used the operations of *rotation* (of the plane) and *reflection (of the plane) in a straight line.*

On the one hand, the said operation of *rotation* can always be symbolized as follows (Figure 6.3):



Thus, the rotation of an object about a point A by $-\pi$ radians followed by the rotation of the same object about a point B by $\pi/6$ radians will be symbolized as follows:

$$R_B^{\pi/6} \circ R_A^{-\pi}$$

The rotation of an object, in 3-space, about a straight line named λ will be symbolized in a similar way:

$$R_\lambda^\pi$$

and so on.

The operation of *reflection* of an object in a straight line v can be symbolized as follows:

$$S_v$$

for **s**ymmetry in a straight line.

The congruence motion of a rectangle number 2 can, thus, be symbolized as follows:

$$R_O^{\pi+2\pi k},\ k \in \mathbb{Z}$$

but we can all agree that the following symbolic version of the same operation is good enough:

$$R_O^{\pi}$$

Thus, the long and awkward but easy to decipher phrase:

the rotation of the rectangle about its center by π radians counterclockwise followed by the reflection of the rectangle in its long axis of symmetry, h, is equivalent to reflecting the rectangle in its short axis of symmetry, v

can be perfectly happily symbolized as:

$$S_h \circ R_O^{\pi} = S_v$$

and just as happily we can continue working with the above symbols.

Moreover, when, in the context of a plane-geometric problem, we need to record a specific algorithm in terms of these specific operations then it is quite normal to keep the above symbols because doing so is supposed to be enlightening and valuable *in that particular setting*.

However.

For the group-theoretic purposes, it is possible and even advisable to compress all of the above symbology even further.

That is, the *no motion* of a rectangle, as the group's identity element, is symbolized with e.

The rotation of a rectangle about its center by $\pi + 2\pi k$ radians can be symbolized with just r.

The reflection of a rectangle in its short axis of symmetry, v, can be symbolized with v.

The reflection of a rectangle in its long axis of symmetry, h, can be symbolized with h.

The elements of the group of symmetries of a rectangle, thus, become:

$$\{e, r, v, h\}$$

which, again, is *just four symbols* and we already know that:

$$r^2 = e,\ v^2 = e,\ h^2 = e$$

and that:

$$vhr = e$$

and so on.

What about the tire rotations group? How can that group be symbolized?

Our readers should already see that the elements of that group are:

$$\{e, f, s, c\}$$

which is *just four symbols*, yet again.

Here the symbol f stands for the **f**ront-to-back tire rotation (naturally), the symbol s stands for the **s**ide-to-side tire rotation and the symbol c stands for the **c**ross-over tire rotation and so on.

Now. On the one hand, we hope that the above symbolization process is not horribly complicated and that our proverbial upper middle school or a lower high school student can comprehend and digest it.

Can these kids comprehend the flips of levers of a light switch? Sure they can - the kids flip the levers on the light switches at home every day.

Can these kids comprehend the swaps of tires on a car? Sure they can - the kids can be escorted out to the parking lot where we chalk the letters A, B, C and D on the actual tires of an actual car and on the asphalt and then we move the tires around right then and there.

Can these kids comprehend the rotations and the flips of an A4 sheet of paper? Sure they can - they just did that in the classroom.

Can these kids comprehend the swaps of the contra dance partners carried out to the music ? Sure they can - they just performed the contra dance figures in the classroom themselves. What could be more fun then *being* a mathematical experiment?

But.

On the other hand, the moment we switch from the *descriptive* narrative to the *symbolic* narrative in a way that was explained above, then the things tend to miraculously get very abstract very fast and the ability of the students to keep up and manipulate that abstraction correctly and briskly will begin to vary. A lot.

For an opening example.

The Group Theoretic Magic On Display

With just the above few symbolization ideas on the books we can already demonstrate some group-theoretic magic.

Namely.

Suppose we play a game in which we, not knowing anything about the underlying nature of the symbols l, r and b, are told that:

$$ee = e,\ ll = e,\ rr = e,\ bb = e$$

and that these symbols represent a group whose identity element is e and in which it is also the case that $r \circ l = b$.

Still knowing absolutely nothing about exactly what sits behind the letters l, r and b we instantly deduce that in that particular group *it must be* the case that:

$$b \circ l = r$$

and we tell that fact to the person who gave us the initial set of data.

When such a person who does know what the expression $b \circ l = r$ actually means in the context of the duplex light switch group goes and checks it out then, oh the magic, that person will discover that *yes*, if we flip the left lever on a duplex light switch and then we flip both levers on that switch then such a composition of two flipping actions will be equivalent exactly to the flip of the right lever only.

We answer the puzzling *how-the-hell-did-you-know* look on the face of the experimenter as follows.

We know that we are dealing with a group on four symbols with the properties spelled out above.

Thus, we mechanically *multiply* the given expression $r \circ l = b$ by the element l from the right side:

$$r \circ l \circ l = b \circ l$$

We remember now that when we *multiply* a given expression this and that by an element such and such, we actually *apply the group operation* to the underlying constituents, whatever these constituents happen to be.

Admittedly, there is a lot going on in just this one step and that is why this chapter is titled **A Difficult, Giant, Leap Forward**.

Even though we will be officially studying the associativity of binary operations only in the next chapter, it should not be too difficult to see that since the binary operation of a group is associative, as we have seen it in five specific exercises already, the meaning of the expression $r \circ l \circ l$ is well-defined because it always amounts to one and the same result, with no ambiguity of any kind:

$$r \circ l \circ l = r \circ (l \circ l) = (r \circ l) \circ l$$

In other words, the specific order in which we decide to carry out the group operation in a string of two consecutive such operations does not affect the overall result.

In the above expressions we used the parentheses () to alter or specify the order in which the given operations are to be carried out.

In this case we choose to carry out the second operation first:

$$l \circ l$$

because we are told that $l \circ l = e$.

Hence, we can rewrite the LHS of the above result as follows:

$$r \circ l \circ l = r \circ (l \circ l) = r \circ e = b \circ l$$

We also know that in any group it must be the case that $r \circ e = r$.

Therefore, we can further simplify the LHS of the above result as follows:

$$r = b \circ l$$

and announce it to the bewildered audience.

Can you now impress your friends by announcing that in that particular group it must also be the case that:

$$r \circ b = l$$

?

We urge our readers to dwell in this little demonstration for as long as it takes to digest what just transpired: in a strange and bizarre but delightful symbolic shuffling session we just discovered a fact that is true without having any clue about the underlying nature of the symbols that we juggled!

But wait. There is more. The theory of groups is only beginning to weave its magic.

If we play a similar game in which someone tells us only that:

$$ee = e,\ ll = e,\ rr = e,\ bb = e$$

is a group on four elements with e being the group's identity, then we can just as, almost, instantly, and by knowing only what we learned so far, prove beyond any reasonable doubt that this group *must be commutative*!

Granted, we do not yet know what exactly does this mysterious sequence of words *a commutative group* mean but trust us, we will learn that soon enough and our readers will have the pleasure of proving this little theorem on their own as an upcoming exercise.

Now an even more involved example.

Welcome To The World of Abstractness

When we earlier told our readers that they will need reams of paper for the upcoming journey we, of course, knew already that, as a part of the ongoing exercises, they will write down the wordy descriptions of all 16 group products for the duplex light switch group, all 16 group products for the tire rotations group, all 16 products for the group of symmetries of a rectangle and yes, all 64 group products for the contra dance group, which the curious travelers, no doubt, extended into the 64 group products of the group of symmetries of a square.

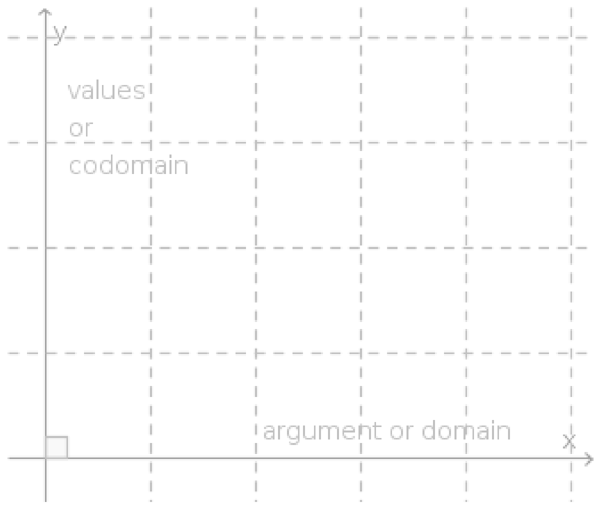
We also told our readers that here and there we will not be spelling out the way in which to accomplish this or that task.

Writing down all these group products should have nudged our readers toward the bright idea that there must be a better way to visualize the ordered pairs of group elements or the group products.

Many correct inductive ideas are obtained *by analogy*.

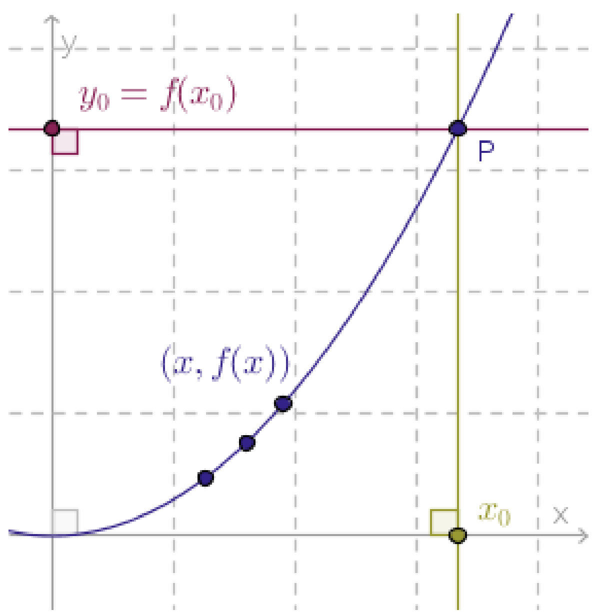
How do we visualize, even if just partially, the behavior of a function $f(x)$, technically a single-variable real-to-real function?

We visualize, even if just partially, the behavior of a function $f(x)$ by introducing the x-axis that carries the geometric representations of the function's *argument* or the *domain* of the function and the perpendicular to it y-axis that carries the geometric representations of the function's *values* or the *codomain* of the function (Figure 6.4):



Then, we represent the specific way in which the values packed into x and $f(x)$ interact with each other by:

- drawing the straight line $x = x_0$ perpendicular to the x-axis
- drawing the straight line $y = f(x_0)$ perpendicular to the y-axis
- and finding the (blue) point P where the above two straight lines intersect (Figure 6.5):



A collection $(x, f(x))$ of P-like points thus obtained gives either a complete or a partial *graph* of the function $f(x)$.

But what *is* the graph of a function $f(x)$?

The graph of a function $f(x)$ is the visualization of how the pairs of real numbers x and $f(x)$ interact with each other.

By analogy, one way to visualize how all n^2 ordered pairs of n elements of which a given group is comprised interact with each other is to pack that n^2 bits of information into *a square table* in a very similar way.

Namely.

Along the proverbial x-axis write down the names of all n elements of a given group in some order in the *one-name-per-cell* fashion and along the proverbial y-axis write down the same names of all n elements of a given group in the same order and in the same fashion (Figure 6.6):

	e	l	r	b
e				
l				
r				
b				

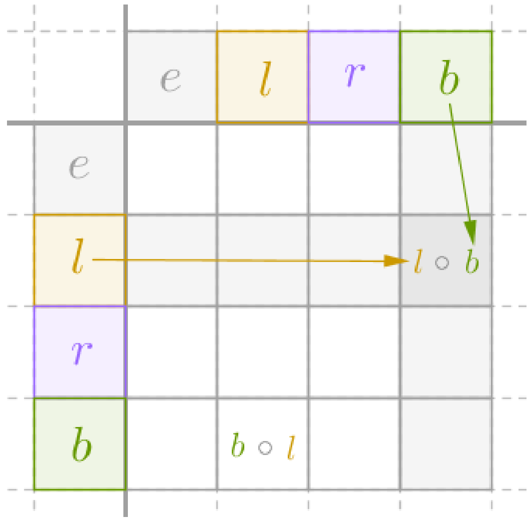
In the above square table we used the names of the elements of the duplex light switch group.

The very top row in such a table plays the role of *the column headers* and the leftmost column in such a table plays the role of *the row headers*.

Fill out the contents of the actual square table by recording the result of each *ordered* group product $p \circ q$ in the *one-result-per-cell* fashion by:

- taking the first or the leftmost element of the product, p, from the respective cell of the row header
- taking the second or the rightmost element of the product, q, from the respective column header and

- by recording the result of that product in the table's cell that is located at the perpendicular intersection of the respective p-row and the respective q-column of the table for all the elements p, q of the group at hand (Figure 6.7):



All and the only two distinct orders in which two elements of a group, p and q, can enter into a group product, $p \circ q$ and $q \circ p$, will be captured in the corresponding two cells of the square table thus constructed.

But we already agreed that:

$$l \circ b = l \cdot b = lb$$

Thus, a square table that captures how the elements of *the light switch group* interact with each other in an order-sensitive way, also known as *a multiplication table* of a group, may look as follows (Figure 6.8):

	e	l	r	b
e	ee	el	er	eb
l	le	ll	lr	lb
r	re	rl	rr	rb
b	be	bl	br	bb

Now go back to your reams of paper and using the results that you generated by solving the earlier exercises, mechanically replace each order-sensitive verbal gymnastics group product above with its *symbolic* equivalent (Figure 6.9):

	e	l	r	b
e	e	l	r	b
l	l	e	b	r
r	r	b	e	l
b	b	r	l	e

Repeat the same exercise for *the group of tire rotations* (Figure 6.10):

	e	f	s	c
e	e	f	s	c
f	f	e	c	s
s	s	c	e	f
c	c	s	f	e

and repeat the same exercise *for the group of symmetries of a rectangle* (Figure 6.11):

	e	r	v	h
e	e	r	v	h
r	r	e	h	v
v	v	h	e	r
h	h	v	r	e

What abstract conclusions can you already draw from the above experiment on your own?

At this point we urge our readers to practice mathematics correctly, as we discussed this in the **What Makes It Perfect** chapter, and resist the temptation of looking out our answer right away – the search for a fruitful mathematical idea that triumphs, after several failures, with a corresponding find is more than rewarding and is one of the joys of mathematics.

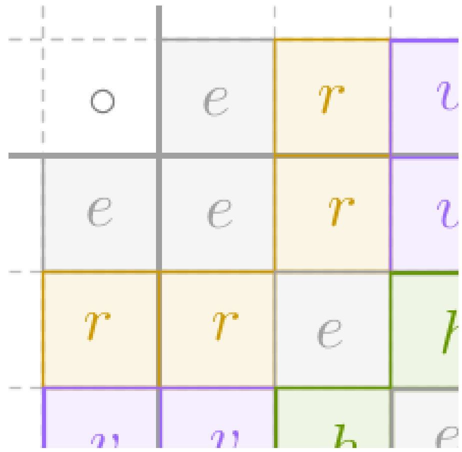
Do not rob yourself of the opportunity to experience that joy.

I. Since the verbal gymnastics is now the thing of the past, the symbolic representation of the group products is much more compact or efficient than its descriptive cousin. The *at-a-glance* view of the group products is nice to have.

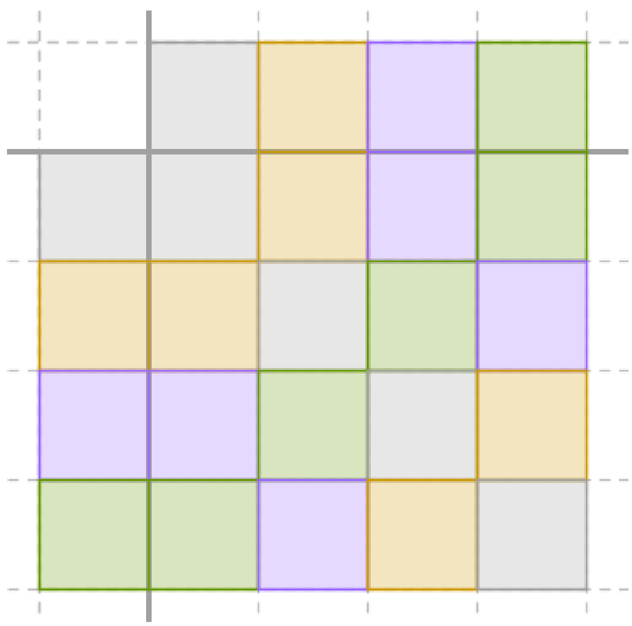
II. The specific underlying nature of the group *elements* in each square table of group products is absent.

III. The specific underlying nature of the group *operation* in each square table of group products is also absent.

However, we could, if we wanted to, record that operation in the upper left cell of the table that has been sitting empty and idle until now (Figure 6.12):



IV. If we stumble upon a bright idea of *removing* or *lifting* all the specific, tortured but suggestive, names of the groups' elements from all these tables then actually and amazingly *one and only one* table will emerge (Figure 6.13):



In other words, when the specific and different across the above groups names of the elements of these groups are removed from the respective tables, *the pattern* of the colored cells of these tables that are left behind is revealed to be one and the same.

That colorful pattern encodes the way in which the elements of the respective groups interact with each other and, thus, that pattern encodes *the structure* of the respective groups.

Consequently, the three seemingly different multiplication tables of the three seemingly different groups, after the removal of the material that is irrelevant in this particular context, turn out to be not that different after all and that is the meaning of our claim that these three groups are *essentially the same*, as all these descriptively different groups are the instances of the Klein four-group.

We can now congratulate ourselves with a victory in the prolonged descriptive-to-symbolic battle that we endured.

Hallelujah. And amen.

V. Thus, it becometh possible to fill in the individual cells of a multiplication table of a group with the absolutely abstract symbols, whose specific underlying nature and the specific underlying nature of the operation that glues these symbols together are completely (!) irrelevant (Figure 6.14):

	a_0	a_1	a_2	a_3
a_0	a_0	a_1	a_2	a_3
a_1	a_1	a_0	a_3	a_2
a_2	a_2	a_3	a_0	a_1
a_3	a_3	a_2	a_1	a_0

VI. If in our travels in:

- the theoretical physics
- the crystallography
- the modern chemistry
- the visual arts
- the performing arts
- the car servicing business and so on

we come across these very specific 16 bits of information or the four symbols $a_{0,1,2,3}$ that interact with each in that specific way then we instantly know that what we are dealing with is an instance of the Klein four-group.

VII. It is possible to study the properties of the Klein four-group, or *the algebra on four symbols*, in a complete isolation from the underlying nature of its elements and of its operation.

Such a study reveals a rich set of interior objects related to a group, such as *a subgroup* of a group, *a factor group of a group*, *the center of a group*, *the centralizer of a group*, *the commutator* of group elements and so on, and a rich set of operations that can be performed with a group and its interior objects, such as *a group homomorphism* or the construction of *the automorphism group* of a group and so on.

Specifically, we will soon learn that the Klein four-group is *commutative* but is not *cyclic*, for example. Thus, the elements of that group that we might come across in theoretical physics or chemistry or the theory of music or anywhere else, will possess these very properties also and when such properties are translated into the local language then they may shine a new and interesting light on the phenomenon of study that would not be apparent otherwise.

In other words, if in the middle school symbolic algebra we could manipulate the symbols mechanically, without paying any attention to the underlying nature of *the numbers* that these symbols represent, in the theory of groups we can manipulate both the symbols as elements *and* the symbols as *operations* mechanically, without paying any attention to the underlying nature of exactly *what* these symbols represent.

Moreover, it is possible to put together a theorem that spells out the exact conditions of when arbitrary n^2 bits of information constitute a group. Such a theorem will give the basis for yet another way to define a (finite) group of n elements and so on.

That wonderful idea of a square table of group products, also, as we mentioned it earlier, known as *a multiplication table of a group*, was put forward in 1854 by the British mathematician Arthur Cayley (1821-1895).

Recall that in our group of rigid symmetries of a non-square rectangle discussion we observed that it is very much possible to study not all the possible congruence motions of a given body but their s*maller part* or *their subset*.

As such, as a fun and informal exercise, we encourage our readers to pull up their records that they generated while working on the **Exercise 5.4.3** of the **I Dance, You Dance, We Dance** discussion and to construct the multiplication table of the group of only the *rotational* congruence motions of a square themselves.

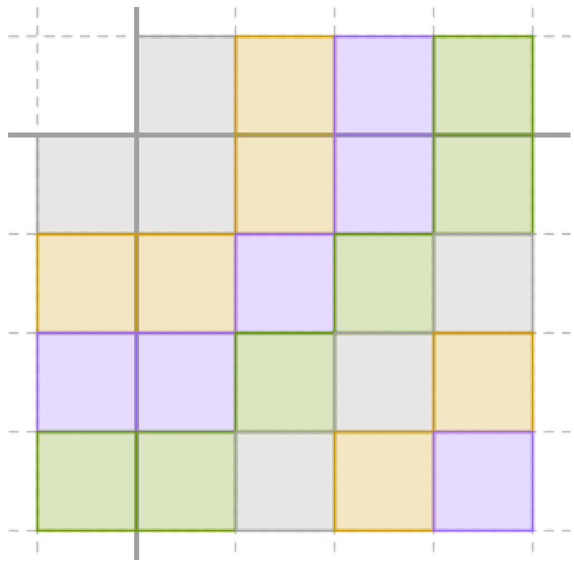
First, do show that the rigid rotations of a square that bring it back into coincidence with itself with the composition of these rotations as the operation form a group.

Second, symbolize that group on your own.

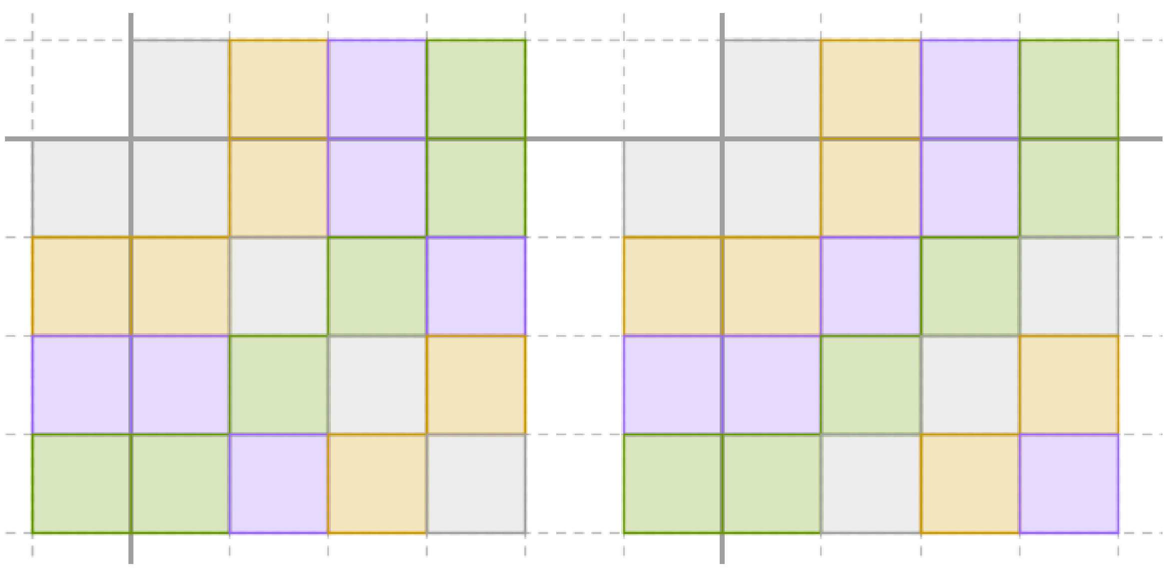
Third, construct the multiplication table of that group as explained above on your own.

Fourth, and it is mandatory, have fun.

As a gentle hint, we will leave our readers with the colorful trace of that group (Figure 6.15):



We, thus, discover that there exists an algebra on also four symbols but this time it is an algebra that is different from the algebra on four symbols that represents the Klein four-group (Figure 6.16):



The colorful pattern of the above leftmost table, associated with the Klein four-group, is very much different from the colorful pattern of the above rightmost, same-size, table, associated with a finite cyclic group C_4.

We already know that such a colorful pattern represents the structure of a group. As such, since the given colorful patterns are different, it follows that the two groups at hand have the structures that are different as well and mathematicians would say that in this particular case the two said groups are not isomorphic with respect to each other or that the two groups at hand are essentially different.

We hope that at this point our readers are already getting this pleasant *we-can-work-with-this* feeling in their tummies and we also hope that the somewhat torturous verbal gymnastics journey that we undertook was worth it.

It is one thing to be told that the symbolization of operations is a powerful concept and it is quite another to experience that process oneself.

And how long did it take us?

Just one week.

But now we consider the door to the theory of groups unlocked and opened just a bit.