# Chapter 6

# Composition, Follow What follows

Just like the concept of *a mapping* in mathematics does not betray our everyday intuition too much or fundamentally, the concept of *a composition of two operations* in mathematics also does not stray too far from the phenomenon of composition of actions or operations in our daily lives when we follow one such action or operation by another.

For example.

Consider a typical scenario when we want to prepare a dish called *Chicken Parmesan* for dinner.

To that end we:

- grab our car keys
- jump into the car
- turn the engine on

- shift the gear into the *drive* position
- navigate our way to the groceries store
- collect the needed ingredients
- get into the checkout line
- pay for the groceries
- get back into the car
- drive home
- prepare the dish of choice and
- consume the dish of choice

In the above list we can distinguish the following compositions of two operations:

- the first composition in this list is the act of following the *grab our car keys* operation by the *jump into the car* operation
- the second composition in the list is the act of following the *jump into the car* operation by the *turn the engine on* operation
- the third composition in the list is the act of following the *turn the engine on* operation by the *shift the gear into the drive position* operation

and so on.

Now let us think together - when we read the above sequence of steps then what characteristic property of their compositions can we discern?

In other words, what abstract observation can we put together here in order to cover this particular everyday sequence of steps and many more other like it?

When we read the above list of operations then we can discern the following characteristic property of their compositions:

*the next step in any one individual composition of the sequence is carried out from the state that was produced by the previous step of that composition*

That is, it becomes possible to *jump into the car* only after we unlock that car with its keys, which we *grabbed* previously.

It becomes possible to *turn the car's engine on* only after we *jumped into the car*.

It makes sense and is possible to *shift the car's gear into the drive position* only after we *turn the car's engine on*.

We can *collect the needed ingredients* at the grocery store only after we *drove there*.

We *get into the checkout line* in the grocery store after we *collect all the ingredients* for our dish and so on.

In contrast, observe that it would be extremely strange to attempt to *collect all the ingredients* without *jumping into the car* prior, wouldn't it.

Likewise, it would be a rather adventurous exercise to attempt to *turn the car's engine on* without having *grabbed the car keys* prior and so on.

Note how we inductively came up with an abstraction that can now be completely unbuckled from the trip to a groceries store and applied in all sorts of different and unrelated contexts.

### *Definition Of*

Technically, then, a composition, firstly and in general, is an operation itself and it is an operation that acts on necessarily *two* entities that are sometimes called *arguments*.

Secondly and tactically, a composition of two operations is the order-sensitive process of *following* one operation by another or the process of *combining* the result of one operation with the result of another operation.

More precisely,

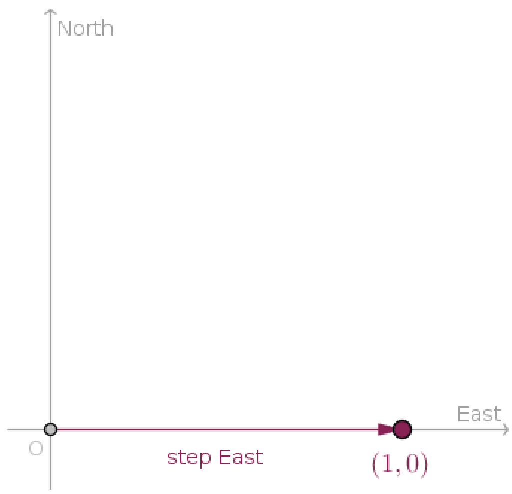
**Definition 9:** a composition "operation A followed by operation B", *in that order*, is an operation that acts on two entities, operation A and operation B, and whose result is obtained according to the following rule:

- the operation A is carried out first
- the result or the output of the operation A becomes the input or the initial state for the operation B
- the operation B is carried out next
- the result or the output of the operation B becomes the result or the output of the composition "operation A followed by operation B"

### *Examples*

**Example 1:** for an opening concrete example, consider the two operations of *make one step East* and *make one step North*.

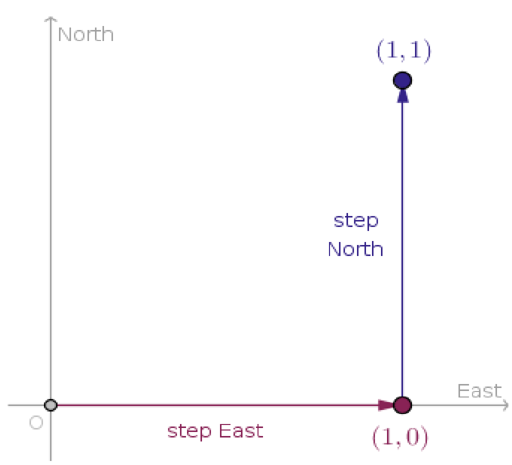
Assuming that we begin carrying out the composition *make one step East* followed by *make one step North* from the origin of an orthogonal coordinate system, then, as per **Definition 9**, we *first* step East once (Figure 6.1):




As a result of carrying out the operation *make one step East* or just *step East*, we find ourselves at the location $(1, 0)$ in the parent plane and, in our lingo that location $(1, 0)$ is *the result* or *the output* or *the final state* of the operation *step East*.

In order to complete this composition, we, as per **Definition 9**, begin carrying out the next operation of *make one step North* not from the origin of the chosen coordinate system but from the location $(1, 0)$ in which we find ourselves after the completion of the operation *step East*.

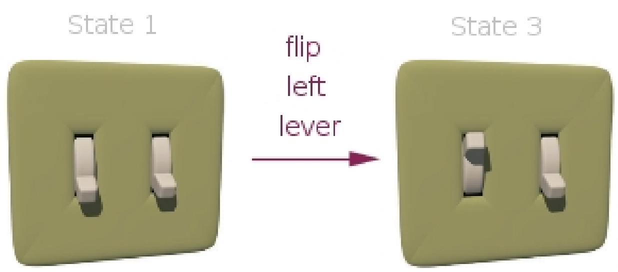
In our lingo, the output or the result or the final state of the operation *step East* becomes *the input* or *the initial state* for the operation *make one step North* or just *step North* (Figure 6.2):



The above point $(1,1)$ in which we, thus, find ourselves is the result or the output or the final state of the *step North* operation and is the result or the output or the final state of the composition *step East followed by step North*.

**Example 2:** consider the composition *flip the left lever followed by flip the right lever* from the earlier duplex light switch group discussion.

Assuming that we begin carrying out that composition when a light switch is in State 1, then, as per **Definition 9**, we first flip the *left* lever of that switch (Figure 6.3):




taking the switch into State 3, which is the result or the output or the final state of the operation *flip the left lever*.

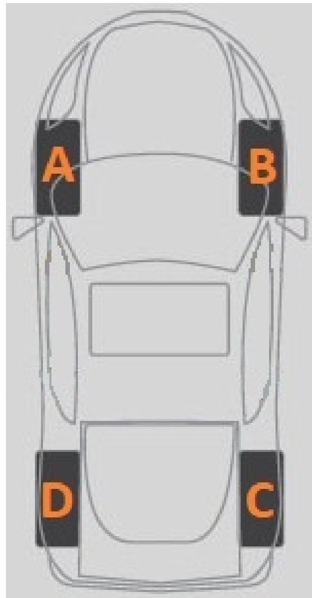
In order to complete this particular composition, we begin carrying out the operation *flip the right lever* not from State 1 but from State 3, which became the input or the initial state for that *flip the right lever* operation (Figure 6.4):


The above State 2 of the light switch thus obtained is the result of the *flip the right lever* operation and the result of the *flip the left lever followed by flip the right lever* composition.

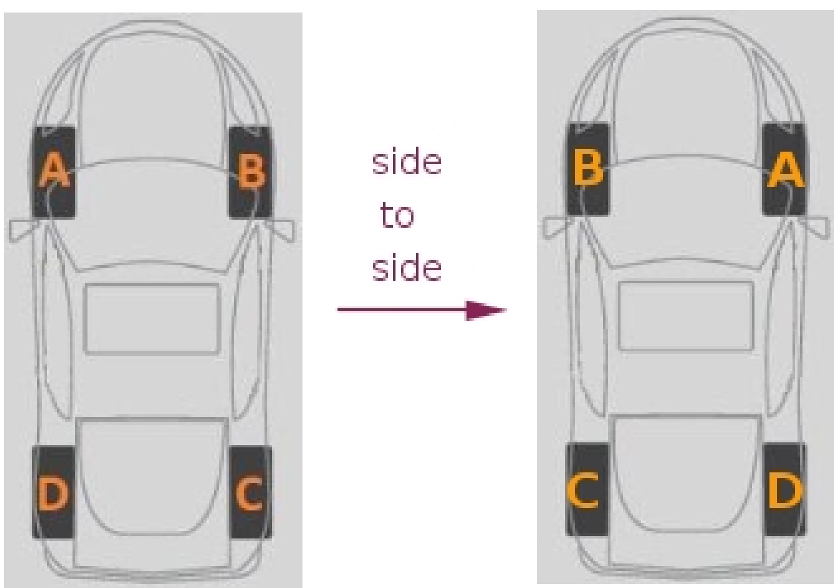
We can now state in a slightly more official way that our duplex light switch group is formed by the said flipping actions and the operation of *composition* of these actions.

**Example 3:** consider the composition *side-to-side tire rotation followed by the cross-over tire rotation* from the earlier tire rotations group discussion.

Assuming that we begin carrying out that composition from the initial state shown in Figure 5.2.8:

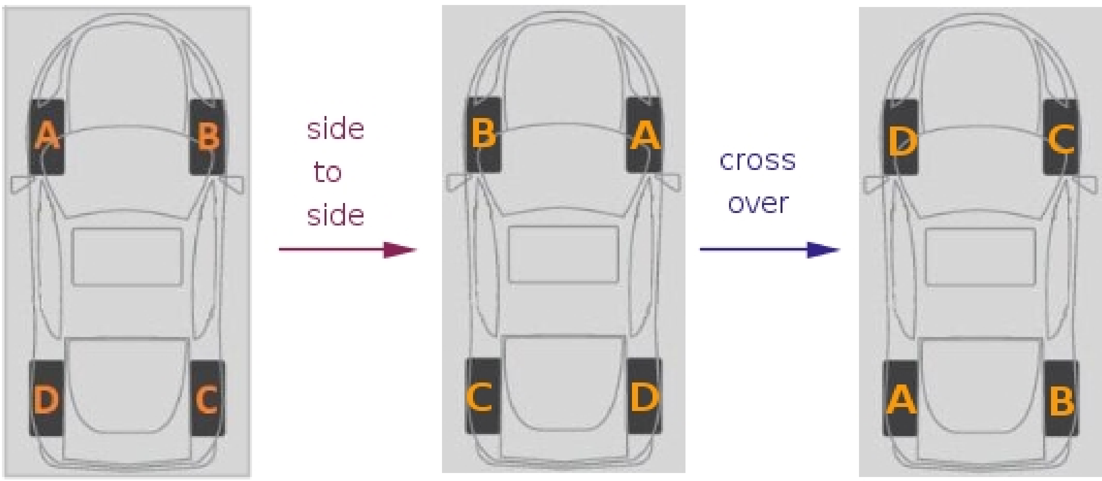



then we *first* rotate the above tires *side-to-side* (Figure 6.5):




taking the said tires into the state shown in the rightmost diagram above - that state is the result or the output or the final state of the *side-to-side tire rotation* operation and it becomes the input or the initial state for the *cross-over tire rotation* operation.

In other words, we apply the *cross-over tire rotation* operation not to the tires in the state depicted by the leftmost diagram above but to the tires in the state depicted in rightmost diagram above (Figure 6.6):




The state of the tires depicted by the rightmost diagram above is the result or the output or the final state of the operation *cross-over tire rotation* and is the result or the output or the final state of the composition *side-to-side tire rotation followed by the cross-over tire rotation*.

We can now state in a slightly more official way that our tire rotations group is formed by the said tire rotations and the operation of *composition* of these rotations.

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**Exercise 6.1:** consider a composition of any two congruence motions of a rectangle of your choice.

Explain in detail how the result of such a composition is obtained.

Restate in a slightly more formal way how the group of symmetries of a rectangle is formed.

**Solution:** is left to the reader. □

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**Exercise 6.2:** consider a composition of any two contra dance figures of your choice.

Explain in detail how the result of such a composition is obtained.

Restate in a slightly more formal way how the group of contra dance figures is formed.

**Solution:** is left to the reader. □

**Exercise 6.3:** consider a composition of any two congruence motions of a square of your choice.

Explain in detail how the result of such a composition is obtained.

Restate in a slightly more formal way how the group of symmetries of a square is formed.

**Solution:** is left to the reader. □

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*Properties Of*

The *properties* of a certain entity should not be confused with what that entity *is*.

Sometimes, but not always, it is possible to define what the entity under consideration is.

For example, while in physics there is no definition of *time*, we know certain properties of *time* - time *flows*, *runs*, etc.

While in mathematics there is no definition of *a set*, of *a geometric point*, of *a straight line* and of *a plane*, we know the certain properties of these entities that cannot be defined (in a finitely recursive way).

For example, if two straight lines *intersect* then a point results and if two planes *intersect* then a straight line results.

From a slightly more philosophical standpoint, the fact that certain, fundamental, entities cannot be defined is never a source of trouble because essentially entities *are* what entities *do*.

In other words, in the big scope of things it is less interesting exactly what such entities are and it is much more interesting what happens when these entities begin *interacting* with each other in certain, prescribed ahead of time, ways.

In the case of a composition of two operations we were lucky enough to be able to eek out its sensible definition. Fine.

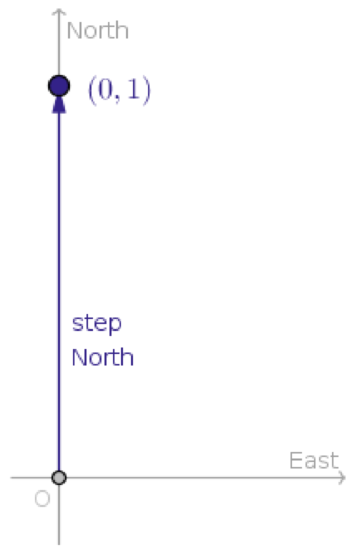
Even though the two upcoming properties of a composition are actually the properties of *an operation*, we will discuss them briefly so that the readers who are new to the idea of a composition understand that these two properties apply to the compositions of operations also.

***Commutativity***

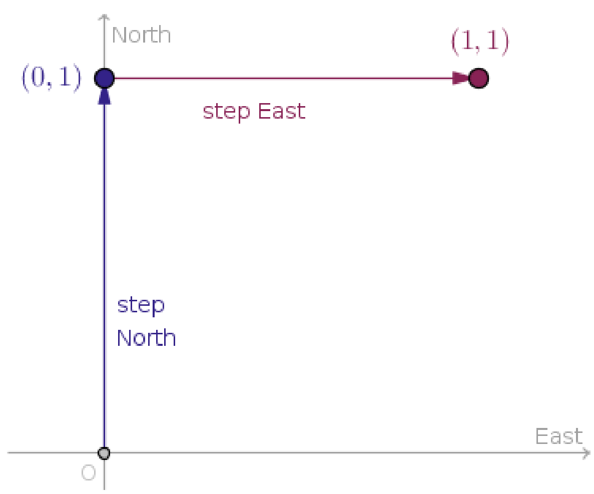
From **Definition 9** it follows right away that, due to its nature, a composition of two operations is highly sensitive to *the order* in which it acts on its two constituents.

What will happen if in our first concrete example of a composition we will reverse the order of the operations that go into that composition?

Well, starting from the origin, we now *step North* first (Figure 6.7):




into the location $(0, 1)$ that now plays the role of the result or the output or the final state of the operation *step North* and, at the same time, becomes the input or the initial state of the operation *step East* (Figure 6.8):




which puts us into the location $(1, 1)$ that is the output or the result of the operation *step East* and, at the same time, is the output or the result of the composition *step North followed by step East.*

We, thus, see that the order of the individual operations that comprise that particular composition does not affect its result.

Namely, the result of the composition *step East followed by step North* is the location $(1, 1)$ and the result of the composition *step North followed by step East* is the same location $(1, 1)$.

Officially then, a composition of two operations is *commutative* if the order of these two operations in that composition does not affect its result.

Many compositions are actually not commutative.

For example, all our *Chicken Parmesan* compositions are not commutative.

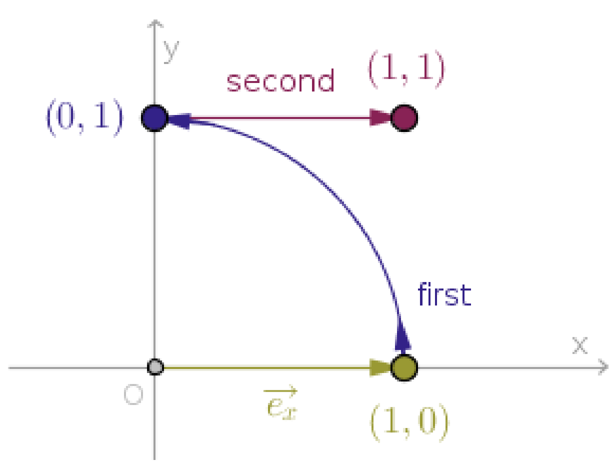
The result of the composition *focus the photo lens* followed by *press the shutter button* is drastically different from the result of the composition *press the shutter button* followed by *focus the photo lens*.

The result of the composition *tune the guitar* followed by *whip out a Paganini caprice on the guitar* is drastically different from the result of the composition *whip out a Paganini caprice on the guitar* followed by *tune the guitar*.

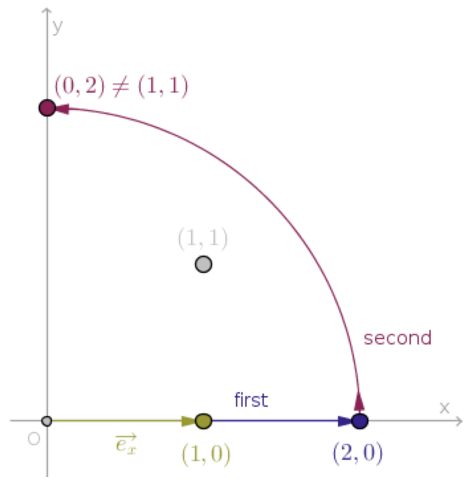
The result of the composition *take your clothes off* followed by *step into the shower* is drastically different from the result of the composition *step into the shower* followed by *take your clothes off*.

For a more mathematical example of a noncommutative composition consider the two fundamental operations of rotation (of the plane) and translation (of the plane).

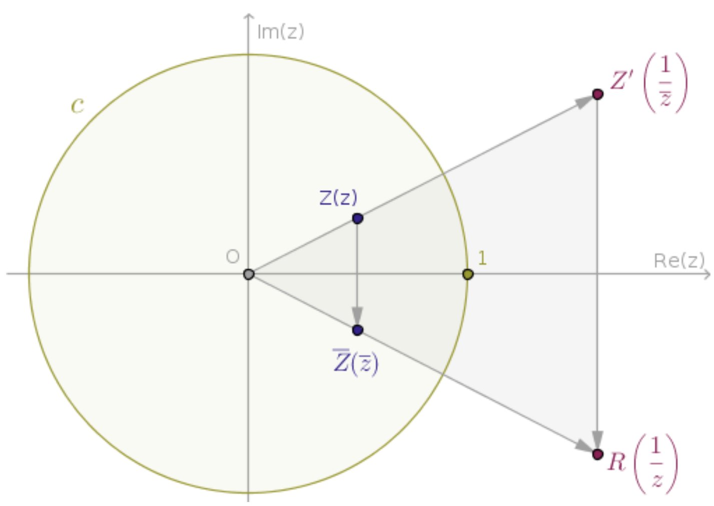
Rotating the point $(1, 0)$ about the origin by quarter-of-a-turn counterclockwise first, will take that point into its image $(0, 1)$ and translating that image point (0,1) along the unit vector $e_x$ second, will take that point into its destination $(1, 1)$ (Figure 6.9):




However, translating the point $(1, 0)$ along the unit vector $e_x$ first, will take that point into its image $(2, 0)$ and rotating that image point $(2, 0)$ about the origin by quarter-of-a-turn counterclockwise second, will take that point into its destination $(0, 2)$ that is totally different from the destination $(1, 1)$ obtained in the previous experiment (Figure 6.10):




For a slightly more sophisticated example of a commutative composition of two operations, targeted at a more mathematically savvy readers, recall that a geometric image of a reciprocal number $1/z$ of an arbitrary nonzero complex number $z$ can be obtained via *a commutative composition* of the operations of *reflection in the real axis* and *the inversion in the origin-centered unit circle with positive power* (Figure 6.11):




In this particular case the order in which the individual reflections are carried out does not affect the result.

Namely, one and the same point in the complex plane $R(1/z)$ will be obtained if the original point $Z(z)$ is reflected in the real axis first and in the origin-centered unit circle (with positive power) - next or, visa versa, if the original point $Z(z)$ is reflected in the origin-centered unit circle (with positive power) first and in the real axis – next.

***Associativity***

When two compositions are glued together back-to-back into a string of consecutive steps and the order of the three entities that these two compositions act on is fixed then another interesting property of a composition reveals itself when we pose the following question:

*under the constraints specified, will the order in which the individual compositions are carried out affect the overall result?*

In our *Chicken Parmesan* experiment the result of the following two consecutive compositions:

- navigate our way to the groceries store
  - followed by
- collect the needed ingredients
  - followed by
- get into the checkout line

depends a lot on the order in which each individual composition is carried out.

The result of the first composition *navigate our way to the food store followed by collect the needed ingredients* is quite predictable - our shopping cart is full of the said ingredients.

The result of the second composition of *get into the checkout line* with the shopping cart full of ingredients is also predictable - we are ready to drive home with all the goods acquired.

However, if we try to carry out the above *second* composition first then we will clearly fail - we cannot even begin collecting the needed ingredients unless we are at the grocery store and we are not at the grocery store because we never drove there in the first place.

Thus, in this particular case the order in which the above two compositions are carried out *does* affect the overall result.

In contrast, consider the following composition of operations:

- chop the cucumbers
  - followed by
- chop the tomatoes
  - followed by
- chop the carrots

If we first chop the cucumbers and then chop the tomatoes then, dumping both chopped vegetables into a salad bowl, nothing can stop us from chopping the carrots next and making a salad.

But we can also first chop the tomatoes and chop the carrots and, dumping them into a salad bowl, we can next chop the cucumbers, thus, making exactly the same salad.

We see that in this particular case the order in which the above two compositions are carried out does *not* affect the overall result.

Thus, officially, if two consecutive compositions that act on any three entities of interest produce one and the same result regardless of the order in which these two compositions are carried out then such a composition is *associative*.

In all the early examples of groups that we worked with we were dealing with the compositions that *are* associative:

- the composition of the flipping actions in the duplex light switch group
- the composition of the tire rotations in the tire rotations group
- the composition of the congruence motions in the group of symmetries of a rectangle
- the composition of the contra dance figures in the contra dance group
- the composition of the congruence motions in the group of symmetries of a square
- the composition of the addition operations in the additive group of integers

*Important*

The above two properties of commutativity and associativity of compositions in particular and operations in general are *independent of each other*.

In the upcoming Associativity discussion we will show the examples of operations that are:

- associative and commutative
- neither associative nor commutative
- associative but not commutative
- not associative but commutative

We, thus, are inching closer to one official and formal definition of a group and we now know that a group operation must be what?

A group operation must be *associative*.