Chapter 2

All, Every, Each, Any, Exists, If P Then Q

At it its very non-romantic and very technical core, the vast majority of results in modern grown-up mathematics are based on the following two, neatly templatizable, constructs, with which we shall become very good friends over the course of our joint adventure:

1) For all entities *this and that*, the property *such and such* holds
2) There exists an entity *this and that* with the property *such and such*

Intuitively we can think about these constructs as being two types of smallest Lego blocks that may be, and often are, combined or snapped together into more elaborate mathematical sentences.

The *order* in which these constructs are combined is of gargantuan importance.

A typical toy example of the first construct from the above template is the following one:

for all real numbers x, the square of x is nonnegative: $x^2 \geqslant 0$

A typical toy example of the second construct from the above template is the following one:

there exists a real number x *that is a root of the polynomial* $x^2 - 1$

A more elaborate example of the first construct from the above template may be fished out of the discipline known as *Fourier Analysis*:

for all real numbers x *such that* $0 < x < 2\pi$*, the quadratic function* x^2 *can be expanded into Fourier series as follows*:

$$x^2 = \frac{4\pi^2}{3} + 4\sum_{n=1}^{+\infty} \frac{\cos(nx)}{n^2} - 4\pi \sum_{n=1}^{+\infty} \frac{\sin(nx)}{n}$$

A more elaborate example of the second construct from the above template may be fished out of the discipline known as *Real Analysis*:

if a single-variable real-to-real function $f(x)$ *1) is defined and continuous on a closed interval* $[a, b]$*; 2) possesses a finite first derivative* $f'(x)$ *everywhere in, at least, the open interval* (a, b)*; 3) takes on the same values at the interval's boundaries:* $f(a) = f(b)$ *then there exits a point* c *interior to the given interval,* $a < c < b$*, such that the said first derivative of the function at that point vanishes:* $f'(c) = 0$

The above result is known as the *Rolle's Theorem*.

All, Every, Each, Any

Both constructs in the above two templates use the so-called *quantifiers* or words that in general reveal the amount of something. In a non-mathematical setting the implied scope or scale of the meaning attached to the word *quantifier* is rather wide and loose, as it can indicate the amounts of something smudged across the entire spectrum:

- *just one and only one* or
- *a few*, here and there or
- *all*

In mathematics, however, the scope of the meaning attached to the word *quantifier* is much narrower as it is limited to just two extremes, see below.

The quantifier *"all"* or *"for all"* is called *universal*, since its purpose is to make a sweeping and a non-exclusive claim and it has the following synonyms (including itself for completeness):

- *for all*
- *for every*
- *for each* and
- *for any*

In order to make it perfectly and unambiguously clear, the four phrases listed above have the same meaning in mathematics and the nuanced differences between them will become clearer with time and experience.

Exists

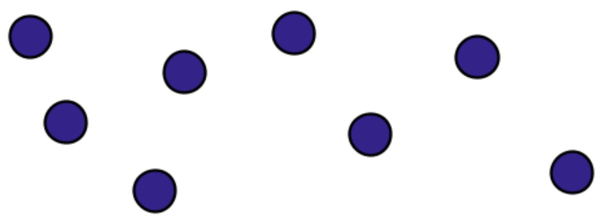
The quantifier *"exists"* is called *existential* and its purpose is to make a claim that something exists.

The following two constructs are the mathematical synonyms of *exists*:

- *there is at least one* and
- *some*

Examples

If we can somehow prove that *all* the circles under consideration are blue (Figure 2.1):



then the following statement:

$$\textit{for all the circles } c \textit{ in the drawing above, it is the case that } \mathrm{Color}\,(c) = \mathrm{blue}$$

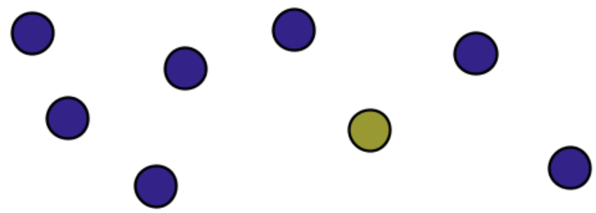
is true.

Intuitively speaking, the universal quantifier *"all"* or *"for all"* suggests *a complete and total absence of exceptions or restrictions.*

In other words, if the above statement about the color of all the circles is indeed true then no matter how hard and how long we look for a circle of a different color in the drawing presented, we will never find even a single one.

Conversely.

all it takes to invalidate or refute the above universal claim is the presence of just one and only one measly circle of a color that is not blue (Figure 2.2):



In this new scenario shown in Figure 2.2, it is no longer true or it is no longer the case that all the circles presented are blue. By visual inspection, we see that there is a non-blue impostor in the mix - it is, no doubt, still a circle, nice try, mister (say we), but the color of that circle is certainly not blue.

Game over - the only green circle shown in Figure 2.2 is known in general as *a counterexample*.

Our earlier universal claim is no longer valid and as mathematicians would put it succinctly:

one is enough

We, thus, see that the *"all"* or *"for all"* claims or requirements in mathematics are very strong on the one hand and are very easy to refute, on the other hand.

Such statements are strong because, once proved and true, they guarantee the absence of exceptions and counterexamples and such claims are easy to refute because one, and only one, counterexample is sufficient for that purpose.

In contrast, if our ambition amounts to only showing that in the last collection of circles presented there exists a green circle then the following existential claim:

there exists a circle in Figure 2.2 that is green

and its synonym:

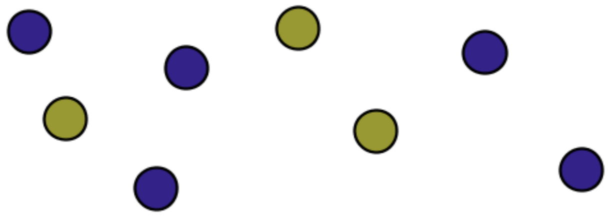
at least one circle in Figure 2.2 is green

or its yet another synonym:

some circles in Figure 2.2 are green

will certainly be true.

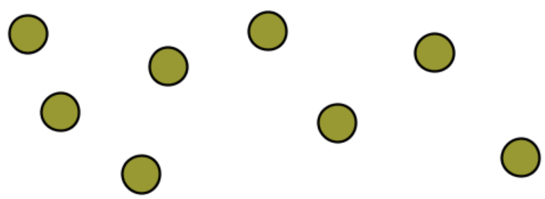
Moreover, if we have more than one green circle in such a collection, say, *three* green circles (Figure 2.3):



then our last existential claim is still true and when the existential quantifier looks at the drawing shown in Figure 2.3 and juxtaposes it against the said claim, it just yawns and, bored, walks away:

yeah, yeah, whatever … who cares … what else is new … the moment I see the first green circle, I go home happy

Even if *all* the circles in the new drawing happen to be green (Figure 2.4):



then our last existential claim, slightly adjusted:

- *there exists a circle in Figure 2.4 that is green*
- *at least one circle in Figure 2.4 is green*
- *some circles in Figure 2.4 are green*

is still true and when the existential quantifier looks at the drawing shown in Figure 2.4 and juxtaposes it against the said claim, it, again, just yawns and briskly walks away happy and not impressed. We, thus, see that the presence of just one specimen with the required properties is enough or, as the mathematicians would put it, is *sufficient* to keep the existential quantifier happy and pacified.

Negation

Purely for the sake of reference, without the relevant deductions and examples of any kind, we state the following rules for negating the above quantifiers.

In general, however, all these rules can be deduced:

- *"all are"* negates into *"at least one is not"*
- *"at least one is not"* negates into *"all are"*
- *"at least one is"* negates into *"none are"*
- *"none are"* negates into *"at least one is"*

The universal quantifier is symbolized as ∀, which in LaTeX is spelled as `\forall`.

The existential quantifier is symbolized as ∃, which in LaTeX is spelled as `\exists`.

However, as a promise to our readers, in this book we will not be relying heavily, if at all, on symbolic manipulations and will instead attempt to convey the fact that mathematics is not about formal symbolic shuffling but is about generation and re-generation and combination and re-combination of *ideas*.

If P Then Q

Together with the so-called *logical connectives* NOT, OR, AND, the IF P THEN Q construct known as *an implication* or, more generally, as *a conditional*, constitutes the fundamental tactical language of modern grown-up mathematics. Such language is studied in detail in the mathematical discipline known as *Formal Logic* or *Mathematical Logic*.

Even though this is not the text where we will study or even rely on and heavily use that language, our readers are nonetheless encouraged to gain some basic understanding of the logical connectives and the implications on their own and out of band.

As unlikely as it may sound, it is nonetheless the case that pretty much the entire body of modern grown-up mathematics is built with just these four logical building blocks of NOT, AND, OR, IF P THEN Q and the two popular quantifiers, the universal quantifier *all* and the existential quantifier *exists*.

Despite the fact that the size of the above tactical mathematical vocabulary is very small, three logical connectives, one implication and two quantifiers, that vocabulary is very powerful and any student who plans to undertake a more or less serious study of mathematics will benefit greatly from investing some amount of time into learning how to read the said language and how to use that language for that language travels far and wide throughout all of mathematics and is applicable outside of mathematics as well, in various computer programming languages, for example.

The book *"Introduction to Logic and to the Methodology of the Deductive Sciences"* by Alfred Tarski from Oxford University Press, fourth edition, 1994, is a good place to start.

Below, mostly as a reference, we show a minimal viable information about the above logical connectives and an implication that all our readers should be aware of before moving on.

Propositions

All logical connectives and an implication are applied to what mathematicians refer as *propositions*.

A proposition is a collection of words and/or mathematical symbols that always has a well-defined logical value of either True or False. Note that the logical values of True and False are left explicitly undefined.

Initially a proposition conveys one and only one coherent thought or fact and then multiple such propositions are spliced into more and more elaborate propositions by the means of logical connectives and implications.

Intuitively we may think about such initial propositions as *atomic* and *indivisible* and we can think of larger and more elaborate such propositions as *compound* propositions that are built up from certain atomic or basic propositions.

It is important to understand what a proposition in mathematics is and what a proposition is not.

A proposition in mathematics is such a collection of words to which a logical value of either True or False can be meaningfully assigned.

That collection of words may be either completely void of any meaningful mathematical content or it may be deep-fried, soaked in and dusted with mathematics all over.

The following sentences are all legal and valid propositions:

- *I live on planet Earth*
- *The sky is blue*
- *the number* 5 *is prime*
- *Elephants can fly*
- *Roses are read*

In negative terms, a proposition is definitely:

- not a question
- not a command
- not a wish
- not a promise
- not an opinion

As such, none of the following collections of words are a proposition, as far as mathematical logic is concerned:

- What time is it?
- Move that chair over there
- I wish I owned a helicopter
- I will most likely meet you tomorrow after lunch
- I do not think that I like purple tulips

In order to capture the notion of *any possible proposition*, mathematicians use *variables*, or single-letter symbols, in the same way as variables, such as x, y and z, are used in order to capture the notion of *any possible number*.

When a variable refers to a proposition then that variable is called *a propositional variable* or *a sentential variable*.

Some authors prefer to designate propositional variables with the upper-case Latin letters, such as P or Q, while other authors prefer to designate propositional variables with lower-case Latin letters, such as p or q.

However, in mathematics there does not exist a codified law that forces the usage of specific letters from specific alphabets for the designations of variables. There do exist long-standing discipline-specific traditions but some such traditions are nothing more than fossilized ignorance.

It is the job of the author of a unit of mathematical text to explain the meaning of symbols used in that text clearly and unambiguously beforehand.

As an example, P is a perfectly fine variable that can designate *a prime number*, $\rho(z)$ is a perfectly fine name of a single-variable function that takes a complex number, z, as its independent variable and x is a perfectly fine name of a variable to designate a propositional variable.

NOT

NOT is *a unary* connective or a connective that acts on exactly one *argument* or *operand* and the result of the application of that connective to its operand is *the negation*, of that operand. The original operand and its negation form *a contradiction* because they have diametrically opposite logical values. If the logical value of the original operand was True then the logical value of that negated operand will be False and vice versa.

If the logical value of the original operand was False then the logical value of that negated operand will be True.

Because the logical value of no statement can be both True and False at the same time, it follows that if the original operand of NOT and its negation are adjoined with the AND logical connective, see below, then a contradiction will result.

The behavior of the application of multiple consecutive operators NOT to one and the same operand mimics that of the $(-1)^n$ arithmetic ($n \in \mathbb{N}$):

- any even number of applications of the logical connective NOT to one and the same operand does not change the logical value of that operand, while
- any odd number of applications of the logical connective NOT to one and the same operand is equivalent to applying the logical connective NOT to that operand exactly once

For example, when we tell our children "*Do not not do it*" then what we really mean is "*Make sure that you do it*".

In the spoken English the operator NOT is applied selectively. When negating the sentence:

Elephants can fly

we do not say:

Not elephants Not can Not fly

We, rather, say:

Elephants cannot fly

In formal structures, however, the operator NOT plows right through the entire sentence to which it is applied and consumes everything in its path voraciously.

For example, if P and Q are any two propositional variables then the sentence P OR Q will be negated into (NOT P) AND (NOT Q), where P became (NOT P), the logical connective OR became AND and Q became (NOT Q).

However, in certain sentences the operator NOT is still applied selectively.

For example:

- the function $f(x) = |x|$ is NOT differentiable at $x = 0$, or
- the number 64 is NOT prime, or
- a distribution is NOT normal

and so on.

AND

AND, also known as *a conjunction*, is a *binary* connective or a connective that acts on exactly two operands which are called *conjuncts*.

The logical value of AND evaluates to True in one case only - when the logical values of both of the arguments of a conjuction are True at the same time.

In all the other cases the logical connective AND evaluates to False (Figure 2.5):

P	Q	P AND Q
T	T	T
T	F	F
F	T	F
F	F	F

For example, the following construct:

Blue turtles solve differential equations AND the number 17 *is prime*

evaluates to False because the logical value of the left operand of the logical connective AND, which is *Blue turtles solve differential equations*, is, clearly, False.

However, this construct:

$$(\pi > 0) \text{ AND } (\pi < 4)$$

evaluates to True, since it is True that $\pi > 0$ and it is also True that $\pi < 4$ and that lengthy sentence can be rendered more compactly as follows:

$$0 < \pi < 4$$

The logical connective AND is negated as follows.

If P and Q are any two propositional variables then the sentence P AND Q is negated into (NOT P) OR (NOT Q), where P became (NOT P), the logical connective AND became OR and Q became (NOT Q).

When translated into a formal notation then all of the following English words are reincarnated as AND:

- but
- however
- while
- though
- even though
- although
- neither … nor …

For example, the following sentence:

It was raining but the sun was out

becomes (*It was raining*) AND (*the sun was out*).

OR

OR, also known as *a disjunction*, is a binary connective as well because it acts on exactly two operands called *disjuncts*.

The logical value of OR evaluates to False in one case only (Figure 2.6):

P	Q	P OR Q
T	T	T
T	F	T
F	T	T
F	F	F

when the logical values of both of the arguments of a disjunction are False at the same time.

In all the other cases OR evaluates to True.

For example, the following construct:

Roses are red OR Alpha Centauri is 4.37 *light years away from Earth* evaluates to True because it so happens that in this case both operands of the logical connective OR, the left operand *Roses are red* and the right operand *Alpha Centauri is* 4.37 *light years away from Earth*, evaluate to True.

While in the spoken English the word OR *sometimes* carries the aura of exclusivity, in a formal language of mathematics the logical connective OR is *always* inclusive.

For example, giving directions to a lost tourist in Manhattan, we can say " … on 42-nd and 8-th turn right or go straight" in order to get to the Central Park Zoo.

In that case the word OR implies exclusivity, since a tourist, being not a good old quantum object, cannot be in two distinct places at once: if this tourist decides to go in one direction then (s)he cannot be traveling in a different direction at the same time.

In a different scenario, the phrase "the knowledge of geometry or algebra is sufficient for taking this course" does not imply exclusivity because a particular student can very well be proficient in both geometry and algebra at the same time.

The logical connective OR is negated as follows.

If P and Q are any two propositional variables then the sentence P OR Q is negated into (NOT P) AND (NOT Q), where P became (NOT P), the logical connective OR became AND and Q became (NOT Q).

IF P THEN Q

IF P THEN Q is a construct that is generally called *a conditional*, which is a logical formation from which causality can be totally absent – such was a decision made and handed down to us by the founders of mathematical logic.

Put slightly differently, literally any clauses can be packed into a conditional.

As an example, the following sentence:

IF (π is irrational) THEN (binary zebras invented the Bessel functions)

is a perfectly legal sample conditional even though there does not exist any meaningful flow of causality from the true fact that the number π is irrational to a nonsensical suggestion that zebras invented the Bessel functions.

That peculiar state of affairs is explained by the fact that the founding fathers of formal logic decided to allow the maximal degree of freedom when it came to the act of connecting two smaller sentences into a single larger sentence in this particular way.

In mathematics, however, the causality or a meaningful flow of causality better always be there.

As such, we have the following formula that binds a conditional construct to a flow of causality:

a conditional + causality = *an implication* or *an inference*

In other words, in mathematics *an implication* or *an inference* is a combination of a, perhaps, loose conditional and a mathematically meaningful flow of causality.

For example, the following construct is an example of *an implication* or *an inference*:

IF ($a \equiv b \pmod{n}$) THEN ($ac \equiv bc \pmod{n}$)

which, in words, says that if two integers are congruent modulo n then the products of these integers with one and the same integer are congruent modulo n as well.

In this particular context, in an IF P THEN Q implication P is known as *a premise*, *a hypothesis* or *an antecedent*, while Q is known as *a consequent* or *a conclusion*.

We also suggest that the newcomers to the world of implications get used to the following fun fact right away:

the logical value of an inference is False in one case only - when Q, the conclusion, is False and P, the antecedent, is True

In all the other cases an inference evaluates to True, as shown in the Figure 2.8 below, even when the antecedent of an inference is False!

Intuitively speaking, that means that *a false premise implies anything*, which is another decision handed down to us by the founding fathers of formal logic.

For example, the following conditional statement:

IF (violets are mammals) THEN ($\pi > 2$)

evaluates to True, despite the fact that overall there is no meaningful flow of causality in that statement and despite the fact that the consequent of that statement, in its decimal expansion the value of the real number π is strictly larger than 2, is True.

The premise of that construct, however, violets are mammals, is False. Therefore, that particular statement evaluates to True no matter how not intuitive it may seem (Figure 2.7):

P	Q	IF P THEN Q
T	T	T
T	F	F
F	T	T
F	F	T

When a conditional is negated then, fundamentally, it becomes the logical connective AND as follows. If P and Q are any two propositional variables then the negation of IF P THEN Q becomes an P AND (NOT Q).

For example, the following sentence:

IF (I have $3 in my pocket) THEN (I will buy a loaf of bread)

is negated into:

(even though or despite the fact that) I have $3 in my pocket AND I will not buy a loaf of bread (nonetheless)

When translated into a formal notation, all of the following English words and phrases are reincarnated as an IF P THEN Q implication:

- P implies Q
- if P then Q
- from P it follows that Q
- P is sufficient for Q
- for P it is necessary that Q
- P only if Q
- for Q it is sufficient that P
- Q is necessary for P
- Q if P
- Q whenever P

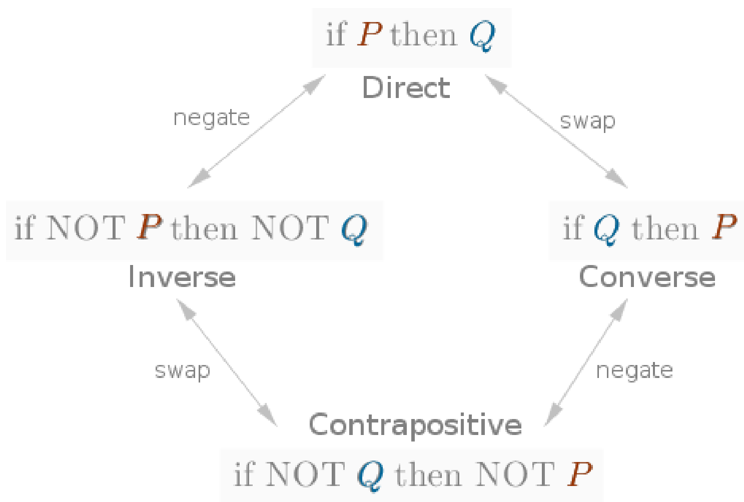
Lastly, if P and Q are any two propositional variables then with these variables fixed, an implication can be folded into the following four related shapes, each of which has its own name and meaning in mathematics.

For the sake of a demonstration, since we have to key off of something, let us refer to the conditional IF P THEN Q as *a direct* statement.

Then, the swap of the two predicates in a direct statement results in *a converse* statement IF Q THEN P.

The negation of the two predicates in a direct statement results in *an inverse* statement IF (NOT P) THEN (NOT Q).

The swap and the negation or the negation and the swap of the two predicates in a direct statement results in *a contrapositive* statement IF (NOT Q) THEN (NOT P) (Figure 2.8):



The conditional constructs that are located diametrically opposite of each in the Figure 2.9 above are logically equivalent:

- the direct statement IF P THEN Q is logically equivalent to its contrapositive IF NOT Q THEN NOT P
- the inverse IF NOT P THEN NOT Q is logically equivalent to the converse IF Q THEN P

As an example, the implication:

IF an integer is odd THEN the square of that integer is odd

is logically equivalent to its contrapositive:

IF a square of an integer is even THEN that integer is even

When would we use one such implication over the other?

It depends. Assume that we need to prove the second implication above but we do not know how can we prove it.

In that case we can construct the contrapositive of that implication and see if it will be easier to prove that contrapositive.

Well, the contrapositive of the second implication does seem easier to prove. Indeed, if we assume that an integer k is odd and is representable as $k = 2n + 1$ for a suitable integer n then the square of such an odd integer will be represented as:

$$k^2 = (2n+1)^2 = 4n^2 + 4n + 1 = 2(2n^2 + 2n) + 1$$

But if n is an integer then n^2 is an integer, $2n^2$ is an integer and $2n$ is an integer. Hence, all these integers can be absorbed into a single integer, m:

$$2n^2 + 2n = m$$

in which case the square of our odd integer k can be represented precisely as an odd integer:

$$k^2 = 2m + 1$$

Thus, we just proved that if an integer is odd then the square of that integer is odd and, since a direct statement and its contrapositive are logically equivalent, it follows that we also proved that if a square of an integer is even then that integer itself is even as well.

Again, we suggest that our readers who are not familiar with the basics of formal logic read and re-read this chapter until all the ideas and notions covered in this chapter become clear.