Chapter 5

Early Examples

Style

The theory of groups is usually taught to the fourth, give or take, year mathematics and theoretical physics majors who have already been exposed to a fair amount of different branches of grown-up mathematics and who already possess some amount of mathematical maturity that can be spoken of.

As such, in a traditional flow of events the theory of groups is presented and developed in a highly *de*ductive manner when abstract and formal definitions are followed by concrete examples and when specific observations are drawn from the generic ones.

One of the reasons why that approach is taken is because it is quietly assumed that all or most of the participating students are in a firm command of a large number of the supporting and foundational facts, concepts and paradigms on which a deductive exposition relies heavily.

Our space, however, is not suitable for such an approach and unleashing a classical tornado and a perfect storm of abstract symbolism onto our readers from the first second serves no purpose at best and will likely make many readers walk away at worst.

That is why we shall *at first* switch **the style** of our presentation around and proceed in an *in*ductive manner when concrete and not too formal examples are followed by the abstract and formal definitions. Rest assured, though, that in due time we will be as formal and rigorous as formal and rigorous gets - just not right away.

As a refresher, the flow of causality in *in*ductive reasoning is opposite with respect to the flow of causality in deductive reasoning. That is, *in*ductive reasoning inhales a particular amount of specific and concrete observations and makes an attempt to draw generic and abstract observations from them. In other words, with inductive reasoning specific and concrete observations play the role of the fodder for general and abstract observations and ideas.

It is interesting that many a time mathematics on a grand scale works precisely in an *in*ductive manner when various people work with various very specific and concrete mathematical objects but then wrap these objects into a larger general and abstract framework of which the good old concrete objects are a natural part.

Content

In the three out of five upcoming examples we will use the everyday objects and activities in order to dispel a popular but grossly inaccurate opinion that mathematics is only about numbers and number crunching.

In the first example we will look at the good old household light switches.

In the second example we will take a ride in a car to a friendly local mechanic in order to rotate the tires.

In the third example we will engage in a highly specialized tango with a rectangle.

In the fourth example we, again, will dance but this time in the *be there or be square* style.

In the fifth example we *will* look at some numbers: negative whole numbers, zero and positive whole numbers.

With our duplex light switch group discussion we will begin the take off from the group-theoretic runway by introducing and gradually developing the discipline-specific vocabulary.

Being the first in the series, the **We See The Light** discussion is rather voluminous and is a lot to take in.

Plan to spend a good couple of days on it and maybe more.

Purpose

In the early examples of groups we shall take a, hopefully, fun and reasonably informal look at the specific examples of groups whose purpose is twofold.

I. On the one hand, with these not very formal examples we want to cultivate the ground and motivate the upcoming formal definitions of what exactly a mathematical group *is*.

The examples that we have picked are fairly representative but they are not all-inclusive. However, even the handful of such sample groups already contains some very deep ideas and it will take us a while to decipher them and get to their bottom.

It should be understood that even though with the upcoming discussions we are making an attempt to showcase *an inductive* reasoning as it usually unfolds in real time, a reasoning that makes accurate generalized predictions based solely on a finite number of concrete and specific examples, the purpose of such examples is purely pedagogical and, as we already mentioned in our **Historical Note** discussion, in real time it took many years and a number of very smart people to see that generality and expose it for the rest of us.

In our toy demonstrations we will go in exactly that, historically accurate, direction from concrete and specific to generic and abstract but we will do so at an incredibly fast, and historically not accurate, pace compressing more than one hundred years of effort into just a small number of days.

II. On the other hand, we would like our readers to experience first-hand the idea that we mentioned in the **Prerequisites** discussion - the symbolization of *numbers*, which all of us went through in middle school in an algebra class, can be extended to the symbolization of mathematical *operations*.

In other words, in our early examples we would like our readers to become the actors on the stage and *to participate* in the very magic of the symbolization of mathematical operations and pretend to be the people who did that many years ago, albeit in a highly time-compressed and a highly orchestrated manner.

To that end, we purposely will be too chatty and too wordy so that our readers can step into the shoes of the ancient Babylonians or the ancient Egyptians who gave such wordy and awkward descriptions to the problems that were *numeric* and *computational* in nature.

Except that we will be giving the wordy and awkward descriptions of the problems that are *not* numerical and that are *operational* in nature.

We will also leave some of the useful ideas off the table by not spelling them out explicitly and by not telling our readers exactly what to do in the hope that our readers will come to these ideas *on their own*, which is always a source of fun and intellectual pleasure.

For example.

In our first exercise we will ask our readers to compute the number of all possible order-sensitive pairs of group elements in the light switch group but we will not tell our readers how to organize the enumeration of all such pairings - *we* will do that the ancient Egyptian, painful by the modern standards, style and our readers are more than welcome to exercise their own wits and see if they can find a better way.

The arrival at such ideas on your own and the making of such miniature discoveries by a student usually facilitates a much better understanding of the material.

Some readers will feel it sooner, some readers will feel it later but all readers will (or should) feel the pain of wordy descriptions of operational tasks. We urge our readers, however, to persevere through that pain - it will not last long and after the small number of the early examples, we will abandon that practice but we hope that our readers will do that on their own *before* we get to the *Difficult Leap Forward* discussion.

By doing so our readers will:

- witness
- experience
- enjoy
- appreciate and
- truly understand

the symbolization of mathematical *operations* that is to follow.

We hope that even this small investment into *slow and thorough* will pay off handsomely later on when the students of the theory of groups will progress *fast and deep* on their own with minimal guidance.

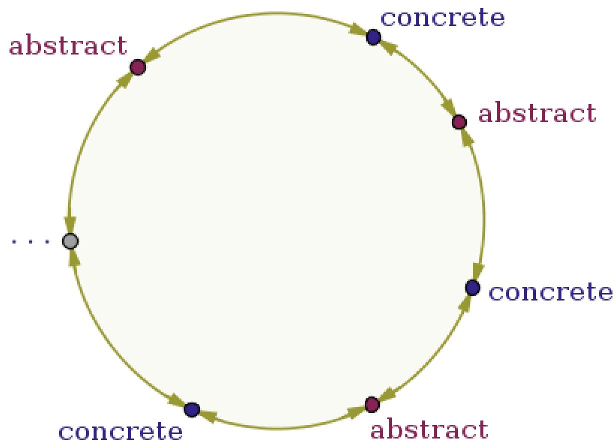
Texture

The exercises in our early examples will not be the typical typical group-theoretic exercises for the reason explained above.

Asking our readers to do the typical, highly abstract, group-theoretic exercises right away is like asking a toddler who just learned how to walk to run a 100-meter dash against an Olympic champion - pointless and meaningless.

During a successful solution of a typical group-theoretic exercise a person who generates such a solution, consciously or unconsciously, not only goes through this very peculiar *concrete-to-abstract-to-concrete-to-abstract* cycle that is always convoluted and always messy, as shown in the Figure 5.1

below, but such a person goes through this very peculiar concrete-to-abstract-to-concrete-to-abstract cycle *very rapidly* (Figure 5.1):



It goes without saying that in order to be able to engage in such a cycle, the ability to jump back and forth and switch between concrete and abstract must be in place.

However, because of the lack of proper training many students, early on, do not possess that skill *yet*.

Thus, the purpose of the specific texture of the exercises in our early examples is to put the minds of our readers through the calisthenics of exactly that back and forth concrete-to-abstract switching process. The importance of this concrete-to-abstract juggling act cannot be fully appreciated by many at this stage. That is perfectly fine.

But it should be ignored by none.

Some of our readers will get the hang of this skill sooner than others. That is fine too. We suggest, however, that even if some of our readers see what is required of them sooner than later, these readers should do all the exercises nonetheless because the said skill will be relied on more and more - not less and less.

The inability to ground oneself in something concrete and tangible while working with something abstract is the source of many failures to proceed further. To some extent, with our early examples we will be providing these very concrete and tangible settings that our readers can come back to and fall on as they move through the higher levels of, potentially confusing and troublesome, abstraction.

This specific skill that we will be developing with these exercises travels well not only throughout the theory of groups but across all of mathematics, especially at its higher end, where it becomes *absolutely crucial* - it is not unheard of for some students to drop out of a course on the theory of groups because the theory of groups seems too abstract for them and it is not unheard of for some students to drop the mathematics major altogether and switch to something else because the high brow mathematics seems too abstract for them.

We, thus, have a lot of work to do and a lot of ground to cover.

But once the key ideas of *a* way in which a group can be defined are understood, we will symbolize these ideas in the **Difficult, Giant, Leap Forward**, **Binary Operations** and **Associativity** discussions and we will formalize them in the **Group Axioms** discussions.

5.1 We See The Light

For the very first fun example or, more technically correct, *model* of a group we will pick a household item that is so popular and ubiquitous that many people, forgetting its importance and utility, simply take it for granted because it *just works* - the trusty old duplex toggle light switch (Figure 5.1.1):

The reason why in the photograph above we showed a duplex light switch with its numerous details is to emphasize the fact that *a light switch as a whole* is a rather complex and elaborate contraption. For the group-theoretic purposes, however, out of the maze of the parts that make up a light switch we will be interested only in its two *toggles* or *levers*:

- *the left* lever and
- *the right* lever

While the levers on light switches come in a variety of shapes and configurations, fundamentally, each light switch lever can be in only one of only two mutually exclusive or diametrically opposite positions or *states*:

- *on* or
- *off*

Thus, via the various states of the individual levers, any duplex light switch as a whole can be in four and only four different states that we list below.

State 1: both levers are *off* (Figure 5.1.2):

State 2: both levers are *on* (Figure 5.1.3):

State 3: the left lever is *on*, while the right lever is *off* (Figure 5.1.4):

State 4: the left lever is *off*, while the right lever is *on* (Figure 5.1.5):

Let us agree that *the act of changing* the current state of any single lever, *left* or *right*, to its opposite state, whatever it is, *on* or *off*, constitutes one *flip* of that lever.

That is, if a given lever is currently in the *on* state then changing the state of that lever to *off* constitutes *one flip* or just *a flip* of that lever.

Likewise, if a given lever is currently in the *off* state then changing the state of that lever to *on* also constitutes *one flip* or just *a flip* of that lever.

Put differently, the flips of the levers of a light switch do not care at all about the concrete states in which these levers happen to be at the moment when these flips stopped by to pay a visit: such flips are hell-bent on *changing* the states of the levers to their opposites and they do not care about anything else.

We now carefully conglomerate the flips of the individual levers into the *flipping actions* that can be carried out with any number of such levers - zero levers, one lever or two levers. The *flipping actions* can also be thought of as the *flipping schemes* that can be applied to a light switch as a whole in order to take it from any one state shown above to any other state shown above.

We, however, will use the word *actions* in *flipping actions* because it captures the group-theoretic essence of the phenomenon at hand more truthfully than the word *schemes*. As such, it turns out that the above four states of a duplex light switch can always be *achieved* or *described* via an interaction of the following four different flipping actions:

- no flips at all (1)
- flip the left lever only (2)
- flip the right lever only (3)
- flip both levers, at once (4)

and no others.

The flipping action number 1 means that we do not flip any levers or we simply do nothing.

The flipping action number 2 means that we flip the left lever only and we flip it exactly once.

The flipping action number 3 means that we flip the right lever only and we flip it exactly once.

The flipping action number 4 means that we flip both the left and the right lever at once and exactly once. Note carefully that even though in this flipping action we act on two levers, it still counts as just one flipping action.

Show Time

Since what follows may sound slippery on the first reading, we pay a special attention to the fine(er) details and we read and re-read these details until they sink in.

A mathematical group can be defined in a multitude of ways. Below, in an informal fashion, we give a blueprint of *an axiomatic* definition of a group.

Neither the duplex light switch itself *nor* its individual levers *are* a group or *form* a group.

Rather, it is *the (unordered) pair* of the following, two, entities that *is a* group or *forms* a group:

- *the collection* of the four flipping actions listed and numbered above and
- *the operation* of gluing these actions into a string of a finite number of necessarily *consecutive flipping actions*: a flipping action followed by another flipping action, followed by another flipping action, followed by another flipping action and so on
  - with the above *followed by* operation having the property that two consecutive such operations that act on any three flipping actions produce the same result regardless of the order in which these two operations are carried out

and that is all that there is to it.

Just kidding.

We shall now gradually unpack and spiral through the fundamental ideas that are baked into the above informal definition of a mathematical group.

Collection

The loose and intuitive notion of the *collection* of four flipping actions has its dedicated name in mathematics and is known as *a set*, a notion that we already met in the **Mappings** discussion.

Thus, from now on we will refer to the above collection of four flipping actions as *the set of flipping actions* named G and we will refer to the flipping actions themselves as *the group elements* or the constituents of a new algebraic structure that we are studying and to which we will be referring as *a (mathematical) group*.

Because we remember that any kind of things can be an element of a set, we should feel comfortable with the idea that not only the traditional, in some sense, things that are described with nouns, such as *a banana* or *a planet*, can be the elements of a set but the entities that are described with *verbs*, such as *flip*, *reflect* or *rotate*, can be the elements of a set just as well.

A sweeping mathematical abstraction for such verbs is *an operation* and it so happens that the group elements in this case are *operations*.

Namely, *an operation* here is *a flipping action* and the four possible flipping actions are listed and numbered above: no flips, flip the left lever only, flip the right lever only, flip both levers.

Put differently, in this particular case *a flipping action* is *a representative* of *an operation* and in the breathtaking idea that:

a flipping action is a representative of an operation

our readers should recognize its middle school algebra analog, according to which:

the symbol a *is a representative of a number*

as we discussed this symbolization concept in our **Prerequisites** section.

We, thus, are inching closer to grasping the far-reaching idea of *symbolization of mathematical operations*.

Consecutive Flipping Actions

After we buy and install a new light switch, we begin using it on the daily bases. If we compress that usage over time then it will look like a series of consecutive flipping actions pouring out of the horn of plenty:

flip the left lever, then flip the left lever, then flip both levers, then flip the right lever, then flip the right lever, then flip the left lever, then flip the left lever …

and so on.

Such a process of gluing the flipping actions into a string of *consecutive* actions, when one action is *followed by* another action, also has its dedicated and precise meaning in mathematics, and we will study that meaning in the upcoming **Composition** discussion, but here, again, that process:

- *plays the role of a concrete instance of* or
- is *a specific example of* or
- is *a representative* of

a generic mathematical operation that is carried out against the flipping actions, the elements of the above set G or the group elements.

What does it mean *an operation that is carried out against the elements of a set*?

An operation that is carried out against the elements of a set means that we, at our will and on our volition, have *defined* or invented such an operation.

In general, on any good day, we can just sit there and mint these *operations* for as long as we want. Intuitively speaking, such operations are cheap - a dime a dozen. Does that mean that *any* operation that we decide to define on a collection of elements can participate in the formation of a group?

No. Not at all.

The candidate operation that can participate in the formation of a mathematical group must possess the following three restrictive properties:

- it has to act on exactly two elements, no more and no less
- it has to produce an element that belongs to the same set to which the two elements that the proposed operation acts on belong
- two consecutive proposed operations that act on three elements must produce the same result regardless of the order in which these operations are carried out

We will study these properties more thoroughly and more formally in the upcoming **Binary Operations** and **Associativity** discussions. For now we just demonstrate these three properties of the *followed by* operation that acts on the flipping actions descriptively.

Altogether

In this particular model of a group *an operation*, namely *followed by*, glues *other operations*, namely the *flipping actions*, together in a carefully orchestrated manner into the duplex toggle light switch group.

Fine Points

It is very important to realize and digest early on that neither the flipping actions themselves and in an isolation nor the *followed by* operation by itself and in an isolation is a group.

The phrase:

the flipping actions are a group

is mathematically meaningless at best and is just plain wrong, inaccurate and ambiguous at worst.

Likewise, the phrase:

the "followed by" operation is a group

is also mathematically meaningless at best and is just plain wrong, inaccurate and ambiguous at worst.

It is the happy and *the indivisible marriage* of the flipping actions, or the elements of a set, *and* the process of following one flipping action by another, or a certain operation defined on the elements of that set, that is a group or forms a group.

In order to drive the point home, we will keep coming back to this idea in our formal exercises later on, but for now it suffices to just memorize the fact that:

- the elements of a candidate set in an isolation from a candidate operation do *not* form a group and
- a candidate operation in an isolation from the elements of a candidate set does *not* form a group

It is:

- the dovetailing or
- the splicing or
- the double helix

of a candidate set *and* a candidate operation that acts on the elements of the candidate set and that has certain restrictive, informally stated above, properties that forms a group.

Thus, a mathematically sound, correct and cultured way to frame a group formation related statement is as follows:

a set this and that with such and such as the operation is a group

or

a set this and that forms a group under the operation such and such

For example, in our case we would say that:

the set of flipping actions with "followed by" as the operation is a group

or

the set of flipping actions forms a group under the operation "followed by"

Moreover, even though it is entirely possible to invent multiple different suitable operations that can act on the elements of a given set, a group, once defined, has *one and only one* such operation - not zero operations, not two operations, not three operations, not seventeen operations and not thirty seven operations but one, only one and exactly one.

That is why such an operation, in English, is referred to as *the* operation of a group or *the* group operation.

Upon even a very short reflection it should be clear now that that coupling of a certain set with a certain operation provides a very rich source for the multitude of groups.

Put differently, a group is such a mathematical construct that is highly sensitive to both entities that make it up - if we fix a certain set and vary the operations that can act on its elements then a given set coupled with one operation may form a group but when the same set is coupled with another operation, it may very well *not* form a group and conversely.

If we fix a certain operation and vary the sets on whose elements that operation can act then a given operation coupled with one set may form a group but when the same operation is coupled with another set, it may very well fail to form a group.

We will solidify the understanding of these fine points in the upcoming exercises.

Basic Properties of a Group

Keeping a sober attitude toward the fact that in real time it took several contributing mathematicians many decades to shake out and distill the following facts, we now take a technical survey of the duplex light switch experiment and list the quintessential properties of a group simply based on one concrete example. Later on most of these properties will become the so-called *group axioms*.

Rules Constancy

I. The list of the four allowed flipping actions, once defined and agreed upon, and the rule according to which these actions can be combined remain constant at all times and never change.

In other words, the flipping actions and a way in which they can be combined, being etched in stone, explicitly spell out what *can be* done with the levers of a light switch and imply that *nothing else* can be done with them.

Intuitively speaking, the rules of a game cannot be changed *during* the game.

Slightly more formally, the rules according to which the candidate operation acts on the elements of the candidate set, once inked, cannot be altered and must remain fixed at all times.

Determinism

II. The result of each flipping action is fully deterministic and is known to us with absolute certainty ahead of time - turning the light switches on and off is not the game of chance and probabilities.

Inaction

III. It is always possible to *do nothing* with a duplex light switch.

Assuming that a duplex light switch is in, say, State 4, but any other state will do just as well, what will be the outcome of applying to such a switch the flipping action number 1 followed by any other flipping action?

The outcome of applying to such a switch the flipping action number 1 followed by any other flipping action is equivalent to applying that latter flipping action to the switch all by itself.

That much should be evident because first we did nothing and then we did something - such a sequence of events should, clearly, result in *doing something*. Moreover, the order in which the flipping action number 1 is applied also makes no difference - we can do nothing first and then do something later or we can do something first and then do nothing later, the result will be the same either way.

While now it may *feel* like there is one and only one way to do nothing in the light switch group, later on we will actually *prove* that there is one and only one way to do nothing in any group. In general, such an ability to do nothing is not a logical conclusion of any kind but *a demand*:

a candidate set that wants to participate in the formation of a group must possess the "do nothing" element

If such a demand is not met then the candidate set that lacks the *do nothing* element cannot participate in the formation of a group.

Why is it so important to have the ability to do nothing in a group?

The ability to do nothing is important because it is coupled with the next basic property of a group.

Reversibility

IV. Every flipping action can be undone or reversed.

Intuitively, if we realize that we turned the light off in the kitchen by mistake, when all along we wanted to turn the light off in the living room, then we can always fix that error by turning the light in the kitchen back on reversing the effect of the undesired flip.

More formally, what flipping action from the set G do we have to apply to a duplex light switch in order to undo the effect of the flipping action number 2?

In order to undo the effect of the flipping action number 2, we carry out the same action again and flip the left lever.

Undone.

Now, exactly what did we accomplish by flipping the left lever of a switch twice in a row?

By flipping the left lever of a switch twice in a row we returned that switch back into its original state, whatever that state was. And if we returned a light switch back into its original state, then what is the other way of saying that?

If we returned a light switch back into its original state then we, equivalently, *did nothing*. But *do nothing* is the flipping action number 1 and that action is certainly in the set of the flipping actions G - lucky us!

Therefore, the reversal of the flipping action number 2 can be described in terms of the defined flipping actions only and there is no need to import any new flipping actions that are not in our set of four already - the existing vocabulary of the flipping actions is sufficient.

That is, *all three* flipping actions can be tied together into a neat little package as follows:

the flipping action number 2 "followed by" the flipping action number 2 is equivalent to the flipping action number 1

We invite our readers to convince themselves that such a neat little package can be put together for *all* the flipping actions.

That is:

the flipping action number 1 "followed by" the flipping action number 1 is equivalent to the flipping action number 1

and:

the flipping action number 3 "followed by" the flipping action number 3 is equivalent to the flipping action number 1 and:

the flipping action number 4 "followed by" the flipping action number 4 is equivalent to the flipping action number 1

We can now give a group-theoretic spelling of the fact that every flipping action can be reversed: every flipping action can be reversed *by a flipping action from the same set* G.

Now let us manually check, is there *any other* flipping action in the set G that can reverse the effect of the flipping action 2?

A scan through the short list of the four flipping actions reveals that *no*, there are no other flipping actions that can reverse the effect of flipping the left lever: doing nothing accomplishes literally nothing, flipping the right lever does not help and flipping both levers makes an even bigger mess.

Thus, we conclude that there is one and only one way to reverse the effect of the flipping action number 2 - carry out that same action again.

Likewise, we verify that there is one and only way to undo each of the effects of:

- the flipping action number 1
- the flipping action number 3 and
- the flipping action number 4

Later on, however, we will *prove* that each flipping action can be reversed by a flipping action in a unique way.

In general, the reversibility of every element of a candidate set in the sense explained above is not a logical conclusion of any kind but *a demand* or *a requirement*:

every element of a candidate set that wants to participate in the formation of a group must be reversible

If such a demand is not met then the candidate set that lacks the *reversibility of its each element* under the chosen operation cannot participate in the formation of a group.

The above four properties of the manner in which a duplex light switch can be operated in a group-theoretic fashion are all fine and dandy.

However, it is the next, fifth, property of such manipulations that brings the cake home and defines two more characteristic properties of a group which we will formalize and symbolize a bit later and which we will examine here in an informal fashion.

Specific Arity and the Independence of the Order of Execution of Two Consecutive Group Operations

V. If we glue *any two* flipping actions from the set of four with one *followed by* operation then the result of such a sequence of two arbitrary flipping actions will always *always* be equivalent to a flipping action *from the same set* G.

What exactly does that mean?

It Takes Two

In such a sequence of flipping actions the *followed by* operation glues together or acts on exactly two elements, no more and no less, because due to the very nature of the *followed by* process that process only makes sense when we specify *two* flipping actions:

*"flip this lever" followed by "flip that lever"**

and no other way of describing such a process can carry a reasonable meaning.

For example, the phrase:

followed by "flip that lever"

or the phrase:

"flip this lever" followed by

or the phrase:

followed by

is incomplete and does not constitute a sensible command.

In vacuum, *an* operation can act on one, or on two, or on three, or on four, or on five elements and so on. However, for the purpose of forming a group it is *required* that a candidate operation acts on exactly two elements, no more and no less.

Thus, this specific property of a group operation to act on exactly two elements is not a logical conclusion of any kind but rather *a demand*:

an operation that wants to participate in the formation of a group must act on exactly two, no more and no less, elements

The number of entities that an operation acts on is known as *arity* and that is why to the above property of a group operation to act on exactly *two* elements we referred earlier as *Specific Arity*.

The group operation of the duplex light switch group, *followed by*, acts on exactly two flipping actions, no more and no less, and, thus, honors the arity demand. However, if a candidate operation does not act on two elements then it cannot participate in the formation of a group.

It Always Comes Back

As we noted earlier, in addition to acting on exactly two elements, the *followed by* operation applied to *any* (!) two flipping actions always produces a result that is equivalent to a flipping action from the same set G. Put differently, in order to describe the result of any flipping action *followed by* any flipping action sequence no new vocabulary of actions is needed - the set of the four flipping actions shown above is enough. Always.

Before we look at the relevant examples, let us ponder the following, natural, question framed as an exercise.

Exercise 5.1.1: if the group operation *followed by* acts on exactly two flipping actions then how many such (distinct) pairs of flipping actions in our duplex light switch group exist?

Solution: first of all, should the pairs of flipping actions that the *followed by* operation can act on be order-sensitive or order-*in*sensitive?

Because we do not want to impose any additional requirements, we will work under the widest possible assumption that the order in which the flipping actions are fed into the *followed by* operation *does* matter.

In other words, we will take it that the result of the sequence *flip this lever* followed by *flip that lever* can very well be different from the result of the sequence that uses the same flipping actions but in reverse order of *flip that lever* followed by *flip this lever*.

Hence, we have to compute the total number of distinct *order-sensitive* pairs of flipping actions under the condition that any flipping action from the set G can be paired up with any flipping action from the same set.

Which is to say that the repetitions in these order-sensitive pairs of flipping actions are allowed, as we have already seen such repetitions in the above *Reversibility* experiments. But in the **Exercise 4.0.1**, where we computed the number of all possible mappings of one finite set to another finite set, we, pretty much, did exactly that by using the Multiplication Counting Principle.

In this exercise, we have exactly 4 choices or 4 flipping actions for the first element that the *followed by* operation will chew on

.

And how many choices do we have for the second element that the *followed by* operation will chew on?

We have also 4 choices because *any* flipping action can be paired up with *any* flipping action, which is another way of saying that in our order-sensitive pairs of actions duplicate actions, when an action is followed by the same action, are allowed.

Thus, by the same Multiplication Counting Principle there will be exactly:

$$4 \cdot 4 = 4^2 = 16$$

different order-sensitive pairs of flipping actions with repetitions allowed and we remark in passing that our reasoning is not really tied to the specific number 4 and it will hold for any finite number of different elements n.

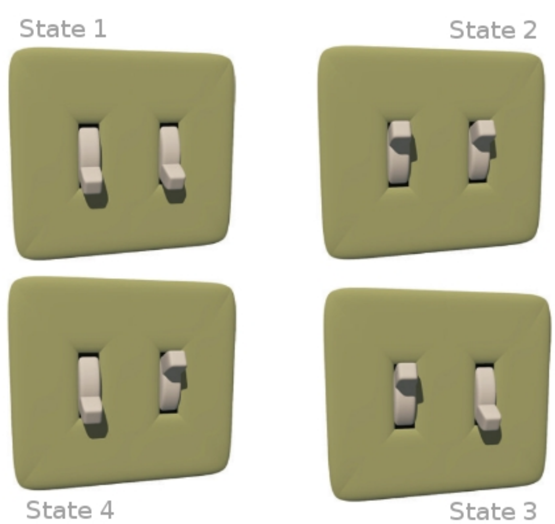
Thus, the number of all possible order-sensitive pairs of elements picked from a set of n with repetitions allowed is equal to:

$$n \cdot n = n^2 \tag{1}$$

for future references. □

It Always Comes Back (Closure)

Now let us pick any state of a duplex light switch to be the *initial* state (Figure 5.1.6):



Which specific state we choose for that purpose makes no difference whatsoever.

Say, we take the **State 4**, as shown in the Figure 5.1.6 above, to be our initial state - in a such a state the left lever is *off* and the right lever is *on*.

Next, as a by-hand experiment, let us construct all $4 \cdot 4 = 16$ order-sensitive pairs of flipping actions that comprise our group and investigate whether the results of such actionable pairs belong to G or not.

In other words, during such investigations we want to see if any pair of flipping actions that we care to choose has an individual-flipping-action synonym that also belongs to G.

In the *Reversibility* experiments we already showed that any flipping action followed by the same flipping action results in the flipping action number 1, which is certainly in the set G.

Thus, pairing each individual flipping action with itself, we find that:

1: *the flipping action number 1 "followed by" the flipping action number 1 is equivalent to the flipping action number 1*

2: *the flipping action number 2 "followed by" the flipping action number 2 is equivalent to the flipping action number 1*

3: *the flipping action number 3 "followed by" the flipping action number 3 is equivalent to the flipping action number 1*

4: *the flipping action number 4 "followed by" the flipping action number 4 is equivalent to the flipping action number 1*

As a side bar, observe that when any one flipping action is paired with itself then the notion of *order* of actions in such a pair looses its meaning.

Very well. Four order-sensitive pairs of actions down, twelve to go.

In the *Inaction* experiments we convinced ourselves that the application of the *do nothing* action to any flipping action, itself including, does not really change anything and results in the original flipping action, which is certainly in G:

5: *the flipping action number 1 "followed by" the flipping action number 2 is equivalent to the flipping action number 2*

6: *the flipping action number 2 "followed by" the flipping action number 1 is equivalent to the flipping action number 2*

7: *the flipping action number 1 "followed by" the flipping action number 3 is equivalent to the flipping action number 3*

8: *the flipping action number 3 "followed by" the flipping action number 1 is equivalent to the flipping action number 3*

9: *the flipping action number 1 "followed by" the flipping action number 4 is equivalent to the flipping action number 4*

10: *the flipping action number 4 "followed by" the flipping action number 1 is equivalent to the flipping action number 4*

Six more order-sensitive pairs of actions down, six to go.

If to the light switch in **State 4** we apply the flipping action number 2 followed by the flipping action number 3 then that switch will be taken to which state?

Well, the *followed by* operation has the following tactical property - *the next* flipping action of the *followed by* operation is applied to the state of the light switch in which it was left by *the previous* flipping action.

In a slightly more technical jargon, two actions glued together by the *followed by* operation work in such a way that *the output* of the first action becomes *the input* for the second action.

Thus, the flipping action number 2 will take the switch from **State 4** to **State 2** and the flipping action number 3 will take the switch from **State 2** to **State 3**, in which the left lever is *on* and the right lever is *off*.

What single flipping action applied to the light switch in **State 4** will take that switch directly into **State 3**?

Why, the flipping action number 4, when we flip *both* levers at once, does exactly that and that action is certainly in G:

11: *the flipping action number 2 "followed by" the flipping action number 3 is equivalent to the flipping action number 4*

But we also see right away that if we swap the order of the above actions and flip the right lever first and flip the left lever next, we will obtain the same result - such a pair of actions will be also equivalent to the flipping action number 4 that is certainly in G:

12: *the flipping action number 3 "followed by" the flipping action number 2 is equivalent to the flipping action number 4*

Two more order-sensitive pairs of actions down, four more to go.

If to the light switch in **State 4** we apply the flipping action number 2 followed by the flipping action number 4 then that switch will be taken to which state?

State 1, when both levers are *off* (show your intermediate work here – talk to yourself, or talk to a wall, or talk to your grandmother, or talk to your cat).

What single flipping action applied to the light switch in **State 4** will take that switch directly into **State 1**?

The flipping action number 3, when we flip the right lever only, and that action is certainly in G:

***13:** the flipping action number 2 "followed by" the flipping action number 4 is equivalent to the flipping action number 3*

Again, we right away see that swapping the order of these two actions and flipping both levers first and flipping the left lever next is equivalent to flipping the right lever only:

***14:** the flipping action number 4 "followed by" the flipping action number 2 is equivalent to the flipping action number 3*

Fourteen order-sensitive pairs of actions down, two to go.

Lastly, if to the light switch in **State 4** we apply the flipping action number 3 followed by the flipping action number 4 then that switch will be taken to which state?

State 2, when both levers are *on* (show your intermediate work here also).

What single flipping action applied to the light switch in **State 4** will take that switch directly into **State 2**?

The flipping action number 2, when we flip the left lever only, and that action is certainly in G:

***15:**the flipping action number 3 "followed by" the flipping action number 4 is equivalent to the flipping action number 2*

Again, we right away see that carrying out these two actions in reverse order, by first flipping both levers and then flipping the right lever, will be equivalent to flipping the left lever only:

***16:** the flipping action number 4 "followed by" the flipping action number 3 is equivalent to the flipping action number 2*

Since we enumerated all the order-sensitive pairs of flipping actions, we are done and we now understand what exactly does it mean that:

if we glue any two flipping actions from the set of four with one "followed by" operation then the result of such a sequence of two arbitrary flipping actions will always always be equivalent to a flipping action from the same set

In general, the above property of a candidate group operation is not a logical conclusion of any kind but *a demand*:

a candidate operation that wants to participate in the formation of a group must always return an element from the same set to which the two elements that such an operation acts on belong

If such a demand is not met then the candidate operation cannot participate in the formation of a group.

Optional (Extra For The Curious)

In grown-up mathematics the phenomenon that we just witnessed above has a dedicated terminology attached to it and is known as *a closure of a set under an operation*. That is, a set this and that is *closed under an operation* (of arity 2) such and such if the said operation acting on any two elements of the said set always returns an element from that set.

As an easy to comprehend example, consider the set of nonnegative whole numbers and the operation of addition of nonnegative whole numbers: if any two nonnegative whole numbers, such as 3 and 47, are added then the result of such an addition is always another nonnegative whole number, 50 in this case. Thus, the set of nonnegative whole numbers *is* closed under the operation of *addition* of nonnegative whole numbers.

However, a straightforward experiment, such as $3–47 = -44$, shows that the set of nonnegative whole numbers is *not* closed under the operation of *subtraction* of nonnegative whole numbers, because, as we just showed, there exist at least one pair of nonnegative whole numbers, the ordered pair $(3, 47)$, such that when the second nonnegative whole number of that pair is subtracted from the first nonnegative whole number of that pair *a negative* whole number, or a number that does not belong to the set of nonnegative whole numbers, results.

Smooth Operation (Associativity)

Still using **State 4** of our light switch as the initial state, let us carry out these three consecutive flipping actions:

- flip the left lever

  - *followed by*
- flip the right lever
  - *followed by*
- * flip both levers

in *two different ways*.

Initially, we will carry out these actions in the geometric order in which they are listed above when the first *followed by* operation acts on *flip the left lever* and *flip the right lever* elements.

The result of this *followed by* operation will be the intermediate **State 3** of the light switch (show your intermediate work).

To the light switch in **State 3** we apply the second *followed by* operation and flip its both levers, putting the switch into its final state, **State 4**. Very well.

Alternatively, starting with the **State 4**, we skip the first *followed by* operation and we right away carry out the second (!) *followed by* operation of flipping the right lever of the switch and then flipping its both levers, taking that switch into the intermediate **State 2**.

To the light switch in **State 2** we apply the first *followed by* operation, flipping its left lever and putting it into its final state, **State 4** – again!

Aha, we see that the above two different orders in which the two consecutive *followed by* operations were executed have both put the light switch into *the same* final state or have both produced the same result.

Note well, however, that while in the above experiment we did change the order in which the consecutive *operations* were carried out, we did *not* change at all the order of *the elements* on which these operations acted as we kept that order strictly *fixed*.

Thus, we just demonstrated the fact that:

two consecutive "followed by" operations that act on any three flipping actions produce the same result regardless of the order in which these two operations are carried out

That is, we now understand what *the independence of the order of execution of two consecutive group operations* really means.

In general, the above property of a candidate group operation is not a logical conclusion of any kind but *a demand*:

a candidate operation that wants to participate in the formation of a group must possess the property that two such consecutive operations acting on any three elements produce the same result regardless of the order in which these operations are executed

If such a demand is not met then the candidate operation cannot participate in the formation of a group.

Optional (Extra For The Curious)

In grown-up mathematics the phenomenon that we just witnessed above has a dedicated terminology attached to it and is known as the *associativity of an operation*.

We will take an initial deep dive into the notion of associativity of operations in its own, **Associativity**, chapter in due time.

We encourage our readers to experiment with other initial configurations of a duplex light switch, using an actual switch if need be, on their own and convince themselves that all the properties of a group that we discussed so far do hold in this case.

Ask your family to be patient and understanding about the flickering lights, after all, you will be *doing* the theory of groups in particular and grown-up mathematics in general:

- jot down the initial configuration of the levers
- flip these levers any which way into their final state
- find that final state on the Figure 5.1.6
- find the one flipping action, from the set of four, that transforms your light switch from its initial to its final state and so on

We observe in passing that the tire rotations group that we worked with in this example is essentially the same as a group that is known officially as *the Klein four-group*, sometimes symbolized as K_4.

Now. The next exercise, marked with an asterisk, will be rather challenging for many readers - do not rush to solve this exercise right away. Let the material of this discussion cook and simmer for a few days.

As the new ideas covered in this discussion sink in, give it a try.

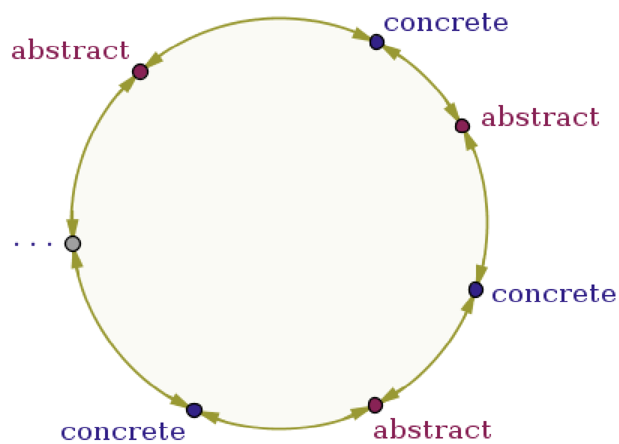
Exercise 5.1.2*: on your own, think of another everyday example of a group that is *essentially the same* as the Klein four-group.

Essentially the same, in this case, means that *your* example of the Klein four-group must have exactly 4 elements in it, no more and no less, and the names of these elements and the name of the operation that acts on these elements may very well be different from the corresponding names that we used in this discussion. However, the general way in which your operation acts on your elements must be exactly the same as the way in which our *followed by* operation acted on tires.

Solution: of this exercise will be a challenge for many because instead of a formal definition of a group, our readers were given only a specific example of a group.

As such, in order to solve this problem correctly, our readers will have to *think like a mathematician* and that is the purpose of this and all the other exercises in this book:

- make an attempt to discern what matters here from what does not matter
- scan a personal web of knowledge for the potential candidates
- fish out an example that fits the model at hand and
- by doing so, cross blades with the process of abstraction
- jumping over the hurdle of going from something concrete to something abstract and then back to something concrete that fits the proposed abstraction, as we mentioned earlier (Figure 5.1):



Admittedly, that is a tall order.

And that is *why* we are doing these exercises - in order to begin developing the ability to do the back and forth from-specific-to-abstract juggling act.

Those readers who managed to solve this problem on their own correctly had to have the following bright idea visit them at some point:

work by analogy

and retrace the steps that we highlighted in the **Abstraction** chapter.

Example 2: consider a coupe - a sports car that is equipped with two doors:

- *the left* door and
- *the right* door

While the doors on cars, and cars themselves, come in a variety of shapes and sizes, fundamentally, each car door can be in only one of only two mutually exclusive or diametrically opposite *states*:

- *open* or
- *closed*

Thus, via the various states of the individual doors, a coupe can be in four and only four different states:

- both doors *closed*
- both doors *open*
- the left door is *open*, while the right door is *closed*
- the left door is *closed*, while the right door is *open*

Let us agree that *the act of changing* the current state of any single door, *left* or *right*, to its opposite state, whatever it is, *closed* or *open*, constitutes one *clop* of that door.

That is, if a given door is currently in the *open* state then changing the state of that door to *closed* constitutes *one clop* or just *a clop* of that door.

Likewise, if a given door is currently in the *closed* state then changing the state of that door to *open* also constitutes *one clop* or just *a clop* of that door.

We now carefully conglomerate the clops of the individual doors into the *clopping actions* that can be carried out with any number of such doors - zero doors, one door or two doors.

We, then, claim that the above four states of a 2-door sports vehicle can be achieved via the interaction of the following four types of *clopping actions*:

- no clops at all (1)
- clop the left door only (2)
- clop the right door only (3)
- clop both doors, at once (4)

By analogy, then, neither the 2-door car itself nor its individual doors themselves *are* or *form* a group that is essentially the same as the Klein four-group.

Rather, it is the (unordered) pair of the following, two, entities:

- the collection of the four clopping actions listed above *and*
- *the operation* of gluing these actions into a string of a finite number of necessarily *consecutive clopping actions*: a clopping action followed by another clopping action, followed by another clopping action, followed by another clopping action and so on

  - with the above *followed by* operation having the property that two consecutive such operations that act on three clopping actions produce the same result regardless of the order in which these two operations are carried out

that forms a group that is essentially the same as the Klein four-group based on an everyday object.

Show by-hand that all five basic properties of a group do hold for the coupe group also. Work by analogy.

Example 3: in the **Example 2** replace the two doors of a sports vehicle with two *drawers of a kitchen cabinet**.

Example 4: in the **Example 2** replace the two doors of a sports vehicle with two *eyes of a person*.

What about these two nice curtains that you have right there in your living room?

And what about these two faucet handles in your kitchen or in your shower?

Run, by-hand, through all the basic properties of a group for all of the above examples. □

Overall, do not stop with just these few examples of the Klein four-group. As you go through your daily routine, use your imagination and make it a fun habit of collecting and jotting down the various specimen of that group and tell your family that, without knowing it, they have been dealing with the Klein four-group for years.

5.2 Group Theory Takes a Ride

Now that we have seen the light, for our second fun example of a group we will also choose an everyday object that is ubiquitous and omnipresent in many countries , a passenger car or just *a car*.

Cars have wheels.

Wheels have tires.

Over the lifetime of a car its tires wear and tear unevenly.

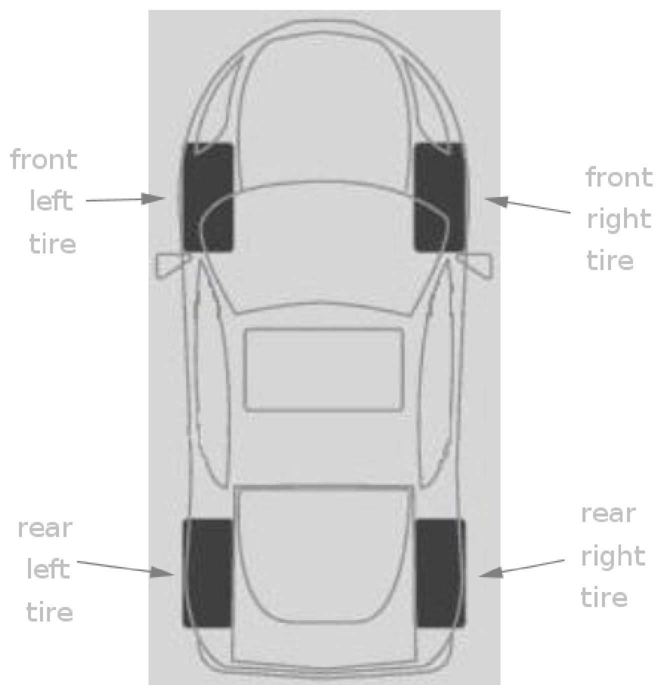
Thus, in order to ensure that the tires on a car wear more evenly and last longer, we, on a more or less regular basis, do the so-called *tire rotation*.

When we do the tire rotation services at the local car shop in real life, we *follow one tire rotation by another tire rotation*.

That is, when we buy a brand new car, we begin driving it right away, using the initial tire configuration chosen by the manufacturer. After, say, the first five or seven thousand miles we do our next tire rotation. After the next five or seven thousand miles we do another tire rotation. After the next five or seven thousand miles we do yet another tire rotation and so on.

If we agree that a typical car has four wheels and that a typical wheel is equipped with just one tire then our typical car will have four tires:

- two *front* tires:
  - *the front left* tire and
  - *the front right* tire
- and two *rear* tires:
  - *the rear left* tire and
  - *the rear right* tire (Figure 5.2.1):



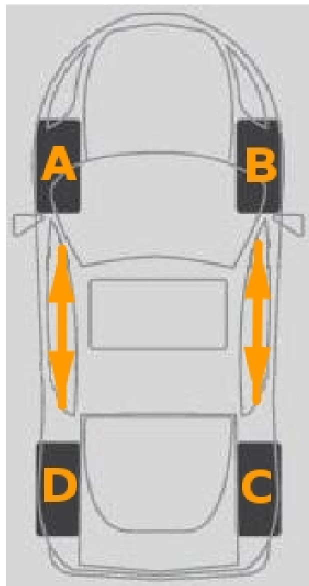
Now let us define the following four types of tire rotations.

I. Type 1: no tire rotations at all.

In that case all four tires stay where they currently are and we *do nothing* with these tires.

II. Type 2: the *front-to-back* tire rotation.

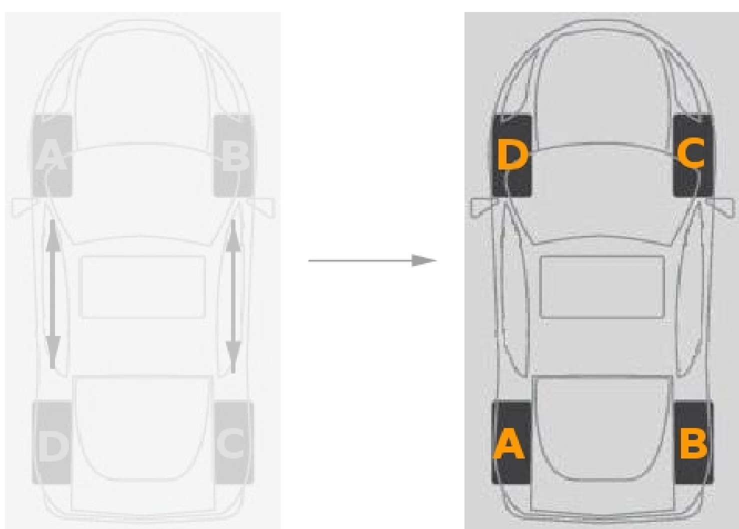
The *front-to-back* tire rotation means that we swap the two front tires, named as A and B (Figure 5.2.2):



with the two rear tires, named as D and C respectively:

- the tire named A trades places with the tire named D and
- the tire named B trades places with the tire named C

The result of a front-to-back tire rotation is shown below (Figure 5.2.3):



Observe that we have used these specific tire names for demonstration purposes. In general, the *front-to-back* tire rotation means that whatever tires happen to be at the front of the vehicle are swapped with whatever tires happen to be at the back of the vehicle at the moment of the swap, correspondingly and at the same time.

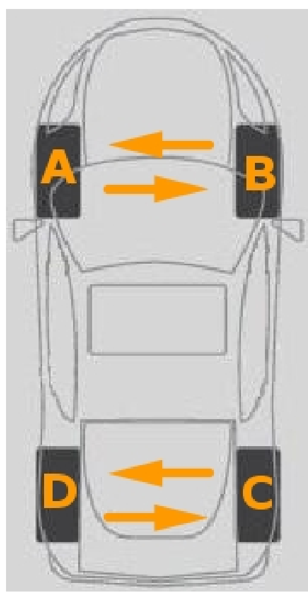
One important technical wrinkle that our readers should keep in mind here is the following.

It is not the case that only one pair of tires, such as the tire named A and the tire named D, for example, is swapped during a front-to-back tire rotation, while the other pair of tires is left untouched.

On the contrary, when a front-to-back tire rotation is performed then *both* (!) pairs of tires, the tires named A and D and the tires named B and C trade places as was described above *at the same time*.

III. Type 3: the *side-to-side* tire rotation.

The *side-to-side* tire rotation means that we swap the two *left* tires named as A and D (Figure 5.2.4):

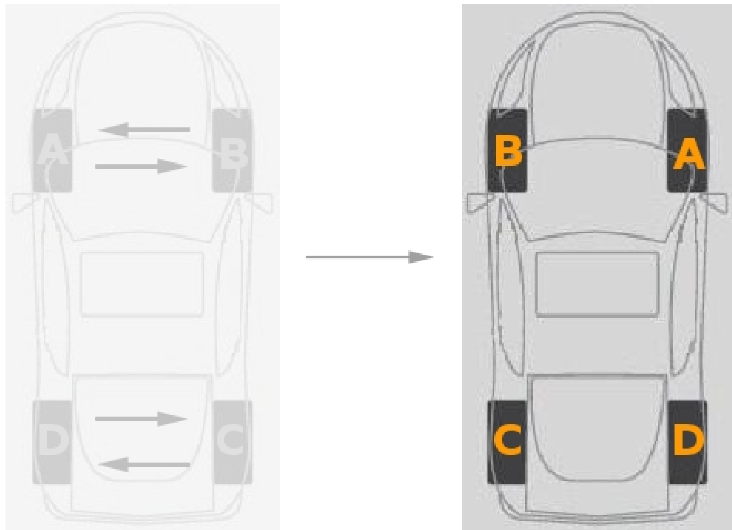


on the left with the two right tires named as B and C respectively:

- the tire named A trades places with the tire named B and
- the tire named D trades places with the tire named C

Again, we have used these specific tire names for demonstration purposes. In general, the *side-to-side* tire rotation means that whatever tires happen to be on the left side of the vehicle are swapped with whatever tires happen to be on the right side of the vehicle at the moment of the swap, correspondingly and at the same time.

The result of *a side-to-side* tire rotation is shown below below (Figure 5.2.5):

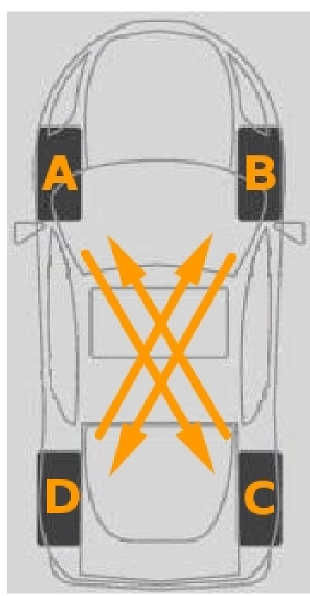


As before, it is not the case that only one pair of tires, such as the tire named A and the tire named B, for example, is swapped during a side-to-side tire rotation, while the other pair of tires is left untouched.

On the contrary. When a side-to-side tire rotation is performed then *both* (!) pairs of tires, the tires named A and B and the tires named D and C trade places as was described above *at the same time*.

IV. Type 4: the *cross-over* tire rotation.

The *cross-over* tire rotation means that we swap the tires named as A and C (Figure 5.2.6):

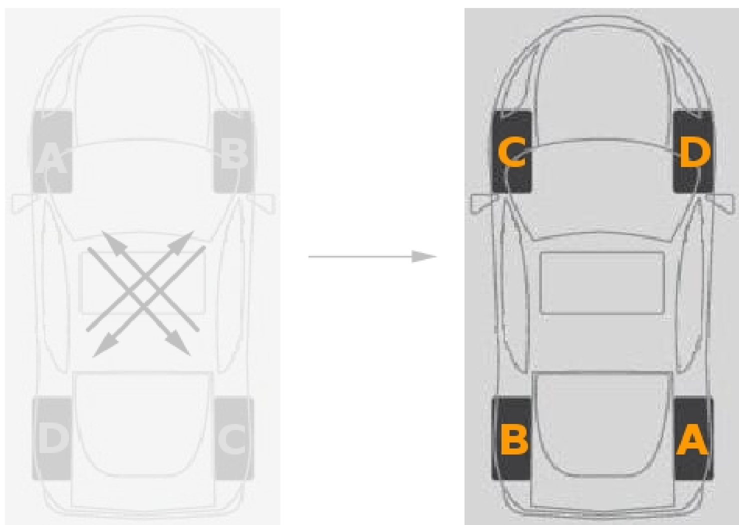


on the left in the North-West to South-East direction with the tires named as B and D in the North-East to South-West direction:

- the tire named A trades places with the tire named C and
- the tire named B trades places with the tire named D

We have used these specific tire names for demonstration purposes. In general, the *cross-over* tire rotation means that whatever tire happens to be in the North-Western position is swapped with whatever tire happens to be in the South-Eastern position and, likewise, whatever tire happens to be in the North-Eastern position is swapped with whatever tire happens to be in the South-Western position. Both of the above swaps are carried out at the same time.

The result of the cross-over tire rotation is shown below (Figure 5.2.7):



We hope that our readers understand by now that it is not the case that only one pair of tires, such as the tire named A and the tire named C, for example, is swapped during a cross-over tire rotation, while the other pair of tires is left untouched.

On the contrary, when a front-to-back tire rotation is performed then *both* pairs of tires, the tires named A and C and the tires named B and D trade places as was described above *at the same time*.

And these are the only four types of tire rotations that we will allow and consider.

Show Time

Neither the car itself *nor* its tires *are* a group or *form* a group.

Rather, it is the (unordered) pair of the following, two, entities that is a group or forms a group:

- *the collection* of the four (types of) tire rotations described above and
- *the operation* of following one tire rotation by another, and then by another, and then by another and so on for a finite number of times
  - with the above *followed by* operation having the property that two consecutive such operations that act on any three tire rotations produce the same result regardless of the order in which these two operations are carried out

It so happens that this is also an example of a group that is essentially the same as the Klein four-group.

We now begin the process of getting used to the texture of the group-theoretic work. Roll up the sleeves. Grab plenty of paper.

Exercise 5.2.1: using the duplex light switch group discussion as the guide, verify that all the basic properties of a group hold in the case of the tire rotations group also.

In addition to developing the skill of being able to rapidly jump back and forth between concrete and abstract, these exercises also show what grown-ups do in real life.

Namely. When mathematicians come across a candidate set and a candidate operation that acts on the elements of that set and they want to know whether that dynamic duo is a group, then one way to find that out is to trawl the said dynamic duo through the upcoming verification process.

Besides, *doing* mathematics is the only way to *learn* mathematics - when we are asked to generate all the ordered pairs of tire rotations, we do generate all the ordered pairs of tire rotations ourselves; when we are asked to show that any two consecutive tire rotations are equivalent to a single tire rotation, we do show that that observation holds for all such pairs by finding all the respective equivalents ourselves and so on.

In other words, as was described in the **What Makes It Perfect** chapter, we practice mathematics correctly.

Solution: of this exercise is expected to be generated by the readers on their own.

Verify the *Rules Constancy* and the *Determinism* property of the tire rotations group.

What is the *inaction* or the *do nothing* element of this group? How many such elements does this group have?

Can every tire rotation be undone?

That is, can every tire rotation be reversed via a tire rotation from the set of the four tire rotations defined above?

For each tire rotation write down the tire rotation that reverses it.

Is such a reversing tire rotation unique or there are others?

Do not forget about the *no rotations* rotation.

How many tire rotations does the *followed by* operation act on?

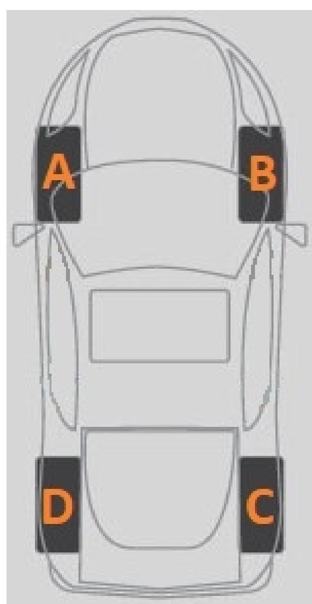
How many ordered pairs of tire rotations exist in this group? List them all.

Verify that *any* two tire rotations glued together with a single *followed by* operation are equivalent to a single tire rotation from the set of four tire rotations defined above.

Write your findings down for each such pair.

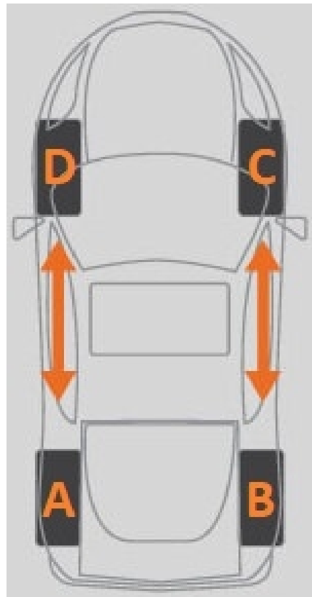
For example.

Say, in its initial state the tires on a car are named as A, B, C and D in a clockwise fashion (Figure 5.2.8):

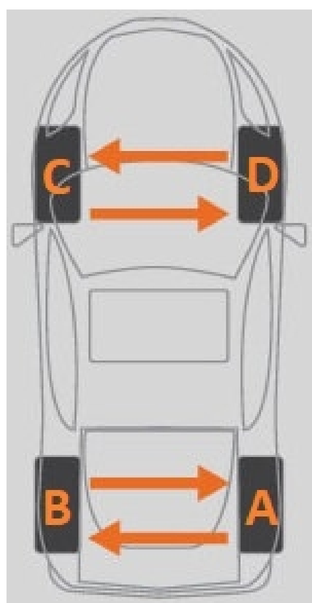


Even though we chose the particular initial configuration of tires shown above, we encourage our readers to play around with a different initial mix of tires and observe the result.

For experimentation purposes, to that initial configuration we apply the *front-to-back* tire rotation (Figure 5.2.9):



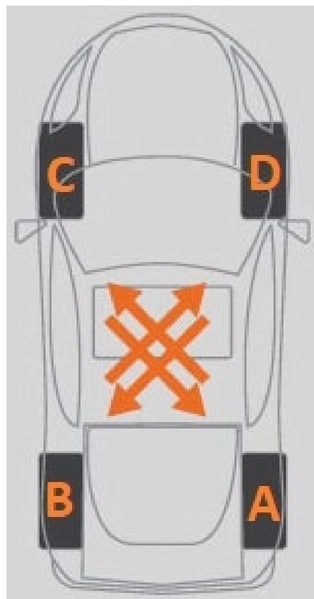
and then, to the said result of the front-to-back tire rotation we apply the *side-to-side* tire rotation (Figure 5.2.10):



Next, we claim that one consecutive application of any two tire rotations, in this experiment, is over and we ask the following question.

What *single* tire rotation from the menu of the four tire rotations defined earlier will transform this car from the initial state shown in Figure 5.2.8 into the final state shown in Figure 5.2.10?

Clearly, it is the *cross-over* tire rotation applied to the car in the initial state depicted in Figure 5.2.8 that will transform this car directly into the final sate shown in Figure 5.2.10 in one go, as shown below (Figure 5.2.11):



Thus, we jot down our finding and by so doing we get a glimpse into the type of work that many refer as *mathematical research*:

the front-to-back tire rotation followed by the side-to-side tire rotation is equivalent to the cross-over tire rotation

or some such and so on.

Verify that two consecutive *followed by* operations that act on any three tire rotations do produce one and the same result regardless of the order in which these two operations are carried out.

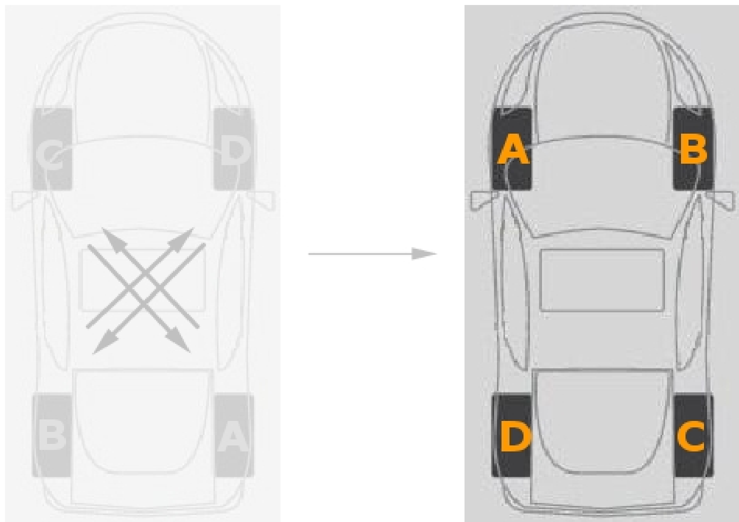
For example, if we follow the above two consecutive tire rotations of *front-to-back* and *side-to-side* by the *cross-over* tire rotation then we will readily see that that, third, tire rotation will take our vehicle from the state shown in Figure 5.2.8 into its original state, as it is shown in the Figure 5.2.8 above.

Thus, we know that the following string of three consecutive tire rotations:

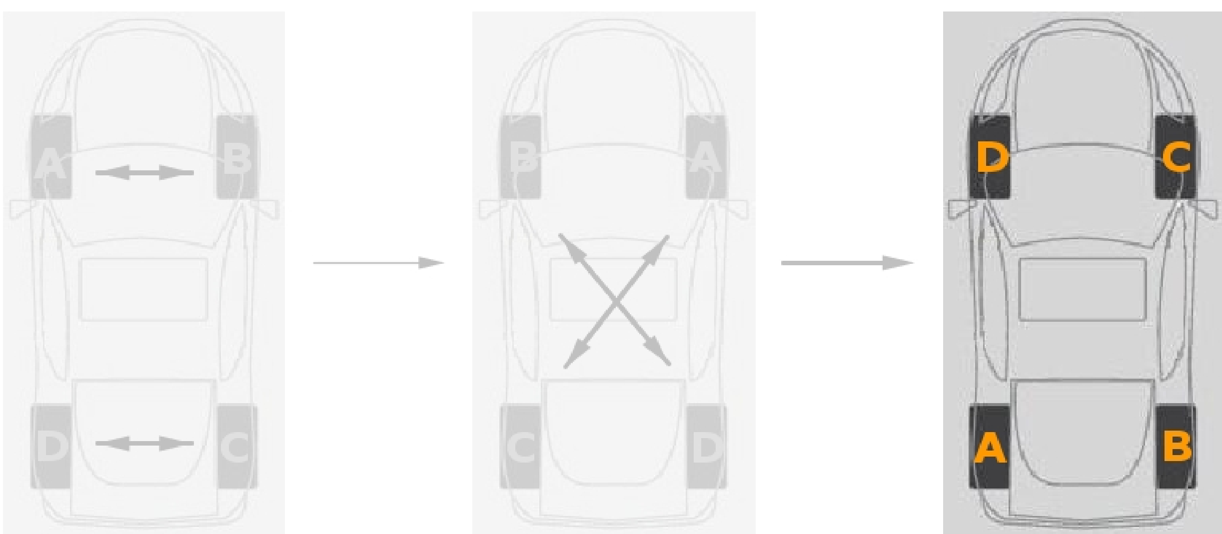
- the front-to-back
  - *followed by*

- the side-to-side
  - *followed by*
- the cross-over

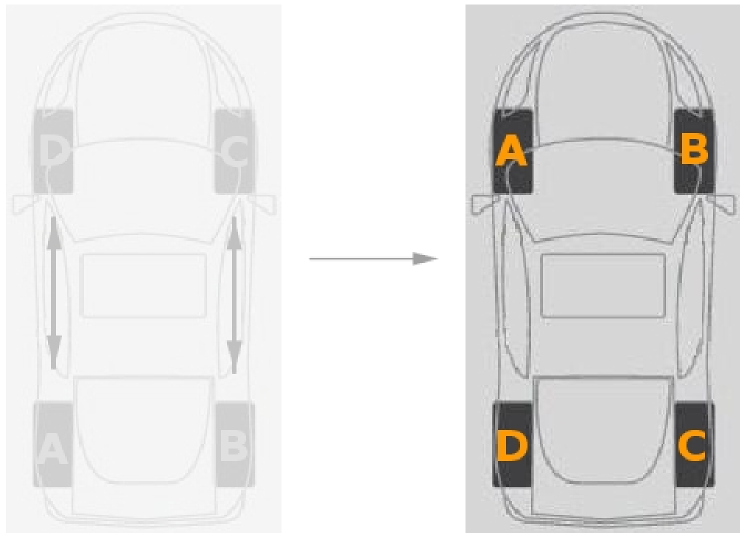
tire rotation puts the tires on a vehicle back into its original configuration or state (Figure 5.2.12):



Now, we change the order in which the *followed by* operations are carried out, while keeping the order of the individual tire rotations intact, and we right away execute the second such operation by doing the *side-to-side* tire rotation followed by the *cross-over* tire rotation, taking a vehicle from the state shown in Figure 5.2.8 into the following, intermediate, state shown below (Figure 5.2.13):



Next, to the intermediate state of the vehicle shown in the Figure 5.2.13 above we apply the *front-to-back* tire rotation, putting the vehicle into the following, final, state that is exactly the same final state into which the vehicle was put when the two consecutive *followed by* operations were carried out in the original, listed above, order (Figure 5.2.14):



Thus, we just demonstrated the fact that:

two consecutive "followed by" operations that act on any three tire rotations produce the same result regardless of the order in which these two operations are carried out

and so on.

Again, do note carefully that in the above experiment we did change the order of the followed by *operations* but we did not change the order of the *elements* that these operations acted on - geometrically or top-to-bottom, the order of the specific tire rotations, or the group elements, was fixed and remained the same both times. ☐

The tire rotations group just described, by the way, is also a group that is essentially the same as the Klein four-group.

Comparison

Observe that while in the duplex light switch group from our previous discussion the flipping actions could act on zero, one or two individual pieces that made up the whole, in the tire rotations group, barring the trivial rotation that does not move any tires around, *all four* individual pieces that made up a whole must be acted on, in unison.

Moreover, these two groups are formed by distinctly different things - what can light switches and car tires possibly have in common?

Yet, we are told that these two groups are *essentially the same.*

It stands to reason, then, that there must be some pretty powerful abstraction that can swallow and ignore all these visual, verbal and physical differences between how the duplex light switch and the tire rotation groups are described and formed.

The *essentially the same* concept travels well throughout all of mathematics and plays an important role in the so-called Category Theory.

When, for example, we compare two *prime numbers* then such a comparison seems to be conceptually easy because in some sense we are comparing two very simple or very basic objects neither of which can be broken down multiplicatively any further.

When we want to compare two planar triangles then all of a sudden such a comparison stops being easy because the two objects that we are comparing are *composite* - they consist of a number of simpler objects, line segments.

Does it make sense to compare these two planar triangles?

It certainly does.

But when can we claim that these two triangles are *essentially the same*?

What properties of a triangle can guide us through such a comparison process?

Their color? The solid or the dashed lines with which they are drawn? Their location, in different planes, perhaps? Their orientation in space?

Ah, none of the above - we now know that two planar triangles are essentially the same if they have the same *structure*.

But what is *structure*? The structure of a mathematical object that is.

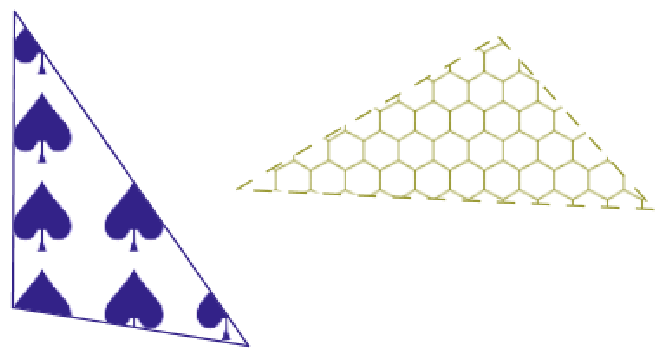
The *structure* of a certain whole is an essential way in which the smaller and simpler parts that form that whole *interact with each other under certain rules that are deemed important and interesting in a particular context*.

In its strictest sense, two *congruent* triangles are *essentially the same*, despite the fact that these two triangles can be of different color, that they can be drawn with different types of straight lines, that they can be in different locations in space and that they can be oriented differently.

A planar triangle is a closed figure formed by three line segments or three *sides* joined edge-to-edge in such a way that the three respective pairs of these sides form three interior *angles*. Thus, the triangle's angles and sides are its smaller and simpler parts that lock in or define the structure of *that* triangle. Exactly how do the triangle's angles and sides define its structure?

The four basic criterions, **SSS**, **SAS**, **ASA** and **AAS**, capture the structure of a triangle or *a way in which the simpler objects that make it up interact with each other*.

If, for example, the three sides of one triangle are equal in length to the three corresponding sides of another triangle then these two triangles are congruent or *essentially the same*, like the triangles shown below, despite the fact that some of their features seem to be drastically different (Figure 5.2.15):



The faces of the two triangles are filled with different images.

The boundaries of the two triangles are drawn in different styles, dashed versus solid.

The two triangles occupy totally different areas in the ambient plane and do not intersect.

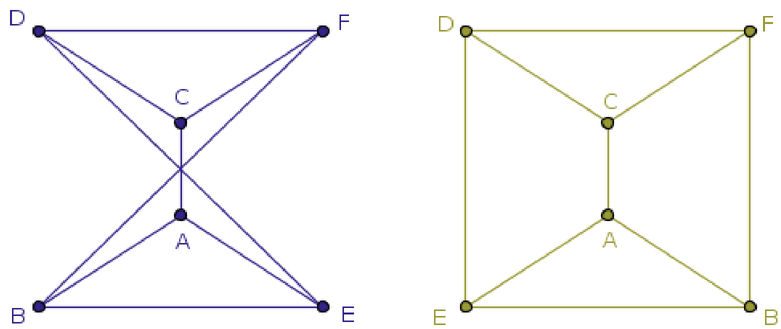
The two triangles have totally different orientation: the obtuse angle of one of these triangles points North, while the obtuse angle of the other triangle points in the South-Eastern direction.

Moreover, if we numb down the sensitivity to what exactly interests us in this or that particular situation, in a more relaxed sense, two *similar* triangles may be deemed essentially the same.

In which case the three basic criterions, **AA**, **SAS** and **SSS**, capture the structure of a triangle or *a way in which the simpler objects that make a triangle up interact with each other* under such interesting geometric transformations as rotation, reflection, homothety and parallel translation.

In the theory of graphs it makes perfect sense to inquire if two *graphs* are *essentially the same* or not.

Just like triangles in plane geometry, the two sample graphs depicted below may *appear* to be different but upon a deeper inspection they turn out to have the same structure and, thus, are essentially the same (Figure 5.2.16):



Technically, two graphs are *essentially the same* if there exists a bijection between their vertex sets with the property that whenever two vertices are adjacent in either graph, the corresponding two vertices in the other graph are adjacent also.

Thus, if we grab the two graphs shown in Figure 5.2.16 and trawl them through the above definition then we will see that yes, no matter which two blue adjacent vertices of the blue graph we consider, their green siblings-by-name in the green graph will be adjacent as well.

A mathematical synonym for *essentially the same* is *isomorphic*.

In either case, essentially the same or similar triangles are *isomorphic* to each other and essentially the same graphs are *isomorphic* to each other.

Likewise, in the theory of groups *the language* that is used to describe two specific groups may be very different but *the structure* of these two groups may very well turn out to be the same.

When the structures of two specific groups are the same then these groups are said to be *isomorphic to each other* or just *isomorphic*.

Thus, based on the examples of groups that we have looked at so far, our tire rotations group is isomorphic to the dual light switch group and conversely. The dual light switch group is isomorphic to the tire rotations group.

Later on, in our **Group Homomorphisms** and **Group Isomorphisms** discussions we will give a precise definition of when two groups are *isomorphic* or *structurally the same*.

It will take us a while to get there but the good news is that our readers who have worked their way through all of our **Mappings** discussions in general, including the discussion of bijections in particular, are already in a good shape because they have made an important first step toward understanding the idea of group isomorphism.

In order to provoke more thought on the matter and further demonstrate the idea of *descriptively different but structurally the same*, in our next discussion we look at another example of a group that is essentially the same as the Klein four-group and that comes form a totally different world.

5.3 Rectangle, Anyone?

In our third example of a group we will switch gears ever so slightly as we look at an example of a group that in its nature is more *explicitly* mathematical than our first two groups but that is still decidedly not arithmetic.

In this discussion we will tango with a non-square rectangle in a highly specific way by only considering the motions of bodies that are *rigid* and that bring the rigid frame, boundary or perimeter of choice, a rectangle, back into *coincidence with itself*.

Definition 8: a body is (absolutely) *rigid* if and only if the distance between any of its two points remains constant at all times.

Why So Rigid?

Why are we considering the motions of rigid objects only?

We are considering the motions of rigid objects only because if we open the door of possible objects too wide and bring in non-rigid bodies also then the classification of the potential motions becomes a major hurdle and at the moment we are not equipped with the sophisticated enough mathematical apparatus to overcome that hurdle.

For example, when we somehow move a non-rigid rectangle then in that process of motion we can fix its two diagonals in their length and mutual juxtaposition and then we can *squeeze* one of the rectangle's sides in one non-uniform way, and then we can squeeze another side of the moving rectangle in a different non-uniform way, and then we can *stretch* yet another side of the rectangle in yet another non-uniform way and so on.

Because we have fixed the diagonals of a moving rectangle, such a figure will still be a rectangle at the end of an experimental motion but the sides of that rectangle can be modified non-rigidly in an enormous number of different ways.

There will be the time and there will be the place for the grown-up mathematical study of such wildly different types of motions, but that time is not now and that place is not here.

Thus, from now on, when and if we use the word *body* in the context of motions and transformations, it should be understood that the said body is *rigid* in the above sense, unless noted otherwise.

Coincidence With Itself

A motion that brings a (rigid) body back into coincidence with itself is called *a congruence motion* or *a symmetry*.

In order to understand how exactly does a motion that brings a body back into coincidence with itself work, we, grabbing only one Euclidean tool, a straightedge, construct the following *physical* model.

I. Take a heavy or *project* A4 sheet of paper that will be our *model* of a rigid rectangle.

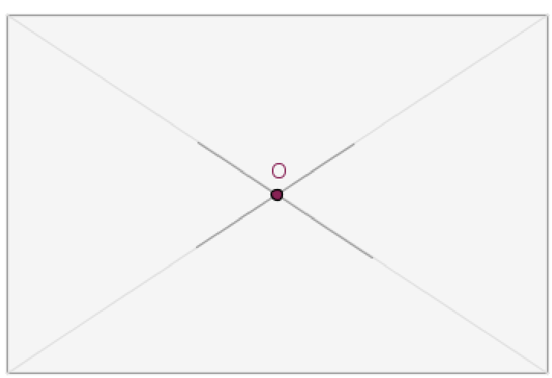
If our readers can find a thin but rigid enough piece of see-through plastic then that would be even better because the markings on such a piece of transparent plastic can be done on one side only.

We remember that every square is a rectangle, but not every rectangle is a square.

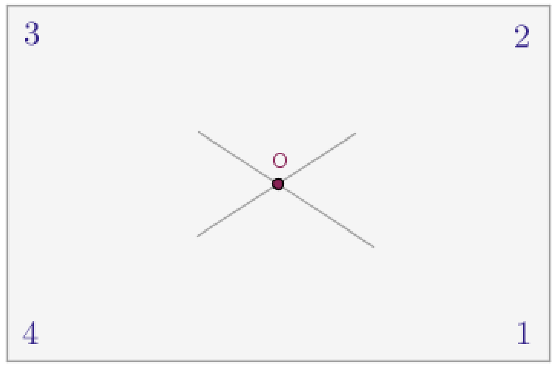
In this particular discussion we want to make sure that our rectangle is *not* a square.

For the duration of this experiment we will be referring to this model of a rigid rectangle as *the paper rectangle*.

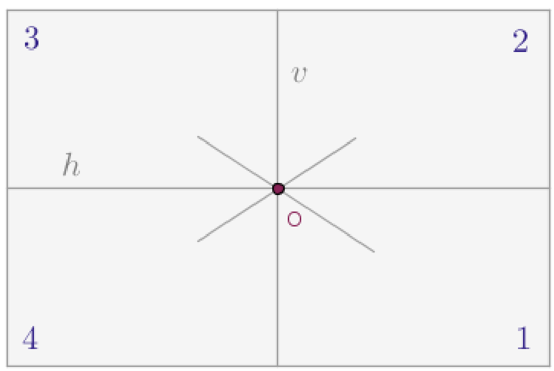
II. On *both sides* of the sheet of paper lightly draw just enough of its two diagonals in order to locate the rectangle's center, O (Figure 5.3.1):



III. On *both sides* of the sheet of paper, *inside* of the rectangle that is, write down the names of its vertices, making sure that each given vertex has *the same* name on either side of the sheet of paper(Figure 5.3.2):



IV. Still on both sides of the sheet of paper mark the dead middles of the short sides of the rectangle and connect these midpoints with the straight line h and then mark the dead middles of the long sides of the rectangle and connect these midpoints with the straight line v (Figure 5.3.3):



If we were precise enough with our paper, pencil and straight edge manipulations then the straight line h and the straight line v should both pass exactly through the rectangle's center O:

- the straight line v, that will play the role of the rectangle's *short* axis of symmetry, should be parallel to the short sides of the said rectangle, while
- the straight line h, that will play the role of the rectangle's *long* axis of symmetry, should be parallel to the long sides of the rectangle

V. Grab a throwaway piece of cardboard paper that is *larger* than the chosen sheet of paper.

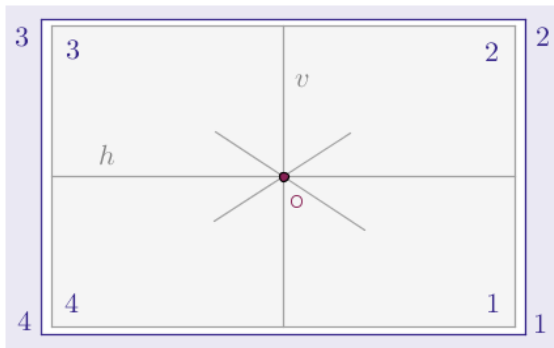
VI. Punch a whole in the paper rectangle through the point O and, using a pin, attach the sheet of paper to the cardboard background.

Rotate the rectangle about the point O back and forth any which way.

Decide on a random position of the sheet of paper that you like.

Holding the paper rectangle firmly in-place, trace that rectangle *on the cardboard* and then write down the names of the *corresponding* vertices of the rectangle on the cardboard also.

In the diagram below the cardboard is portrayed with the solid blue color (Figure 5.3.4):



For the duration of this experiment we will be referring to the above blue rectangle drawn on the cardboard as *the cardboard rectangle* or *the fixed rectangle*.

We are now ready to experiment with congruent motions.

Rotation

In order to completely describe the transformation of the plane known as *a rotation*, *three* independent parameters must be specified:

- *the center* of rotation
- *the amount* of rotation or the so-called *angular displacement* or just *the angle* and
- *the direction* of rotation

If the direction of rotation is *counterclockwise* then:

- that direction is taken to be *positive* and
- the sign of the amount of rotation is taken to be *the plus*, $+$, which, more often than not, is omitted but is implied

For example, in the absence of any context, the phrase:

rotate an object this and that about a point/axis such and such by π *radians*

implies a rotation by π radians in the positive or counterclockwise direction.

If the direction of rotation is *clockwise* then:

- that direction is taken to be *negative* and
- the sign of the amount of rotation is taken to be the minus, –

When and if a rotation is symbolized, the direction of rotation can be and often is attached to the amount of rotation.

For example, in the absence of any context, the phrase:

rotate an object this and that about a point/axis such and such by $-\pi$ *radians*

implies a rotation by π radians in the negative or clockwise direction.

When the direction of rotation is explicitly specified along with *the signed* amount of rotation then the interpretations of such rotations should be kept consistent with the above rules.

For example, the phrase:

rotate an object this and that about a point/axis such and such by $-\pi$ *radians counterclockwise*

should be interpreted as a rotation by π radians in the negative or clockwise direction and the phrase:

rotate an object this and that about a point/axis such and such by $-\pi$ *radians clockwise*

should be interpreted as a rotation by π radians in the positive or counterclockwise direction.

Descriptively, then, *a rotation* of a plane p about a fixed point, the center of rotation, O, by an angle θ is a transformation of the plane, p, into itself under which every point P of p is taken into a point P' in p in such a way that the distance between the center of rotation, O, and the point P, |OP|, is equal to the distance between the image point P' and O, $|OP'|$:

$$|OP| = |OP'|$$

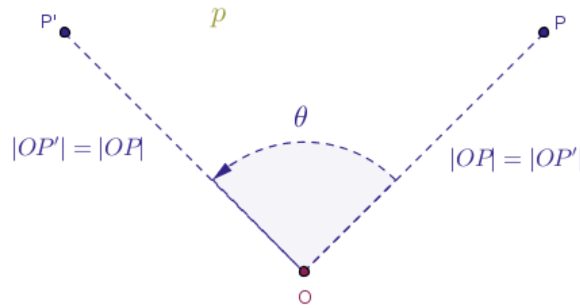
and the magnitude and the direction of the angle POP' is equal to the magnitude and the direction of the angle θ:

$$\angle POP' = \theta$$

Observe that the above symbolization of the angles at hand, $\angle POP' = \theta$, implies not only the equality of the *magnitudes* of these angles but also the equality of the *directions* of these angles.

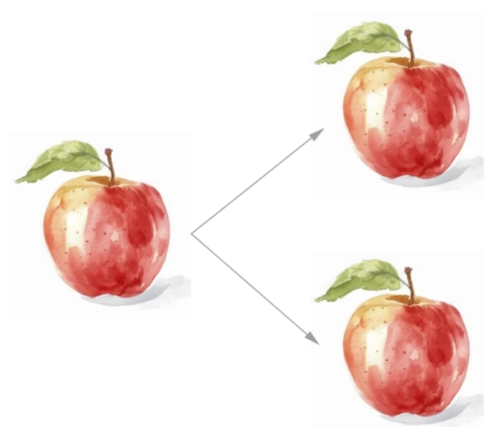
Effectively, the above definition of a rotation of a plane means that under such a transformation of the plane into itself, every ray that emanates from the center of rotation is *turned* by the same angular amount and in the same direction.

Moreover, every rotation of the plane about a fixed point leaves one and only one point, the center of rotation, *fixed,* while moving all the other points of the plane in a nontrivial fashion (Figure 5.3.5):



We observe in passing that, as a transformation, rotations of the *plane* are *commutative*, which means that such transformations can be executed in any order without affecting the result, while rotations of the *space*, which is a much more involved business, are not.

The key to understanding the concept of congruence motion is the idea of *splitting of the object at hand into two separate like copies of each other* (Figure 5.3.6):



Once the two such copies of an object are envisioned or created, we first make these two copies coincide with each other completely, which, effectively, makes the two said copies *look like* a single object.

The initial position of such two coincident objects in space is completely irrelevant and can be chosen at will.

However, once such an initial position is picked, it becomes and remains a fixed reference for all the consequent motions.

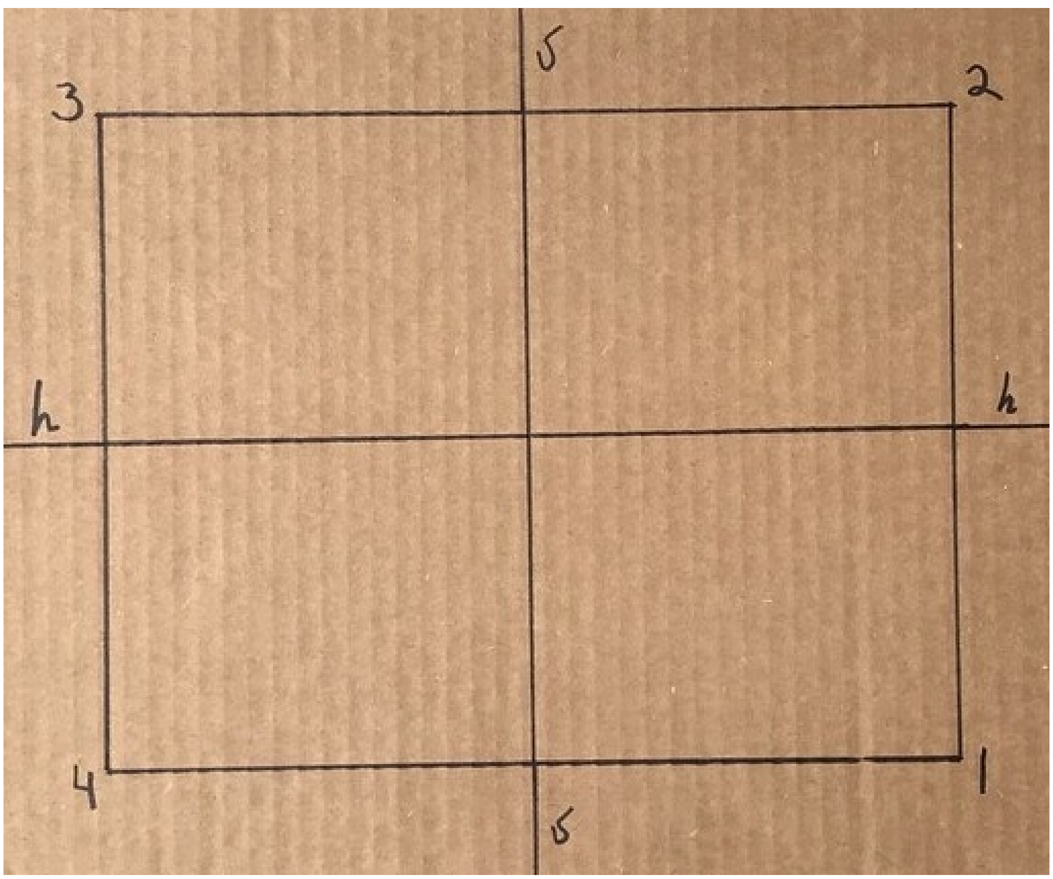
Next, we keep one copy of the object *fixed* and *immovable*, in some sense, and we move the second copy of the object with respect to the first copy of the object in such a way that at the end of the said motion the traveling copy of the object comes back into coincidence with its fixed twin.

Observe that this corner of mathematics intersects with *physics* because physics deals with *reference frames* as well and such a reference frame, in physics, can be chosen arbitrarily.

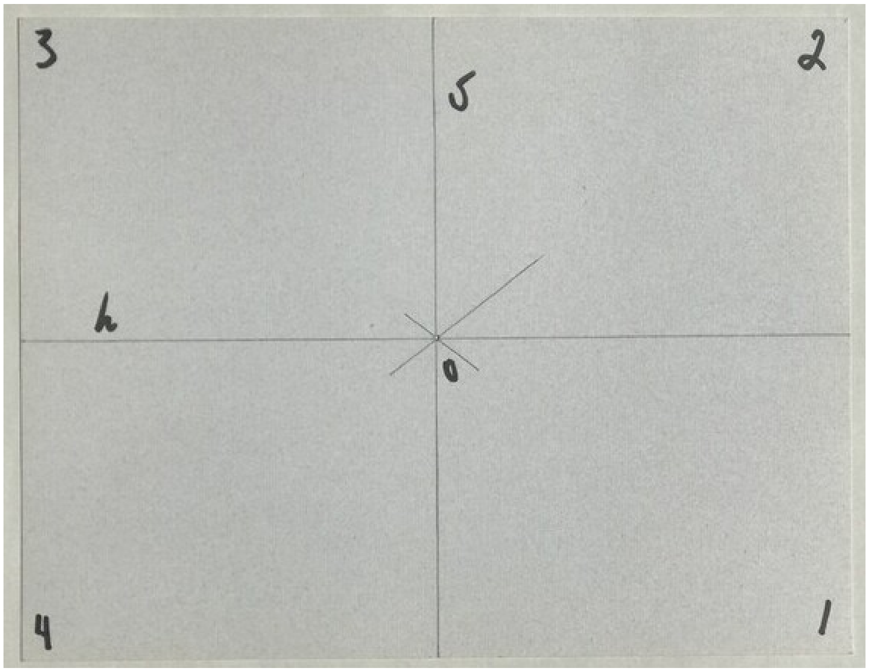
Moreover, in physics we would say that we make the cardboard play the role of a reference frame, a coordinate system and a clock, with respect to which the motion of the other copy of a rigid rectangle is tracked.

In mathematics, however, we, in general, do not care about the clock or the temporal aspect of motion.

In any case, in our physical model the rectangle that is traced on the cardboard background plays the role of the fixed, reference, copy (Figure 5.3.7):



and the sheet of the A4 paper plays the role of the moving copy of a rectangle (Figure 5.3.8):



Note that we have drawn both *the fixed* axes of symmetry h and v of a rectangle on the cardboard workbench that will remain stationary at all times and *the dynamic* axes of symmetry h and v on the moving rectangle as well.

We have drawn the dynamic axes symmetry primarily to help us see what goes where during the upcoming transformations.

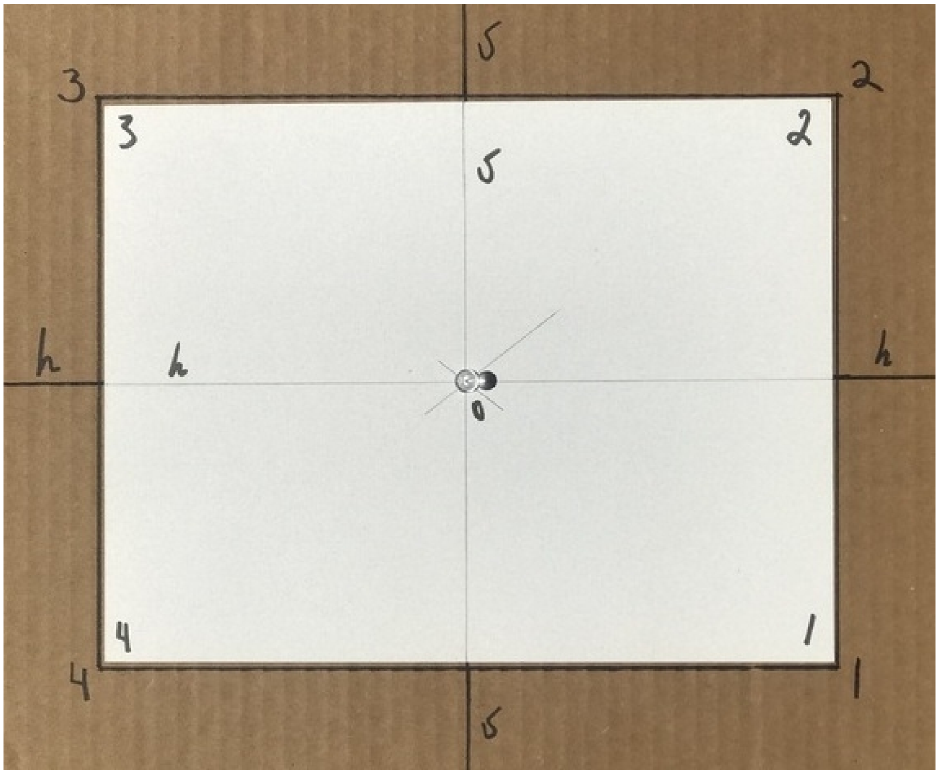
However, it is important to understand right away that in this and all other similar exercises all the *reflections* of a rectangle and similar figures must be done *with respect to the fixed axes of symmetry* that are drawn on the stationary workbench.

Next, make the sheet of paper coincide with its twin rectangle on the cardboard in such a way that the names of the vertices of the paper rectangle exactly match the names of the corresponding vertices of the cardboard rectangle.

The state of the paper rectangle as it relates to the cardboard rectangle depicted above will be *the initial state*.

When we asked our readers to rotate the paper rectangle any which way, we wanted to demonstrate the idea that *the particular* initial state of the body of interest plays no role in the congruence motion(s) of that body whatsoever - that initial state can be completely arbitrary.

However, and this is very important, once such a state is picked, then it remains fixed and becomes a reference, with respect to which all the consequent motions of an object are tracked (Figure 5.3.9):



We now pose the following question:

what type of a rigid motion of the paper rectangle will bring it back into coincidence with the fixed cardboard rectangle?

Well, one trivial motion that satisfies the above coincidence requirement comes to mind right away - no motion at all.

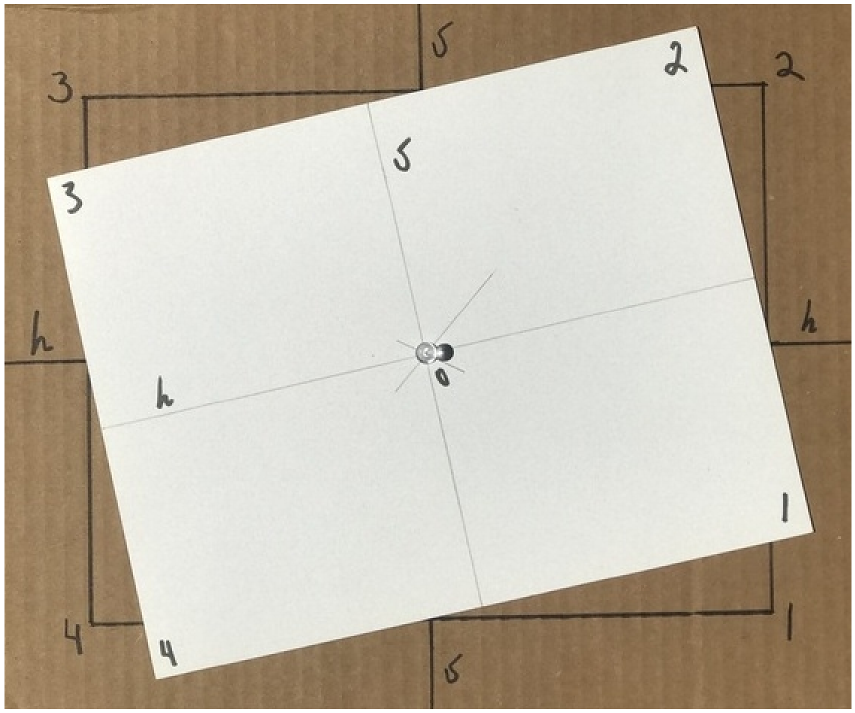
Fine.

What else?

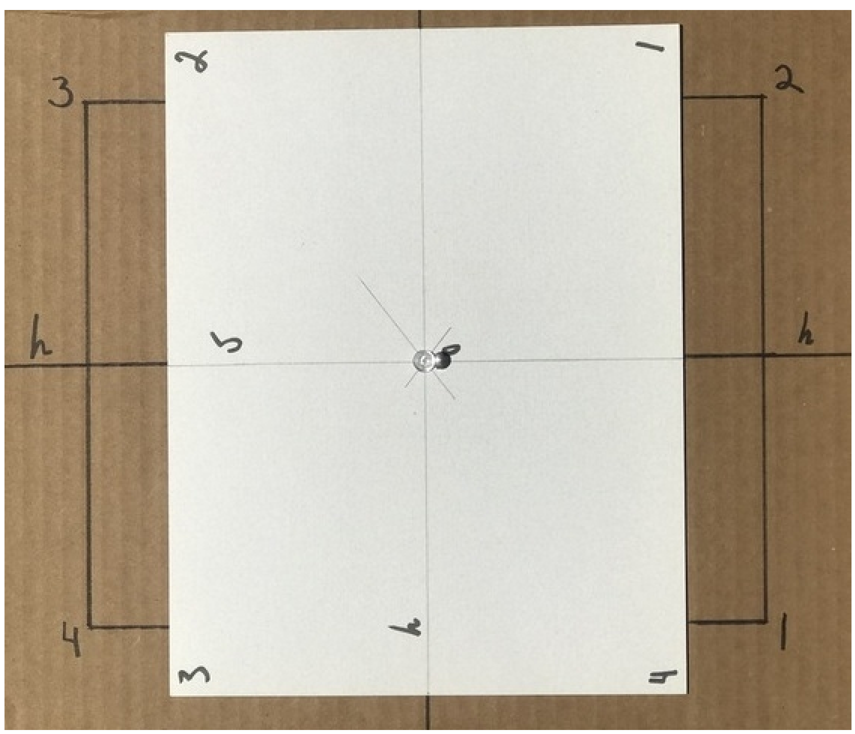
If we *rotate* the paper rectangle about its center O by, say, less than one quarter of a complete revolution in either direction, will such a rotation bring the paper rectangle into coincidence with the cardboard rectangle?

In other words, will such a rotation make the vertices of the paper rectangle coincide with the vertices of the stationary or reference, cardboard, rectangle?

Apparently, no, not at all (Figure 5.3.10):

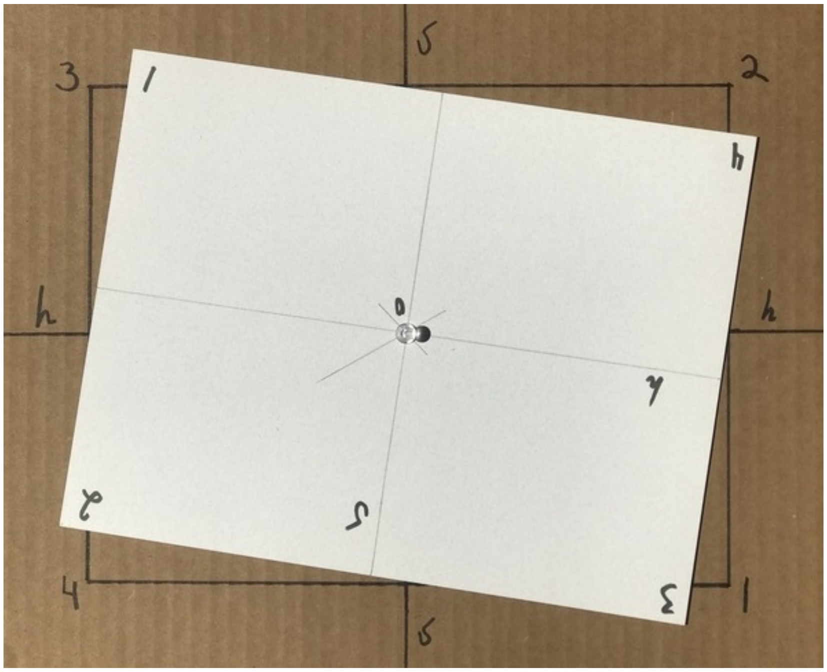


What about one complete quarter of a turn? Nope, no luck with this rotation either (Figure 5.3.11):

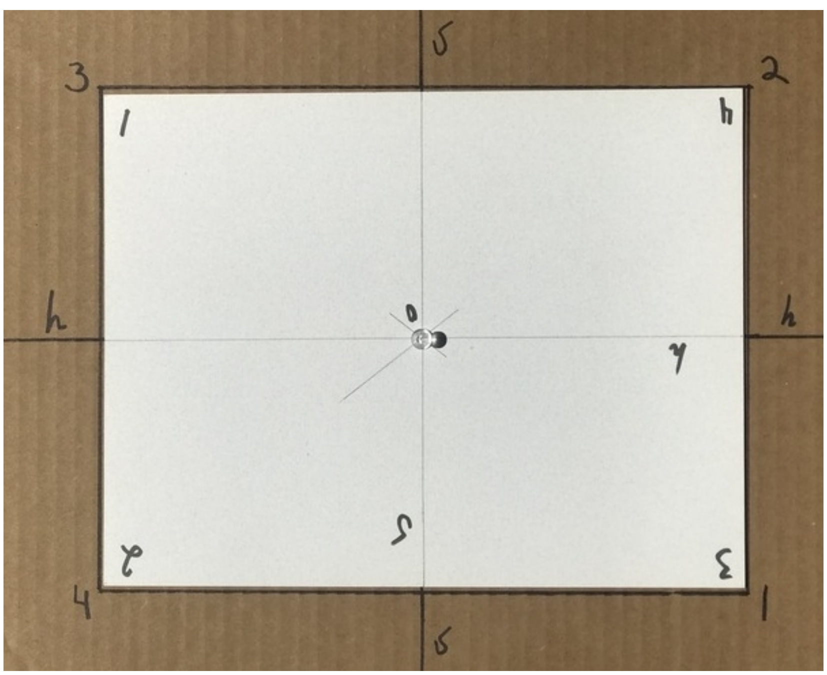


What about a rotation that is slightly larger than one quarter of a complete revolution?

Still no cigar (Figure 5.3.12):



What if we rotate the paper rectangle about its center O by exactly two quarters of one complete revolution or by $180°$, which is π radians (Figure 5.3.13):



Aha! We just found one more, new and distinct, congruence rotation of a rigid rectangle that brings such a rectangle into coincidence with itself.

That congruence motion shuffles the names of the vertices of the moving rectangle with respect to the names of the vertices of the fixed rectangle in some specific way.

Namely.

The vertex of the fixed, cardboard, rectangle named 1 is now lined up with the vertex of the traveling, paper, rectangle named 3. Therefore, the rotation of the rectangle about its center, starting from a totally arbitrary position, by $\pm\pi$ radians, takes its vertex 1 into its vertex 3.

Intuitively speaking, after the specified congruence motion, 1 *became* 3.

Shuffling *the numbers* around symbolically is something that we do know how to do - we simply use the left-to-right arrow as a replacement of the word *into* and we drop all the surrounding context:

$$1 \to 3$$

and so on.

In general, then, the rotation of the rectangle about its center, starting from a totally arbitrary position, by $\pm\pi$ radians, takes:

- its vertex 1 into the vertex 3:
$$1 \to 3$$
- its vertex 2 into the vertex 4:
$$2 \to 4$$
- its vertex 3 into the vertex 1:
$$3 \to 1$$
- its vertex 4 into the vertex 2:
$$4 \to 2$$

but the above shuffle of the names of the rectangle's vertices does not change the fact that the rectangle now coincides with itself shape-wise: what we started with prior to any rotation was *a rectangle* and what we wound up with after a particular rotation is still, essentially the same, rectangle.

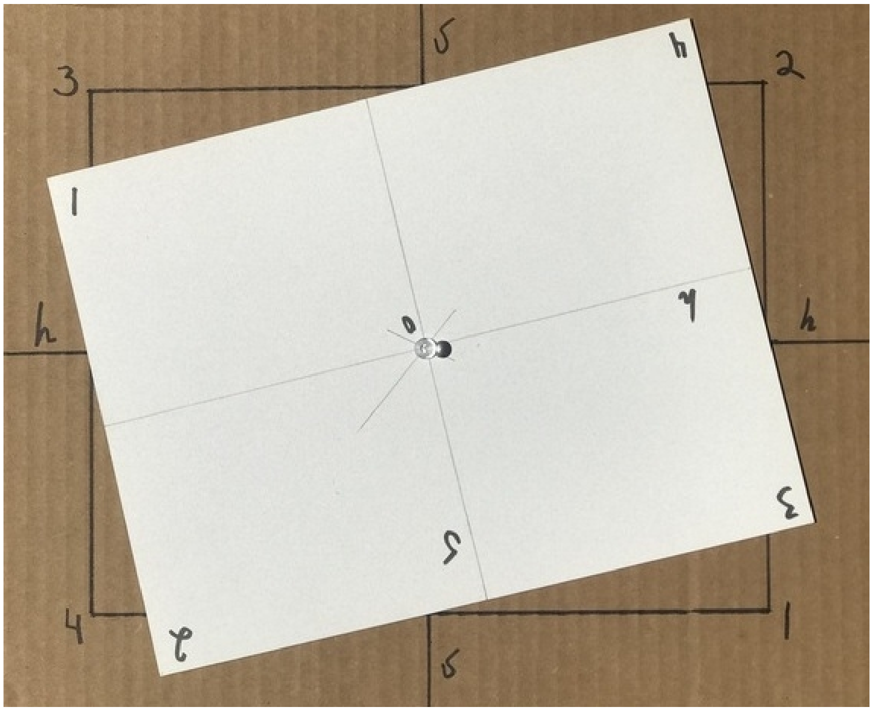
Very well.

We managed to find one congruence motion of a rigid rectangle.

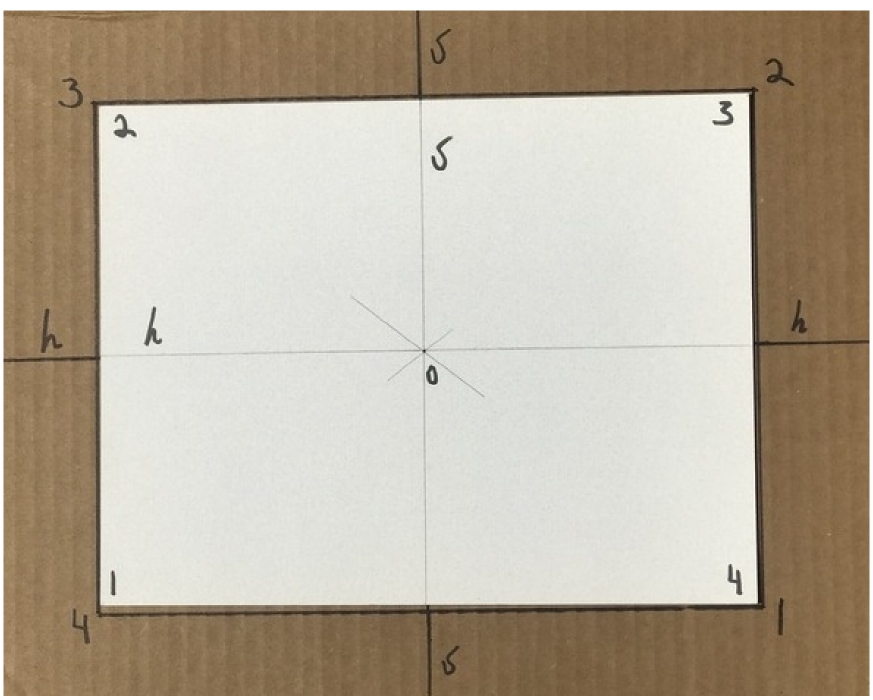
Keep experimenting.

If we rotate the paper rectangle about its center O by a bit more than π radians in either direction, will such a rotation bring that paper rectangle into coincidence with the cardboard rectangle?

Not at all (Figure 5.3.14):



Only when we rotate the paper rectangle about its center O by $2 \cdot \pi$ radians, will it coincide with the cardboard rectangle again (Figure 5.3.15):



But such a rotation of a rectangle about its center by 2π radians or 360° in particular, or by $\pm 2\pi$ radians or $\pm 360^\circ$ in general, is equivalent to *no rotation* at all, since in one non-stop or *continuous* motion we started from and came back to the initial state of the rectangle in which the names of the vertices of both the fixed and the moving rectangles match.

Moreover, we can say the same thing about the rotation of a rectangle about its center by *any integral multiple of* 2π radians.

We, thus, conclude that no other in-plane rotations will make the paper rectangle coincide with the cardboard rectangle.

Thus, the only (non trivial) rotations that *will* bring a rectangle into coincidence with itself will be the rotations about the rectangle's center by half-turns or π radians in either direction.

Hence, formally, any rotation of a rectangle about its center from the following infinite set A:

$$A = \{\pi + 2\pi k,\ k = 0, \pm 1, \pm 2, \pm 3, \ldots\}$$

will bring a rectangle into coincidence with itself.

In order to elaborate on the above notation, if in the above set A we pick, say, $k = -1$ then the rotation of a rectangle about its center by $\pi - 2\pi = -\pi$ radians, or the half-turn of the said rectangle about its center in the *negative* or *clockwise* direction, will bring that rectangle into coincidence with itself.

If in the above set A we pick, say, $k = 3$ then the rotation of a rectangle about its center by $\pi + 6\pi = 7\pi$ radians, or seven half-turns of the said rectangle about its center in the *positive* or *counterclockwise* direction, will bring that rectangle into coincidence with itself and so on.

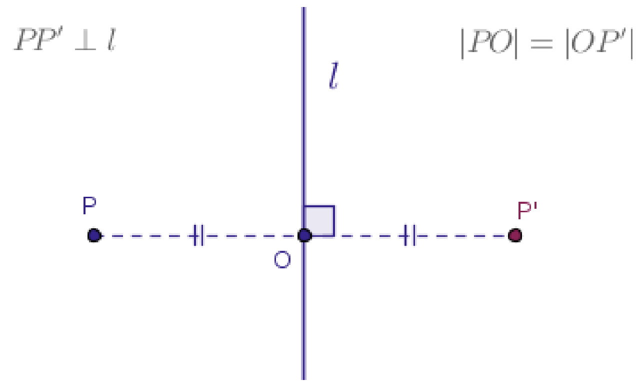
Is that it?

Are there any *other* transformations of the plane that can bring a rectangle in that plane into coincidence with itself?

Upon further reflection we realize that a transformation known as *a line symmetry* or *a reflection in a straight line* will also bring a rectangle into coincidence with itself.

We remind our readers that a reflection of the plane, p, in a straight line, l, is a transformation of the plane, p, into itself under which every point P of the plane p is taken into its image point P' such that the line segment PP' 1) is perpendicular to the straight line l and 2) is cut by l exactly in half.

From the above definition of reflection of the plane in a straight line it follows that the straight line that passes through the points P and P' intersects the straight line l under the right angle at the point O in such a way that the distance from the point P to the point O is equal to the distance from the point O to the point P' (Figure 5.3.16):

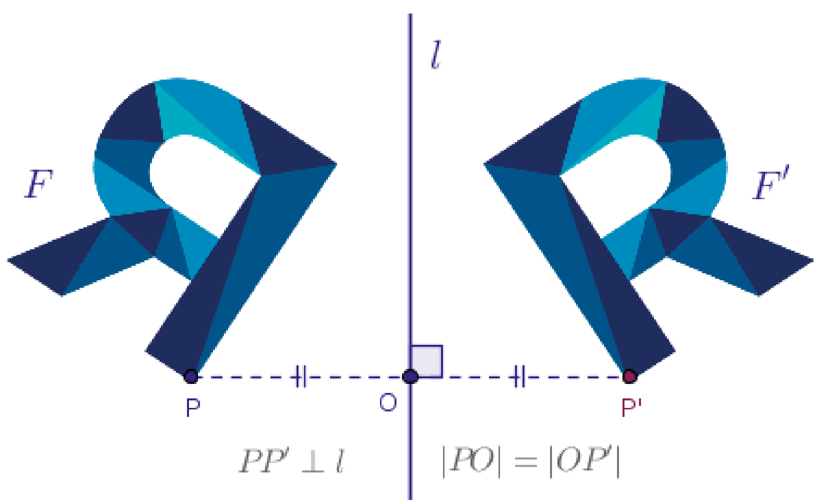


Moreover, from the above definition of reflection of the plane in a straight line it follows that if the point P' is the image of the point P under that transformation then, conversely, the point P is the image of the point P' under the same transformation and, in general, it is said that the points P and P' in that case are the reflective images of each other.

As an unofficial exercise, we suggest that our readers try to construct a reflective image of a given point P in a given straight line l using Euclidean tools, a straightedge and a compass, alone.

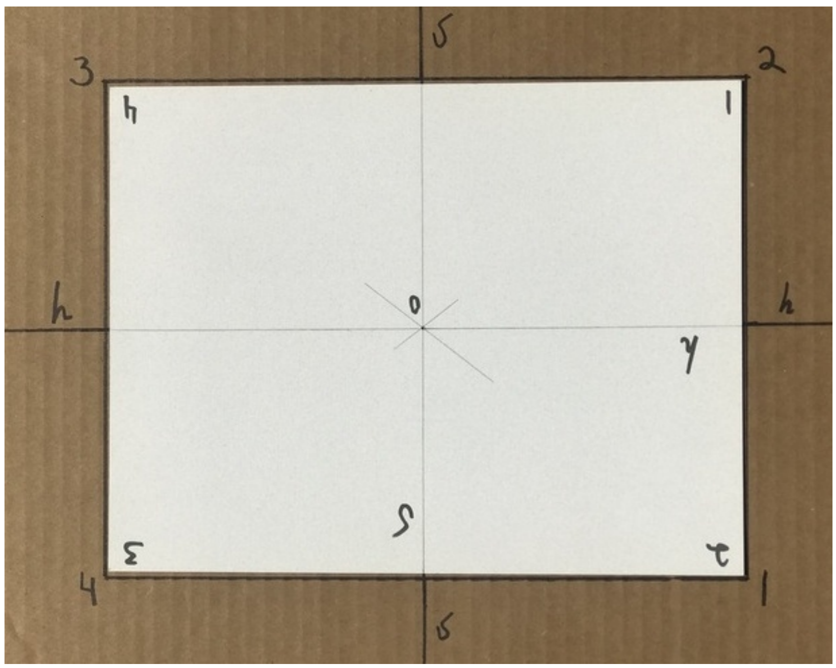
For curious travelers we remark in passing that the plane can also be reflected into itself with respect to other curves, such as a circle, an ellipse, a hyperbola, a parabola and so on.

In order to construct a reflective image F' of an arbitrary planar figure F, it is sufficient to construct a reflective image of *every point of* F (Figure 5.3.17):



Lastly, under a reflection of the plane in a straight line, l, the only points of that plane that remain fixed are the points that make up the said straight line, l, while all other points of the plane are moved in a non trivial fashion.
Consequently, if we reflect our rectangle in the straight line v then such a motion will bring the rectangle into coincide with itself.

In our physical model we simply separate the paper rectangle from the cardboard, lift that paper rectangle up and flip it or rotate it about the straight line v by π radians in either direction and then we place the flipped over paper rectangle back on top of the cardboard rectangle ((Figure 5.3.18):



Such *a flip* about or such *a reflection* of the rectangle in the straight line v leaves the rectangle's long axis of symmetry, h, and the rectangle's short axis of symmetry, v, intact or fixed but shuffles the names of its vertices in some specific way.

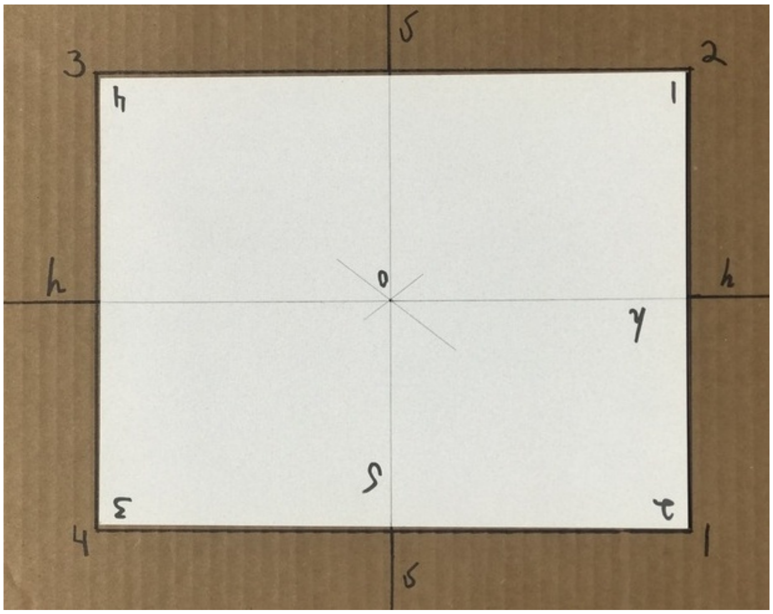
Namely. The reflection of the rectangle in the straight line v takes:

- the vertex 1 into the vertex 4: $1 \to 4$
- the vertex 2 into the vertex 3: $2 \to 3$
- the vertex 3 into the vertex 2: $3 \to 2$
- the vertex 4 into the vertex 1: $4 \to 1$

but that shuffle does not change the fact that the rectangle now coincides with itself shape-wise.

Likewise, if we reflect our rectangle in the straight line h then such a motion will also bring the rectangle into coincide with itself.

In our physical model we simply unbuckle the paper rectangle from the cardboard, lift that paper rectangle up and flip or rotate it about the straight line h by $\pm\pi$ radians and then we place the flipped paper rectangle back on top of the cardboard rectangle (Figure 5.3.19):



Again, the reflection of the rectangle in the straight line h leaves the rectangle's axes of symmetry h and v fixed but swaps the rectangle's upper vertices with their lower siblings as follows:

- the vertex 1 is taken into the vertex 2:
$$1 \to 2$$
- the vertex 2 is taken into the vertex 1:
$$2 \to 1$$
- the vertex 3 is taken into the vertex 4:
$$3 \to 4$$
- the vertex 4 into the vertex 3:
$$4 \to 3$$

but that shuffle does not change the fact that the rectangle now coincides with itself shape-wise.

At this point we are done - no other rigid motions of the plane will take a rectangle into coincidence with itself other than the following ones:

- no motion at all (1)
- any rotation of a rectangle about its center from the infinite set $A = \{\pi + 2\pi k,\ k = 0, \pm1, \pm2, \pm3, ...\}$ (2)
- the reflection of a rectangle in its short axis of symmetry, v in our experiment, (3)

- the reflection of a rectangle in its long axis of symmetry, h in our experiment, (4)

We will be referring to the above list as *the set of the congruence motions of a rectangle*.

Technically, the congruence motion of a rectangle number 1 can be rephrased in a way that is similar to the shape of the respective congruence motion number 2:

any rotation of a rectangle about its center from the infinite set $E = \{0 + 2\pi k,\ k = 0, \pm 1, \pm 2, \pm 3, \ldots\}$ (1)

but we will leave it the way it is shown in our set for now.

For the sake of brevity, after the official definition is given and after enough context that avoids any potential ambiguity is established, it is customary to refer to the above second congruence motion of a rectangle as just *the rotation of a rectangle about its center by* $\pm\pi$ *radians* or, even simpler, *the rotation of a rectangle about its center by* π *radians*.

In our experiment we gave *the short* axis of symmetry of a rectangle a name of v, hinting that it is a **v**ertical, and we gave *the long* axis of symmetry of a rectangle a name of h, hinting that it is a **h**orizontal because we chose to fix this specific orientation of a rectangle.

However, we would like to stress one more time the fact that we introduced these designations purely for pedagogical and illustrative purposes - congruence motions do not care in the least about the initial position of a body and about how the body's parts are named.

But once we fix an initial position of a body at hand and once we name its parts, then:

- such an initial position becomes *and stays fixed* and
- the relative positions of the *named parts* of the moving body are also locked in

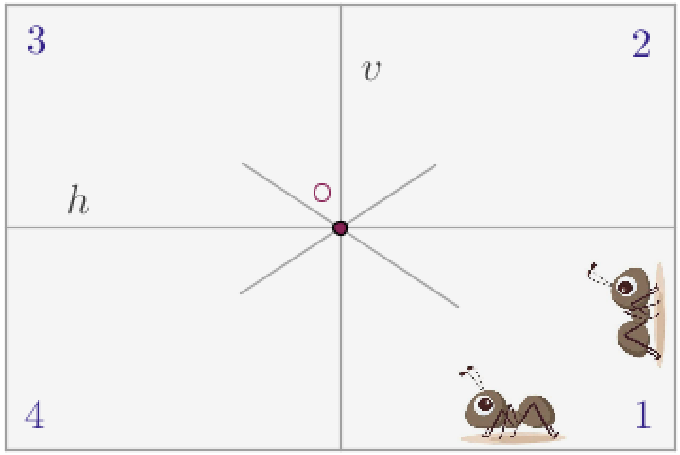
all the way through the experiment in its entirety and at all times.

In order to elaborate on the second bullet above, imagine that an ant is crawling along the perimeter of that rectangle in either direction starting from the rectangle's vertex named 1.

In general, the vertices of a rectangle can be named in any which way.

However.

Once we name the vertices of such a rectangle in a particular way, as is shown in Figure 5.3.2, for example, then no matter which rigid congruence motion of the rectangle we execute, the ant will *always* see the vertex named 2 or the vertex named 4 *before* (s)he sees the vertex named 3 and the short and the long axes of the rectangle's symmetry, v and h respectively, will always remain perpendicular to each other and will always remain parallel to the rectangle's corresponding sides and so on (Figure 5.3.20):



In other words, *the relative* positions of the named vertices and of the named axes of symmetry of the rectangle cannot and will not change under no congruence motion of that rectangle.

Hence, and that is a mighty important *hence*, the above set of the congruence motions of a rectangle is universal in the following sense.

There are infinitely many rectangles that are not squares and the above set of the congruence motions that we established based on one specific rectangle applies to *all* such rectangles!

Put differently, *any* rectangle that is not a square will be brought back into coincidence with itself by the congruence motions from the above set *regardless of its exact shape, size and orientation*.

Show Time

Neither a rectangle with its vertices nor the rigid motions that take that rectangle into coincidence with itself are a group or form a group.

Rather, it is *the (unordered) pair* of the following, two, entities:

- the collection of the four congruence motions of a rectangle listed above and
- the operation of following one such congruence motion by another, and then by another, and then by another and so on for a finite number of times
    - with the above *followed by* operation having the property that two consecutive such operations that act on any three congruence motions of a rectangle produce the same result regardless of the order in which these two operations are carried out

that forms the so-called *group of rigid symmetries of a rectangle*, which also happens to be a group that is essentially the same as the Klein four-group.

In this particular context the phrase *rigid symmetry of an object* should be understood in the sense of self-coincidence and the respective congruence motions of the object at hand.

Thus far, we have worked with these three groups:

- the duplex toggle light switch group
- the tire rotations group and now
- the group of rigid symmetries of a rectangle

that are *descriptively* starkly different but *structurally*, we are told, are essentially the same.

We already know the grown-up name for this *descriptively-different-but-structurally-essentially-the-same* phenomenon - such groups are *isomorphic* to each other.

We do not yet know what exactly does it mean for two groups to be isomorphic, at some point of our journey we will hone in on the group-theoretic criterion of *a group isomorphism*, but in an exercise that is upcoming very soon we will already take our first stab at it.

Now that we saw congruence motions in action, we note that a number of interesting and useful things can be done with them.

As one example, the idea of self coincidence and congruence motions extends naturally from 2-space into 3-space: it is possible to study the explained above rigid symmetries of solid objects such as tetrahedrons, cubes, octahedrons, dodecahedrons and the likes. We will do exactly that in the **Group Homomorphisms and Isomorphisms** chapter.

As another example, it is perfectly fine to study not all the possible congruence motions of a given object but rather limit the menu of their choices based on a neat criterion.

One such straightforward criterion can be the selection of the congruence motions of only *one specific type* - we, for example, can study only the *rotations* of a figure and no other types of rigid motions.

As a matter of fact, we will do just that when we will be introducing the notion of *a subgroup* of a group.

In order to solidify our understanding of congruence motions, we will revisit them in the next discussion in which we will study the square or contra dancing group.

However useful this tactical maneuvering with congruence motions may be, it pales in comparison with the depth and the scope of the recent mathematical discovery into which such congruence motions organically dovetail.

Namely.

In the early 2000s group theorists have completed the classification of the so-called *finite simple groups* which happen to be the atoms or the building blocks from which all other finite groups are made and despite the deceptive word *simple* that made it into the name of this type of groups, these groups are of immense depth, variety and beauty.

While we do not yet possess enough group-theoretic knowledge to fully grasp the significance of the classification of finite simple groups, let the above fact be an inspiration and a motivation to study the subject matter further.

Exercise 5.3.1: using the above duplex light switch group discussion as the guide, verify that all the basic properties of a group hold in the case of the group of rigid symmetries of a rectangle.

Solution: is expected to be generated by our readers on their own.

Verify the Rules Constancy and the Determinism property of the group of symmetries of a rectangle.

What are the group elements in this case?

What is the *inaction* or the *do nothing* element of this group? How many such elements does this group have?

Can every congruence motion of a rectangle be undone?

That is, can every congruence motion of a rectangle be reversed via a congruence motion from the set of the four defined above?

For example, the rotation of a rectangle about its center by π radians, counterclockwise, followed by exactly the same rotation will produce what result?

For each congruence motion of a rectangle write down the congruence motion that reverses it. Is such a reversing motion unique or there are others? Do not forget about the *no motions* motion.

How many congruence motions of a rectangle does the *followed by* operation act on?

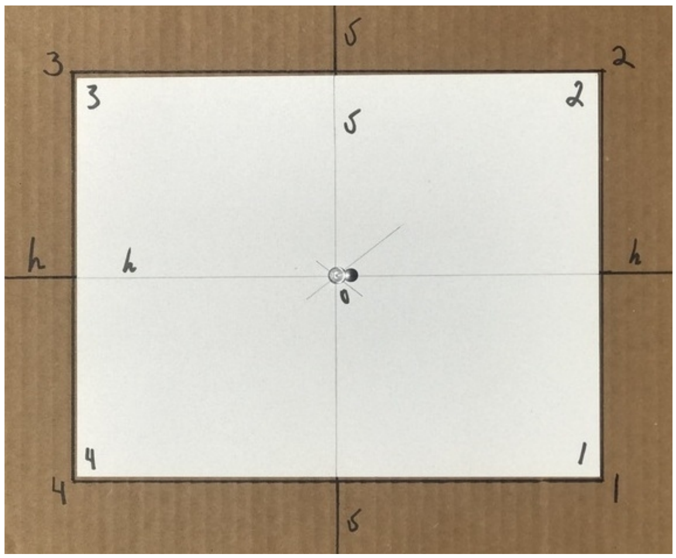
How many ordered pairs of such congruence motions exist in this group?

List them all.

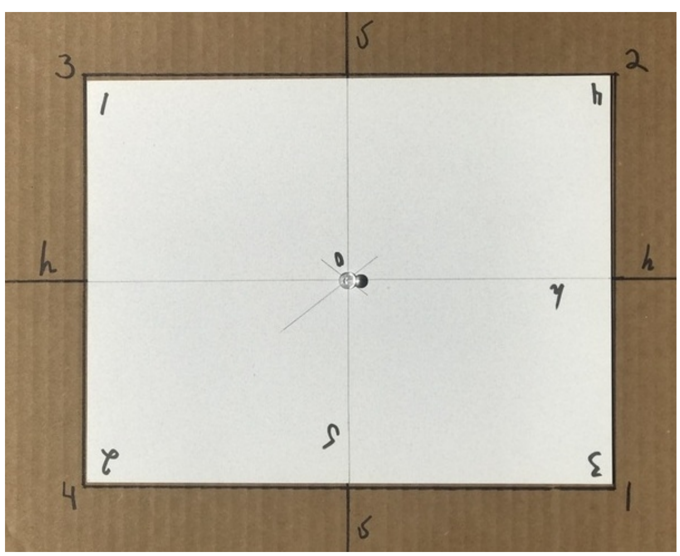
Verify that *any* two congruence motions of a rectangle glued together with a single *followed by* operation are equivalent to a single congruence motion of that rectangle from the set of four.

Write your findings down for each such pair.

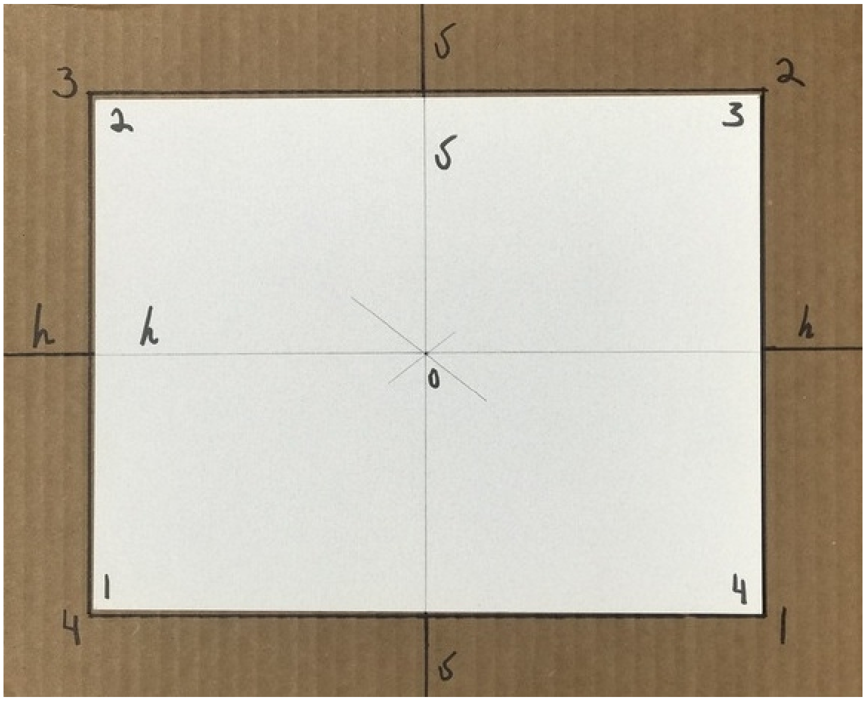
For example, assume that from the initial position shown in Figure 5.3.9 above:



our rectangle is rotated by π radians about its center, counterclockwise, into the intermediate state shown in Figure 5.3.13 earlier:



and then, from that intermediate state of the rectangle shown in Figure 5.3.13, the rectangle is reflected in its long axis of symmetry h that is *drawn on the stationary cardboard workbench* into the state shown below (Figure 5.3.21):



What single congruence motion of a rectangle will take the rectangle from the state shown in Figure 5.3.9 directly into the state shown in Figure 5.3.21 above?

By scanning the set of the congruence motions of a rectangle, we see that the congruence motion number 3 or the reflection of the rectangle in its short axis of symmetry, v, will take the rectangle from the state shown in Figure 5.3.9 directly into the state shown in Figure 5.3.21, because that reflection leaves the axes of symmetry of our rectangle fixed, while swapping the (logically) top and bottom *right* vertices of that rectangle with their (logically) top and bottom *left* siblings.

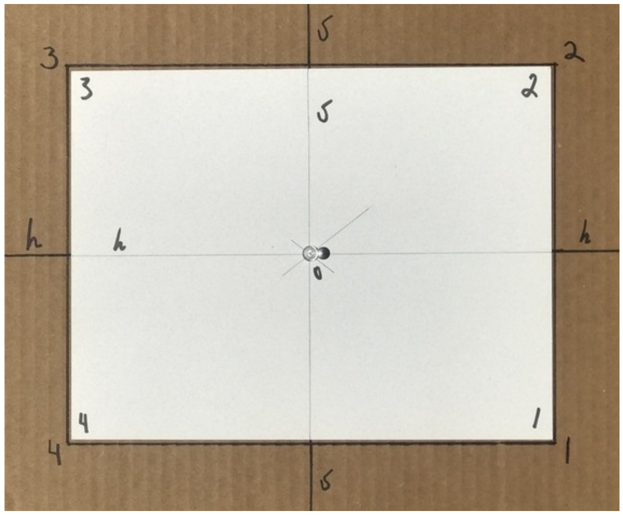
Thus, we jot down:

- the rotation of the rectangle about its center by π radians counterclockwise followed by the reflection of the rectangle in its long axis of symmetry, h, is equivalent to reflecting the rectangle in its short axis of symmetry, v

or something to that extent and so on.

Verify that two consecutive *followed by* operations that act on any three congruence motions of a rectangle do produce the same result regardless of the order in which these two operations are carried out.

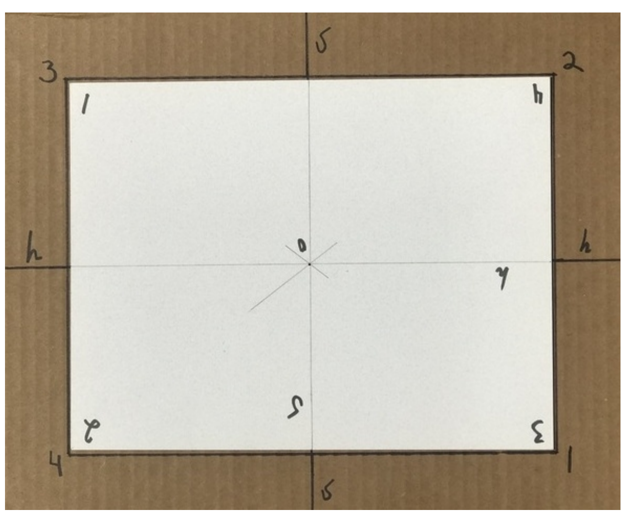
For example, if we follow the above two consecutive congruence motions of rotation and reflection in h by the reflection of the rectangle in its short axis of symmetry, v, then we will readily see that that, third, congruence motion will take our rectangle into its original state already depicted in Figure 5.3.9:



and we, thus, know the result of these two consecutive *followed by* operations:

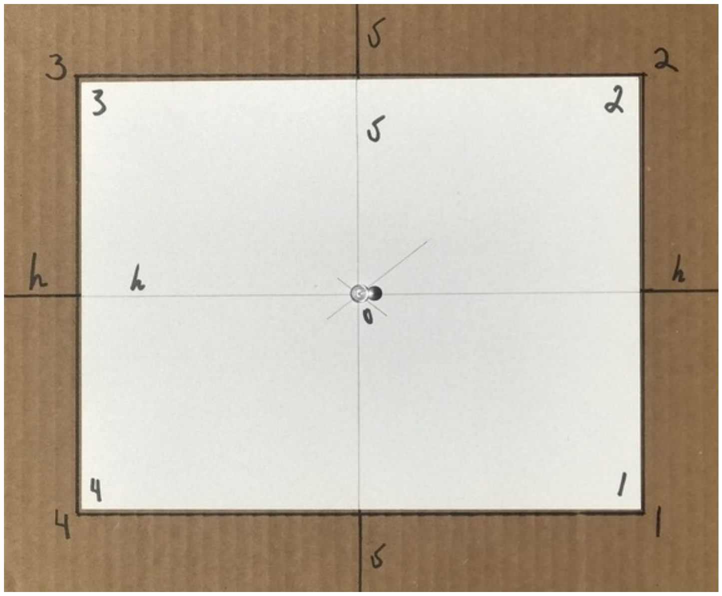
- the rotation by π radians
  - followed by
- the reflection in h
  - followed by
- the reflection in v

But now, alternatively, starting from the original state of the rectangle shown in Figure 5.3.9, we right away reflect that rectangle first in h and then - in v (Figure 5.3.22):



thus, executing the above second *followed by* operation first.

Next, we execute the above first *followed by* operation, rotating the rectangle shown in Figure 5.3.22 above about its center O by π radians, counterclockwise (Figure 5.3.9):



bringing our rectangle into the same final state as the state depicted in Figure 5.3.9 above.

Thus, we just demonstrated the fact that:

two consecutive "followed by" operations that act on any three congruence motions of a rectangle produce the same result regardless of the order in which these two operations are carried out

and so on.

Do note carefully that in the above experiment we did change the order of the followed by *operations* but we did not change the order of the *elements* that these operations acted on - geometrically or top-to-bottom, the order of the specific congruence motions of a rectangle, or the group elements, was fixed and remained the same both times.

We would also like to remind our readers that even though in the case of a rectangle its stationary axes of symmetry did coincide with the axes of symmetry drawn on the moving rectangle, in the upcoming exercises that will employ other geometric figures, such as an equilateral triangle that is not an isosceles triangle or a square, this will no longer be the case.

Consequently, it is best to always, carefully, reflect the given shape in the fixed axes of its symmetry that are drawn on *the stationary cardboard workbench* - otherwise it is way too easy to generate the results that are incorrect.

Based on these incorrect results we will be constructing the so-called *multiplication tables* of the corresponding groups and these tables will make no sense and will be incorrect as well. □

So far we have been studying the congruence motions of a rectangle by looking only at that rectangle's initial and final states and generating the corresponding pairings of the rectangle's vertices across these two states only.

But when we glue several congruence motions into a lengthy string of consecutive actions many beautiful and interesting things happen *in between* the rectangle's initial and final states.

In the next exercise we will slow down a lot by zooming in on exactly what is going on with the intermediate states of a rectangle as it is trawled through a multitude of consecutive congruence motions.

Exercise 5.3.2: putting your physical model of a rigid rectangle in the initial state shown in Figure 5.3.9, determine where will the following *sequence* of congruence motions of a rectangle take its vertices:

- the rotation of the rectangle by πradians about its center counterclockwise
    - followed by
- the reflection of the rectangle in its long axis of symmetry, h
    - followed by
- the rotation of the rectangle by π radians about its center counterclockwise?

including *all* (!) of the intermediate states of the said rectangle.

Write your answers in the shape of the parenthesized two-row table that we already saw in the **Bijection** discussion (Figure 4.3.12):

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

In such a table's top row write down the initial state of the rectangle's vertices as our proverbial ant sees them crawling along the rectangle's perimeter counterclockwise starting from the vertex named 1.

In such a table's bottom row write down the names of the rectangle's vertices into which each named vertex from the top row directly above was taken by that *individual* congruence motion.

Invent various more-than-two-congruence-motions-in-a-row sequences of your own and repeat this exercise with that sequence.

In addition, answer the following questions:

a. Can you discover/think of a rule that can be mechanically applied to all of the above intermediate two-row tables in order to obtain the final two-row table on your own?

b. How can any one two-row table in this exercise be interpreted?

Solution: all in all, we have to execute *three* consecutive congruence motions of a rectangle.

We, thus, have to construct three two-row tables of pairings of the rectangle's vertices - one for each congruence motion of the rectangle.

Motion 1: the rotation by π radians

We already discovered which vertex of a rigid rectangle is taken where by the rotation of that rectangle by $\pm\pi$ radians about its center:

$$1 \to 3$$

$$2 \to 4$$

$$3 \to 1$$

$$4 \to 2$$

Thus, in order to generate the first requested two-row pairings of the vertices of the rectangle in its initial state and after its first rotation by π radians about its center all we have to do is line up the above vertices *vertically* as follows:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \tag{2}$$

and we are done with that motion.

Motion 2: the reflection in h

When we next reflect the rectangle in its long axis of symmetry, h, we begin with the state of the rectangle in which it was left by the previous congruence motion.

But we also already discovered exactly where each named vertex of the rectangle is taken by the reflection in its long axis of symmetry, h:

$$1 \to 2$$

$$2 \to 1$$

$$3 \to 4$$

$$4 \to 3$$

and because the motion of a rectangle is *rigid*, the relative positions of the named vertices of such a rectangle will not change under no congruence motion.

Thus, in order to generate the second, intermediate, two-row pairings of the vertices of the rectangle it is sufficient to take the names of such vertices from the bottom row of the table (**2**) *verbatim*, in the order that these vertices are shown there, and use them as the top row of the second table sought-after:

$$\begin{pmatrix} 3 & 4 & 1 & 2 \\ & & & \end{pmatrix} \tag{3}$$

In order to generate the bottom row of the second table, we simply pair up or merry the names of the vertices in its top row with the names of the vertices from the above *arrowed* or symbolized h-list:

$$\begin{pmatrix} 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix} \tag{4}$$

Such a mechanical construction of the table (**4**) is equivalent to:

- writing the names of the rectangle's vertices from the bottom row of the table (**2**) on the fixed cardboard
- copying the bottom row of the table (**2**) into the top row of the table (**4**)
- applying to the paper rectangle the congruence motion number 4 from the above set, which we already constructed and which we already studied
- reading the resultant pairings off and recording these names of the vertices of the paper rectangle that are lined up against the respective vertices of the cardboard rectangle in the bottom row of the table (**4**)

In other words, in order to construct the second table, which completes the first *followed by* operation, we can either do a mechanical pairing up of the vertices of the rectangle from the discovered list or we can conduct the elaborate copy-and-execute experiment.

The result, however, will be *the same* - recall our ant and recall our emphasis on the fact that the congruence motions of a rectangle work for the rectangle's *arbitrary* initial positions.

Therefore, as a general observation that we will reference in the future, it is perfectly safe to rewrite the table (4) in such a way that the names of the vertices of the rectangle in its top row follow their original order:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \tag{5}$$

which is to say that:

$$\begin{pmatrix} 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

In such a reshuffling of the entries in our two-row table we just want to make sure that we preserve *the pairings of the top and the bottom entries themselves*.

In other words, exactly where within such a table the vertical column of two paired up names is located is *irrelevant*.

The only fact that matters in such a pairing is what name of the rectangle's vertex from the top cell of the column is assigned to what name of the rectangle's vertex in the bottom cell of the column, since such a pairing captures the heart and soul or the essence of this individual congruence motion.

We can think of such vertical columns populated with the respective names of the rectangle's vertices as *dominoes*.

Within the given table each domino can be shuffled into any which position we like.

But the two parts of each such domino, during its shuffling, are *inseparable* (Figure 5.3.23):

$$\begin{pmatrix} 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 3 & 4 & 1 & 2 \end{pmatrix} = \dots$$

How many such (colorful) tables for that congruence motion of a rectangle exist?

Because 4 distinct names of four distinct vertices of a rectangle can be permuted linearly in $4! = 24$ ways (in the table's top row), it follows that it is possible to record or render the pairings of the vertices of a rectangle that corresponds to an individual congruence motion of the said rectangle in the above two-row table format in 24 different but equivalent ways.

Motion 3: the rotation by π radians

When we rotate our rectangle by π radians one last time, then we repeat the above process again. That is. We pick up where we left the rectangle off after the previous congruence motion.

From the corresponding pairings of the rectangle's vertices shown earlier, ignoring the fact that these vertices in the bottom row of the table (**4**) are shuffled in some way, we simply lookup where each such vertex is placed when a rectangle is rotated by π radians about its center and we write the name of that vertex in the bottom row of the table (**6**), see below, directly under its preimage vertex:

$$\begin{pmatrix} 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \end{pmatrix} \tag{6}$$

and we are done.

Again, in general and for future references, for the same reasons explained above, it is also perfectly fine to put the names of the vertices of our rectangle in the top row of the table (**6**) in their original order:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \tag{7}$$

which to say that:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

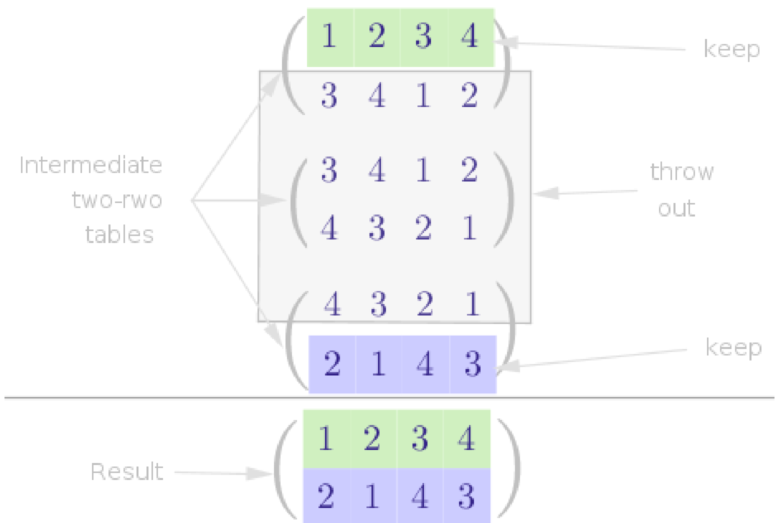
and we also remember that we can construct the table (**6**) by first copying the names of the vertices of the rectangle from the bottom row of the table (**4**) onto the fixed cardboard, by copying the names of the rectangle's vertices from the bottom row of the table (**4**) into the top row of the table (**6**), by applying to the paper rectangle the congruence motion number 2 next and by simply reading off the resultant pairings of the said vertices and recording them in the bottom row of the table (**6**).

We can now answer the question **a**: in order to construct the two-row table that pairs up the names of the vertices of our rectangle in its original and final states, absorbing all the intermediate pairings of the said vertices generated by all the intermediate congruence motions of the said rectangle, stack up all the intermediate tables as follows:

- put the table (**4**) under the table (**2**)
- put the table (**6**) under the table (**4**)

and then from the resultant tower of tables throw out all the intermediate rows, keeping only the top row of the table (**2**) and the bottom row of the table (**6**).

It is that simple (Figure 5.3.24):



Thus, the two-row table that reflects the pairings of the vertices of our rectangle generated by the sequence of the given congruence motions has the following shape:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \tag{8}$$

Verify our computations by simply carrying out the requested sequence of congruence motions *ignoring* all the intermediate pairings of the rectangle's vertices and by simply reading these pairings off of our physical model in its final state:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \tag{9}$$

Yep, our tables (**8**) and (**9**) are a perfect match r*egardless of how the individual vertical columns, or dominoes, are mixed in them*.

Next, we can answer the question **b**: any one two-row table in this exercise, such as the table number (**4**), for example:

$$\begin{pmatrix} 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

can be interpreted as *a bijection of a finite set onto itself.*

Recall that in our last discussion of what a bijection is and what its basic properties are we operated in a rather abstraction fashion.

With this example, however, we can put a name to a face, so to speak, and inhale a certain amount of a concrete meaning in this particular type of mappings.

In any one two-row table in this exercise, such a table's top row represents a set of distinct names of a rectangle's distinct vertices and such a table's bottom row also represents a set of distinct names of the vertices of *the same* rectangle.

Thus, in the bottom row of our tables:

- there are no extra names of some previously forgotten or popped out of nowhere vertices of a rectangle
- no name of a rectangle's vertex from the table's top row is missing
- there are no duplicate names
- the names of the vertices correspond to one and only one name of a vertex in the table's top row

and we should now be able to develop a reasonably truthful intuitive grasp on the bijection mechanism because in this exercise we have a tangible and visual model of it. □

Exercise 5.3.3: using our discussion of the congruence motions of a rectangle and the solution of the exercises **5.3.1** and **5.3.2** as a guide, correlate each element of the group of symmetries of a rectangle with the respective elements of the tire rotations group.

Solution: of this exercise should cause no problems because we already did much of the required legwork.

We know that there are four elements in each group of interest.

I. Taking it from the top, we quickly see that to the congruence motion of a rectangle number 1, no motions, there corresponds the tire rotation number 1, no tire rotations, and conversely.

To the tire rotation number 1, no tire rotations, there corresponds the congruence motion of a rectangle number 1, no motions.

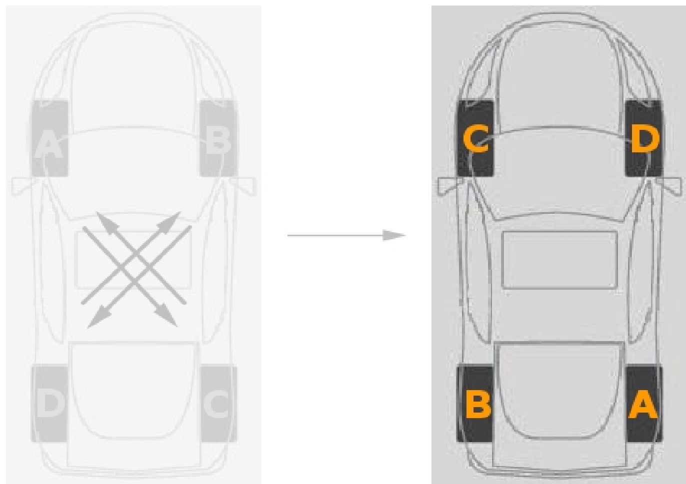
Since it seems that there is one and only one *do nothing* element in each group of interest, it follows that the above correlation establishes a connection not from an element of one group to the element*s*, plural, of another but, rather, the above correlation establishes a connection between one element of one group and exactly one element of another group.

Check.

II. The congruence motion of a rectangle number 2, the rotation of a rectangle about its center by $\pm\pi$ radians, simply throws each vertex of the said rectangle across the respective diagonal of that rectangle.

Hence.

To the congruence motion of a rectangle number 2, the rotation of a rectangle about its center by $\pm\pi$ radians, there corresponds the tire rotation number 4, the cross-over tire rotation (Figure 5.2.7):

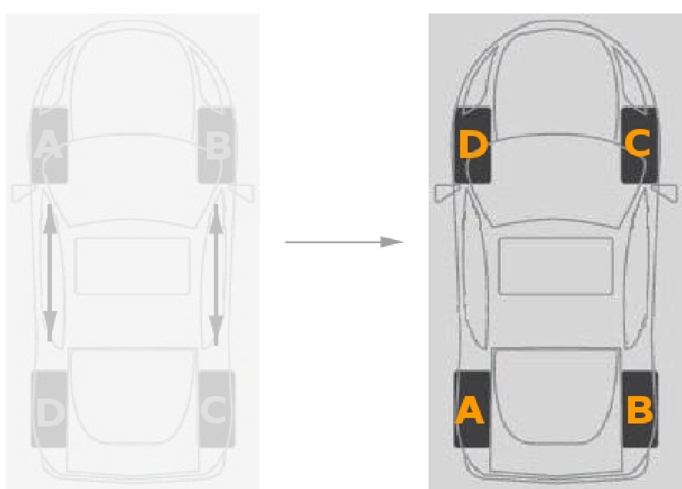


and conversely.

To the tire rotation number 4, the cross-over tire rotation, there corresponds the congruence motion of a rectangle number 2, the rotation of a rectangle about its center by $\pm\pi$ radians.

Check.

III. It should now be perfectly clear that to the congruence motion of a rectangle number 3, the reflection of the rectangle in its *short* axis of symmetry, there corresponds the tire rotation number 2, the front-to-back tire rotation (Figure 5.2.3):

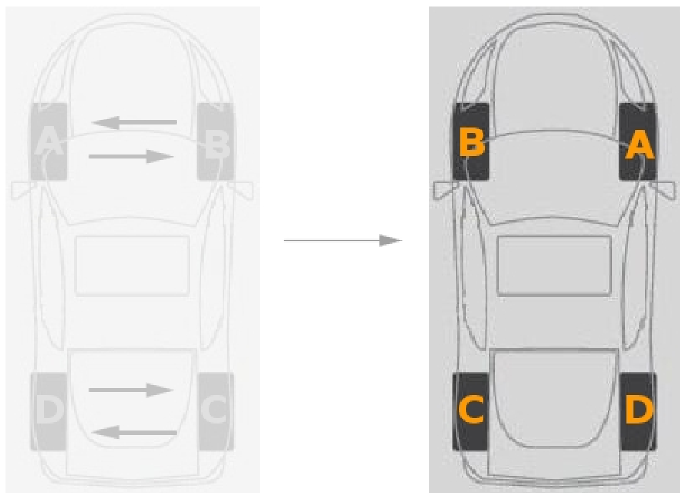


and conversely.

To the tire rotation number 2, the front-to-back tire rotation, there corresponds the congruence motion of a rectangle number 3, the reflection of the rectangle in its *short* axis of symmetry.

Check.

IV. It should also be perfectly clear that to the congruence motion of a rectangle number 4, the reflection of the rectangle in its *long* axis of symmetry, there corresponds the tire rotation number 3, the side-to-side tire rotation (Figure 5.2.5):



and conversely.

To the tire rotation number 3, the side-to-side tire rotation, there corresponds the congruence motion of a rectangle number 4, the reflection of the rectangle in its *long* axis of symmetry.

Check. □

Exercise 5.3.4*:

- pick any two congruence motions of a rectangle
- glue them together with the respective *followed by* operation
- obtain its result and
- locate the tire rotation that that resultant congruence motion of a rectangle corresponds to
  - then
- locate the two tire rotations that correspond to the two congruence motions of a rectangle that you picked earlier
- glue these respective tire rotations together with the corresponding f*ollowed by* operation and

- jot down the result

What can you say about the two resultant tire rotations thus obtained?

Solution: of this exercise should be generated just like we play the game of checkers - hop, hop, hop.

Namely.

Say, we pick:

- the rotation of a rectangle about its center by π radians and
- the reflection of a rectangle in its *long* axis of symmetry, h

as two arbitrary congruence motions of a rectangle.

From the solution of the **Exercise 5.3.1** we know that:

the rotation of a rectangle about its center by π radians counterclockwise followed by the reflection of the rectangle in its long axis of symmetry, h, is equivalent to reflecting the rectangle in its short axis of symmetry, v

and, thus, the reflection of a rectangle in its short axis of symmetry, v, constitutes the result of the first experiment.

Next.

From the solution of the **Exercise 5.3.3** we know that:

to the reflection of the rectangle in its short axis of symmetry, v, there corresponds the front-to-back tire rotation

Hence, that front-to-back tire rotation constitutes *the first resultant tire rotation.*

Next.

From the solution of the **Exercise 5.3.3** we know that:

- to the rotation of a rectangle about its center by $\pm\pi$ radians there corresponds the cross-over tire rotation and that
- to the reflection of the rectangle in its *long* axis of symmetry, h, here corresponds the side-to-side tire rotation

From the solution of the **Exercise 5.2.1** we know that:

the cross-over tire rotation followed by the side-to-side tire rotation is equivalent to the front-to-back tire rotation

and, thus, that front-to-back tire rotation constitutes *the second resultant tire rotation*.

Huh.

Would you look at that?

The two resultant tire rotations came out to be exactly the same!

To summarize.

If we agree to call the respective tire rotations *the images* of the congruence motions of a rectangle in the sense established in the **Exercise 5.3.3** then we demonstrated that:

the image of a sequence of any two congruence motions of a rectangle is equal to the sequence of the individual images of these two congruence motions

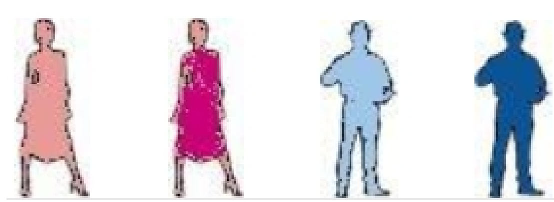
Tuck away this result in a corner of your memory, as we will come back to it in our **Group Homomorphisms and Isomorphisms** chapter. □

It may be difficult to read the conclusion of the **Exercise 5.3.4** in its purely verbal form but once we work our way through the **Difficult, Giant, Leap Forward** chapter, that conclusion will taken on a very short and sweet symbolic form that will be the workhorse of our study of group homomorphisms and isomorphisms.

5.4 I Dance, You Dance, We Dance

Another example of a mathematical group that is decidedly not arithmetic in nature waltzes in from the land of square or contra dancing.

In this experiment we will limit ourselves to just two couples - two ladies, dressed in the shades of pink, and two gentlemen, dressed in the shades of blue (Figure 5.4.1):



On the dance floor these four distinct people will form a certain fixed square most of the time except when they rhythmically, on beat and to the music, trade places in a highly orchestrated way.

Trading places rhythmically, on beat and to the music, in a highly orchestrated way is just another way of saying that our four heroes will be executing the so-called *dance figures* by:

- starting with a square formation
- moving about in a prescribed way and then
- forming the original square again, possibly standing in its different corners

In the tire rotations group we saw that the tires on an average vehicle form *a rectangle* that is not a square.

In the congruence motions of a rectangle group we also dealt with a rectangle and, because every square is a rectangle but not every rectangle is a square, we insisted that we will study the said congruence motions of *not* a square also.

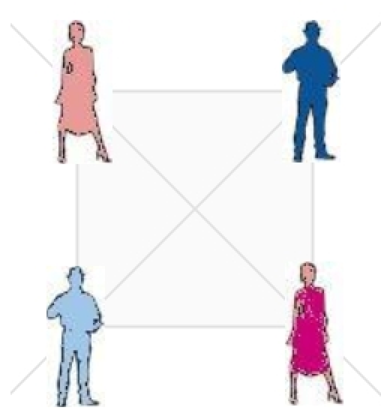
Here, however, we change our stance and demand that the above dancers form not just any good old rectangle but, specifically, *a square*. Let us agree to refer to such a square as *the home square*.

The size and the orientation of the home square anchored to a certain convenient reference, such as the edge of the stage or the theater's curtains or some such, are completely irrelevant.

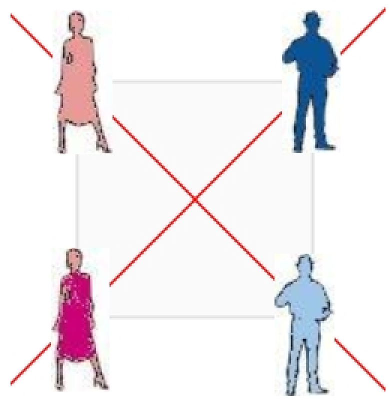
However, in terms of the relative gender mix our participants will obey the following rule at all times regardless of what type of dance figures they perform:

the dancers of the same gender are positioned on the same diagonal of the home square

In order to illustrate the above rule in positive terms, the diagram below shows a legal contra dancing arrangement of the two couples (Figure 5.4.2):



In order to illustrate the above constraint in negative terms, the diagram below shows *an illegal* contra dancing arrangement of the two couples (Figure 5.4.3):



In order to be perfectly clear, the arrangement of the two couples shown in the above Figure 5.4.3 in contra dancing is *not* allowed.

Exercise 5.4.1: explain in your own words, what is the mathematical equivalent of the above *same-gender-same-diagonal* requirement?

Solution: the mathematical equivalent of the above *same-gender-same-diagonal* requirement is the permanency of *the relative positions* of the names of the home square's vertices, once these names are assigned.

If the vertices of a square are named as, say, $ABCD$ then such a naming locks in or fixes the relative positions of these vertices and dovetails into the concept of rigid congruence motions that we already studied.

That is.

Such a locking in guarantees that no matter what specific type of a rigid congruence motion is applied to such a square, the square's vertex named A will always be opposite the square's vertex named C and conversely, and that the square's vertex named B will always be opposite the square's vertex named D and conversely. □

Now imagine that as *dance designers* we marked the home square on the floor of our dance studio with four decals - labels that can be glued to and peeled off a surface easily.

We, next, ponder: hm, how can we entertain an audience by moving any two couples around in such a way that they always obey the *same-gender-same-diagonal* constraint?

Upon some experimentation, that our readers are encouraged to carry out on their own already, we realize that a good amount of entertainment can be provided if we limit the menu of choices to only *swapping the positions* which the dancers occupy in the corners of the home square at the moment, subject to the standing constraint.

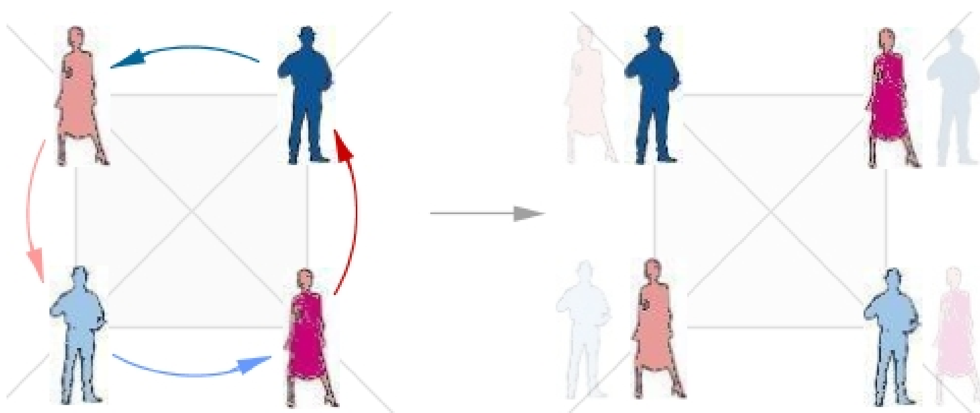
Thus, we come up with the following *eight* different group-theoretic dance figures, which may or may not be the actual figures customarily used in various regions/countries.

1. Dance figure 1: no swaps at all.

In order to execute this figure the dancers, staying at the vertices of the home square, can, for example, jump up in unison.

2. Dance figure 2: circle right *once*.

In order to execute this figure, the dancers, facing the center of the home square, in one continuous motion trace an imaginary, co-centric with the home square, circular arc in the direction in which points their right hand until they reach the very *next for them* vertex of the home square (Figure 5.4.4):



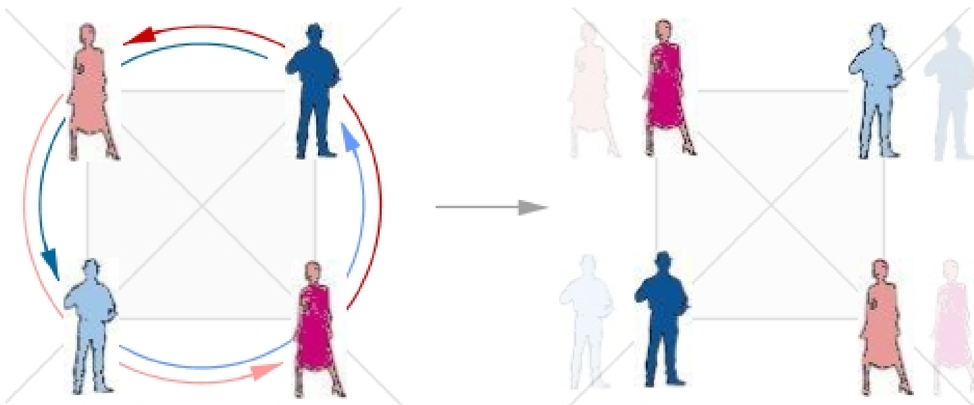
In general, it does not really matter in which specific direction, clockwise or counterclockwise, our dancers execute the above figure.

Solely for the purpose of definitiveness we chose the *counterclockwise* direction for that figure but our readers are free to experiment with the clockwise direction on their own.

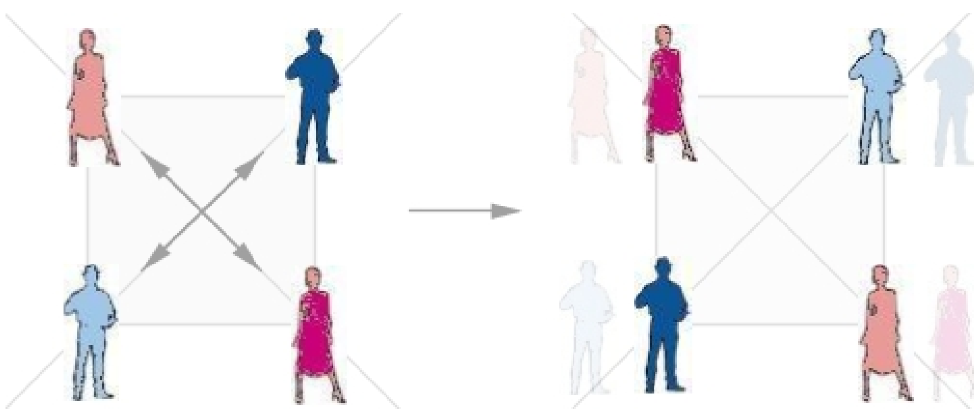
Consequently, as the net effect of executing the dance figure 2, each person above rotates about the center of the home square by one quarter of a complete turn counterclockwise.

3. Dance figure 3: circle right *twice*.

In order to execute this figure the dancers, facing the center of the home square, in one continuous motion trace an imaginary, co-centric with the home square, circular arc in the direction in which points their right hand until they reach the *opposite for them* vertex of the home square (Figure 5.4.5):



We could have designed a dance figure called *right and left through* in which our pairs trace different trajectories, swapping their positions along the respective diagonals of the home square, but wind up in the same places (Figure 5.4.5.1):



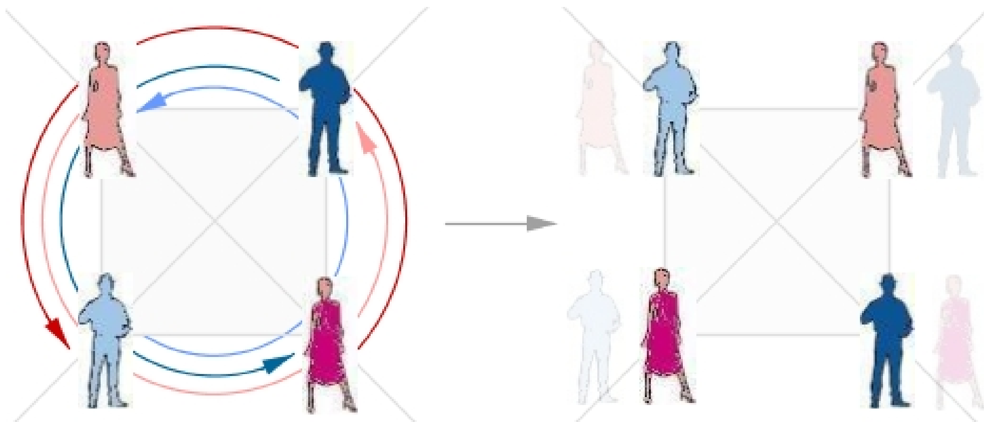
In practice this is actually *an official*, so to speak, name of this particular dance figure.

However, in order to make the solution of the upcoming exercise more tractable, we opted to go with the more suggestive and revealing *circle twice* figure instead.

Either way, as a net effect of executing the dance figure 3, each person rotates about the center of the home square by one half of a complete turn, again, counterclockwise, for definitiveness.

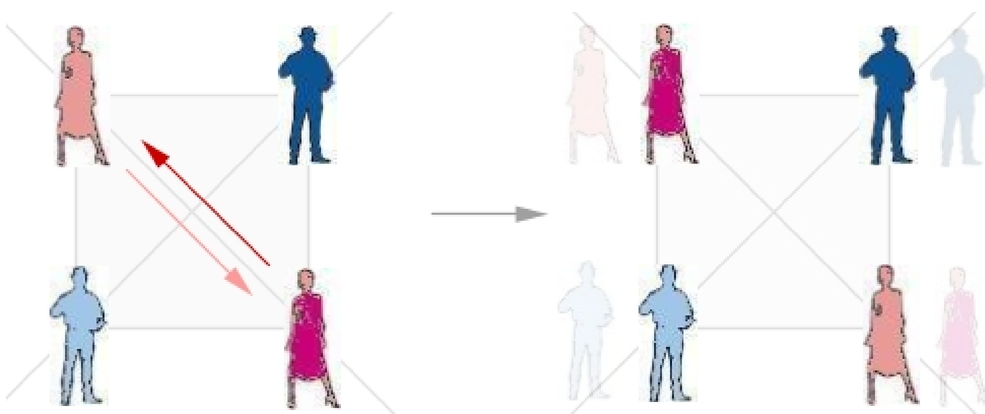
4.Dance figure 4: circle right *thrice*.

In order to execute this figure the dancers, facing the center of the home square, in one continuous motion trace an imaginary, co-centric with the home square, circular arc in the direction in which points their right hand until they reach the vertex of the home square that was immediately to *their left* (Figure 5.4.6):



5. Dance figure 5: ladies chain.

In order to execute this figure *only the ladies* of the dance troupe swap their positions at the vertices of the home square by graciously moving along its corresponding diagonal (Figure 5.4.7):



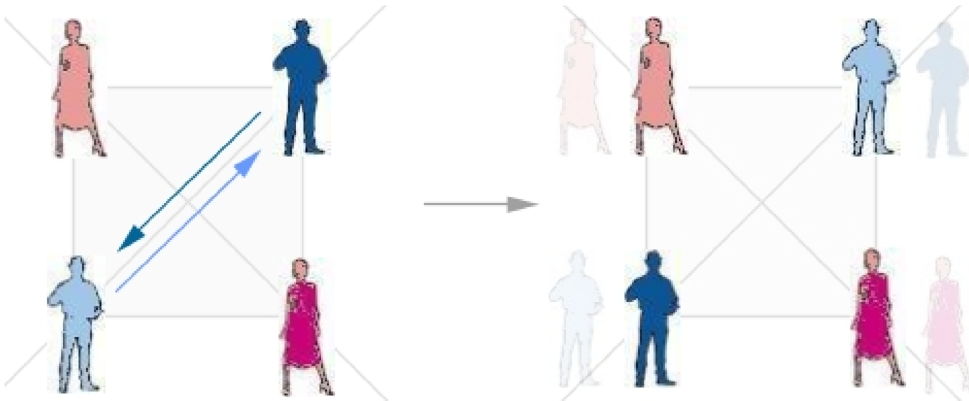
During the execution of this dance figure the gents of the dance troupe essentially stay where they were – they may wave their hands vigorously or move about within the compounds of a small area.

However, as far as the theory of groups is concerned, each gentlemen of the pair remains fixed at the current corner of the home square.

In the next figure the ladies of the dance troupe will do the same.

6. Dance figure 6: gents allemande.

In order to execute this figure *only the gentlemen* swap their positions at the vertices of the home square by graciously moving along its corresponding diagonal (Figure 5.4.8):

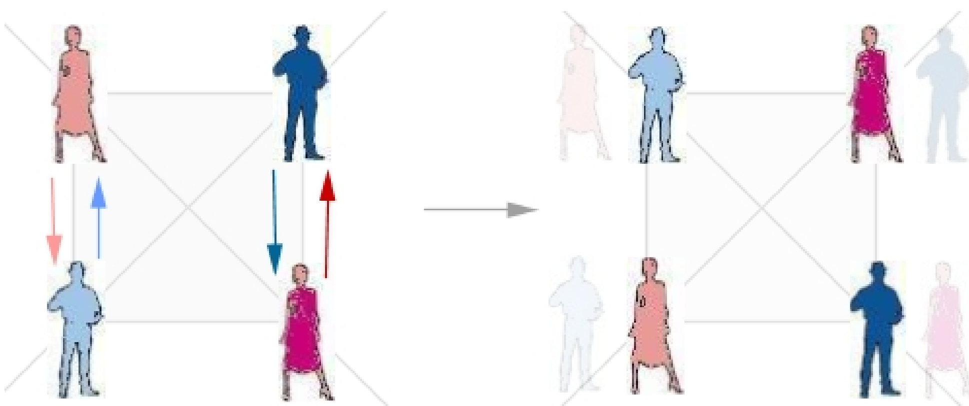


Again, as a result of the execution of this dance figure, ladies stay where they were.

7. Dance figure 7: California twirl.

In order to execute this figure the dancers, facing the center of the home square, swap their positions as follows:

- each lady trades places with the gent who is positioned on her *right* and, equivalently
- each gent trades places with the lady who is positioned on his *left* (Figure 5.4.9):

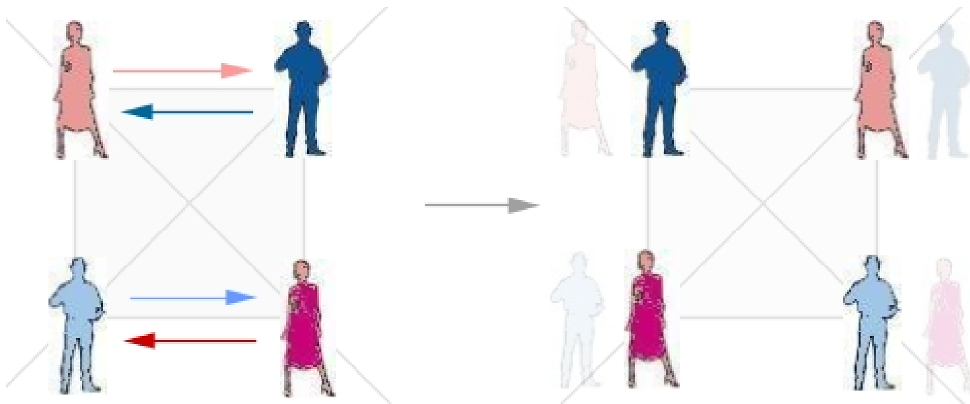


8. Dance figure 8: swing on side.

This dance figure is similar to the California twirl except that with swing on side our dancers pick a different opposite gender partner to trade places with.

Namely, in order to execute this figure, the dancers, facing the center of the home square, swap their positions as follows:

- each lady trades places with the gent who is positioned on her *left* and, equivalently
- each gent trades places with the lady who is positioned on his *right* (Figure 5.4.10):



These are all the contra dance figures that we will define.

Show Time (literally and figuratively)

Neither the dancers themselves nor the dance figures that they execute are a group or form a group.

Rather, it is *the (unordered) pair* of the following, two, entities:

- *the collection* of the eight dance figures listed and numbered above and
- *the operation* of following one such dance figure by another dance figure, and by another dance figure, and by another dance figure and so on for a finite number of times
  - with the above *followed by* operation having the property that two consecutive such operations that act on any three dance figures produce the same result regardless of the order in which these two operations are carried out

that forms a group which, as it turns out, is isomorphic to the the so-called *group of rigid symmetries of a square*, which is an example of a so-called *dihedral group* symbolized as D_4.

Exercise 5.4.2: using the duplex light switch group discussion as the guide, verify that all the basic properties of a group hold in the case of this specific group of contra dance figures also.

Solution: is expected to be generated by our readers on their own.

Verify the Rules Constancy and the Determinism property of the contra dancing group.

What is the *inaction* or the *do nothing* element of this group? How many such elements does this group have?

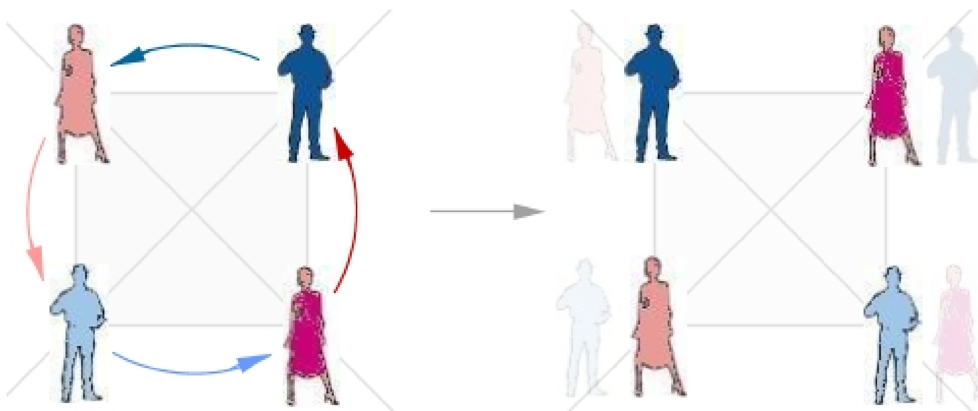
Can every contra dance figure defined above be undone?

That is, can every contra dance figure be reversed via a dance figure from the set of the eight figures defined above?

For each dance figure write down the dance figure that reverses it.

Print/draw the silhouettes shown in this exercise, cut and color them accordingly, roll up your sleeves and go to group-theoretic work.

As a sample solution, the dance figure number 2 puts the dancers into the state shown in Figure 5.4.4:



Ponder, which dance figure from the set of eight executed by the dancers located in the corners of the home square as depicted by the above rightmost diagram will bring them back into the positions depicted by the above *leftmost* diagram?

Clearly, if, starting from their current positions shown in the rightmost diagram in Figure 5.4.4, the dancers will execute the figure number 4 of *circling right thrice*, then they all will be taken to their original corners of the home square as depicted by the leftmost diagram in Figure 5.4.4 and we can write:

the dance figure number 2 "followed by" the dance figure number 4 is equivalent to the dance figure number 1

and so on.

Is such a reversing dance figure unique or there are others? Do not forget about the *no swaps* figure.

Observe that if we are discovering the reversing figures in the order in which these figures are defined above and we already know that the figure number 5 is reversed by the same figure and that the figure number 6 is also reversed by the same figure then can we write down the reversing moves for the figures number 7 and 8 mechanically and without even thinking about it?

No, not really. Since we already know the reversing moves for all the figures numbered 1 through 6, we see that the figures number 7 and 8 never showed up in that list and we know that we are working with *a group*, what we can instantly conclude is that either these figures are reversed by themselves or the figure number 7 is reversed by the figure number 8 and conversely, the figure number 8 is reversed by the figure number 7.

After a straightforward verification, we see that the latter is the case.

Next, think about how many dance figures does the *followed by* operation act on?

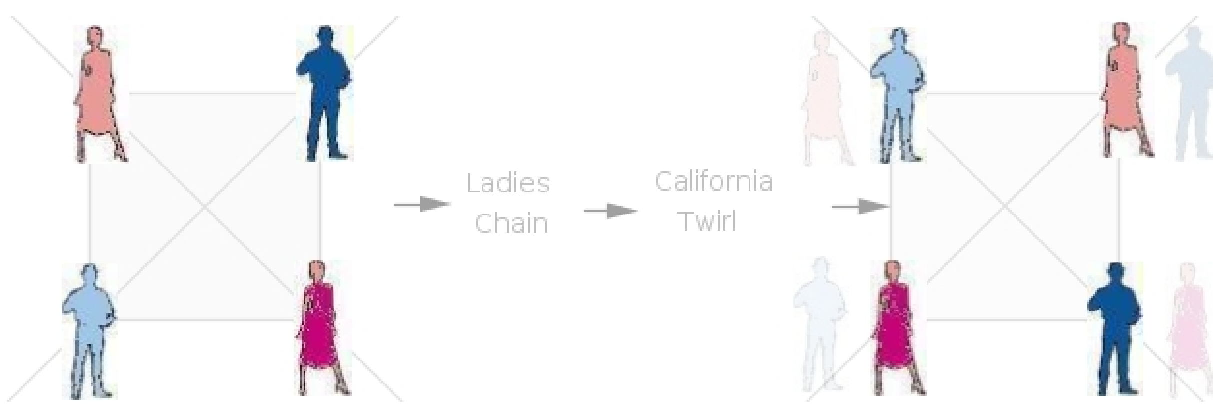
How many ordered pairs of dance figures exist in this group? List them all - yes, this will be a little bit of an effort and that effort should be suggestive of something.

On a separate note, we promised that in this text we will introduce our readers to the way in which the grown-up mathematics is done. Well, it turns out that grown-ups do a huge amount of dirty and messy work that never makes it into the spotlight of the polished academic texts.

However, the cost of the said polish is exactly the dirtiness and the messiness of the throwaway work. Here we get *a tiny* glimpse into such work.

Verify that *any* two dance figures glued together with a single *followed by* operation are equivalent to a single dance figure from the set of eight. Write your findings down for each such pair.

For example, the ladies chain figure followed by the California twirl figure will put the dancers from their initial state shown in Figure 5.4.2 into the following state (Figure 5.4.11):



Let us think, which dance figure from the set of eight will take the dancers from the state shown in Figure 5.4.2 directly into the state depicted in the rightmost diagram above?

Clearly, the dance figure number 4, circle right thrice, will take the dancers from the state shown in Figure 5.4.2 directly into the state depicted in the rightmost diagram of Figure 5.4.11:

the dance figure number 5 followed by the dance figure number 7 is equivalent to the dance figure number 4

and so on.

For the confirmation of the fact that two consecutive *followed by* operations that act on any three dance figures will produce the same result regardless of the order in which these two operations are carried out, study the following sample string of dance figures applied to the initial state of the troupe depicted in Figure 5.4.2:

- ladies chain
  - followed by
- California twirl
  - followed by
- gents allemande

For the first experiment, show that the above sequence of dance figures will take the dancers into the following final state (Figure 5.4.12):

For the second experiment, demonstrate that the alternative order in which the above dance figures are executed, *California twirl* followed by *gents allemande* first and only then the *ladies chain* figure applied to the result of the previous sequence, will take the dancers into the same final state depicted in Figure 5.4.12.

Show all your intermediate states.

Conclude that:

two consecutive "followed by" operations that act on any three dance figures produce the same result regardless of the order in which these two operations are carried out

and so on. □

Exercise 5.4.3: using the discussion of the congruence motions of a rectangle, the solutions of the **Exercises 5.3.1**, **5.3.2**, **5.3.3** and the current discussion of the contra dance group as a guide, generate the corresponding set of congruence motions of *a square* with all the respective diagrams.

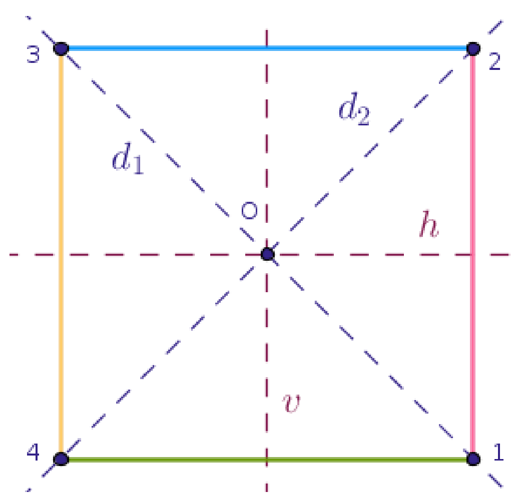
Are these congruence motions of a square a group?

Coupled with which operation the above set of congruence motions of a square becomes a group?

Solution: should cause no difficulties if we replace our distinct dancers with the distinct named vertices of a square and keep in mind the *same-gender-same-diagonal* constraint.

Those readers who still do not see how our dance figures port into the congruence motions of a square are encouraged to make a physical model of a square in exactly the same way in which we made a physical model of a rectangle earlier.

Assuming that we also name the two diagonals of a square as d_1 and d_2 and the other two axes of symmetry of a square as v and h (Figure 5.4.13):



then the following set will be the set of all eight congruence motions of a square.

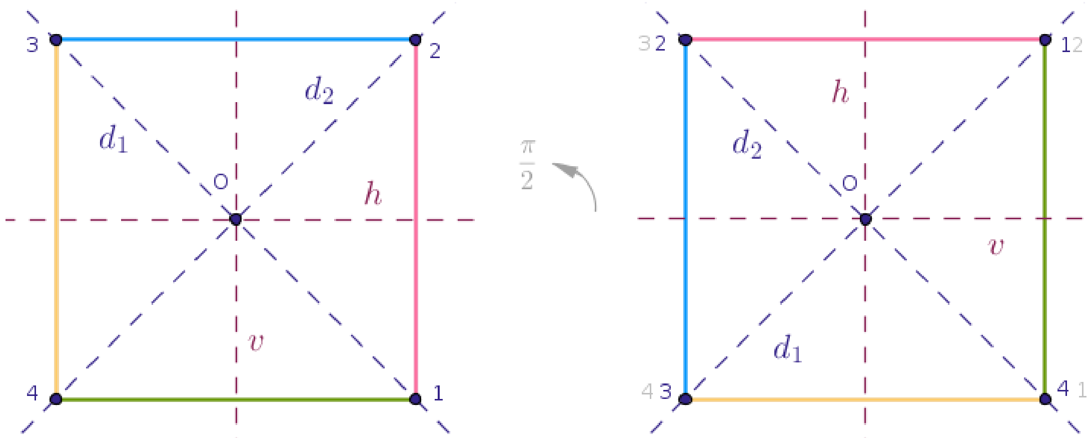
Motion 1: no motion at all or any rotation of a square about its center, O, from the infinite set E such that:

$$E = \{0 + 2\pi k,\ k = 0, \pm 1, \pm 2, \pm 3, \ldots\}$$

Motion 2: any rotation of a square about its center from the infinite set A such that:

$$A = \left\{\frac{\pi}{2} + 2\pi k,\ k = 0, \pm 1, \pm 2, \pm 3, \ldots\right\}$$

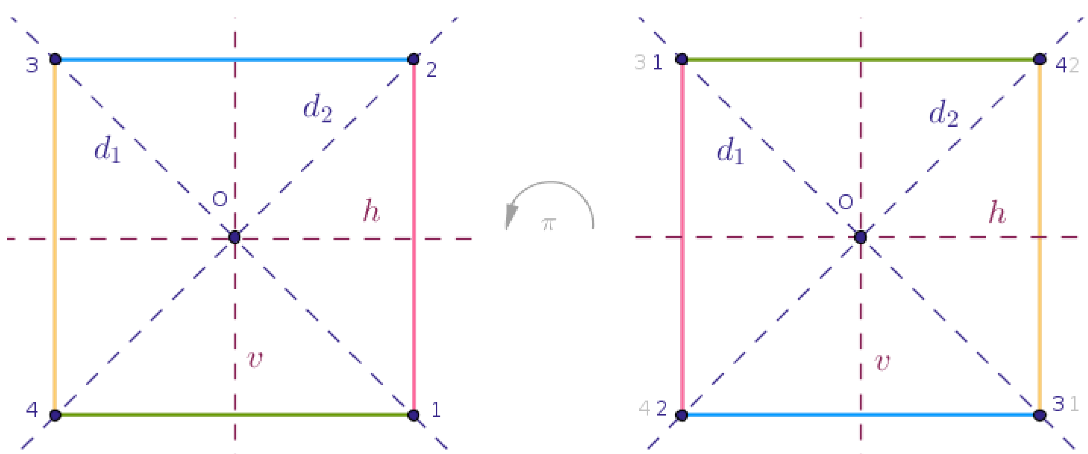
resulting, essentially, in one counterclockwise quarter of a turn of a square (Figure 5.4.14):



Motion 3: any rotation of a square about its center from the infinite set B such that:

$$B = \{\pi + 2\pi k,\ k = 0, \pm 1, \pm 2, \pm 3, \ldots\}$$

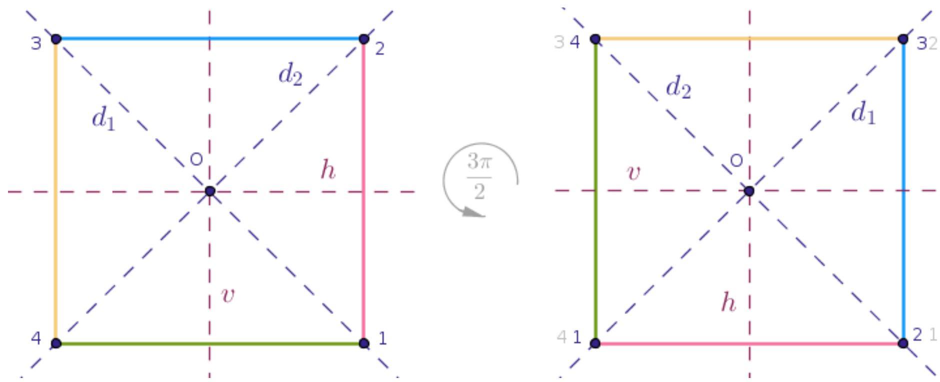
resulting, essentially, in one counterclockwise half-a-turn of a square (Figure 5.4.15):



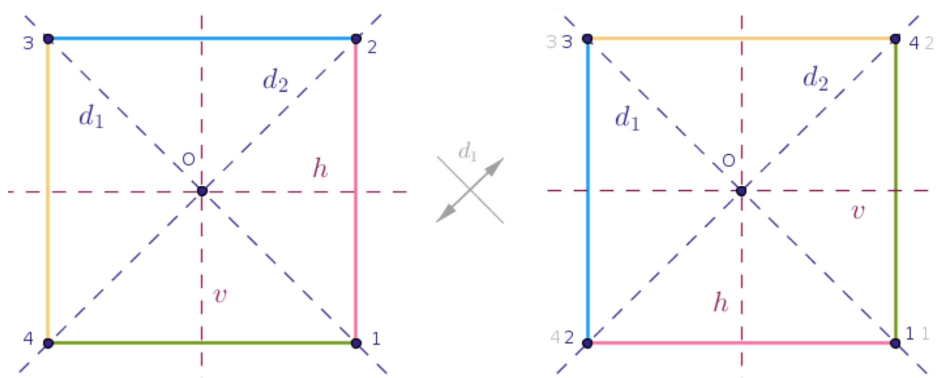
Motion 4: any rotation of a square about its center from the infinite set C such that:

$$C = \left\{ \frac{3\pi}{2} + 2\pi k,\ k = 0, \pm 1, \pm 2, \pm 3, \ldots \right\}$$

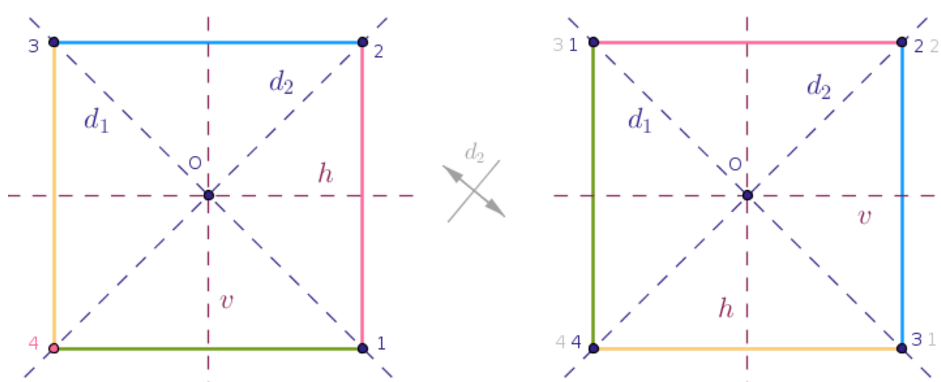
resulting, essentially, in one counterclockwise three-quarters-of-a-turn of a square (Figure 5.4.16):



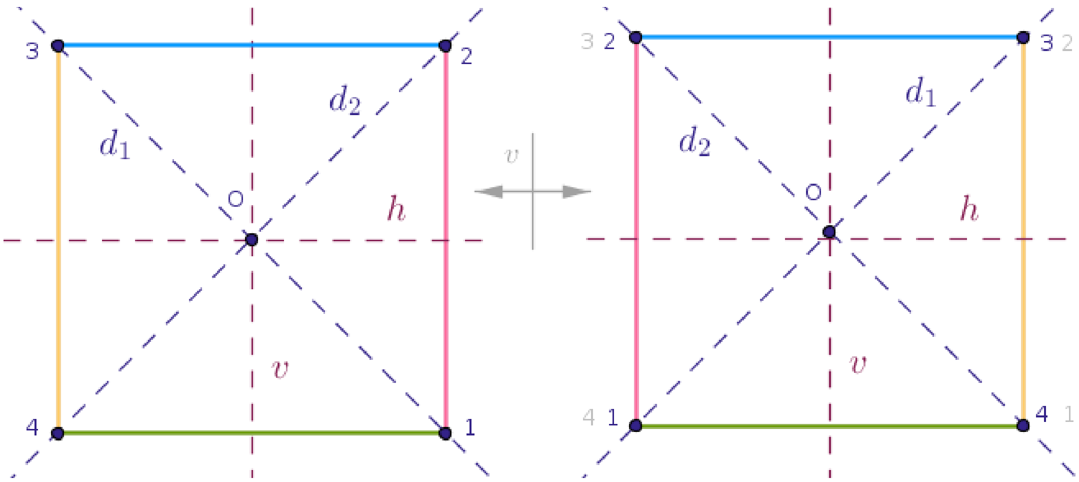
Motion 5: the reflection of a square in one of its diagonals, d_1 (Figure 5.4.17):



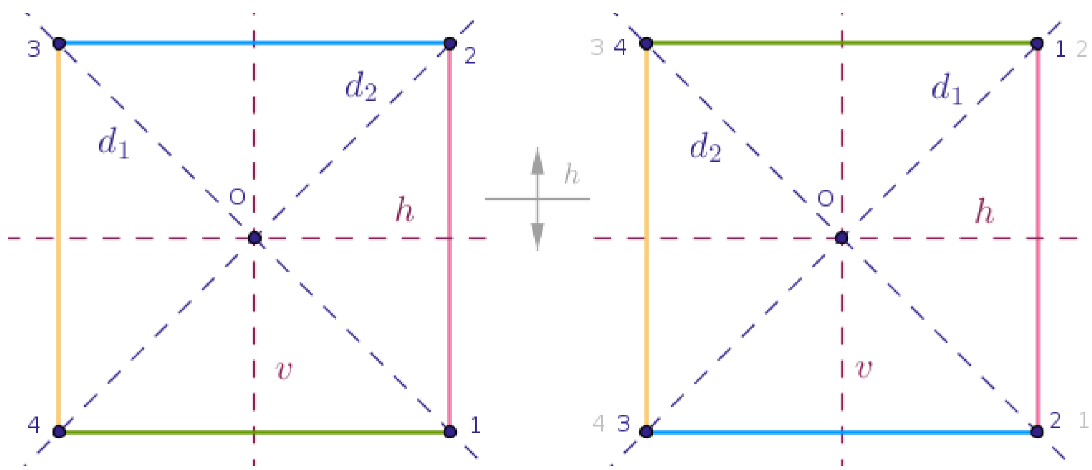
Motion 6: the reflection of a square in its other diagonal, d_2 (Figure 5.4.18):



Motion 7: the reflection of a square in its one axis of symmetry that passes through the middles of its two opposite sides, v (Figure 5.4.19):



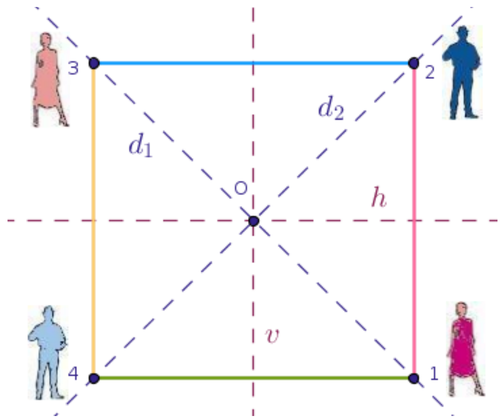
Motion 8: the reflection of a square in its other axis of symmetry that passes through the middles of its two other opposite sides, h (Figure 5.4.20):



The above set of the congruence motions of a square is not a group. Rather, it is the above set of the congruence motions of a square together with the *followed by* operation that acts on the elements of the said set is a group or *the group of rigid symmetries of a square*, which is an example of *a dihedral group*, designated with D_4. □

Exercise 5.4.4: correlate each element of the contra dancing group with the elements of the above group of rigid symmetries of a square.

Solution: if we agree to we correlate the dancers to the named vertices of a square as shown below (Figure 5.4.21):



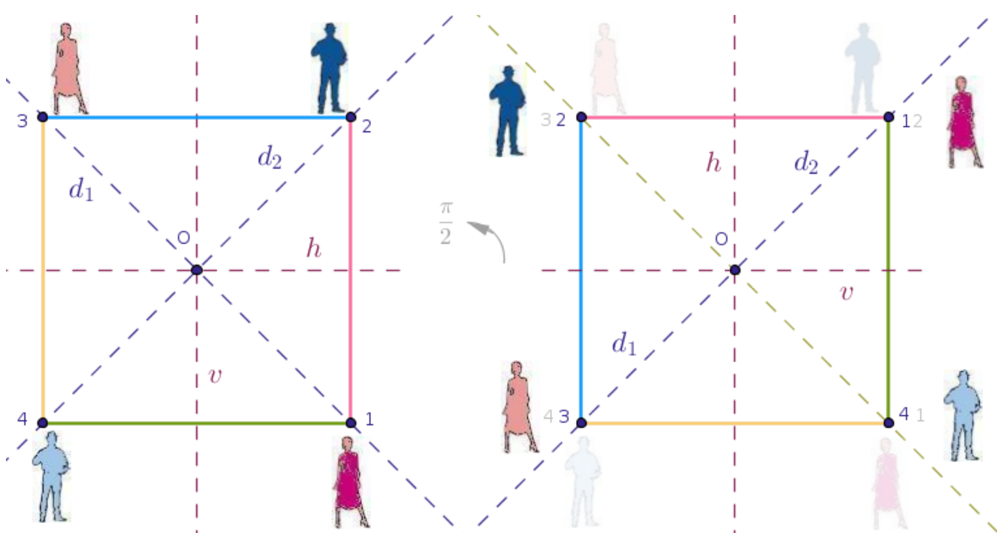
then the solution of this exercise should be straightforward and should cause no problems.

1. To the contra dance figure number 1, *no swaps*, there corresponds the congruence motion of a square number 1, *no motions*, and conversely. To the congruence motion of a square number 1, *no motions*, there corresponds the contra dance figure number 1, *no swaps*.

2. To the contra dance figure number 2, *circle right once*, there corresponds the congruence motion of a square number 2, any rotation of a square about its center from the infinite set A:

$$A = \left\{ \frac{\pi}{2} + 2\pi k,\ k = 0, \pm 1, \pm 2, \pm 3, \ldots \right\}$$

which, essentially, is one counterclockwise quarter of a turn of a square (Figure 5.4.22):



and conversely.

To the congruence motion of a square number 2, any rotation of a square about its center from the infinite set A:

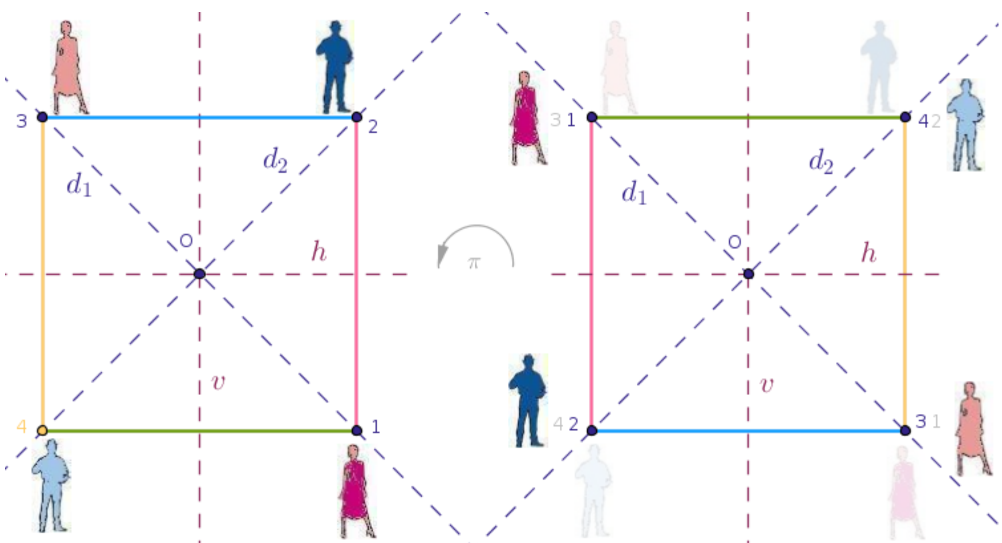
$$A = \left\{ \frac{\pi}{2} + 2\pi k,\ k = 0, \pm 1, \pm 2, \pm 3, \ldots \right\}$$

there corresponds the contra dance figure number 2, *circle right once*.

3. To the contra dance figure number 3, *circle right twice* or its close cousin *right and left through*, there corresponds the congruence motion of a square number 3, any rotation of a square about its center from the infinite set B:

$$B = \{\pi + 2\pi k,\ k = 0, \pm 1, \pm 2, \pm 3, \ldots\}$$

which, essentially, is one counterclockwise half a turn of a square (Figure 5.4.23):



and conversely. To the congruence motion of a square number 3, any rotation of a square about its center from the infinite set B:

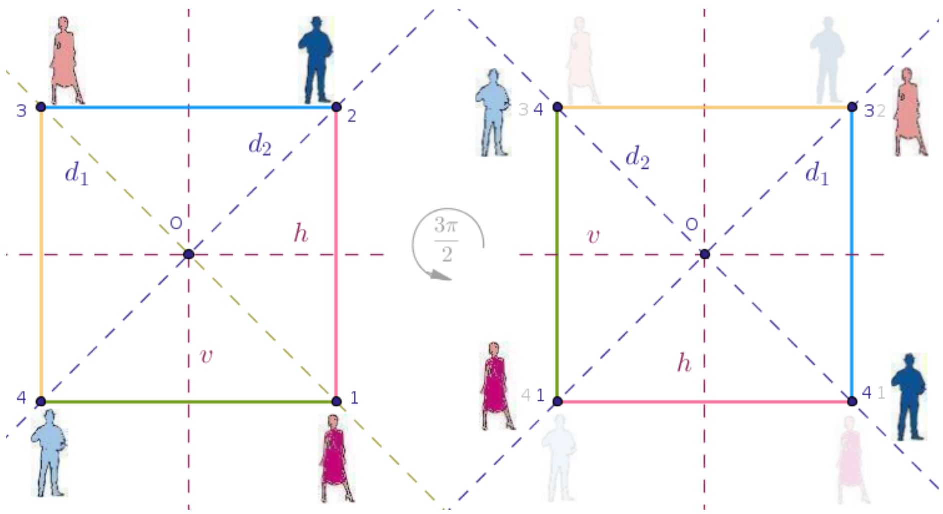
$$B = \{\pi + 2\pi k,\ k = 0, \pm 1, \pm 2, \pm 3, \ldots\}$$

there corresponds the contra dance figure number 3, *circle right twice* or its close cousin *right and left through*.

4. To the contra dance figure number 4, *circle right thrice*, there corresponds the congruence motion of a square number 4, any rotation of a square about its center from the infinite set C:

$$C = \left\{ \frac{3\pi}{2} + 2\pi k,\ k = 0, \pm 1, \pm 2, \pm 3, \dots \right\}$$

which, essentially, is one counterclockwise three-quarters-of-a-turn of a square (Figure 5.4.24):

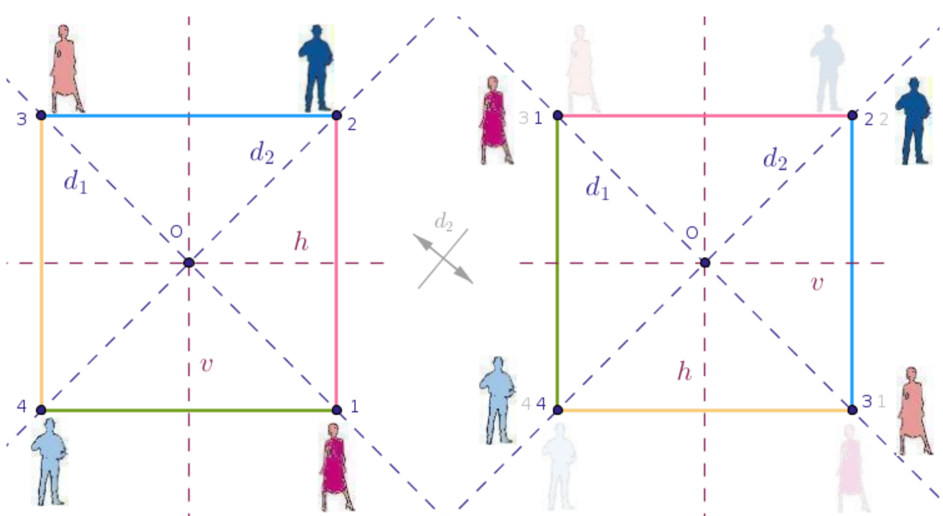


and conversely. To the congruence motion of a square number 4, any rotation of a square about its center from the infinite set C:

$$C = \left\{ \frac{3\pi}{2} + 2\pi k,\ k = 0, \pm 1, \pm 2, \pm 3, \dots \right\}$$

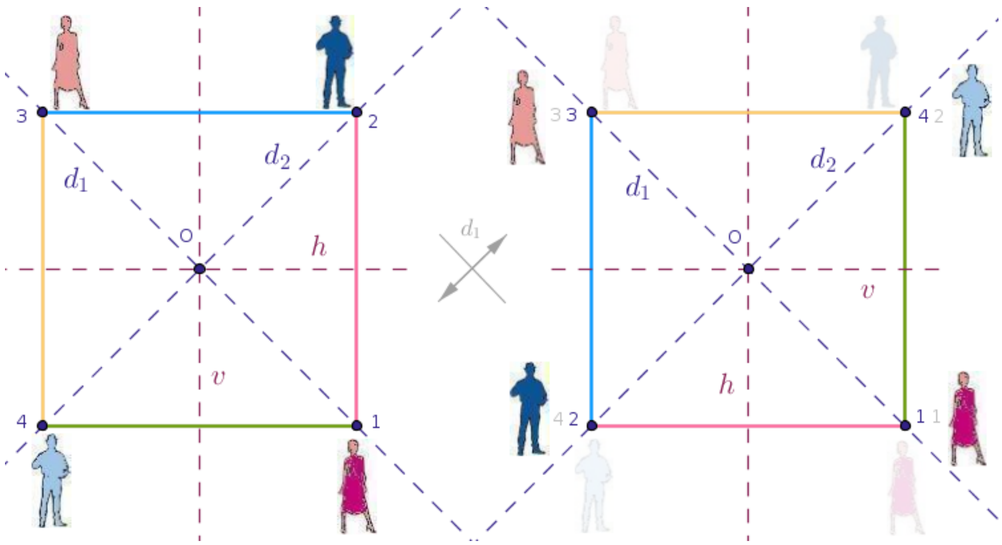
there corresponds the contra dance figure number 4, *circle right thrice*

5. To the contra dance figure number 5, *ladies chain*, there corresponds the congruence motion of a square number 6, the reflection of a square in its other diagonal, d_2 (Figure 5.4.25):



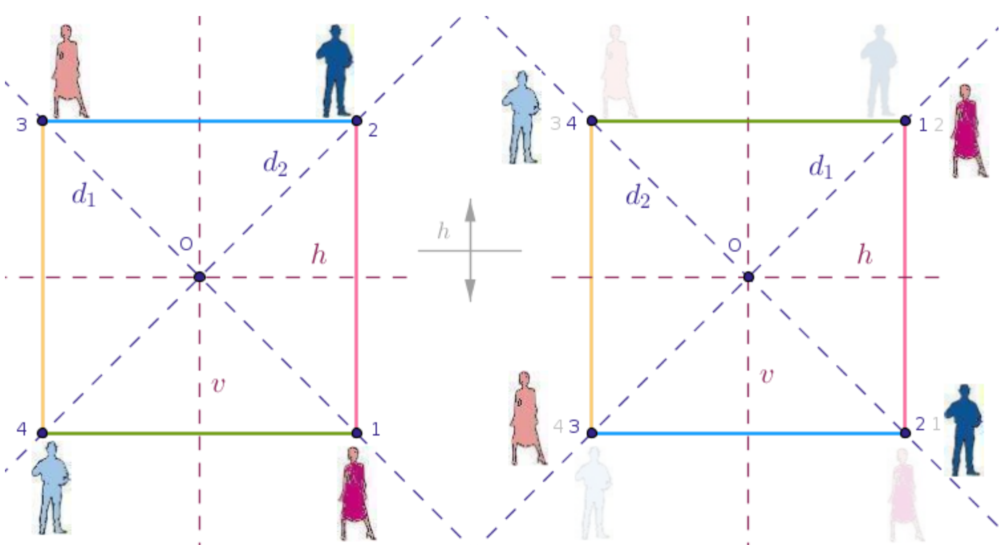
and conversely. To the congruence motion of a square number 6, the reflection of a square in its other diagonal, d_2, there corresponds the contra dance figure number 5, *ladies chain*.

6. To the contra dance figure number 6, *gents allemande*, there corresponds the congruence motion of a square number 5, the reflection of a square in its one diagonal, d_1 (Figure 5.4.26):



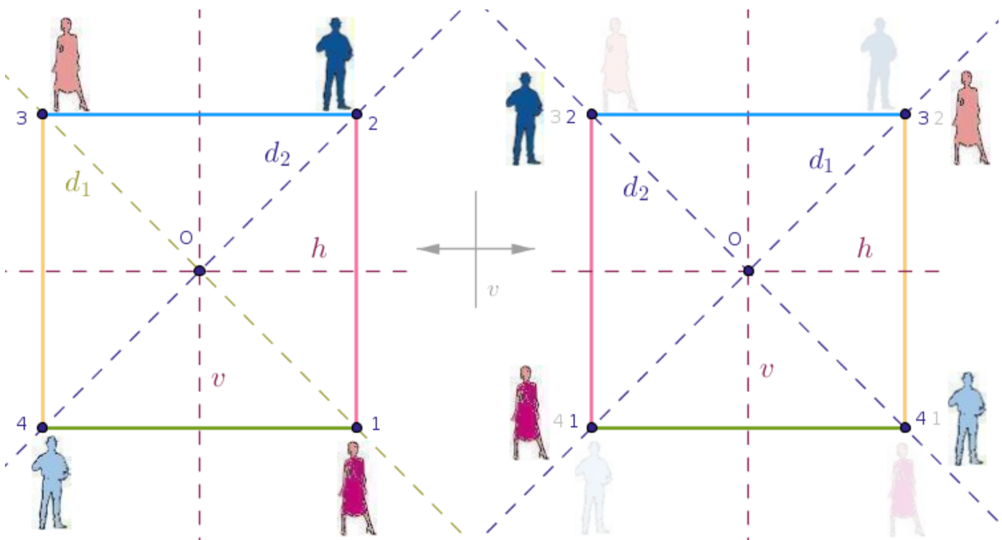
and conversely. To the congruence motion of a square number 5, the reflection of a square in its one diagonal, d_1, there corresponds the contra dance figure number 6, *gents allemande*.

7. To the contra dance figure number 7, *California twirl*, there corresponds the congruence motion of a square number 8, the reflection of a square in its other axis of symmetry, h, that passes through the middles of its two other opposite sides (Figure 5.4.27):



and conversely. To the congruence motion of a square number 8, the reflection of a square in its other axis of symmetry, h, that passes through the middles of its two other opposite sides, there corresponds the contra dance figure number 7, *California twirl*.

8. To the contra dance figure number 8, *swing on side*, there corresponds the congruence motion of a square number 7, the reflection of a square in its axis of symmetry, v, that passes through the middles of its two opposite sides (Figure 5.4.28):



and conversely. To the congruence motion of a square number 7, the reflection of a square in its axis of symmetry, v, that passes through the middles of its two opposite sides, there corresponds the contra dance figure number 8, *swing on side*. □

The purpose of the above *correlate-the-elements-of-the-groups-such-and-such* is to gradually lead our readers up to the comprehension of the notion of *group isomorphism*. As an example from the last exercise, observe that the square dancing group is all about people who dance, while the group of rigid symmetries of a square is all about rotations and reflections of a square and descriptively *people* who *dance* are vastly different from *rotations*, *reflections* and *squares*.

Yet, this way or that way, *structurally*, we are told, these two groups are essentially the same.

5.5 Negative Three Cows

In our last early example of a group we will, finally, look at something that is explicitly arithmetical in nature. Here come *numbers*. Yay!

The discovery of many mathematical concepts and entities that we know, love and use these days without even thinking about it was once a major intellectual battle in the past.

It took, for example, a while to introduce the number *zero* and to the storied discovery/invention of that number it is possible to dedicate a couple of good-sized chapters of their own.

However, it seems to be rather easy to justify the need for the number zero - when we look at that field over there and see no cows in it then it is quite natural to say that there are zero cows out there.

But how can we look at a herd of two cows and, with a straight face, claim that there are *negative* three cows out there?

Ah. That is where *the negative* (whole) numbers enter the picture.

It turns out that negative numbers are a very convenient model of various phenomena in the observable reality.

In the mechanics of liquids and gases there is *negative buoyancy*, *neutral buoyancy* and *positive buoyancy*. Each such type of buoyancy can be naturally described with negative numbers, zero and positive numbers.

Moreover, if we drop a small sphere of uniform density into a liquid/gas then, depending on the type of buoyancy that such a point mass exhibits, the so-called *reduced mass* of a system of a finite number of masses can very well be modeled with negative numbers, zero and positive numbers.

There are also negative temperatures, zero-degree temperature and positive temperatures.

Growing up as a kid in the 1970-ies Moscow and having to wait for a bus at 7 o'clock on a fine January windy and snowy morning when it is -30° Celsius outside, for ten winters in a row, I can personally vouch for the fact that *negative* numbers do exist. And *no*, there were no *snow days* in Moscow when the school is closed because there are a few snowflakes out there.

It also turns out that the everyday concept of *a debt* can be modeled with negative whole numbers beautifully.

When a person A owes a person B three cows then it can be said that the person A owes a debt to the person B and, thus, is in the possession of negative three cows.

When and if the person A acquires, say, five cows then via a straightforward arithmetic of:

$$5 + (-3) = 2$$

we see that the net worth of the person A is two (positive) cows.

Now consider the set of all integers $\mathbb{Z}$.

Such a set is comprised of:

- negative whole numbers or the whole numbers that are less than zero
- zero and
- positive whole numbers or the whole numbers that are greater than zero:

$$\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

We claim that the set of integers $\mathbb{Z}$ with the good old *integer addition* as the operation defined on that set forms the so-called *additive group of integers*.

That is, the traditional operation of addition as it is defined on the set of Natural numbers is the one and the only action that can be carried out against any two elements of the set $\mathbb{Z}$.

If in the previous early examples of groups *the elements* of a group were:

- the flipping actions
- the tire rotations
- the congruence motions of a rectangle
- the contra dance figures and
- the congruence motions of a square

then in this example the elements of a group are *the integers* of the set $\mathbb{Z}$.

If the number of elements in all of the previously studied groups was *finite* then the number of elements in the additive group of integers is not finite or *infinite* - that is fine, in the upcoming exercise we should be able to adjust to that fact accordingly.

If in all of the previous examples *the group operation* was *followed by* then in this case the group operation is *the integer addition* that is symbolized as $+$.

Now let us try

Exercise 5.5.1: using the duplex light switch group discussion as the guide, verify that all the basic properties of a group hold in the case of the additive group of integers.

Solution: is expected to be generated by our readers on their own.

Verify the Rules Constancy and the Determinism property of the additive group of integers.

What are the group elements in this case?

Integers.

How many integers does *the integer addition* operation act on?

Two.

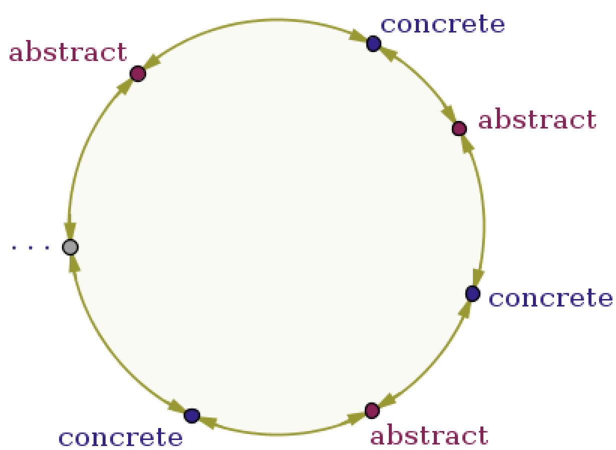
What is the *inaction* or the *do nothing* element of this group?

By analogy, the *do nothing* element of this group is zero - when we add zero to any element k of the set $\mathbb{Z}$, zero including, we obtain that original element or that original integer k, which effectively means that we *did nothing* to the integer k:

$$k + 0 = 0 + k = k$$

Moreover, there is one and only one zero in $\mathbb{Z}$.

Going through the exercise of jumping back and forth between the concrete and the abstract (Figure 5.1):



we will now use this and all the previous exercises in order to carry out the main program of **abstraction** and cleave away all the irrelevant minutiae, while keeping the essence of the basic entities that make a group.

To that end, we can now say that *the do nothing* element of a group is such an element of that group which under its pairing with any element of that group via the group operation does not change the element of the group that it is paired with.

Officially, such a *do nothing* element will soon become *the identity element of a group* and under one type of Group Axioms it is possible to bake the uniqueness of the group's identity right into the axioms themselves. Under another type of Group Axioms it is possible to leave the uniqueness of the group's identity element out of the axioms but *prove* its uniqueness right away. Under yet another type of Group Axioms it is possible to split the identity element of a group into *the left identity* and *the right identity* and then prove that the two identities are actually one and the same.

Can every integer of the set $\mathbb{Z}$ be acted on by the group operation in such a way that the *do nothing* element is produced?

Yes, sure.

Every element k of the set $\mathbb{Z}$ has its buddy element named $(-k)$ that, intuitively and geometrically speaking, sits on the other side of the *do nothing* element, zero, at exactly the same distance from it:

$$k + (-k) = (-k) + k = 0$$

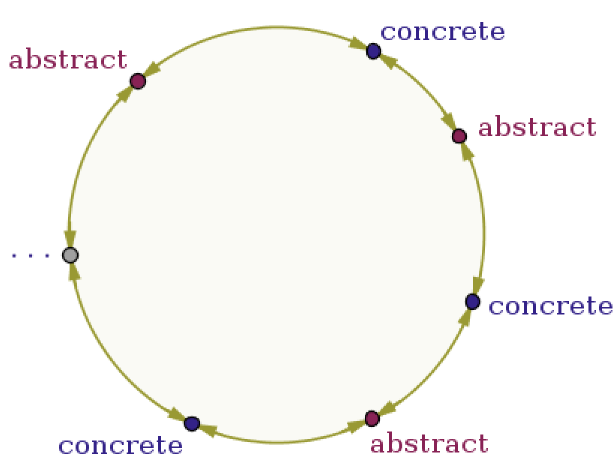
and there is one and only one such buddy element for a given integer k.

Intuitively speaking, every integer can be *undone* or *reversed* or *sent back to zero* by a particular integer from the same set. But what is *zero* in this group?

Zero in this group is the (unique) *do nothing* element.

In other words, every integer of the set $\mathbb{Z}$ can be taken by the group operation into the *do nothing* element of that group by a unique integer.

Going through the exercise of jumping back and forth between the concrete and the abstract (Figure 5.1):



we can now say that when we pick a particular element of a group then that element can be taken by the group operation into the identity element of that group by one and only one element of the group.

Officially, a unique element of a group that takes a particular element of that group into the group's identity via the group operation is *the inverse element* or just *the inverse* of that particular element of the group.

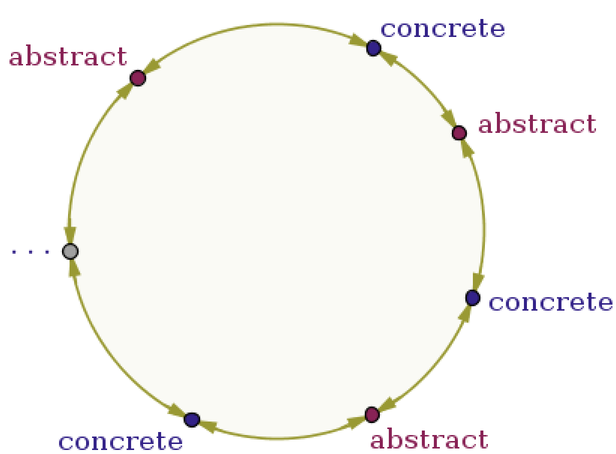
Again, under one type of Group Axioms it is possible to bake the uniqueness of the inverse elements right into the axioms themselves. Under another type of Group Axioms it is possible to leave the uniqueness of the inverses out of the axioms but *prove* the said uniqueness right away. Under yet another type of Group Axioms it is possible to split the inverses into *the left inverse* and *the right inverse* and then prove that the two inverses of a given element of a group are actually one and the same element.

Thus, formally speaking, every integer in the additive group of integers has a unique *additive inverse*.

What is the result of the integer addition of any two integers?

A particular integer, which, clearly, belongs back to the set $\mathbb{Z}$.

Going through the exercise of jumping back and forth between the concrete and the abstract (Figure 5.1):



we can now say that an operation that:

- acts on exactly two elements of a set and

- for each ordered pair of such elements produces a uniquely determined element that belongs to the same set to which the two elements that it acts on belong

will soon and officially become *a binary operation* defined on the set such and such.

We now can think of a binary operation defined on a set as a mapping that assigns a unique element of that set to each ordered pair of elements of that set.

Do two consecutive *integer addition* operations that act on any three integers produce the same result regardless of the order in which these two operations are carried out?

They certainly do.

If a, b and c are any three elements of the set $\mathbb{Z}$ then, using the parenthesis () as the order of execution modifiers, we know that the following equalities always hold:

$$a + b + c = (a + b) + c = a + (b + c)$$

For example:

$$(-3) + (-5) + (-11) = ((-3) + (-5)) + (-11) = (-3) + ((-5) + (-11)) = (-19)$$

Again, note carefully that while in the above experiment we did change the order of the integer addition *operations*, we did not change the order of *the elements* that these operations acted on.

Geometrically or left-to-right, the order of the specific integers, or the group elements, was fixed and remained the same both times:

$$(-3) +_1 (-5) +_2 (-11) = ((-3) +_1 (-5)) +_2 (-11) = (-3) +_1 ((-5) +_2 (-11)) = (-19)$$

In the first part of our experiment we at first added the elements (-3) and (-5), carrying out the leftmost integer addition operation marked as $+_1$ and obtaining the unique element (-8) as the intermediate result, and only then we carried out the rightmost integer addition marked with $+_2$, obtaining the unique element (-19) as the overall result.

In the second part of our experiment we at first added the elements (-5) and (-11), carrying out the *rightmost* integer addition operation marked as $+_2$ and obtaining the unique element (-16) as the intermediate result, and only then we carried out the first integer addition marked with $+_1$, also obtaining the same unique element (-19) as the overall result.

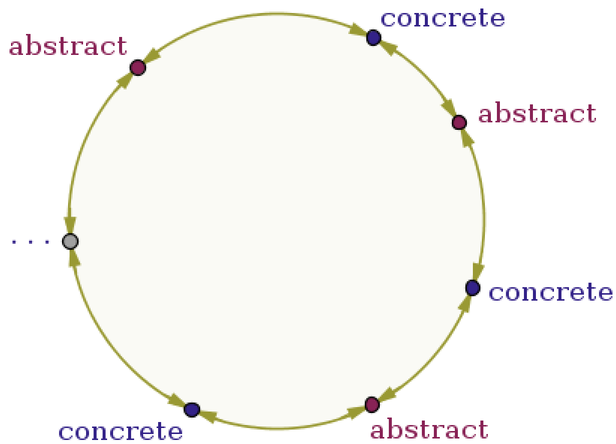
Consequently, carrying out the two *addition of integers* operations against three integers in two distinct ways, we obtained one and the same result, (-19).

Thus, we have informally demonstrated that the following unordered pair of independent but inseparable entities:

- the set of integers $\mathbb{Z}$ and
- the operation of *addition* defined on the elements of that set

is a group also known as *the additive group of integers*. □

The reason why in this discussion we, purposefully, kept injecting the same *vicious-concrete-to-abstract-circle* image every chance we had (Figure 5.1):



is to demonstrate the fact that the earlier made observation about the process depicted in that image was not *idle* - the theory of groups is shrapnelled and impregnated with that process, meaning that that process is not going anywhere any time soon.

Quite on the contrary, as we wrap up our early examples and get ready to make the **Difficult, Giant, Leap Forward**, we will be relying on that process more and more and the deeper the students of the theory of groups wade into the group-theoretic waters, the thicker and the heavier that process becomes.

It can be safely stated that the success or the failure of an individual in taking on the study of the theory of groups hinges on the ability of that individual to engage in the above *back-and-forth-concrete-to-abstract* process rapidly and effortlessly.

While it is impossible to teach the skill of *seeing the abstractions through the forest of concrete* in just five days, we have laid out these five early examples in a highly detailed manner exactly because we wanted to showcase the above process *in action*.

Our descriptive or non symbolic narrative cast in *the-ancient-Egyptians* mold vividly demonstrates how the process of abstraction works:

- we begin with something very concrete, something full of all sorts of details, something that is potentially *chaotic*
- with the Razor of Occam we cleave away all the irrelevant details from that something
- out of the primordial soup of details, with the Group Theoretic Spoon, we fish out and scoop up only what is needed or what is *essential* for our purposes and then
- we do the good old switcheroo by claiming that the essence thus obtained is *universal*, in some sense, and is applicable elsewhere

Moreover, the abstraction in general delivers the following miracle: the context of the above *elsewhere* can very well be totally disconnected from the original context of *something* that was the starting point.

For example, the descriptive differences across the light switch levers and their flips, car tires and their rotations and rectangles and their rigid congruence motions are stark. Yet, we are told that the *groups* that these respective entities form are essentially the same.

Getting a handle on the concrete-to-abstract cyclic process requires careful and *slow* work that methodically, day after day, grinds the relevant details into a fine powder which is then taken intravenously - there is no need to rush around and we certainly hope that that is how our readers approached the early examples in this space: carefully, deliberately and slowly.

This, perhaps, is one of the paradoxes in mathematics, and not in mathematics only - the slower we go initially, the faster we get to the destination.

If, after reading these couple of paragraphs, our readers realized *oh, man, I didn't do that*, then we recommend that these readers go back, start from square one and rework all five early examples the right way.

The Dawn Before the Attack

As a short technical summary of all the early examples and exercises that our readers duly worked their way through themselves, we now have an informal and intuitive grasp on:

- the identity element of a group
- an inverse element of a given element of a group and
- a binary operation on a set

The Identity Element

In the duplex light switch group the identity element of that group is the flipping action number 1 or no flips.
In the tire rotations group the identity element of that group is the tire rotation type number 1 or no tire rotations.

In the group of symmetries of a rectangle the identity element of that group is the congruence motion number 1 or no motions.

In the square dancing group the identity element of that group is the dance figure number 1 or no partner swaps.

In the group of symmetries of a square the identity element of that group is the congruence motion number 1 or no motions.

In the additive group of integers the identity element of that group is *zero*.

Note well that *the specific* identity elements across the groups that we have studied are different and these differences are explicitly spelled out above.

However.

In all of these groups *the local* identity element of each group possesses the following property that is one and the same across all the groups:

any element of each respective group paired up with that group's local identity element via that group's local group operation corresponds to itself

and, after all the exercises that we have done, that much should be clear.

An Inverse Element

In the duplex light switch group every flipping action has its unique inverse or can be reversed by a unique flipping action that is a group element itself.

In the tire rotations group every tire rotation has its unique inverse or can be reversed by a unique tire rotation that is a group element itself.

In the group of symmetries of a rectangle every congruence motion of a rectangle has its unique inverse or can be reversed by a unique congruence motion of that rectangle that is a group element itself.

In the square dancing group every dance figure has its unique inverse or can be reversed by a unique dance figure that is a group element itself.

In the group of symmetries of a square every congruence motion of a square has its unique inverse or can be reversed by a unique congruence motion of that square that is a group element itself.

In the additive group of integers every integer k has its unique additive inverse or a unique *equidistant-from-zero* image $(-k)$ that is a group element itself.

Again, note well that *the specific* inverse elements across the groups that we have studied are different and these differences are explicitly spelled out above.

However.

In all of these groups *the local* inverse elements of each group possess the following property that is one and the same across all the groups:

any element of each respective group paired up with that element's inverse via that group's local group operation corresponds to that local group's identity element

and, after all the exercises that we have done, that much should be clear.

A Binary Operation

In the duplex light switch group the binary operation that glues each respective ordered pair of flipping actions together and produces a unique flipping action as a result is *followed by*.

In the tire rotations group the binary operation that glues each respective ordered pair of tire rotations together and produces a unique tire rotation as a result is *followed by.*

In the group of symmetries of a rectangle the binary operation that glues each respective ordered pair of congruence motions together and produces a unique congruence motion as a result is *followed by*.

In the square dancing group the binary operation that glues each respective ordered pair of dance figures together and produces a unique dance figure as a result is *followed by*.

In the group of symmetries of a square the binary operation that glues each respective ordered pair of congruence motions together and produces a unique congruence motion as a result is *followed by*.

In the additive group of integers the binary operation that glues each respective ordered pair of integers together and produces a unique integer as a result is *addition of the said integers*.

Again, note well that *the specific* binary operations across the groups that we have studied are different, *followed by* versus *integer addition*, and these differences are explicitly spelled out above.

However.

In all of these groups *the local* binary operation of each group possess the following two properties that are the same across all the groups:

1) acting on each ordered pair of elements of each respective group, that group's local binary operation produces a unique for that group element that belongs to the same underlying set of that group from which the said pair of elements came from
2) two consecutive group's local binary operations that act on any three elements of that group produce the same result regardless of the order in which these two operations are carried out

and, after all the exercises that we have done, that much should be clear.

In our early examples of groups we started with something that is fun, tangible, visual and easy to comprehend and latch on to.

We shall now begin the process of gradual migration from concrete and specific toward generic and abstract.

As we suggested earlier, the readers who feel lost and confused by the upcoming abstract lines of reasoning and an unavoidable perfect storm of abstract symbolic shuffling here and there should readily fall back on the early examples and use these examples in order to find a path from specific to abstract - the importance of the ability to do so rapidly and effortlessly cannot be overstated.

Be realistic, such an ability does not come in one day.

Stay with and work at it.