A Historical Note

More often than not, with a precision that many people expect or at least have a fuzzy idea of what it should be, it is quite difficult to pin down an exact point in time when a given vaguely defined branch of mathematics is actually born simply because many such areas in mathematics are interconnected in various and, perhaps, unexpected ways and because the development of the basic ideas that are built upon is smudged across time and, often, space.

There are exceptions, however.

Taking the following historical fact at its face value, apparently, the vaguely defined branch of mathematics known as *the calculus of variations* was born on one fine January afternoon in the year of 1697 in a span of a small number of hours when the English scientist Isaac Newton (1642–1726) solved, in an off-the-cuff manner, the so-called *brachistochrone problem* posed by the continental mathematician Johann Bernoulli (1667–1748).

Even though Isaac Newton requested that his solution is to be published anonymously, the lion, in the words of Johann Bernoulli, was recognized by his claw.

On the other hand, the distance in time between an informal concept of a mathematical limit that shows up in the work of Isaac Newton in 1687 and the formal and rigorous epsilon-delta definition of a limit attributed to the German mathematician Karl Weierstrass (1815–1897) is more than one hundred and fifty years and the process of the crystallization of the said rigorous definition included such mathematicians as Agustin-Louis Cauchy (1789–1867), who gave a purely *verbal* definition of a limit, and Bernard Bolzano (1781–1848). But did the evolution of the notion of a mathematical limit really take the proverbial one hundred and fifty years?

More than two thousand years ago the Greek mathematician Archimedes estimated the value of the transcendental number π by:

- trapping a circle, or a difficult to solve problem, between an inscribed and a circumscribed regular polygon, each of which represented an easy to solve problem, and, consequently
- by incrementing the number of sides of the said regular polygons

At its essential core, Archimedes walked around, toyed with and touched but never explicitly spelled out the word *limit* in the mathematical sense with which we infuse that word now.

But, even two thousand years later, in many a modern textbook on the subject the above trapping or squeezing process is often used to illustrate the idea of a limit at an introductory, loose and intuitive level.

However useful and omnipresent the tactical computations that flow out of the concept of a mathematical limit are, it is *the (germ of an) idea* and *the play of abstract thought* that matter to mathematicians and mathematics the most.

Moreover, it is a sad fact that in many high schools in many countries the emphasis in the study of integral calculus is completely misplaced - instead of stressing, studying and analyzing the value and the power of the trapping or the *Divide and Conquer* idea, an unjustified amount of time and effort is wasted on the computational minutia that obscures and keeps the actual precious gem of a mathematical thought out of sight.

In that respect, the concept of a mathematical limit was born at least two thousand years ago and after some downtime in its development, it culminated with our favorite formulation:

f*or any strictly positive real number epsilon there exists another strictly positive real number delta such that as soon as a particular varying quantity is delta-close to one number of interest, another varying quantity dependent on the first is epsilon-close to another number*

to which we refer as *the limit* of the latter varying quantity.

A polished garden path through which the modern students of mathematics are taken in various courses hides the actual way in which the grown-up mathematics is done because the real time business of creating the grown-up mathematics is dirty and messy and it includes many stutters, blind alleys, false starts, dead ends and unrealized hopes.

Thus, on the one hand, it is simply impractical and impossible to expose the said students to the tortured bricks with which the said polished garden path is paved. On the other hand, in the words of Albert Einstein (1879–1955):

“There is always a certain charm in tracing the evolution of theories in the original papers; often such study offers deeper insights into the subject matter than the systematic presentation of the final result polished by the words of many contemporaries”.

Please see the pages 376–377 of the book *“The Ultimate Quotable Einstein”* collected and edited by Alice Calaprice.

The elimination of the real-time development of a given technical discipline from the respective curricula tends to leave some inquisitive students befuddled at the motivation of where did this strange but neat prepackaged result come from and why does it have this particular shape.

The development of the modern theory of groups absorbs the above observations.

In a certain, trivial, sense the moment people arrived at the notion of negative integers, zero and positive integers, they already were dealing with a so-called *infinite additive commutative group of integers* without realizing it.

At a much more limiting scale, as we shall discover soon enough, just the two integers −1 and +1 alone already form a so-called *finite multiplicative commutative group (of order 2)* in which each element is its own so-called *inverse*.

In a non-trivial sense, just like the ancient Greek mathematician Archimedes before him, the Swiss mathematician Leonard Euler (1707–1783) walked around, toyed with and touched the concept of a mathematical group without realizing it around the year of 1758 in his version of a proof of Fermat’s little theorem, in which Euler relied on the existence of inverse elements.

In a similar pattern, while studying the so-called *permutations* of roots of algebraic equations circa 1771, the French mathematician Joseph-Louis Lagrange (1736–1813) walked around, toyed with and touched the so-called group S_n of permutations of n elements and some of its so-called *subgroups* without realizing it.

It is highly likely that it was the young French mathematician Galois (1811–1832) who first caught the group-theoretic tiger by the tail and truly grasped the concept of a mathematical group when in 1831 he gave an outstanding general explanation of exactly when an algebraic equation is solvable by radicals.

Some time around 1872 the German mathematician Felix Klein (1849–1925) proposed the idea that each *type* of geometry is actually characterized by *a group* of certain transformations.

A Historical Note

Looking into the rear-view mirror of time, it can be said that the modern theory of groups was born at the intersection of the theory of numbers, geometry and the study of the roots of algebraic equations. That birth of the theory of groups was spread over about one hundred years as the various contributing mathematicians worked with concrete and highly specific instances of groups under loose assumptions and without explicitly stating exactly how a group is defined.

For example, starting from the year of 1815, the French mathematician Cauchy, whom we already met earlier, worked only with groups whose elements are represented in terms of *permutations*, which we will study at length.

We now know that a mathematical group can be defined in a number of distinct ways, one of which is axiomatic.

In modern textbooks on the subject such an axiomatic definition of a group is shown in one fell, short and sweet, crisp swoop as the three group axioms:

- that of *associativity*
- that of *the identity element* and
- that of *inverses*

However, these cut-and-dried group axioms did not originate in the mind of a single mathematician in one day in one sitting in their final instance. Quite on the contrary. It took more than one hundred years and several contributors for the essential features of the group axioms to gradually crystallize and come into a sharp and polished focus known to and expected by us today.

As we shall soon see, even these group axioms can be formulated in a number of different ways – it is possible to bake the uniqueness of the identity element and of the inverses right into them, it is possible to leave the uniqueness property of these elements out and *prove* it later on, it is possible to splice both left and right identities and left and right inverses into the group axioms or it is possible to frame these axioms using the one-sided identity and inverses and prove the existence of the remaining, other-sided, identity and inverses separately and later and so on.

The word *group* itself was, apparently, explicitly introduced in 1832 by Galois, who also made the first and a very important step of unbuckling the concept of a group from the shackles of the concrete when he showed that groups can be defined without using permutations as elements.

It was the British mathematician Arthur Cayley (1821–1895) who, in 1854, first penned a modern-like set-theoretic definition of a group that was completely divorced from the concrete and specific nature of the underlying elements of a group - a giant leap forward in and of itself. Cayley next pioneered the absolutely breathtaking idea that:

the structure of a group depends solely on how the binary operation that acts on the ordered pairs of its elements is defined

That awe-inspiring idea lies at the root of the remarkable power and reach of the theory of groups.

Moving forward chronologically, not only the theory of groups, as well as abstract algebra, was developed further as stand-alone discipline, but it was also taken to spectacular heights in theoretical physics by Emmy Noether (1882–1935), a remarkable woman whose name, due to the myopic policies of men, is not exposed and known as well as it deservedly should be.

Admittedly, we do not really understand what the above few paragraphs mean and we cannot comprehend and appreciate its scope and significance even more so.

To some extent, this little body of work is dedicated to deciphering the above ideas, laying them bare, unlocking and opening the door into the theory of groups.

Whether our readers decide to walk through that door is entirely up to them and to those readers who will decide to do so we wish the best of luck.