Chapter 1

What Makes It Perfect

The adage *practice makes perfect* is, unfortunately, slightly off mark because it fails to reveal a crucial detail - only *a correct* practice makes perfect. Even if we decide to practice eight hours a day every day for years on end, if our practice, however enthusiastic, is incorrect then little or no progress will be achieved.

When it comes to technical exercises in hard sciences, such as mathematics, theoretical physics or theoretical computer science, there is certainly an incorrect way of doing them and there is certainly a correct way of doing them.

Doing It Wrong

A definitely *incorrect* way of doing the upcoming technical exercises is to read the problem statement and then to read the presented solution right away, without applying any personal effort. Such an approach to doing technical exercises leads nowhere. It is a complete waste of time, it may as well be not undertaken to begin with and we shall quickly understand why next.

A realistic understanding of the material at hand comes only through a personal struggle with new ideas and notions and all exercises in this book are designed with precisely that purpose in mind.

Conversely, simply reading a solution of a technical problem constructed by someone else relegates us to the status of mere spectators and not the participants of the game.

Doing It Right

A definitely correct way of doing the upcoming technical exercises is for us to commit to the academic struggle from minute one of day one and to make an honest, recorded on paper, attempt to generate at least *some* solution steps for the problem at hand on our own.

The Loop

If, after a reasonable number of attempts to solve a given problem, we feel that we are stuck and cannot finish the solution of that problem on our own then the only correct way to proceed is:

- to *not* (!) look up the presented solution in its entirety but, instead,
- to look up only *the minimum viable* amount of information from the shown solution that will help us get unstuck
- at which point we:
  - mark the steps generated by not us clearly
  - turn away from the solution presented and
  - go back to generating our own solution until we either
    - complete such a solution in its entirety on our own or
    - we get stuck again

If we get stuck again then we should repeat the above *look up only the minimum viable helpful information* step and then we should proceed completely on our own until we either finish the solution ourselves or we get stuck again.

At which point we look up only the smallest amount of information presented in the problem's solution and make an honest attempt at completing the solution of the current exercise on our own and so on.

All At Once

After we realize that we have solved the given problem completely, we grab a blank piece of paper and without the earlier stutters, zigzags and false starts we write our solution down in one fell, uninterruptible and confident swoop.

When we work on a technical problem our mind often scatters its attention to various details of varying scope. Some such details are rather small, while other such details may be larger.

The above scattering of our attention during the search for a solution of a problem at hand results in a rather chaotic view of the solution of that problem.

The purpose of the *All At Once* step is to create a unified and a coherent view of the problem that we just solved and of the solution that we just generated, a view in which no single detail sticks out too much and all the technical details work in concert and in harmony with each other, subordinate to the common theme – this solution of this exercise.

Retrospect

We, next, put our tortured, awkward and ugly solution next to the polished and sanitized solution offered by this book and for each place where we got stuck we answer the following questions.

Exactly *why* did we get stuck? Was it because we forgot a theorem or an important detail in a definition? Or was it because we remembered all the theory well but we missed a clever tactical, problem-solving maneuver?

Clearly, if we forgot an important theoretical information then we need to go back to the source and hit the books – we need to find that information in the text and understand why did we miss it and what steps should we take in order to improve our ability to fish out purely theoretical facts from the web of our mathematical knowledge.

If we missed a problem-solving tactical maneuver then we want to make sure that we can summarize that maneuver's essence, such as *Divide and Conquer*, for example, in one concise paragraph or less and commit the context in which that maneuver was applied to memory: *this* approach worked *here*.

In either case, we look back at our problem-solving experience and make a mental note of what worked, what did not work and why, how were the problematic points addressed and we take the results of such an analysis to our *next* problem-solving session where we do *the next* technical exercise.

Abstract and Generalize

After the battle for a solution is over and all of the above points have been addressed and buttoned up, we want to abstract ourselves from this concrete and specific instance of the problem and make an attempt to generalize it by peeling our minds away from the local minutiae and finding the areas of the problem that can be reused elsewhere, in a context that may be similar to or different from the context of the just solved problem.

Such a generalization of a problem with the consequent templatization of its solution increases the radius of its application as it allows us to solve *similar but different problems*. As a part of the abstraction and generalization step, we also would like to find other problems that may be worded differently but have an essentially the same solution (see exercise 4.3 for an example of such a comparison).

We will discuss the definition of *abstraction* and some of its properties soon.

Memorialize

If we carefully save the paper trail of all of our trials and errors, failures and successes then in just six to ten month we can look back at our work and learn a lot of interesting things about ourselves, possibly including the things that we did not even suspect that we possessed.

In a small number of years such a paper trail may, and likely will, show an evolution of how we become more mathematically mature.

We, thus, see that fundamentally there are actually *two* loops or two *cycles* in a correct way of solving technical problems in hard sciences:

- the immediate, *inner*, loop of generating our own solution steps, getting stuck, looking up the minimal viable information in order to get unstuck, proceeding on our own and so on and
- the global, *outer*, loop of stepping back, retrospecting and reflecting, while inhaling the entire experience, analyzing and reconciling its failures and successes

To many readers the above stop-and-go process will likely seem very unnatural and difficult at first.

That is more than fine.

The most likely reason why many of our readers may feel that way is because unfortunately they were never exposed to a correct way of solving correct technical exercises in the first place.

The correct technical *exercises* are carefully worded in a way that, on the one hand, completely avoids the templatized plug and chug solutions and, on the other hand, makes or forces the student to rely on her/his creative juices at all times.

As such, most of our exercises are designed in a way that facilitates the readers to generate their own lines of reasoning. For example, instead of the wording:

Show that the n-th roots of unity form a commutative group under the operation so and so

our exercise will be worded as follows:

Can the n-th roots of unity form a group?

As we shall see in the near future, this particular formulation of a problem presents its own set of challenges to a student and an expected answer to such a question must include a proper justification (a proof) of why or why not the objects at hand can or cannot form a group and a demonstration.

In general, in this book we will pay little to no attention to the raw symbolic manipulations that are carried out only for their own sake but we will place all the significance on the creative and mathematically cultured lines of reasoning generated by the readers themselves.

Thus, if the above correct problem-solving process seems unnatural and difficult then it is actually a good thing because it tells us that our readers are actually trying to do the right thing by unlearning the old and incorrect habits of solving technical problems and learning the new and correct way of solving such problems, which is always a struggle.

Do resist the urge to read the ready-made solutions presented by us in their entirety - that step alone will nudge our readers toward relying on their own creative juices more and more.

It is also perfectly fine to initially get stuck a lot and to look up the solution steps presented by us often.

However, our claim is that with the above *correct* practice, over time, the following phenomenon will start taking place - our readers will get stuck less and less, they will have to look up the solution steps presented by us less and less frequently and, eventually, they will be able to generate their own solutions and their own proofs that, we hope, will be better, in some ways, than the ones presented by us.

Rest assured, the correct practice of doing the upcoming technical exercises described above *will be* difficult because it is designed to be difficult; it will take a lot of time, paper, patience and perseverance to master. However, the intellectual rewards harvested on the other side of that effort and time invested will be more than worth it.

Especially early on it will be very tempting to disregard our advice on how to solve technical problems correctly. It will be very temping to cut corners and to take shortcuts. However, we urge our readers to resist that temptation precisely because the formation of correct habits for solving and addressing technical problems takes its place early on and that is why it is very important to not rush through a psychologically uncomfortable and an unfamiliar process.

On the contrary. This is not a race. There is no *need* to rush. Let the new ideas and notions seep in and settle at as slow a pace, as needed. Shortcuts taken and corners cut now will inevitably manifest themselves later, at the least opportune moment. At which point the old material must be revisited and the time dedicated to the re-excavation of that material will be the time that is spent not as efficiently as it could have been.

In other words, we suggest to our readers to not get distracted by and disappointed with a seemingly slow rate of the initial progress, because, again, the initial progress is designed to be slow, methodical and deliberate.

Among its many, vaguely defined, mathematical brethren, the theory of groups presents a reasonably unique challenge for the newcomers because that theory is rather abstract. It is one thing to manipulate such highly visual and even tangible mathematical objects as points, straight lines, circles, spheres and tetrahedrons. It is an entirely different thing to reason with confidence about *abstract elements* and *abstract operations*, neither of which can be seen or touched and both of which are the first class citizens of the theory of groups.

As such, we urge our readers to, first and foremost, be honest with themselves and to not buckle under the pressure of an artificial need to show quick but shallow progress – it is best to quietly dig the heels in and make a stubborn commitment to endure the difficulties and stay the course no matter how challenging it gets.

In that endeavor we wish all of our readers the best of luck.