Prerequisites

From a less technical and more of a philosophical perspective, the basic prerequisite for taking on the study of the theory of groups, or any other technical discipline for that matter, is an intellectual openness to new ideas, the willingness to step out of one's comfort zone and the absence of fear to initially fail a few times before finding a more or less solid footing in the discipline of choice.

From a more technical perspective, we made every reasonable effort possible in order to make this elementary introduction to the theory of groups accessible to as wide an audience as possible.

We think that just about the only fundamental discipline-specific prerequisite for a successful study of the introduction to the theory of groups is simply a firm grasp on the idea of *symbolism* in mathematics - we argue that any person who has such a skill will be able to comprehend our introduction and benefit from it for the following reason.

Symbolism

In a typical academic journey, in elementary school we study the basic arithmetic operations of, say, *addition*, *subtraction* and *multiplication*.

A characteristic property of such *arithmetic* manipulations amounts to the fact that they are all firmly grounded in the explicit instances of a very specific and concrete type of mathematical objects known as *counting numbers* or just *numbers* if we are not too picky.

As an example, here is what we study in elementary school:

$$1 + 2 = 3, \qquad 2 \cdot 3 = 6, \qquad 7 - 6 = 1$$

One side effect of working directly with such naked numbers is the fact that even the solutions to similar but different computational problems must be spelled out anew every time. It, thus, becomes very cumbersome and difficult to capture, express and convey the underlying *sameness* or the underlying *similarity* that unites and runs through all such problems.

By the same token, it becomes equally cumbersome and difficult to convincingly demonstrate *the difference* between problems that are inherently different.

As an example, a purely verbal description of a recipe according to which a volume of a right circular cone may be computed can be algorithmically correct and flawless but such a description will likely be rather lengthy and will require more time to parse and comprehend than its purely symbolic counterpart.

Let us do that comparison together.

As a purely verbal definition:

the volume of a right circular cone may be computed by , first, multiplying the circular constant by the square of the radius of the cone's base and, then, by multiplying the result of the previous multiplication by one third of the cone's height

In the above verbal definition of the volume of a right circular cone we also have to explain, purely verbally, the meaning of the circular constant, the number π. But because we do not have the luxury of mathematical symbology, that last symbol, π, is not available to us.

As such, whatever language we decide to use in order to describe what exactly does *the circular constant* mean, the vocabulary of that language cannot use any symbols in it at any time, recursively.

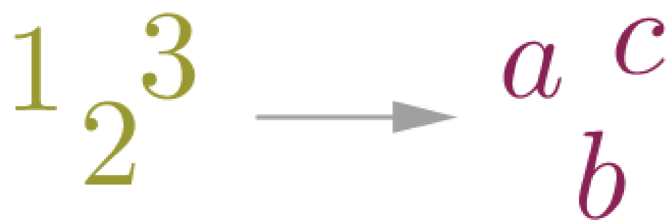
In contrast, a purely symbolic definition of the same entity (Figure 0.1):

$$V_{\text{cone}} = \frac{1}{3}\ \pi r^2 h$$

is quite succinct if we agree on the meaning of its symbols ahead of time.

That explanation of the meaning of the symbols in the above formula does have to be mostly verbal, with the exception of the usage of the symbols themselves, but that is an inevitable and a small price to pay for the benefits and the gains that follow.

As such, in middle school, while keeping the basic menu of arithmetic operations more or less the same, we make a difficult and a very important transition away from concrete numbers to single-letter *symbols* (Figure 0.2):

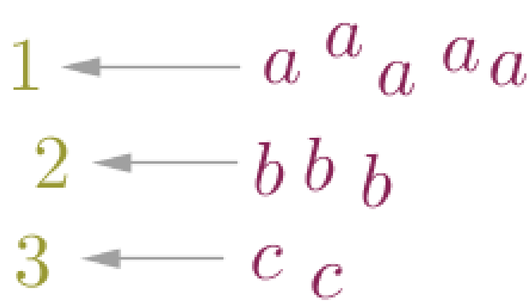


By doing so, we begin manipulating the *algebraic* expressions that glue together some number of such symbols with the arithmetic operations (Figure 0.3):

$$a^2 - b^2 = (a - b)(a + b)$$

In the process, we learn the following three very important concepts birthed by the transition to symbolism in mathematics.

I. Symbols in algebraic expressions become *representatives* of the underlying numbers and if we replace each symbol in an algebraic expression with a specific number uniformly and consistently, in the sense that every occurrence of a given symbol is replaced with the same chosen number (Figure 0.4):



then, upon the application of such a replacement or substitution to all the symbols in a given expression, we will get a ready-made recipe of what operations must be carried out with these numbers and in which order.

In other words, we learn that the above process of uniform replacement of given symbols with the chosen numbers *reduces* an algebraic expression to a mere, concrete and specific, arithmetic computation that we already know and love.

II. Even though there exist infinitely many algebraic identities, such as the one shown in Figure 1.2, for example, it is possible to establish a small number of small and simple basic or *elementary* identities, known otherwise as *the rules* or the algebraic *axioms*, such as the following one (Figure 0.5):

$$(a+b)\cdot c = a\cdot c + b\cdot c$$

out of which all other identities follow.

III. We also discover that no matter how large, intimidating, complex and convoluted a transformation of one algebraic expression into another might be, it can always be carried out as a combination of the above elementary identities *mechanically* and *without worrying at all about the true nature of the numbers* that may eventually animate the participating symbols.

At the far end of the symbolization process in mathematics there lies an important realization that:

numbers are not the only entities that can be symbolized - other things can be symbolized just as well

For example, toward the end of middle school we learn that the entities known in physics as *forces* may be symbolized also. A force in physics is not a naked, stand-alone, single number that we are used to but *a vector quantity* and vectors are certain *collections* of numbers.

Meanwhile, vectors can be *added* together and such an addition of vectors follows the same basic rules of the good old algebraic addition of numbers (Figure 0.6):

$$\overrightarrow{A} + \overrightarrow{B} = \overrightarrow{C}$$

In other words, even though we do have to adjust the technical meaning of the addition of two vectors, symbolically such an addition of vectors follows exactly the same template that we use when we add numbers together.

Therefore, the said forces in physics can be manipulated computationally according to the basic rules of algebra with equal success and under the umbrella of the same powerful idea of peeling or detaching ourselves from the concrete, instantaneous or local, nature of these forces - a force of static friction, or a force of sliding friction, or a force of reaction or a normal force, or a force of tension, or a force of medium resistance or a force of gravity, none of that matters in our computational maneuvers because a vector, is a vector, is a vector.

We, thus, learn the following important idea:

if certain entities can be symbolized in a way that obeys the algebraic axioms then computationally such entities can be manipulated mechanically without paying any attention to their true nature

From that point onward there is a reasonably short leap of imagination to another related idea that, just about, anything can be folded into a sweeping and an all-encompassing notion of *an element*, which is a notion that sits at the foundation of the discipline known as *the set theory*, which, itself, sits at the foundation of modern mathematics.

Symmetry

In addition to the concept of mathematical symbolism, the notion of *symmetry* permeates the theory of groups.

Extracting and unearthing useful algebra out of that notion of symmetry is, admittedly, an exercise that does require certain powers of abstract cognition and group theory is a coat that wears the better, the more different layers of mathematics the students of the theory of groups are exposed to and we more than encourage our readers to voraciously study as many such different mathematical disciplines as they possibly can.

However, we argue that if our proverbial middle school student has enough abstract cognition power to comprehend the above *algebraic* ideas described in **I**, **II** and **III** that stem from *the object* symbolization process then such a student is already equipped with everything that is needed to begin studying *the theory of groups*.

The next big hurdle to overcome in that study is the idea that not only all sorts of objects can be symbolized but, by analogy, all sorts of *operations* themselves can be symbolized just as well!

In some instantaneous, local or tactical sense the theory of groups does to the notion of *operations* what middle school algebra does to the notion of *numbers*.

The symbolization of operations allows the theory of groups to manipulate these operations without any regard to their true, underlying, nature and to study their common properties.