Tangent Lines to Two Circles

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Figure: "Bike chain guard part" [1]

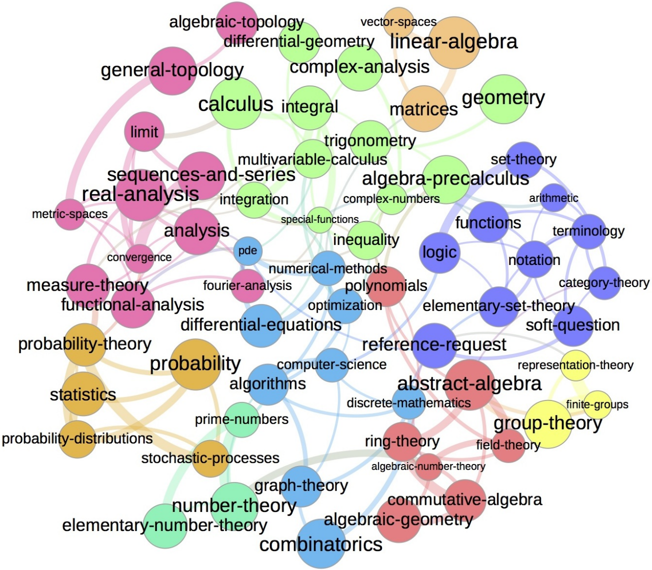
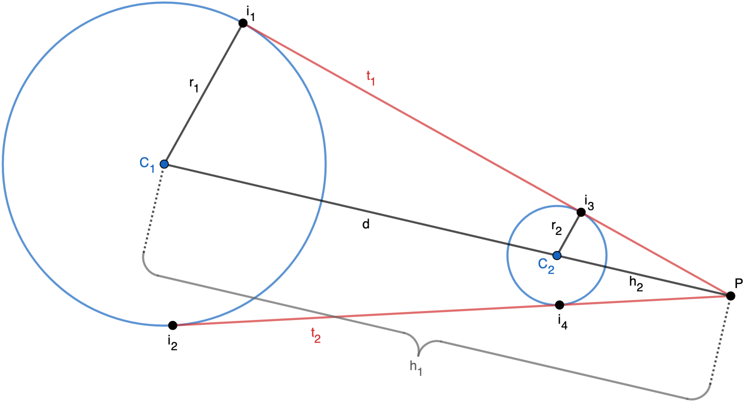
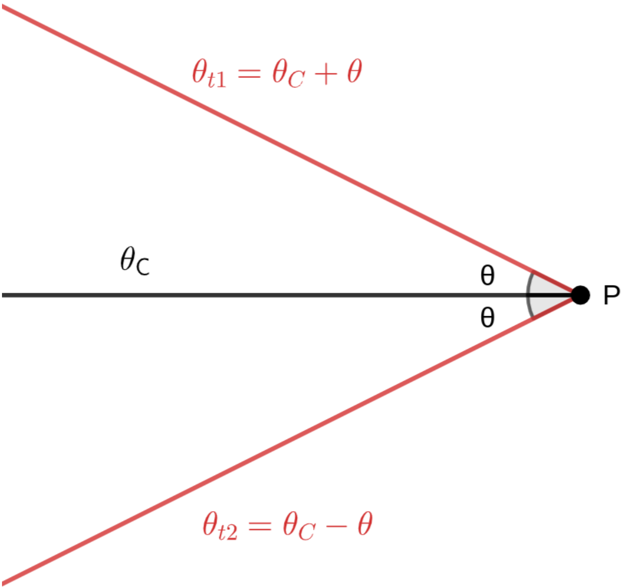


Figure: "tag-graph-map-of-stackexchange" [2]



i1
t1
r1
C1
d
i3
r2
C2
h2
P
i4
i2
t2
h1

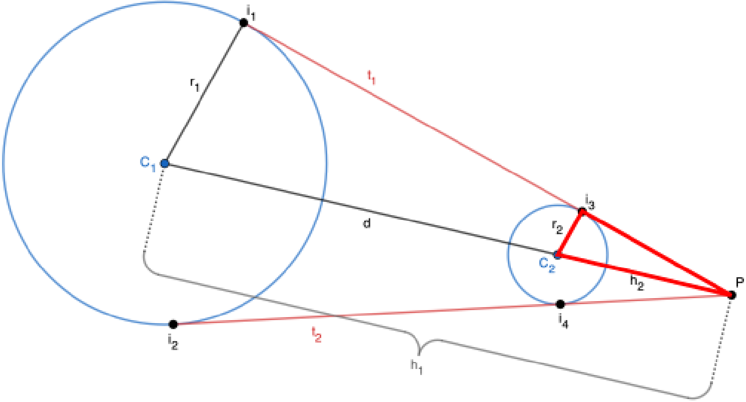


$\theta_{t1} = \theta_C + \theta$
θ_C
θ
θ
P
$\theta_{t2} = \theta_C - \theta$

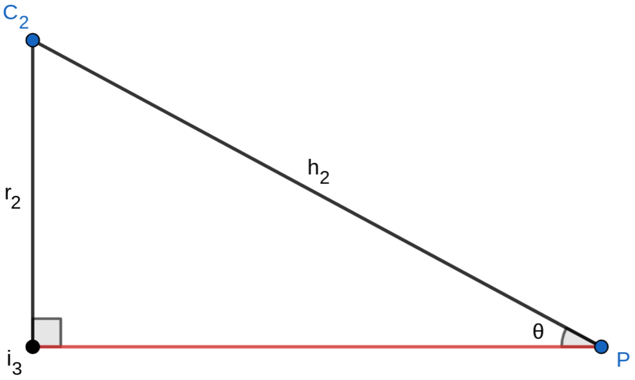
$$\frac{r_1}{r_2} = \frac{h_1}{h_2}$$

$$\frac{r_1}{r_2} = \frac{h_2 + d}{h_2}$$

$$
\begin{aligned}
\frac{r_1}{r_2} &= \frac{h_2 + d}{h_2} \\
r_1 h_2 &= r_2(h_2 + d) \\
r_1 h_2 &= r_2 h_2 + r_2 d \\
r_1 h_2 &= r_2 h_2 + r_2 d \\
r_1 h_2 - r_2 h_2 &= r_2 d \\
h_2(r_1 - r_2) &= r_2 d \\
h_2 &= \frac{r_2 d}{r_1 - r_2}
\end{aligned}
$$



i_1
r_1
t_1
C_1
d
i_3
r_2
C_2
h_2
P
i_4
i_2
t_2
h_1


C_2
h_2
r_2
θ
i_3
P

$$\sin\theta = \frac{r_2}{h_2}$$

$$\theta = \text{arcsin}\left(\frac{r_2}{h_2}\right)$$

$$\theta_t = \theta_C \pm \theta$$

$$\theta_t = \arctan\left(\frac{y_1 - y_2}{x_1 - x_2}\right) \pm \arcsin\left(\frac{r_2}{h_2}\right)$$

$$h_2 = \frac{r_2 d}{r_1 - r_2}$$

$$\theta_t = \arctan\left(\frac{y_1 - y_2}{x_1 - x_2}\right) \pm \arcsin\left(\frac{r_2}{\frac{r_2 d}{r_1 - r_2}}\right)$$

$$\theta_t = \arctan\left(\frac{y_1 - y_2}{x_1 - x_2}\right) \pm \arcsin\left(\frac{r_2(r_1 - r_2)}{r_2 d}\right)$$

$$\theta_t = \arctan\left(\frac{y_1 - y_2}{x_1 - x_2}\right) \pm \arcsin\left(\frac{r_1 - r_2}{d}\right)$$

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$\theta_t = \arctan\left(\frac{y_1 - y_2}{x_1 - x_2}\right) \pm \arcsin\left(\frac{r_1 - r_2}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}\right)$$

$$S_t = \tan\left(\arctan\left(\frac{y_1 - y_2}{x_1 - x_2}\right) \pm \arcsin\left(\frac{r_1 - r_2}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}\right)\right)$$

$$\tan(\alpha \pm \beta) = \frac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha)\tan(\beta)}$$

$$\tan(\alpha \pm \beta) = \frac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha)\tan(\beta)}$$

$$S_t = \tan\left(\arctan\left(\frac{y_1 - y_2}{x_1 - x_2}\right) \pm \arcsin\left(\frac{r_1 - r_2}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}\right)\right)$$

$$\alpha = \arctan\left(\frac{y_1 - y_2}{x_1 - x_2}\right)$$

$$\beta = \arcsin\left(\frac{r_1 - r_2}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}\right)$$

$$\tan(\alpha) = \tan\left(\arctan\left(\frac{y_1 - y_2}{x_1 - x_2}\right)\right)$$

$$\tan(\alpha) = \frac{y_1 - y_2}{x_1 - x_2}$$

$$\tan\left(\arcsin\left(\frac{r_1 - r_2}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}\right)\right) = \tan(\beta)$$

$$\tan(\arcsin(x)) = \frac{x}{\sqrt{1 - x^2}}$$

$$\tan(\beta) = \frac{\frac{r_1 - r_2}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}}}{\sqrt{1 - \left(\frac{r_1-r_2}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}}\right)^2}}$$

$$\tan(\beta) = \frac{\frac{r_1 - r_2}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}}}{\sqrt{1 - \frac{(r_1-r_2)^2}{(x_1-x_2)^2+(y_1-y_2)^2}}}$$

$$\tan(\beta) = \frac{\frac{r_1 - r_2}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}}}{\sqrt{\frac{(x_1-x_2)^2+(y_1-y_2)^2}{(x_1-x_2)^2+(y_1-y_2)^2} - \frac{(r_1-r_2)^2}{(x_1-x_2)^2+(y_1-y_2)^2}}}$$

$$\tan(\beta) = \frac{\frac{r_1 - r_2}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}}}{\sqrt{\frac{(x_1-x_2)^2+(y_1-y_2)^2-(r_1-r_2)^2}{(x_1-x_2)^2+(y_1-y_2)^2}}}$$

$$\tan(\beta) = \frac{\frac{r_1 - r_2}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}}}{\sqrt{\frac{(x_1-x_2)^2+(y_1-y_2)^2-(r_1-r_2)^2}{(x_1-x_2)^2+(y_1-y_2)^2}}}$$

$$\tan(\beta) = \frac{\frac{r_1 - r_2}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}}}{\frac{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2-(r_1-r_2)^2}}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}}}$$

$$\tan(\beta) = \frac{r_1 - r_2}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2-(r_1-r_2)^2}}$$

$$\tan(\alpha) = \frac{y_1 - y_2}{x_1 - x_2}$$

$$\tan(\beta) = \frac{r_1 - r_2}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 - (r_1 - r_2)^2}}$$

$$\tan(\alpha \pm \beta) = \frac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha)\tan(\beta)}$$

$$S_t = \frac{\frac{y_1-y_2}{x_1-x_2} \pm \frac{r_1-r_2}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2-(r_1-r_2)^2}}}{1 \mp \frac{y_1-y_2}{x_1-x_2} \cdot \frac{r_1-r_2}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2-(r_1-r_2)^2}}}$$

$$S_t = \frac{\frac{y_1 - y_2}{x_1 - x_2} \pm \frac{r_1 - r_2}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 - (r_1 - r_2)^2}}}{1 \mp \frac{(y_1 - y_2)(r_1 - r_2)}{(x_1 - x_2)\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 - (r_1 - r_2)^2}}}$$

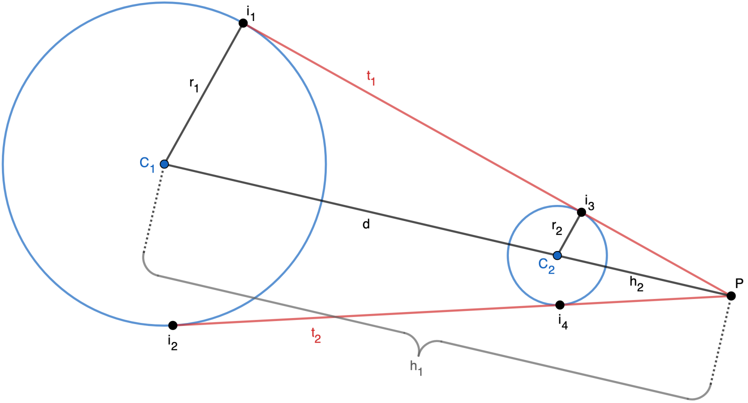
$$S_t = \frac{\frac{(y_1-y_2)\sqrt{(x_1-x_2)^2+(y_1-y_2)^2-(r_1-r_2)^2}}{(x_1-x_2)\sqrt{(x_1-x_2)^2+(y_1-y_2)^2-(r_1-r_2)^2}} \pm \frac{(x_1-x_2)(r_1-r_2)}{(x_1-x_2)\sqrt{(x_1-x_2)^2+(y_1-y_2)^2-(r_1-r_2)^2}}}{\frac{(x_1-x_2)\sqrt{(x_1-x_2)^2+(y_1-y_2)^2-(r_1-r_2)^2}}{(x_1-x_2)\sqrt{(x_1-x_2)^2+(y_1-y_2)^2-(r_1-r_2)^2}} \mp \frac{(y_1-y_2)(r_1-r_2)}{(x_1-x_2)\sqrt{(x_1-x_2)^2+(y_1-y_2)^2-(r_1-r_2)^2}}}$$

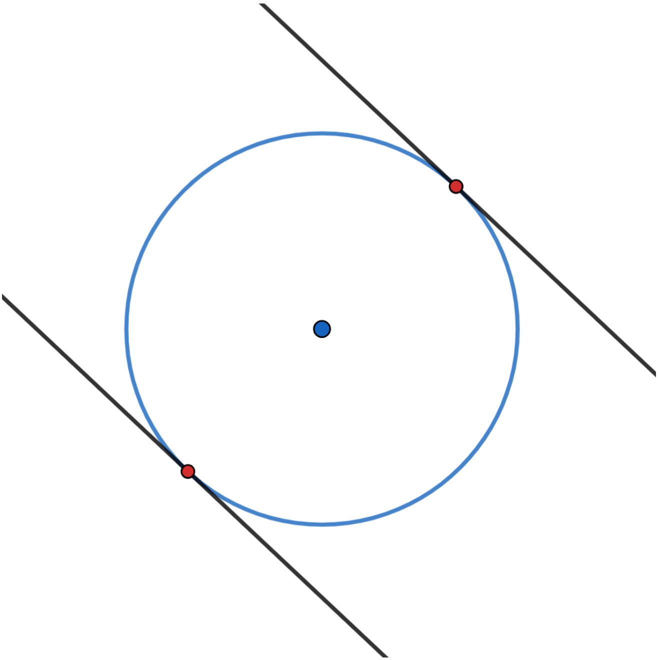
$$S_t = \frac{(y_1 - y_2)\sqrt{(x_1-x_2)^2+(y_1-y_2)^2-(r_1-r_2)^2} \pm (x_1-x_2)(r_1-r_2)}{(x_1 - x_2)\sqrt{(x_1-x_2)^2+(y_1-y_2)^2-(r_1-r_2)^2} \mp (y_1-y_2)(r_1-r_2)}$$

$$S_t = \tan\left(\arctan\left(\frac{y_1-y_2}{x_1-x_2}\right) \pm \arcsin\left(\frac{r_1-r_2}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}}\right)\right)$$

$$a = x_1 - x_2 \qquad b = y_1 - y_2 \qquad c = r_1 - r_2$$

$$S_t = \frac{b\sqrt{a^2 + b^2 - c^2} \pm ac}{a\sqrt{a^2 + b^2 - c^2} \mp bc}$$


i_1
t_1
r_1
C_1
d
i_3
r_2
C_2
h_2
P
i_4
i_2
t_2
h_1



$$
\begin{aligned}
(x - x_1)^2 + (y - y_1)^2 &= r_1^2 \\
(y - y_1)^2 &= r_1^2 - (x - x_1)^2 \\
y - y_1 &= \pm\sqrt{r_1^2 - (x - x_1)^2} \\
y &= \pm\sqrt{r_1^2 - (x - x_1)^2} + y_1
\end{aligned}
$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[y] = \frac{\mathrm{d}}{\mathrm{d}x}\left[\pm\sqrt{r_1^2 - (x - x_1)^2} + y_1\right]$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \pm\frac{\mathrm{d}}{\mathrm{d}x}\left[\sqrt{r_1^2 - (x - x_1)^2}\right]$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \pm\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(r_1^2 - (x - x_1)^2\right)^{\frac{1}{2}}\right]$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \pm\frac{1}{2}\left(r_1^2 - (x - x_1)^2\right)^{-\frac{1}{2}}\left(\frac{\mathrm{d}}{\mathrm{d}x}\left[r_1^2 - (x - x_1)^2\right]\right)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \pm\frac{1}{2\sqrt{r_1^2 - (x - x_1)^2}}\left(-\frac{\mathrm{d}}{\mathrm{d}x}\left[(x - x_1)^2\right]\right)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \pm\frac{1}{2\sqrt{r_1^2 - (x - x_1)^2}}\left(2(x - x_1)(1)\right)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \pm\frac{2(x - x_1)}{2\sqrt{r_1^2 - (x - x_1)^2}}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \pm\frac{x - x_1}{\sqrt{r_1^2 - (x - x_1)^2}}$$

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} = S_t &= a \\
r_1 &= b \\
x = x_i &= c \\
x_1 &= d
\end{aligned}
$$

$$a = \pm \frac{c-d}{\sqrt{b^2-(c-d)^2}}$$

$$\pm a\sqrt{b^2-(c-d)^2} = c-d$$

$$\left(\pm a\sqrt{b^2-(c-d)^2}\right)^2 = (c-d)^2$$

$$a^2(b^2-(c-d)^2) = (c-d)^2$$

$$a^2(b^2-(c^2-2cd+d^2)) = c^2-2cd+d^2$$

$$a^2(b^2-c^2+2cd-d^2) = c^2-2cd+d^2$$

$$a^2b^2-a^2c^2+2a^2cd-a^2d^2 = c^2-2cd+d^2$$

$$a^2b^2-a^2c^2+2a^2cd-a^2d^2-c^2+2cd-d^2 = 0$$

$$-a^2c^2-c^2+2a^2cd+2cd+a^2b^2-a^2d^2-d^2 = 0$$

$$c^2(-a^2-1)+c(2a^2d+2d)+(a^2b^2-a^2d^2-d^2) = 0$$

$$c = \frac{-(2a^2d+2d)\pm\sqrt{(2a^2d+2d)^2-4(-a^2-1)(a^2b^2-a^2d^2-d^2)}}{2(-a^2-1)}$$

$$(2a^2d + 2d)^2 - 4(-a^2 - 1)(a^2b^2 - a^2d^2 - d^2)$$
$$4a^4d^2 + 8a^2d^2 + 4d^2 - 4(-a^4b^2 + a^4d^2 + a^2d^2 - a^2b^2 + a^2d^2 + d^2)$$
$$4a^4d^2 + 8a^2d^2 + 4d^2 + 4a^4b^2 - 4a^4d^2 - 4a^2d^2 + 4a^2b^2 - 4a^2d^2 - 4d^2$$
$$(4a^4d^2 - 4a^4d^2) + (8a^2d^2 - 4a^2d^2 - 4a^2d^2) + (4d^2 - 4d^2) + 4a^4b^2 + 4a^2b^2$$
$$4a^4b^2 + 4a^2b^2$$

$$c = \frac{-(2a^2d + 2d) \pm \sqrt{4a^4b^2 + 4a^2b^2}}{2(-a^2 - 1)}$$

$$c = \frac{-2d(a^2 + 1) \pm \sqrt{4a^4b^2 + 4a^2b^2}}{-2(a^2 + 1)}$$

$$c = \frac{-2d(a^2 + 1) \pm 2ab\sqrt{a^2 + 1}}{-2(a^2 + 1)}$$

$$c = \frac{d(a^2 + 1) \mp ab\sqrt{a^2 + 1}}{a^2 + 1}$$

$$c = \frac{d(a^2 + 1)}{a^2 + 1} \mp \frac{ab\sqrt{a^2 + 1}}{a^2 + 1}$$

$$c = d \mp \frac{ab\sqrt{a^2 + 1}}{a^2 + 1}$$

$$c = d \mp \frac{ab(a^2 + 1)^{\frac{1}{2}}}{(a^2 + 1)^1}$$

$$c = d \mp ab(a^2 + 1)^{-\frac{1}{2}}$$
$$c = d \mp ab \cdot \frac{1}{(a^2 + 1)^{\frac{1}{2}}}$$
$$c = d \mp \frac{ab}{\sqrt{a^2 + 1}}$$

$$x_i = x_1 \mp \frac{S_t r_1}{\sqrt{S_t^2 + 1}}$$

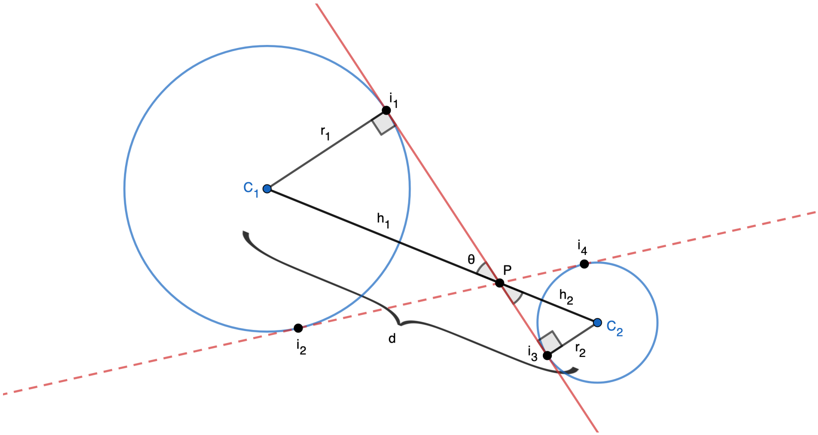
$$y_i = \pm\sqrt{r_1^2 - (x_i - x_1)^2} + y_1$$

$$S_t = \frac{(y_1 - y_2)\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 - (r_1 - r_2)^2} \pm (x_1 - x_2)(r_1 - r_2)}{(x_1 - x_2)\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 - (r_1 - r_2)^2} \mp (y_1 - y_2)(r_1 - r_2)}$$

$$x_i = x_1 \mp \frac{S_t r_1}{\sqrt{S_t^2 + 1}}$$

$$y_i = \pm\sqrt{r_1^2 - (x_i - x_1)^2} + y_1$$

$$y = S_t(x - x_i) + y_i$$

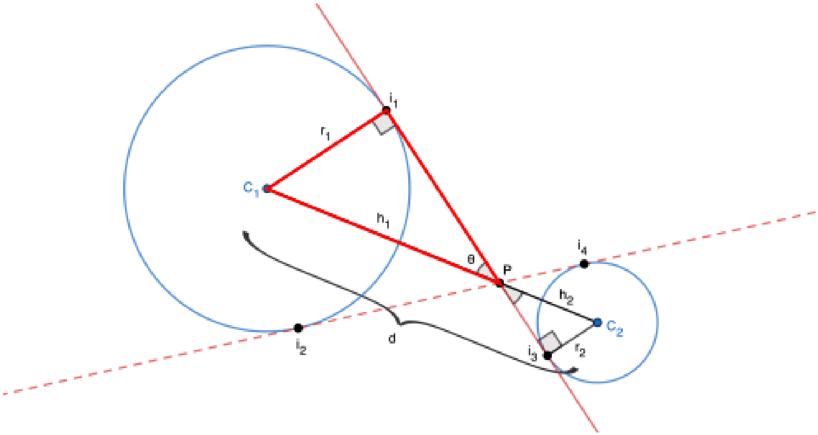


i_1
r_1
C_1
h_1
θ
P
i_4
h_2
C_2
i_2
d
i_3
r_2

$$\frac{r_2}{r_1} = \frac{h_2}{h_1}$$

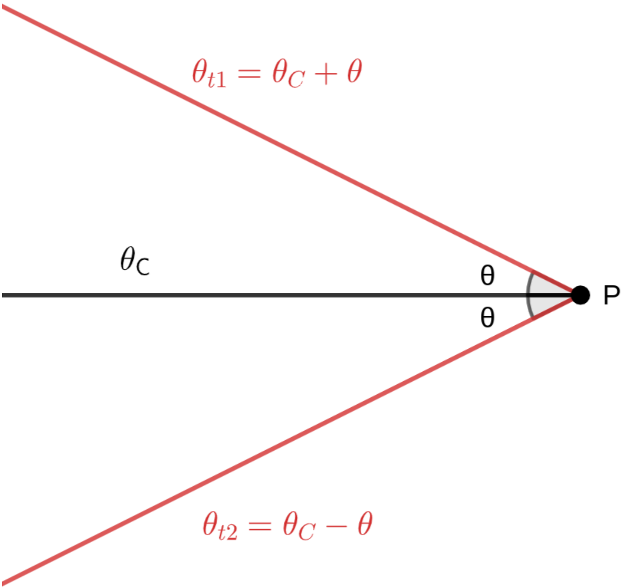
$$\frac{r_2}{r_1} = \frac{d - h_1}{h_1}$$

$$
\begin{aligned}
\frac{r_2}{r_1} &= \frac{d - h_1}{h_1} \\
\frac{h_1 r_2}{r_1} &= d - h_1 \\
h_1 r_2 &= r_1 d - r_1 h_1 \\
h_1 r_2 + r_1 h_1 &= r_1 d \\
h_1(r_2 + r_1) &= r_1 d \\
h_1 &= \frac{r_1 d}{r_2 + r_1}
\end{aligned}
$$



i1
r1
C1
h1
θ
P
i4
h2
C2
i2
d
i3
r2

$$
\begin{aligned}
\sin(\theta) &= \frac{r_1}{h_1} \\
\sin(\theta) &= \frac{r_1}{\frac{r_1 d}{r_2 + r_1}} \\
\sin(\theta) &= \frac{r_1(r_2 + r_1)}{r_1 d} \\
\sin(\theta) &= \frac{r_2 + r_1}{d} \\
\theta &= \arcsin\left(\frac{r_2 + r_1}{d}\right)
\end{aligned}
$$



$\theta_{t1} = \theta_C + \theta$
θ_C
θ
θ
P
$\theta_{t2} = \theta_C - \theta$

$$\theta_t = \theta_C \pm \theta$$

$$\theta_t = \arctan\left(\frac{y_1 - y_2}{x_1 - x_2}\right) \pm \arcsin\left(\frac{r_2 + r_1}{d}\right)$$

$$S_t = \tan\left(\arctan\left(\frac{y_1 - y_2}{x_1 - x_2}\right) \pm \arcsin\left(\frac{r_2 + r_1}{d}\right)\right)$$

$$S_{\text{outer tangents}} = \tan\left(\arctan\left(\frac{y_1-y_2}{x_1-x_2}\right) \pm \arcsin\left(\frac{r_1-r_2}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}}\right)\right)$$

$$S_{\text{inner tangents}} = \tan\left(\arctan\left(\frac{y_1-y_2}{x_1-x_2}\right) \pm \arcsin\left(\frac{r_1+r_2}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}}\right)\right)$$

$$a = x_1 - x_2 \qquad b = y_1 - y_2 \qquad c = r_1 \pm r_2$$

$$S_t = \frac{b\sqrt{a^2 + b^2 - c^2} \pm ac}{a\sqrt{a^2 + b^2 - c^2} \mp bc}$$

$$x_i = x_1 \mp \frac{S_t r_1}{\sqrt{S_t^2 + 1}} \qquad y_i = \pm\sqrt{r_1^2 - (x_i - x_1)^2} + y_1$$

$$y = S_t(x - x_i) + y_i$$

Image Attribution

- bukk. Via Wikimedia Commons. URL: https://commons.wikimedia.org/wiki/File:Bike_chain_guard_part.JPG.
- Piotr Migdał. URL: https://github.com/stared/tag-graph-map-of-stackexchange.

Resources

Go to https://romanhn.github.io/creations/tangents.html for:

- A transcript of the script
- A copy of the slideshow

I may add other resources later, like the .tex file for the slideshow and a interactive Desmos graph to demonstrate my system of equations.