Extended Essay in Mathematics

Graphing Tangent Lines to Two Circles

How can we mathematically model tangent lines to two circles on the coordinate plane?

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1 Introduction

A couple years ago, I sometimes liked to make art in Desmos, an online graphing calculator. One of my almost-finished pieces was a bicycle. I had written the equation for each line that made up the drawing of the bike, and I could even adjust the seat height and the pedal lengths with sliders. However, one feature of the bike was missing: the chain that would have connected the gears. The chains on a bicycle resemble two circles of different sizes with two outer tangent lines connecting them, so that's what I needed to graph in order to complete the image.

Figure 1: A bicycle chain, resembling circles with tangent lines [1]

However, I didn't know how to write the exact equation for a line tangent to two circles, and I could not find an answer to the problem online. Unfortunately, I was unable to solve this problem at the time, and I have since lost the original Desmos file.

In this paper, I will find a general system of equations for the outer tangents of two circles. First, I will use geometry and trigonometry to find the slopes of the tangents, and then, I will use calculus to find points on circle 1 which the tangents cross through. Finally, as an extension, I will adapt the system to generate equations for the inner tangents. The question I'd like to answer is "How can we mathematically model tangent lines to two circles on the coordinate plane?".

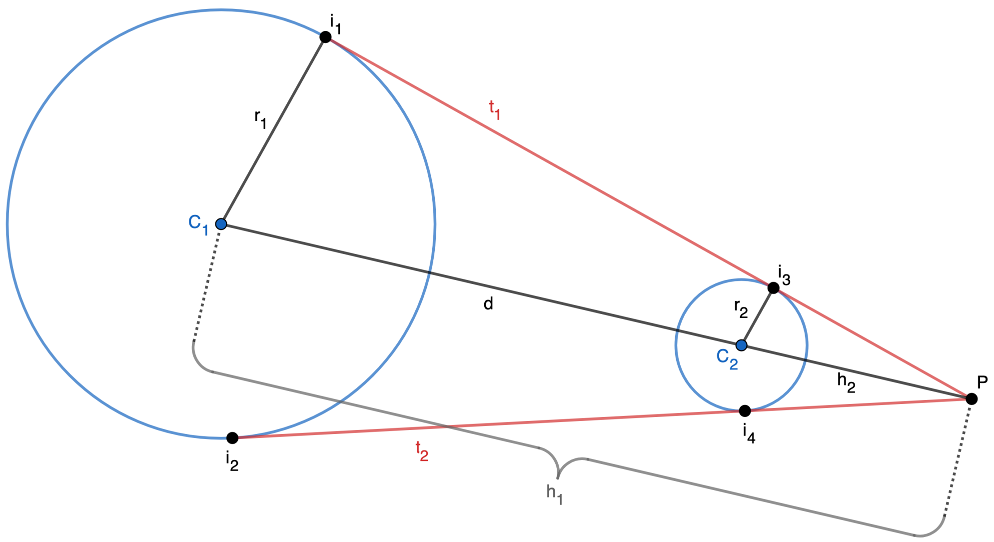
Much of geometry, and mathematics as a whole, is made up of equations derived from what is previously known. Basic situations such as tangent lines to circles deserve systems of equations that one can simply plug in values for. For example, a quick way to generate the graph of a line tangent to two circles could be useful for a programmer coding the graphics for a game, a physicist working with force vectors, or a drafter precisely defining outlines in a technical drawing.

2 External tangent lines

There are two circles:

- Circle 1 with center $C_1 = (x_1, y_1)$ and radius r_1
- Circle 2 with center $C_2 = (x_2, y_2)$ and radius r_2
- $r_1 \geq r_2$

Figure 2: External tangents to two circles



A line is drawn between the centers of the circles, called $C(x)$, and both external tangent lines $t_1(x)$ and $t_2(x)$ are drawn. The points of intersection between the tangent lines and circle 1 are $i_1 = (x_{i1}, y_{i1})$ and $i_2 = (x_{i2}, y_{i2})$. The points of intersection on circle 2 are $i_3 = (x_{i3}, y_{i3})$ and $i_4 = (x_{i4}, y_{i4})$.

All three lines intersect at point P. The angle between the first tangent line and the line that connects the centers of the circles is $\angle C_1Pi_1$.

2.1 Slopes of the external tangents

Angle $\angle C_1i_1P$ is a right angle because it is the angle between a tangent to a circle and a radius that passes through the point of tangency. The same is true for $\angle C_2i_3P$.

Because these right angles are congruent, and because $\triangle C_1Pi_1$ and $\triangle C_2Pi_2$ share the angle $\angle C_1Pi_1$, this makes the triangles similar.

Because these triangles are similar, the ratio between corresponding sides is constant. Therefore,

$$\frac{r_1}{r_2} = \frac{h_1}{h_2}$$

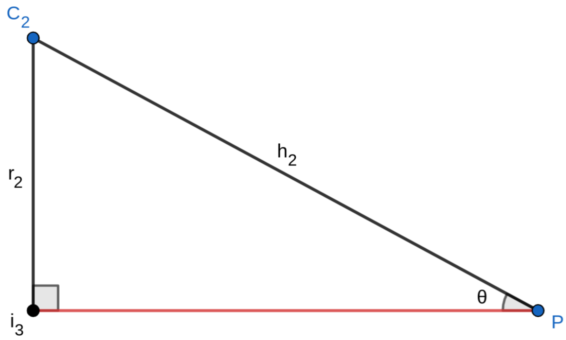
We can substitute h_1 with the two lengths that make it up: $h_2 + d$. Then, by rewriting the equation, we can solve for h_2.

$$\begin{aligned}
\frac{r_1}{r_2} &= \frac{h_2 + d}{h_2} \\
r_1h_2 &= r_2(h_2 + d) \\
r_1h_2 &= r_2h_2 + r_2d \\
r_1h_2 &= r_2h_2 + r_2d \\
r_1h_2 - r_2h_2 &= r_2d \\
h_2(r_1 - r_2) &= r_2d \\
h_2 &= \frac{r_2d}{r_1 - r_2}
\end{aligned}$$

Now that h_2 is defined, we can use trigonometry to define the angle between the line that connects the circles' centers and the first tangent line. This angle is $\angle C_2Pi_3$, or θ.

Consider triangle $\triangle C_2 i_3 P$.

Figure 3: Triangle $\triangle C_2 i_3 P$



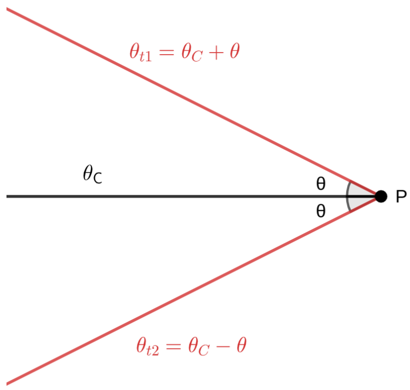
Because of the definition of the sine of an angle (the ratio between the opposing side and the hypotenuse in a right triangle), $\sin\theta = \frac{r_2}{h_2}$. By taking the inverse sine of each side, we can define θ:

$$\theta = \arcsin\left(\frac{r_2}{h_2}\right)$$

To find the angle of the first external tangent line, we can add θ to the angle of $C(x)$ with respect to the x-axis. Similarly, we can subtract θ from the angle of $C(x)$ to find the angle of the second tangent line.

$$\theta_t = \theta_C \pm \theta$$

Figure 4: Angles of external tangents and $C(x)$



But how do we know the angle of $C(x)$? The slope of $C(x)$ can be found by using the slope formula and substituting with the coordinates of the centers of the circles. Then, we can convert this slope into an angle by finding its inverse tangent.

$$\theta_C = \arctan\left(S_C\right)$$
$$\theta_C = \arctan\left(\frac{\Delta y}{\Delta x}\right)$$
$$\theta_C = \arctan\left(\frac{y_1 - y_2}{x_1 - x_2}\right)$$

Now, we can substitute the values of θ_C and θ:

$$\theta_t = \theta_C \pm \theta$$
$$\theta_C = \arctan\left(\frac{y_1 - y_2}{x_1 - x_2}\right)$$
$$\theta = \arcsin\left(\frac{r_2}{h_2}\right)$$
$$\theta_t = \arctan\left(\frac{y_1 - y_2}{x_1 - x_2}\right) \pm \arcsin\left(\frac{r_2}{h_2}\right)$$

Then, we can substitute for h_2 and simplify:

$$h_2 = \frac{r_2 d}{r_1 - r_2}$$
$$\theta_t = \arctan\left(\frac{y_1 - y_2}{x_1 - x_2}\right) \pm \arcsin\left(\frac{r_2}{\frac{r_2 d}{r_1 - r_2}}\right)$$
$$\theta_t = \arctan\left(\frac{y_1 - y_2}{x_1 - x_2}\right) \pm \arcsin\left(\frac{r_2(r_1 - r_2)}{r_2 d}\right)$$
$$\theta_t = \arctan\left(\frac{y_1 - y_2}{x_1 - x_2}\right) \pm \arcsin\left(\frac{r_1 - r_2}{d}\right)$$

We can then substitute for d. d is the distance between the centers of the circles, so we can use the distance formula with C_1 and C_2 as points.

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$
$$\theta_t = \arctan\left(\frac{y_1 - y_2}{x_1 - x_2}\right) \pm \arcsin\left(\frac{r_1 - r_2}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}\right)$$

Because the tangent of the angle of a line produces its slope, we can take the tangent of both sides to generate an equation for both of the slopes of the external tangent lines:

$$S_t = \tan\left(\arctan\left(\frac{y_1 - y_2}{x_1 - x_2}\right) \pm \arcsin\left(\frac{r_1 - r_2}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}\right)\right)$$

This equation, which uses trigonometric functions inside of a trigonometric function, is not very elegant. To rewrite it, we can use a trigonometric identity for the tangent of a sum or difference: [3]

$$\tan(\alpha \pm \beta) = \frac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha)\tan(\beta)}$$

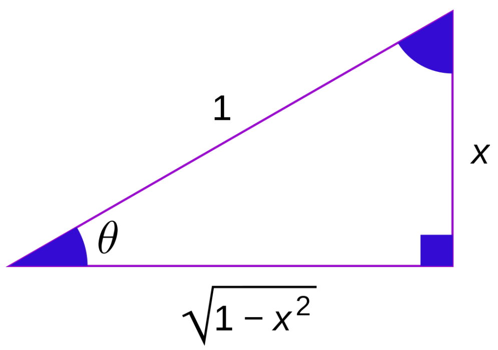
In this case, $\alpha = \arctan\left(\frac{y_1 - y_2}{x_1 - x_2}\right)$ and $\beta = \arcsin\left(\frac{r_1 - r_2}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}\right)$. For ease of explanation, we'll first find $\tan(\alpha)$ and $\tan(\beta)$ separately, then substitute them into the identity.

The tangent of α is simple to find, because the arctangent and the tangent functions cancel each other out:

$$\tan(\alpha) = \tan\left(\arctan\left(\frac{y_1 - y_2}{x_1 - x_2}\right)\right)$$
$$\tan(\alpha) = \frac{y_1 - y_2}{x_1 - x_2}$$

However, the tangent of β is trickier because the tangent function doesn't cancel out the arcsine function. To remedy this, we can use an identity for the tangent of an arcsine, as demonstrated by figure 5.

Figure 5: Right triangle demonstration of $\tan(\arcsin(x))$ [2]



In figure 5, the sine of angle θ is the ratio between the lengths of the opposite side and the hypotenuse, or $\frac{x}{1} = x$. Therefore, the arcsine of this ratio equals θ:

$$\arcsin(x) = \theta$$

The tangent of θ, or in other words, the tangent of $\arcsin(x)$, is equal to the ratio between the opposite side and the adjacent side. The length of the adjacent side can be found using the Pythagorean theorem: $\sqrt{1 - x^2}$. Therefore,

$$\tan(\arcsin(x)) = \frac{x}{\sqrt{1 - x^2}}$$

We can now use this identity to find $\tan(\beta)$:

$$\tan(\beta) = \tan\left(\arcsin\left(\frac{r_1 - r_2}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}\right)\right)$$

$$\tan(\beta) = \frac{\frac{r_1 - r_2}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}}}{\sqrt{1 - \left(\frac{r_1 - r_2}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}}\right)^2}}$$

$$\tan(\beta) = \frac{\frac{r_1 - r_2}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}}}{\sqrt{1 - \frac{(r_1-r_2)^2}{(x_1-x_2)^2+(y_1-y_2)^2}}}$$

$$\tan(\beta) = \frac{\frac{r_1 - r_2}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}}}{\sqrt{\frac{(x_1-x_2)^2+(y_1-y_2)^2}{(x_1-x_2)^2+(y_1-y_2)^2} - \frac{(r_1-r_2)^2}{(x_1-x_2)^2+(y_1-y_2)^2}}}$$

$$\tan(\beta) = \frac{\frac{r_1 - r_2}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}}}{\sqrt{\frac{(x_1-x_2)^2+(y_1-y_2)^2-(r_1-r_2)^2}{(x_1-x_2)^2+(y_1-y_2)^2}}}$$

$$\tan(\beta) = \frac{\frac{r_1 - r_2}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}}}{\frac{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2-(r_1-r_2)^2}}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}}}$$

$$\tan(\beta) = \frac{r_1 - r_2}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 - (r_1 - r_2)^2}}$$

Now that we know $\tan(\alpha)$ and $\tan(\beta)$, we can return to rewriting the slope equation using the identity for the tangent of a sum or difference.

$$S_t = \tan\left(\arctan\left(\frac{y_1 - y_2}{x_1 - x_2}\right) \pm \arcsin\left(\frac{r_1 - r_2}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}\right)\right)$$

$$\tan(\alpha \pm \beta) = \frac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha)\tan(\beta)}$$

$$S_t = \frac{\frac{y_1 - y_2}{x_1 - x_2} \pm \frac{r_1 - r_2}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 - (r_1 - r_2)^2}}}{1 \mp \frac{y_1 - y_2}{x_1 - x_2} \cdot \frac{r_1 - r_2}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 - (r_1 - r_2)^2}}}$$

$$S_t = \frac{\frac{y_1 - y_2}{x_1 - x_2} \pm \frac{r_1 - r_2}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 - (r_1 - r_2)^2}}}{1 \mp \frac{(y_1 - y_2)(r_1 - r_2)}{(x_1 - x_2)\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 - (r_1 - r_2)^2}}}$$

We can rewrite each term with a common denominator.

$$S_t = \frac{\frac{(y_1 - y_2)\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 - (r_1 - r_2)^2}}{(x_1 - x_2)\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 - (r_1 - r_2)^2}} \pm \frac{(x_1 - x_2)(r_1 - r_2)}{(x_1 - x_2)\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 - (r_1 - r_2)^2}}}{\frac{(x_1 - x_2)\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 - (r_1 - r_2)^2}}{(x_1 - x_2)\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 - (r_1 - r_2)^2}} \mp \frac{(y_1 - y_2)(r_1 - r_2)}{(x_1 - x_2)\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 - (r_1 - r_2)^2}}}$$

Finally, multiplying each side by the common denominator yields the final equation for the slopes of the external tangent lines:

$$S_t = \frac{(y_1 - y_2)\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 - (r_1 - r_2)^2} \pm (x_1 - x_2)(r_1 - r_2)}{(x_1 - x_2)\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 - (r_1 - r_2)^2} \mp (y_1 - y_2)(r_1 - r_2)}$$

At first glance, this equation doesn't appear any simpler than the one involving nested trigonometric functions. However, by applying the following substitutions, the symmetry and elegance of this equation becomes clear:

$$a = x_1 - x_2 \qquad b = y_1 - y_2 \qquad c = r_1 - r_2$$

$$S_t = \frac{b\sqrt{a^2 + b^2 - c^2} \pm ac}{a\sqrt{a^2 + b^2 - c^2} \mp bc}$$

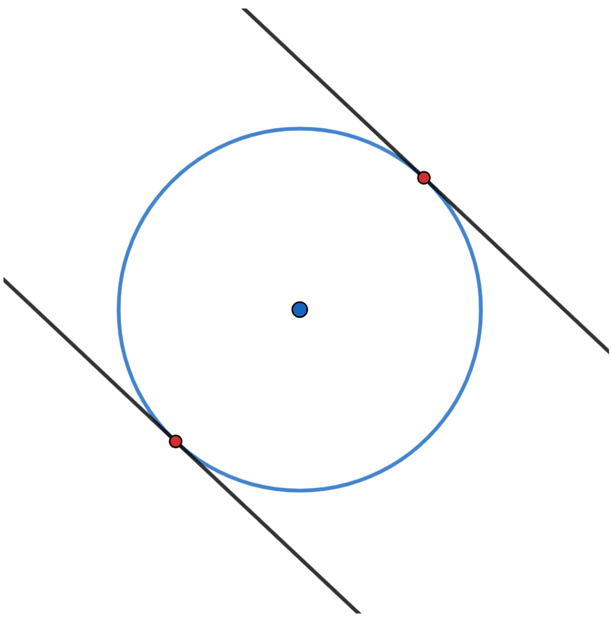
2.2 Points on the external tangents

In order to define an exact equation for the external tangents, defining the slopes isn't enough. We also need a point on each of the slopes, and the easiest points to use are the points of tangency on circle 1.

A point of tangency on circle 1 will have an instantaneous slope equal to that of the tangent line it corresponds to. Therefore, to figure out which points on circle 1 can qualify as points of tangency, we need to find the derivative (the slope at each point) of the equation of circle 1 and set it equal to the tangent's slope, and then solve for the x-value of the point of tangency. This can then be fed back into the circle equation to define the other coordinate of the point of tangency, which would complete the definition of the tangent line function.

This method presents one key problem, however: a circle equation is not a function, and so its derivative would not be a function either. At each x-value in the domain of the graph of the circle, there are two possible y-values and two possible slopes, and as a result, there are two points on circle 1 that could qualify as points of tangency for each tangent line. Because there are two tangent lines, this makes four possible points of tangency, but only two of them are correct.

Figure 6: Two points of tangency on a circle that have the same instantaneous slope



Ultimately, I decided to accept the fact that this approach would generate not two, but four possible external tangents, and someone using this equation would have to manually decide which definitions to use.

The first step is to find the derivative of the circle equation. However, this equation is not defined simply in terms of y, as it's based on the Pythagorean theorem.

$$(x - x_1)^2 + (y - y_1)^2 = r_1^2$$

Implicit differentiation would not work in this case, so we need to rewrite the circle equation in terms of y before we differentiate it with respect to x.

$$\begin{aligned}
(x - x_1)^2 + (y - y_1)^2 &= r_1^2 \\
(y - y_1)^2 &= r_1^2 - (x - x_1)^2 \\
y - y_1 &= \pm\sqrt{r_1^2 - (x - x_1)^2} \\
y &= \pm\sqrt{r_1^2 - (x - x_1)^2} + y_1
\end{aligned}$$

Now, we can find the derivative of this equation to find the slope of any given point on circle 1.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}[y] &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\pm\sqrt{r_1^2 - (x - x_1)^2} + y_1\right] \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \pm\frac{\mathrm{d}}{\mathrm{d}x}\left[\sqrt{r_1^2 - (x - x_1)^2}\right] \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \pm\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(r_1^2 - (x - x_1)^2\right)^{\frac{1}{2}}\right]
\end{aligned}$$

We can use the chain rule for derivatives to continue.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \pm\frac{1}{2}\left(r_1^2-(x-x_1)^2\right)^{-\frac{1}{2}}\left(\frac{\mathrm{d}}{\mathrm{d}x}\left[r_1^2-(x-x_1)^2\right]\right)\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \pm\frac{1}{2\sqrt{r_1^2-(x-x_1)^2}}\left(-\frac{\mathrm{d}}{\mathrm{d}x}\left[(x-x_1)^2\right]\right)\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \pm\frac{1}{2\sqrt{r_1^2-(x-x_1)^2}}\left(2(x-x_1)(1)\right)\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \pm\frac{2(x-x_1)}{2\sqrt{r_1^2-(x-x_1)^2}}\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \pm\frac{x-x_1}{\sqrt{r_1^2-(x-x_1)^2}}
\end{aligned}$$

This resulting equation gives the relationship between the slope of the tangent line ($\frac{\mathrm{d}y}{\mathrm{d}x}$) and the x-coordinate of the point of tangency on circle 1.

We need to rewrite this equation in terms of x, because in this case, x is the x-coordinate of the point of tangency, which will be used in the tangent line equation. However, the algebra that enables us to do this is long and confusing. To help make the process easier to read, we will temporarily reassign each variable to a single letter each:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} = S_t &= a\\
r_1 &= b\\
x = x_i &= c\\
x_1 &= d
\end{aligned}$$

Essentially, we can rewrite the equation as a sum of terms, which can be rearranged into a quadratic equation with multi-term coefficients. These can be plugged into the quadratic formula in order to solve for c, which represents x_i.

$$
\begin{aligned}
a &= \pm\frac{c-d}{\sqrt{b^2-(c-d)^2}} \\
\pm a\sqrt{b^2-(c-d)^2} &= c-d \\
\left(\pm a\sqrt{b^2-(c-d)^2}\right)^2 &= (c-d)^2 \\
a^2(b^2-(c-d)^2) &= (c-d)^2 \\
a^2(b^2-(c^2-2cd+d^2)) &= c^2-2cd+d^2 \\
a^2(b^2-c^2+2cd-d^2) &= c^2-2cd+d^2 \\
a^2b^2-a^2c^2+2a^2cd-a^2d^2 &= c^2-2cd+d^2 \\
a^2b^2-a^2c^2+2a^2cd-a^2d^2-c^2+2cd-d^2 &= 0 \\
-a^2c^2-c^2+2a^2cd+2cd+a^2b^2-a^2d^2-d^2 &= 0 \\
c^2(-a^2-1)+c(2a^2d+2d)+(a^2b^2-a^2d^2-d^2) &= 0
\end{aligned}
$$

Now that the equation is in the form of a quadratic, we can use the quadratic formula to determine the solutions for c.

$$
c = \frac{-(2a^2d+2d) \pm \sqrt{(2a^2d+2d)^2-4(-a^2-1)(a^2b^2-a^2d^2-d^2)}}{2(-a^2-1)}
$$

For readability, we'll simplify the radicand in the quadratic formula separately:

$$
\begin{gathered}
(2a^2d+2d)^2-4(-a^2-1)(a^2b^2-a^2d^2-d^2) \\
4a^4d^2+8a^2d^2+4d^2-4(-a^4b^2+a^4d^2+a^2d^2-a^2b^2+a^2d^2+d^2) \\
4a^4d^2+8a^2d^2+4d^2+4a^4b^2-4a^4d^2-4a^2d^2+4a^2b^2-4a^2d^2-4d^2 \\
(4a^4d^2-4a^4d^2)+(8a^2d^2-4a^2d^2-4a^2d^2)+(4d^2-4d^2)+4a^4b^2+4a^2b^2 \\
4a^4b^2+4a^2b^2
\end{gathered}
$$

Then, we can substitute the radicand back into the quadratic formula and continue simplifying.

$$
\begin{aligned}
c &= \frac{-(2a^2d + 2d) \pm \sqrt{4a^4b^2 + 4a^2b^2}}{2(-a^2 - 1)} \\
c &= \frac{-2d(a^2 + 1) \pm \sqrt{4a^4b^2 + 4a^2b^2}}{-2(a^2 + 1)} \\
c &= \frac{-2d(a^2 + 1) \pm 2ab\sqrt{a^2 + 1}}{-2(a^2 + 1)} \\
c &= \frac{d(a^2 + 1) \mp ab\sqrt{a^2 + 1}}{a^2 + 1} \\
c &= \frac{d(a^2 + 1)}{a^2 + 1} \mp \frac{ab\sqrt{a^2 + 1}}{a^2 + 1} \\
c &= d \mp \frac{ab\sqrt{a^2 + 1}}{a^2 + 1} \\
c &= d \mp \frac{ab(a^2 + 1)^{\frac{1}{2}}}{(a^2 + 1)^1} \\
c &= d \mp ab(a^2 + 1)^{-\frac{1}{2}} \\
c &= d \mp ab \cdot \frac{1}{(a^2 + 1)^{\frac{1}{2}}} \\
c &= d \mp \frac{ab}{\sqrt{a^2 + 1}}
\end{aligned}
$$

We can now reverse the temporary substitutions to give us a definition of the x-coordinate of the point of tangency on circle 1.

$$
a = S_t \qquad b = r_1 \qquad c = x_i \qquad d = x_1
$$

$$
x_i = x_1 \mp \frac{S_t r_1}{\sqrt{S_t^2 + 1}}
$$

This equation is in terms of the slope of the tangent line, the x-coordinate of the center of circle 1, and the radius of circle 1. However, it's important to note that the tangent's slope is itself defined by other given values in the problem.

Earlier, we manipulated the circle equation so that it was in terms of y, which allowed us to find its derivative with respect to x. We can substitute our definition of the x-coordinate of the point of tangency into this circle equation, allowing us to define the corresponding y-coordinate.

$$y_i = \pm\sqrt{r_1^2 - (x_i - x_1)^2} + y_1$$

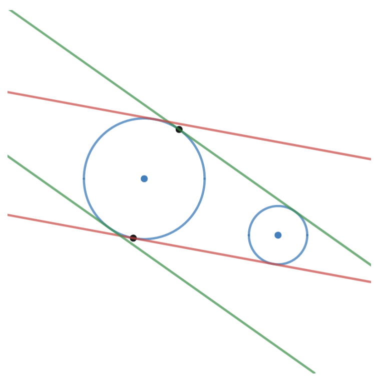
2.3 Final external tangent model

The slopes of the tangent lines, the x-coordinates of the points of tangency, and the y-coordinates of the points of tangency have all been defined in terms of the original parameters of the problem. By using the point-slope equation $f(x) = m(x - x_1) + y_1$, the final system of equations can be displayed:

$$\begin{aligned}
S_t &= \frac{(y_1 - y_2)\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 - (r_1 - r_2)^2} \pm (x_1 - x_2)(r_1 - r_2)}{(x_1 - x_2)\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 - (r_1 - r_2)^2} \mp (y_1 - y_2)(r_1 - r_2)} \\
x_i &= x_1 \mp \frac{S_t r_1}{\sqrt{S_t^2 + 1}} \\
y_i &= \pm\sqrt{r_1^2 - (x_i - x_1)^2} + y_1 \\
y &= S_t(x - x_i) + y_i
\end{aligned}$$

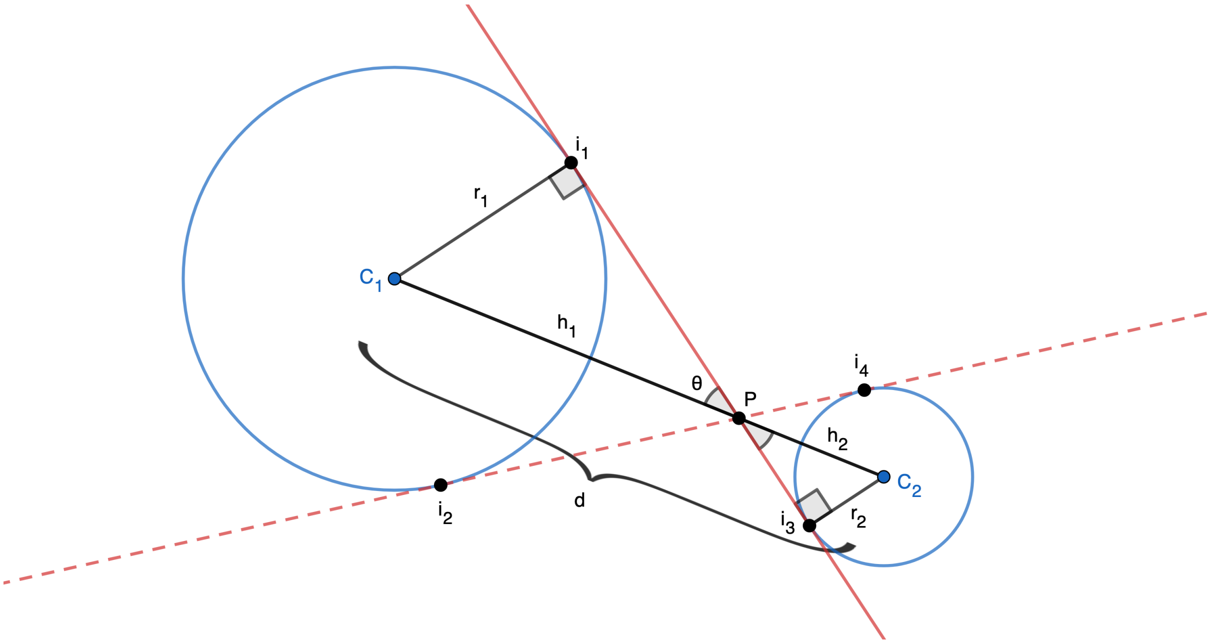
Unfortunately, because of the ambiguity of the plus-or-minus operators in the system of equations, two of the possible equations generated are not correct, and so the correct equations must be chosen manually.

Figure 7: Four lines generated by the external tangent system



3 Internal tangent lines

Figure 8: Internal tangents to two circles



The process for modelling the internal tangents to two circles is very similar to the process for external tangents. In figure 2, two similar triangles were used to write a ratio involving the radii and the hypotenuses. In figure 8, triangles $\triangle C_1Pi_1$ and $\triangle C_2Pi_3$ are similar because they share the vertical angle θ, and they both have a right angle. This allows us to equate the following ratios between corresponding sides:

$$\frac{r_2}{r_1} = \frac{h_2}{h_1}$$

We can rewrite h_2 as $d - h_1$ and then solve for h_1:

$$\frac{r_2}{r_1} = \frac{d - h_1}{h_1}$$
$$\frac{h_1 r_2}{r_1} = d - h_1$$
$$h_1 r_2 = r_1 d - r_1 h_1$$
$$h_1 r_2 + r_1 h_1 = r_1 d$$
$$h_1(r_2 + r_1) = r_1 d$$
$$h_1 = \frac{r_1 d}{r_2 + r_1}$$

This definition of h_1 allows us to find θ using trigonometry. We can express the sine of θ as the ratio between the opposite side in the right triangle and the hypotenuse:

$$\sin(\theta) = \frac{r_1}{h_1}$$

Then, we can substitute for h_1 and solve for θ.

$$\sin(\theta) = \frac{r_1}{\frac{r_1 d}{r_2 + r_1}}$$
$$\sin(\theta) = \frac{r_1(r_2 + r_1)}{r_1 d}$$
$$\sin(\theta) = \frac{r_2 + r_1}{d}$$
$$\theta = \arcsin\left(\frac{r_2 + r_1}{d}\right)$$

Just like the external tangents, the angle of the internal tangents is simply the angle of $C(x)$, plus or minus the angle θ (see figure 4). Because we already found the angle of $C(x)$ (called θ_C) when we modeled the external tangents, we can use a similar formula as before:

$$\begin{aligned}
\theta_t &= \theta_C \pm \theta \\
\theta_C &= \arctan\left(\frac{y_1 - y_2}{x_1 - x_2}\right) \\
\theta_t &= \arctan\left(\frac{y_1 - y_2}{x_1 - x_2}\right) \pm \arcsin\left(\frac{r_2 + r_1}{d}\right)
\end{aligned}$$

Then, because the tangent of the angle of a line equals the slope of the line, we can take the tangent of each side to find the slope of the internal tangent lines.

$$S_t = \tan\left(\arctan\left(\frac{y_1 - y_2}{x_1 - x_2}\right) \pm \arcsin\left(\frac{r_2 + r_1}{d}\right)\right)$$

Finally, we can substitute d with the formula for the distance between C_1 and C_2.

$$S_t = \tan\left(\arctan\left(\frac{y_1 - y_2}{x_1 - x_2}\right) \pm \arcsin\left(\frac{r_1 + r_2}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}\right)\right)$$

This is very similar to the formula for the slope of the external tangent lines. In fact, the sole difference between the formulas is that the external tangent equation uses the difference between the circles' radii, while the internal tangent equation uses the sum of the radii.

$$\begin{aligned}
S_{\text{outer tangents}} &= \tan\left(\arctan\left(\frac{y_1 - y_2}{x_1 - x_2}\right) \pm \arcsin\left(\frac{r_1 - r_2}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}\right)\right) \\
S_{\text{inner tangents}} &= \tan\left(\arctan\left(\frac{y_1 - y_2}{x_1 - x_2}\right) \pm \arcsin\left(\frac{r_1 + r_2}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}\right)\right)
\end{aligned}$$

Conveniently, this difference is consistent throughout the rest of the problem. At no point during the problem-solving process involving the external tangents was the term $r_1 - r_2$ separated or modified, which means that all we need to do to adapt our model to use for the internal tangents is to replace $r_1 - r_2$ with $r_1 + r_2$. Or, to represent both set of tangents in the same system of equations, we can use $r_1 \pm r_2$.

If we rearrange the previous system of equations for external tangent lines, as well as replace $r_1 - r_2$ with $r_1 \pm r_2$, the updated system of equations can be displayed. We can mathematically model tangent lines to two circles on the coordinate plane using the following system of equations:

$$a = x_1 - x_2 \qquad b = y_1 - y_2 \qquad c = r_1 \pm r_2$$

$$S_t = \frac{b\sqrt{a^2 + b^2 - c^2} \pm ac}{a\sqrt{a^2 + b^2 - c^2} \mp bc}$$

$$x_i = x_1 \mp \frac{S_t r_1}{\sqrt{S_t^2 + 1}} \qquad y_i = \pm\sqrt{r_1^2 - (x_i - x_1)^2} + y_1$$

$$y = S_t(x - x_i) + y_i$$

Note that in each of the first three lines in the system of equations, there are one or more uses of the plus-or-minus or minus-or-plus operator. This means that there are $2^3 = 8$ possible linear functions that this system could generate, depending on which option (plus or minus) one chooses for each line in the system. Only four of these functions, however, will represent actual tangents, internal or external, to the circles.

4 Limitations of the model

As previously noted, the biggest limitation in this model is that it's ambiguous: only half of the lines it produces are real tangent lines, so someone using the model would need to manually choose which set to use. Additionally, the model is limited to tangent lines to circles. One possible extension of the problem could be to generalize the model to include tangent lines to ellipses. Finally, there are some situations in which the model breaks – for example, when one or more of the tangent lines would be vertical, the model produces an undefined slope.

5 Conclusion

This general system of equations for tangents to circles might help someone working in a math-related field to precisely define these tangents, rather than needing to estimate them. For example, a programmer writing the code for the graphics in a video game or an art program could use this system if they needed to generate tangent lines precisely, or if they needed to generate them multiple times.

A drafter needs to be precise in their drawings because the smallest error or ambiguity could be magnified when their design is manufactured at a large scale. They may, for example, know that the wall of a building is tangent to two predefined circular corners, but they might not know the precise angle of the wall to label on their diagram. Using this system, they could calculate the exact dimensions of the wall to make their diagram more robust.

The process of creating a model for tangent lines to circles demonstrates an important feature of mathematics as a whole: the interconnectedness of different fields in the subject. In just this one problem, techniques from a variety of areas of mathematics were used, including...

- geometry, to identify relationships between variables in the problem and solve for certain values;
- trigonometry, to work with angles, translate them into the coordinate plane, and help with algebraic manipulation;
- calculus, to find the slope of the circle's graph at any given point; and
- algebra, to manipulate the variables and provide a backbone for the problem-solving process.

Oftentimes, the divisions between fields of mathematics are given too much emphasis. Math is less of a collection of related topics and more of a singular area of knowledge with broad, overlapping sections. Much like the areas in a human brain, the boundaries between these sections can be fuzzy. While it can be useful to, for example, divide math lessons into distinct units, this can often lead students to overlook the conceptual fibers that hold mathematics together. As they solve practice problems that focus on a single concept or unit at a time, they might not realize that the true potential of math lies in the combination of different techniques.

References

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