

Introduction to Program Synthesis (WS 2024/25)

Chapter 2 - Computer Programs

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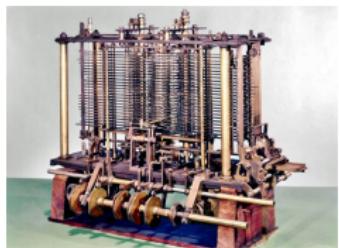
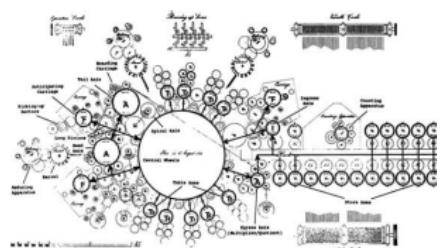
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Computer Programs: Historical Background

Analytical Engine

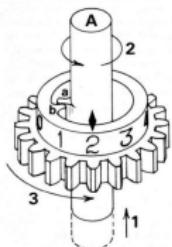
- ▶ Early version of a digital mechanical general-purpose computer
- ▶ First proposed by Charles Babbage (1791 - 1871) in 1837
- ▶ Never actually built to full extend
 - ▶ Would have been the first mechanical universal computer
- ▶ Concept provides many essential features used by modern digital computers
 - ▶ Arithmetic logic unit, control flow (i.e. conditional branching and looping), integrated memory, ...



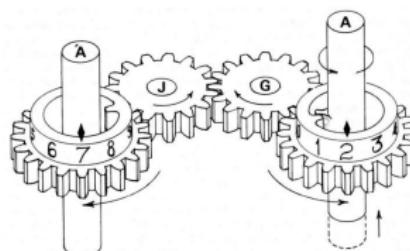
Computer Programs: Historical Background

Analytical Engine

- ▶ Data storage and transfer: *Figure wheels* can be rotated to any of the 10 digit positions (0 - 9)
 - ▶ Multiple *digit wheels* on a axis are used to store numbers
- ▶ Basic addition of digits is achieved with rotations and its propagation via pinions
 - ▶ A digit is given off by one figure wheel and received by the other figure wheel
 - ▶ Process is done simultaneously for all digits of the corresponding numbers



(a) Figure wheel



(b) Basic addition

Computer Programs: Historical Background

Analytical Engine

- ▶ $1,000 \times 50$ -digit numbers (50 gears) can be saved in the store (addressed from 0 to 999)
 - ▶ $1,000 \times 8$ Byte (double) ≈ 8 Kilobyte storage memory
- ▶ Decimal fraction calculations (fixed point arithmetic) can be handled but not stored

Putting numbers in the store:

```
1 N001 +000000000000000000000000000000000000000000000000000000000000000  
2 N275 +00000000000000000000000000000000000000000000000000000000010000  
3 N302 -0000000000000000000000000000000000000000000000000000000001321
```

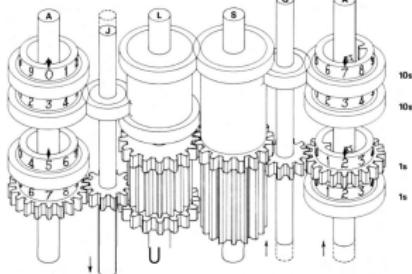
Syntax simplification:

```
1 N1 0  
2 N275 10000  
3 N302 -1321
```

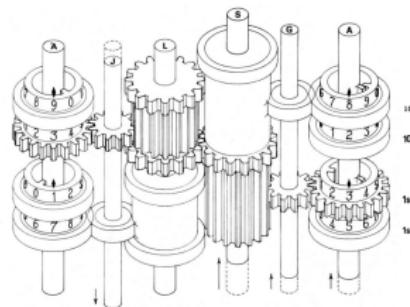
Computer Programs: Historical Background

Analytical Engine

- ▶ Digitwise parallel addition and multiplication/division via vertical transfer gearing
 - ▶ Transfer with pinions to propagate motions between vertical levels
- ▶



(a) Parallel digitwise addition

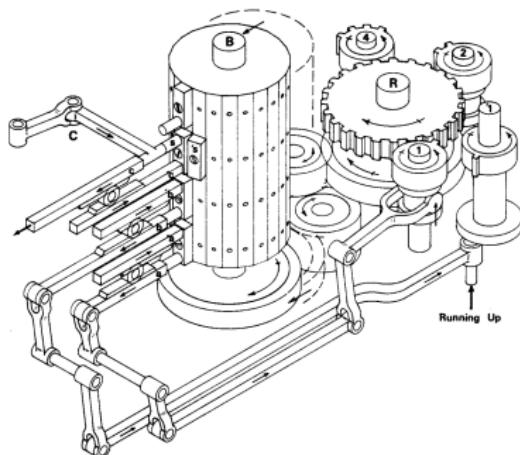


(b) Multiplication and division by 10

Computer Programs: Historical Background

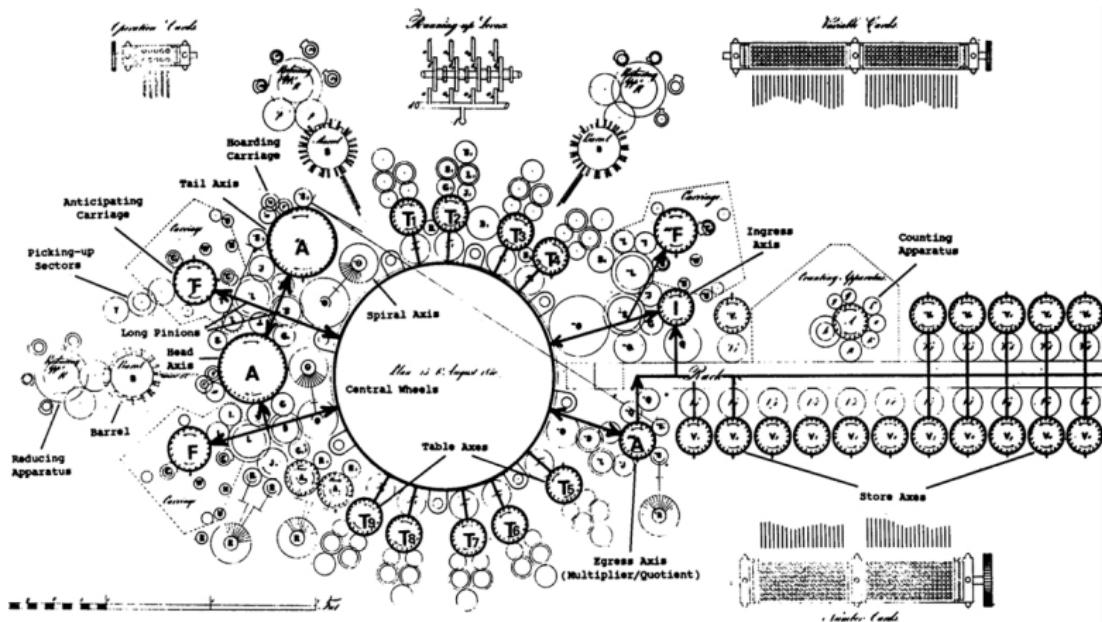
Analytical Engine

- ▶ Internal arithmetic operations of the Mill were performed by barrels
- ▶ The operations were controlled with studs attached to the barrels
- ▶ Can be considered as an early version of a microprogram/code by modern standards
 - ▶ Equivalent of a arithmetic logic unit (ALU)



Computer Programs: Historical Background

Analytical Engine



Computer Programs: Historical Background

Analytical Engine

- ▶ The *mill* ("CPU") was capable of performing four basic arithmetic operations:
 - ▶ Addition, Subtraction, Multiplication and Division
- ▶ Conditional jumping allows looping and branching
 - ▶ From todays perspective the Analytical Engine is considered to be Turing-complete

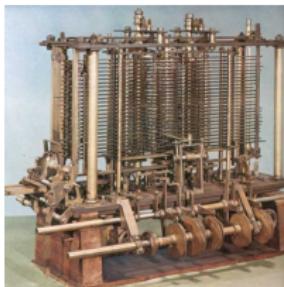


Figure: Trial model of the mill built by Charles Babbage (1870)

Computer Programs: Historical Background

Analytical Engine

Essential components of the mill:

- ▶ Ingress Axes and Egress Axis
- ▶ Primed Axes
- ▶ Run-Up Lever

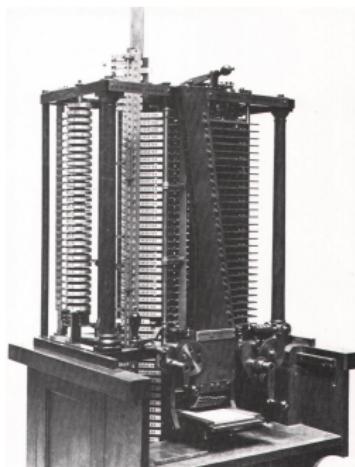


Figure: Trial model of the mill built by Henry Babbage (1910)

Computer Programs: Historical Background

Analytical Engine

- ▶ Operands for arithmetic operations are stored in two **Ingress Axes**
- ▶ The result is stored in an **Egress Axis**
- ▶ The axes of the mill can be seen as an equivalent of **registeres and accumulators**

Computer Programs: Historical Background

Analytical Engine

- ▶ Ingress Axis 1 and the Egress Axis had an additional *primed axis*
- ▶ The primed axis mechanism makes it possible to store remainders of division
- ▶ Example $22 \div 7 = 3, 142857 \rightarrow 3$ will be stored on Egress Axis Primed Axis, the remainder 1 on the Egress Axis

Computer Programs: Historical Background

Analytical Engine

- ▶ Program cards were intended to be used to store and execute programs:
- ▶ Punched card system based on those invented by Joseph Marie Jacquard
- ▶ Three types of program cards were used:
 - ▶ **Number Cards** Transfer of numerical values in the store at the specified wheel number (index)
 - ▶ **Variable Cards** Transfer of values from the Store to the Mill and vice versa
 - ▶ **Operation Cards** Selection of arithmetic operations the Mill is meant to perform on the transferred arguments



Figure: Punched card used for the analytical engine

Computer Programs: Historical Background

Analytical Engine

Table: Set of instructions

Mnemonic	Description
Nx y	Put number y in store x
Lx	Load number from store axis to Mill Ingress Axis, don't zero store column
Zx	Load from from Store axis to Mill Ingress Axis, zero store column
Sx	Transfer from Mill Egress to store column
< n	<i>Stepping up</i> (shift left) by n digits
> n	<i>Stepping down</i> (shift right) by n digits
CF	Advance (skip forward)
CB	Back (skip backward and repeat)
+	Always advance ("F") or back ("B") cards
?	Advance ("F") or back ("B") only if the Mill's run-up lever is set
+ - * /	Mill operation (Add, Sub, Mul, Div)
B	Rings a bell to get the attention of the attendant
H	Halts the engine

Computer Programs: Historical Background

Analytical Engine

- ▶ Performing fixed point arithmetic requires scaling of the dividend by 10 to the power of the number of decimal places
 - ▶ Needs to be performed before the calculation
- ▶ The product of the multiplication must be then divided by the same scale to adjust the decimal point
- ▶ Shifting numbers the given number of decimal places can be accomplished with *stepping up or down* operations
 - ▶ Equivalent of bitshifting operations (left shift, right shift) in modern computers

Example

Calculate the expression $(4000 \times 2.5) \div 28$ with a program that can be executed on the engine.

Computer Programs: Historical Background

Analytical Engine

```
1 // Load numbers into the store
2 N0 4000000000
3 N1 2500000
4 N2 28000000
5
6 // Load N0 and N1 into the mill and multiply
7 X
8 L0
9 L1
10
11 //
12 >6
13 S3
14
15 //
16 ÷
17 L3
18 <6
19 L2
20 S4 '
```

Listing: Stepping Up and Down

Computer Programs: Historical Background

Analytical Engine

- ▶ Repetitive processes can be performed with the *backing and advancing* operations
- ▶ Early version of programming loops for mathematical purposes

Example

Compute the factorial of $n = 4$

$$n! = \prod_{i=1}^n i \quad n! = n \times (n - 1) \times (n - 2) \dots \times 1$$

Computer Programs: Historical Background

Analytical Engine

Listing: Solution without backing

```
1 N0 4
2 N1 1
3 N2 1
4 x
5 L2
6 L0
7 S2
8 -
9 L0
10 L1
11 S0
12 x
13 L2
14 L0
15 S2
16 -
17 L0
18 L1
19 S0
20 x
21 L2
22 L0
23 S2
```

Listing: Solution with backing

```
24 N0 4
25 N1 1
26 N2 1
27 x
28 L1
29 L0
30 S1
31 -
32 L0
33 L2
34 S0
35 L2
36 L0
37 CB?11
```

Computer Programs: Historical Background

Analytical Engine

- Babbage sketched a series of 26 code fragments for the engine between 1836 and 1840
- First attempt to *implement* complex algorithms with a computer.
- Sketched down as tables → first program trace
- Scope of the sketches: linear algebra, polynomial multiplication and division, recursion, astronomical formulas, conditional looping and branching

Number	Elimination between two equations of the second degree - which is equivalent to take away $x^2 - b^2$ by $x^2 - c^2$							Taking the value of x from the equation $\frac{b^2 - c^2}{a^2 - b^2} = \frac{y^2 - z^2}{x^2 - b^2}$
	1. Add	2. Note	3. Add	4. Add	5. Add	6. Add	7. Add	
1	$x - b^2$	θ	$t_1 - b^2$	$t_2 - b^2$	$t_3 - b^2$	$t_4 - b^2$	$t_5 - b^2$	$x = \frac{t_5 - b^2}{t_1 - b^2}$
2	$x - b^2$	$b^2 - b^2$	θ					$x = \frac{t_5 - b^2}{t_1 - b^2} = \frac{t_5 - b^2}{t_1 - b^2}$
3	$b^2 - b^2$	θ	$b^2 - b^2$	θ				
4	$x - b^2$	$b^2 - b^2$	θ					
5	$-b^2 - b^2$	$b^2 - b^2$	$b^2 - b^2$	θ				
6	$-b^2 - b^2$	$b^2 - b^2$	θ					
7	$-b^2 - b^2$	$b^2 - b^2$	θ					

Composition

$x = \frac{t_5 - b^2}{t_1 - b^2} = \frac{t_5 - b^2}{t_1 - b^2} = \frac{t_5 - b^2}{t_1 - b^2} = \frac{t_5 - b^2}{t_1 - b^2}$

Number of operations $= 5$

Number of additions $= 2$

Total number of operations $= 2$

Number of variables $= 3$

Number of working variables $= 3$

Total number of variables required $= 2$

Figure: Table of the BAB L26 program sketch.

Computer Programs: Historical Background

Analytical Engine

- ▶ The program sketch BAB L26 can be considered the first written computer program
- ▶ The code fragment is dated to August 1837 in the Babbage archive
- ▶ Algorithmic solution of a system of two linear equations with two variables:

$$ax + by + c = 0$$

$$a'x + b'y + c = 0$$

$$x = \frac{bc' - b'c}{b'a - ba'}$$

$$y = (-ax - c)/b$$

Computer Programs: Historical Background

Analytical Engine

# Operations	Function	Intermediate Result	Mill			Store		
			+ax	+by	c	+a'x	+b'y	c
		V1	V2	V3	V4	V5	V6	V7
1	×	$b'a$	0			0		$b'a$
2	×	$b'c$		$b'c$	0			
3	×	ba'			0 ba'	0		
4	×	bc'			bc'		0	
5	-	$bc' - b'c$	0	0		$bc' - b'c$		
6	-	$b'a - ba'$		$b'a - ba'$	0			0
7	÷	$\frac{bc' - b'c}{b'a - ba'}$	0		= x	0		

Table: Program Table of BAB L26

Computer Programs: Historical Background

Analytical Engine

- ▶ Ada Lovelace (1815 - 1852) was a British mathematician who worked with Charles Babbage
- ▶ Wrote a concrete program for the analytical machine that calculates of the 8th Bernoulli number
- ▶ She is commonly considered to be the world's first programmer
 - ▶ However, concrete programme examples were found in Babbage's notes several years earlier



Diagram for the computation by the Engine of the Numbers of Bernoulli. See Note G. (page 221 of ms.)	
Index	Working Variables
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_1
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_2
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_3
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_4
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_5
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_6
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_7
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_8
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_9
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{10}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{11}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{12}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{13}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{14}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{15}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{16}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{17}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{18}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{19}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{20}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{21}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{22}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{23}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{24}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{25}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{26}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{27}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{28}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{29}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{30}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{31}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{32}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{33}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{34}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{35}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{36}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{37}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{38}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{39}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{40}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{41}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{42}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{43}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{44}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{45}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{46}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{47}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{48}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{49}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{50}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{51}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{52}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{53}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{54}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{55}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{56}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{57}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{58}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{59}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{60}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{61}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{62}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{63}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{64}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{65}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{66}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{67}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{68}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{69}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{70}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{71}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{72}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{73}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{74}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{75}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{76}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{77}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{78}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{79}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{80}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{81}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{82}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{83}
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$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{92}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{93}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{94}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{95}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{96}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{97}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{98}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{99}
$\frac{1}{2} \frac{1}{2} \frac{1}{2}$	b_{100}

Computer Programs: Historical Background

Analytical Engine

- ▶ Her program sketch *Note G* calculates a number that she called B_7
 - ▶ Nowadays known as the eighth Bernoulli number
- ▶ Demonstration of the capability of the engine to perform more complex mathematical calculations
- ▶ Her program calculated it with by using a recursive equation:

$$B_7 = -1(A_0 + B_1A_1 + B_3A_3 + B_5A_5)$$

Computer Programs: Historical Background

Analytical Engine

- ▶ B and A represent factors of the calculation that have been discovered by Bernoulli:
 - ▶ B1 to B7 are Bernoulli numbers
 - ▶ A0 to A5 represent coefficients that Bernoulli obtained by using Pascal's Triangle:

$$A_0 = -\frac{1}{2} \cdot \frac{2n-1}{2n+1}$$

$$A_1 = \frac{2n}{2}$$

$$A_3 = \frac{2n(2n-1)(2n-2)}{2 \cdot 3 \cdot 4}$$

$$A_5 = \frac{2n(2n-1)(2n-2)(2n-3)(2n-4)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}$$

Computer Programs: Historical Background

Definition (Computer Program)

Computer Programs: Sums of Powers and Bernoulli Numbers

- ▶ First appeared in Jakob Bernoulli's (1654-1705) post humorous publication *Ars Conjectandi* in 1713
 - ▶ Independently discovered by Japanese mathematician Seki Takakazu in 1712
- ▶ Solutions to sums of integer powers have been studied for centuries
- ▶ Basic form of the series is defined as:

$$1^k + 2^k + 3^k + \dots + n^k = \sum_{p=1}^n p^k$$



... Atque si ponit ad ultiores gradationes potestas pergeat, levique negotio sequentur admodum intercedunt facti:
Solutio Potestorum
$$\begin{aligned}J_1(n) &= \frac{1}{2}n^2 + \frac{1}{2}n + \frac{1}{4}n \\J_2(n) &= \frac{1}{3}n^3 + \frac{3}{2}n^2 + \frac{1}{2}n \\J_3(n) &= \frac{1}{4}n^4 + \frac{2}{3}n^3 + \frac{1}{3}n^2 - \frac{1}{3}n \\J_4(n) &= \frac{1}{5}n^5 + \frac{5}{4}n^4 + \frac{1}{2}n^3 - \frac{1}{2}n^2 + \frac{1}{4}n \\J_5(n) &= \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{4}n^4 + \frac{1}{3}n^3 - \frac{1}{3}n^2 - \frac{1}{2}n \\J_6(n) &= \frac{1}{7}n^7 + \frac{2}{3}n^6 + \frac{1}{2}n^5 + \frac{1}{3}n^4 - \frac{1}{2}n^3 - \frac{1}{2}n^2 + \frac{1}{4}n \\J_7(n) &= \frac{1}{8}n^8 + \frac{1}{3}n^7 + \frac{1}{2}n^6 + \frac{5}{4}n^5 + \frac{1}{3}n^4 - \frac{1}{2}n^3 - \frac{1}{2}n^2 + \frac{1}{4}n \\J_8(n) &= \frac{1}{9}n^9 + \frac{1}{2}n^8 + \frac{1}{3}n^7 + \frac{1}{2}n^6 + \frac{1}{3}n^5 + \frac{1}{4}n^4 - \frac{1}{2}n^3 - \frac{1}{2}n^2 + \frac{1}{4}n \\J_9(n) &= \frac{1}{10}n^{10} + \frac{1}{3}n^9 + \frac{1}{2}n^8 + \frac{1}{3}n^7 + \frac{1}{2}n^6 + \frac{5}{4}n^5 + \frac{1}{3}n^4 - \frac{1}{2}n^3 - \frac{1}{2}n^2 + \frac{1}{4}n \\J_{10}(n) &= \frac{1}{11}n^{11} + \frac{1}{2}n^{10} + \frac{1}{3}n^9 + \frac{1}{2}n^8 + \frac{1}{3}n^7 + \frac{1}{2}n^6 + \frac{1}{3}n^5 + \frac{1}{4}n^4 - \frac{1}{2}n^3 - \frac{1}{2}n^2 + \frac{1}{4}n\end{aligned}$$

Quo iure quod legimus progressionis adhuc ultraebras expectare, eadem enim continuare potest absque his successivis additionibus: Sicut enim et pro potestis caputibus exponeat, illi summa censum n^m seu

$$\begin{aligned}\left[n^m \right] &= \frac{1}{m+1}n^{m+1} + \frac{1}{m+1} \cdot \frac{1}{2}m(n-1) \cdot \frac{m-1}{2} \cdot \frac{m-2}{2} \cdot \frac{m-3}{2} \cdots \\&\quad + \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \cdot Cn^{m-3} \\&\quad + \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}{3 \cdot 4 \cdot 5 \cdot 6} \cdot C(n-1) \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot Dn^{m-7} \cdots & \text{et ita deinceps,}\end{aligned}$$

exponitur potestis ipsum a continet numero brutorum, quicquid per unitatem ad v. vel m. Litterae capitulis A, B, C, D &c. ostendit coefficientes ultimorum terminorum pro $f(n)$, $f(n^2)$, $f(n^3)$, $f(n^4)$, $f(n^5)$, $f(n^6)$.

$$A = \frac{1}{2}, B = -\frac{1}{30}, C = \frac{1}{42}, D = -\frac{1}{160}$$

Computer Programs: Sums of Powers and Bernoulli Numbers

- ▶ $\sum_{p=1}^n p^k$ with $k \in \mathbb{N}$ and $p \in \mathbb{Z}$ is also known as Faulhaber's formula
 - ▶ Named after the early mathematician Johann Faulhaber (1580 - 1635)
 - ▶ Polynomial expression of p -th power sums of the first n positive integers
- ▶ Faulhaber obtained equations that can be used to solve sum of powers up to $k = 17$
- ▶ Among his results were equations for the sums of odd powers



Tabula: Durch alle Ziffern füllt auf die Ziffernreihenfolge / durch ein Merk- würdiges Rechnen leichtesten Aufgaben zu lösen.																
1	1	2	3	4	5	6	7	8	9	0	1	2	3	4	5	6
1	1	2	3	4	5	6	7	8	9	0	1	2	3	4	5	6
1	4	9	16	25	36	49	64	81	100	121	144	169	196	225	256	289
1	6	15	24	33	42	51	60	69	78	87	96	105	114	123	132	141
1	9	21	33	45	57	69	81	93	105	117	129	141	153	165	177	189
1	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192
1	15	30	45	60	75	90	105	120	135	150	165	180	195	210	225	240
1	18	36	54	72	90	108	126	144	162	180	198	216	234	252	270	288
1	21	42	63	84	105	126	147	168	189	210	231	252	273	294	315	336
1	24	48	72	96	120	144	168	192	216	240	264	288	312	336	360	384
1	27	54	81	108	135	162	189	216	243	270	297	324	351	378	405	432
1	30	60	90	120	150	180	210	240	270	300	330	360	390	420	450	480
1	33	66	99	132	165	198	231	264	297	330	363	396	429	462	495	528
1	36	72	108	144	180	216	252	288	324	360	396	432	468	504	540	576
1	39	78	117	156	195	234	273	312	351	390	429	468	507	546	585	624
1	42	84	126	168	210	252	294	336	378	420	462	504	546	588	630	672
1	45	90	135	180	225	270	315	360	405	450	495	540	585	630	675	720
1	48	96	144	192	240	288	336	384	432	480	528	576	624	672	720	768
1	51	102	153	204	255	306	357	408	459	510	561	612	663	714	765	816
1	54	108	162	216	270	324	378	432	486	540	594	648	702	756	810	864
1	57	114	171	228	285	342	399	456	513	570	627	684	741	798	855	912
1	60	120	180	240	300	360	420	480	540	600	660	720	780	840	900	960
1	63	126	189	252	315	378	441	504	567	630	693	756	819	882	945	1008
1	66	132	198	264	330	396	462	528	594	660	726	792	858	924	990	1056
1	69	138	207	276	345	414	483	552	621	690	759	828	897	966	1035	1104
1	72	144	216	288	360	432	504	576	648	720	792	864	936	1008	1080	1152
1	75	150	225	300	375	450	525	600	675	750	825	900	975	1050	1125	1200
1	78	156	234	312	390	468	546	624	702	780	858	936	1014	1092	1170	1250
1	81	162	243	321	400	480	560	640	720	800	880	960	1040	1120	1200	1280
1	84	168	252	330	410	490	570	650	730	810	890	970	1050	1130	1210	1290
1	87	174	261	340	420	500	580	660	740	820	900	980	1060	1140	1220	1300
1	90	180	270	350	430	510	590	670	750	830	910	990	1070	1150	1230	1310
1	93	186	279	359	439	519	599	679	759	839	919	999	1079	1159	1239	1319
1	96	192	288	368	448	528	608	688	768	848	928	1008	1088	1168	1248	1328
1	100	200	300	400	500	600	700	800	900	1000	1100	1200	1300	1400	1500	1600

Computer Programs: Sums of Powers and Bernoulli Numbers

- ▶ For $k = 1$ we have $a = \sum_{k=1}^n k^1 = \frac{n(n+1)}{2}$
- ▶ For $k = 3$ we then have: $\sum_{k=2}^n k^2 = \frac{n^2(n+1)^2}{4} = a^2$
- ▶

Computer Programs: Sums of Powers and Bernoulli Numbers

- ▶
- ▶
- ▶ Expansion from $k = 0$ up to $k = 6$:

$$\sum_{k=1}^n k^0 = \frac{1}{1} (n)$$

$$\sum_{k=1}^n k^1 = \frac{1}{2} (n^2 + \frac{1}{2} n)$$

$$\sum_{k=1}^n k^2 = \frac{1}{3} (n^3 + \frac{3}{2} n^2 + \frac{3}{6} n)$$

$$\sum_{k=1}^n k^3 = \frac{1}{4} (n^4 + \frac{4}{2} n^3 + \frac{6}{6} n^2 + 0n)$$

$$\sum_{k=1}^n k^4 = \frac{1}{5} (n^5 + \frac{5}{2} n^4 + \frac{10}{6} n^3 + 0n^2 - \frac{5}{30} n)$$

$$\sum_{k=1}^n k^5 = \frac{1}{6} (n^6 + \frac{6}{2} n^5 + \frac{15}{6} n^4 + 0n^3 - \frac{15}{30} n^2 + 0n)$$

$$\sum_{k=1}^n k^6 = \frac{1}{7} (n^7 + \frac{7}{2} n^6 + \frac{21}{6} n^5 + 0n^4 - \frac{35}{30} n^3 + 0n^2 + \frac{7}{42} n).$$

Computer Programs: Sums of Powers and Bernoulli Numbers

- ▶ The coefficients of Faulhaber's formula in closed form include the Bernoulli numbers B_i ;
- ▶ Series of coefficients of $p = 1$ for all k
- ▶ First eight Bernoulli numbers $B_0 - B_7$:

$$B_0 = 1$$

$$B_1 = \frac{1}{2}$$

$$B_2 = \frac{1}{6}$$

$$B_3 = 0$$

$$B_4 = -\frac{1}{30}$$

$$B_5 = 0$$

$$B_6 = \frac{1}{42}$$

$$B_7 = 0,$$

Computer Programs: Sums of Powers and Bernoulli Numbers



References