

Introduction to Program Synthesis (SS 25)

Exercise - Church Numerals

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Church Numerals

α -Conversion

$$\lambda x. \lambda y. x(y(z)) \rightarrow_{\alpha} \lambda \textcolor{red}{a}. \lambda y. \textcolor{red}{a}(y(z)) \quad \{a/x\}$$

$$\lambda a. \lambda y. a(y(z)) \rightarrow_{\alpha} \lambda a. \lambda \textcolor{red}{b}. a(\textcolor{red}{b}(z)) \quad \{b/y\}$$

$$\lambda a. \lambda y. a(y(z)) \rightarrow_{\alpha} \lambda a. \lambda b. a(b(z)) \quad \{c/z\}$$

$$\lambda z. \lambda a. \lambda y. a(y(z)) \rightarrow_{\alpha} \lambda a. \lambda b. a(y(z)) \quad \{c/z\}$$

$$\lambda \textcolor{red}{c}. \lambda a. \lambda y. a(y(z)) \rightarrow_{\alpha} \lambda a. \lambda b. a(y(\textcolor{red}{c})) \quad \{c/z\}$$

Church Numerals

β -Reduction

$$\begin{aligned}((\lambda x. \lambda y. y)((\lambda z. z)(\lambda y. \lambda x. y)))a &\rightarrow_{\beta} ((\lambda x. \lambda y. y)((\lambda z. z)(\lambda y. \lambda x. y)))a \\ &\rightarrow_{\beta} ((\lambda x. \lambda y. y)((\lambda y. \lambda x. y)))a \\ &\rightarrow_{\beta} (\lambda y. y)a \\ &\rightarrow_{\beta} a\end{aligned}$$

Church Numerals

Arithmetic

$$f^{\circ n} = \underbrace{f \circ f \circ \dots \circ f}_{n \text{ times}}$$

$$\text{succ} = f = f^{n+1} = f(f^n x)$$

$$C_0 = \lambda f. \lambda x. x$$

$$C_1 = \lambda f. \lambda x. fx$$

$$C_2 = \lambda f. \lambda x. f(fx)$$

$$C_3 = \lambda f. \lambda x. f(f(fx))$$

$$C_4 = \lambda f. \lambda x. f(f(f(fx)))$$

$$C_n = \lambda f. \lambda x. f^n x$$

Church Numerals

Arithmetic: Addition

$$\text{plus} = f^{m+n}(x) = f^{\circ m}(f^{\circ n}x))$$

$$\begin{aligned}\mathcal{C}_{3+4} &= \lambda f. \lambda x. \mathcal{C}_3 f (\mathcal{C}_4 f x) \\ &= \lambda f. \lambda x. (\lambda f_3. \lambda x_3. f_3(f_3(f_3 x_3))) f (\lambda f_4. \lambda x_4. f_4(f_4(f_4(f_4 x_4)))) f x) \\ &= \lambda f. \lambda x. f(f(f(\lambda f_4. \lambda f_4. f(f_4(f_4(f_4 x_4)))) f x)))) \\ &= \lambda f. \lambda x. f(f(f(f(f(f(f x))))))) \\ &= \mathcal{C}_7\end{aligned}$$

Church Numerals

Arithmetic: Multiplication

$$\text{mult} = f^{\circ m * n}(x) = (f^{\circ m})^{\circ n}(x)$$

$$\begin{aligned}\mathcal{C}_3 f &= (\lambda f_3. \lambda x. f_3(f_3(f_3 x))) f \\ &= \lambda x. f(f(f x))\end{aligned}$$

$$\begin{aligned}\mathcal{C}_{2*3} &= \lambda f. \lambda x. \mathcal{C}_2 (\mathcal{C}_3 f) x \\ &= \lambda f. \lambda x. \mathcal{C}_2 (\lambda x_3. f(f(f x_3))) x \\ &= \lambda f. \lambda x. (\lambda f_2. \lambda x_2. \textcolor{red}{f}_2(\textcolor{red}{f}_2 x_2) (\lambda x_3. f(f(f x_3)))) x \\ &= \lambda f. \lambda x. (\lambda x_3^a. f(f(f x_3^a))) ((\lambda x_3^b. f(f(f x_3^b))) x) \\ &= \lambda f. \lambda x. f(f(f((\lambda x_3^b. f(f(f x_3^b))) x))) \\ &= \lambda f. \lambda x. f(f(f(f(f(f x)))))) \\ &= \mathcal{C}_6\end{aligned}$$