(1) a) $\lim\limits_{x\to 0} \dfrac{\dfrac{(\sqrt{\cos 2x + \operatorname{sh} 2x} - e^{x})}{x} + }{\dfrac{\ln \operatorname{ch} x}{\operatorname{sh} x} - \dfrac{\arcsin x}{2}}$

$\dfrac{2x(2-x)}{2+x}$

РАСПИШЕМ ОТДЕЛЬНЫЕ ФУНКЦИИ.

$\operatorname{sh} 2x = 2x + \dfrac{4x^3}{3} + o(x^3)$

$\cos 2x = 1 - 2x^2 + o(x^3)$

$e^x = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + o(x^3)$

$\operatorname{ch} x = 1 + \dfrac{x^2}{2} + o(x^3)$

$\ln\left(1 + \dfrac{x^2}{2}\right) = \dfrac{x^2}{2} - \dfrac{x^4}{8} + \dfrac{x^6}{24} + o(x^3)$

$\operatorname{sh} x = x + \dfrac{x^3}{6} + o(x^3) = \dfrac{6x + x^3}{6}$

$\arcsin x = x + \dfrac{x^3}{6} + o(x^3)$

ПОДРОБНО РАЗБЕРЁМ $\dfrac{4x - 2x^2}{2+x}$

$\exists f(x) = \dfrac{4x - 2x^2}{x+2}$ $\quad f(0) = 0$

$f'(x) = \dfrac{(4-4x)(x+2) - (4x - 2x^2)}{(x+2)^2} = \dfrac{4x + 8 - 4x^2}{(x+2)^2}$

$= \dfrac{\cancel{8x} - \cancel{4x} + 2x^2}{} = \dfrac{8 - 8x - 2x^2}{(x+2)^2} = \cancel{\dfrac{2(6 - 2x + x^2)}{x+2}}$

$= \cancel{4}$ $\quad f'(0) = 2$

$$f''(x) = \frac{(-8-4x)(x+2)^2 - 2(8-8x-2x^2)(x+2)}{(x+2)^4}$$

$$= \frac{(x+2)(-8x-\cancel{4x^2}-16-8x-16+\cancel{16x}+\cancel{4x^2})}{(x+2)^{\cancel{4}\,3}}$$

$$= \frac{-32}{(x+2)^3} \qquad f''(0) = -\frac{32}{8} = -4$$

$$f'''(x) = \frac{32 \cdot 3}{(x+2)^4} \qquad f'''(0) = \frac{32 \cdot 3}{16} = 6$$

$$\frac{4x-2x^2}{x+2} = 0 + 2x - 2x^2 + x^3 + o(x^3)$$

$$\cancel{\lim_{x\to 0} \frac{\sqrt{\cos 2x + \sin 2x} - 1 - x - \frac{x^2}{2} - \frac{x^3}{6}}{}}$$

$$\sqrt{\cos 2x + \sin 2x} = \sqrt{1 + 2x - 2x^2 + \frac{4}{3}x^3}$$

$$\cancel{\exists\, g(x) = \sqrt{1 + 2x - 2x^2 + \frac{4}{3}x^3}}$$

$$\cancel{g(0) = 1}$$

$$\cancel{g'(x) = -\frac{(2 - 4x + 4x^2)}{\sqrt{1 + 2x - 2x^2 + \frac{4}{3}x^3}}}$$

$$\cancel{g'(0) = -2}$$

$$\cancel{g''(x) = \frac{-(8x-4)\sqrt{1+2x-2x^2+\frac{4}{3}x^3} - \frac{(2-4x+4x^2)}{}}{1 + 2x - 2x^2 + \frac{4}{3}x^3}}$$

$$3)\ g(x) = \sqrt{\cos 2x + \operatorname{sh} 2x}$$

$$g(0) = 1$$

$$g'(x) = -\frac{(-2\sin 2x + 2\operatorname{ch} 2x)}{\sqrt{\cos 2x + \operatorname{sh} 2x}}$$

$$g'(0) = \frac{2}{1} = 2$$

$$g''(x) = \frac{(4\cos 2x + 4\operatorname{sh} 2x)\sqrt{\cos 2x + \operatorname{sh} 2x}}{\cos 2x + \operatorname{sh} 2x}$$

$$- \frac{\frac{(2\sin 2x - 2\operatorname{ch} 2x)^2}{\sqrt{\cos 2x + \operatorname{sh} 2x}}}{} =$$

$$\frac{(4\cos 2x - 4\operatorname{sh} 2x)(\cos 2x + \operatorname{sh} 2x)}{(\cos 2x + \operatorname{sh} 2x)^{1,5}}$$

$$-\frac{4(2\sin 2x - 2\operatorname{ch} 2x)^2}{} =$$

$$\frac{4(\cos^2 2x - \operatorname{sh}^2 2x) - 4(\sin 2x - \operatorname{ch} 2x)^2}{(\cos 2x + \operatorname{sh} 2x)^{1,5}}$$

$$g''(0) = 0$$

$$g'''(0) = \frac{-8\cdot 2\cos 2x \sin 2x - 16\operatorname{sh} 2x \operatorname{ch} 2x}{(\cos 2x + \operatorname{sh} 2x)^3}$$

$= 8(\sin 2x - \sin x)(\cos 3x - \sin 3x)/(\cos 2x$

$+ \sin 2x)^{1/2} - O(\ldots)$ $= 1 \cdot \frac{8}{1}$

$\sqrt{\cos 2x + \sin 2x} = 1 + 2x + 0 + \frac{2}{3}x^3$

$$\lim_{x \to 0} \frac{\cancel{1} + 2x + \frac{4}{3}x^3 - \cancel{1} - x - \frac{x^2}{2} - \frac{x^3}{6}}{x}$$

$$\frac{6\left(\frac{x^2}{2} - \frac{x^4}{2} + \frac{x^6}{24}\right)}{6x + x^3} - \frac{6x + x^3}{12}$$

$$\frac{+2x - 2x^2 + x^3}{} = \lim_{x \to 0} \frac{1 - \frac{x}{2} + \frac{4}{6}x^2 + 2x}{\frac{36x^2 - 9x^4 + 3x^6}{12(6x + x^3)}}$$

$$\frac{-2x^2 + x^3}{-(6x + x^3)^2} = \lim_{x \to 0} \frac{\left(1 + \frac{3x}{2} - \frac{5}{6}x^2 + x^3\right)}{36x^2 - 9x^4 + 3x^6}$$

$$\frac{\cdot 12 \cdot (6x + x^3)}{-\cancel{36x^2} - 12x^4 - x^6} = \lim_{x \to 0} \frac{\cancel{x}(1 + \ldots)}{\cancel{x}(2x^5 - 2x^3)}$$

$$\frac{\cdot 12(6 + x^2)}{} = \frac{72}{0} = \infty$$

б) $\lim\limits_{x\to\infty} \left(x\ln(1+x) - x\ln x + \operatorname{arctg}\frac{1}{2x}\right)^{x^2\operatorname{arctg}x} =$

$e^{\lim\limits_{x\to\infty} x^2\operatorname{arctg}x \ln(x\ln(1+x) - x\ln x + \operatorname{arctg}\frac{1}{2x})}$

Вычислим

$\lim\limits_{x\to\infty} x^2\operatorname{arctg}x \ln(x\ln(1+x) - x\ln x + \operatorname{arctg}\frac{1}{2x})$

$\cancel{x\ln(1+x) = x^2 - \frac{x^3}{2} - \frac{x^4}{3} + o(x^3)}$

$\operatorname{arctg}\frac{1}{2x} = \frac{1}{2x} - \frac{1}{24x^3}$

~~Разложим $\ln x$~~

$\cancel{f(x) = \ln x \quad f(0) =}$

$x\ln(1+x) - x\ln x = x(\ln(1+x) - \ln x) = x\ln\left(\frac{1+x}{x}\right) = \ln(1+\frac{1}{x})$

$\ln(1+\frac{1}{x}) = \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} + o(x^3)$

$$\ln\left(x\ln\left(1+\frac{1}{x}\right)+\operatorname{arctg}\frac{1}{2x}\right)=$$

$$\ln\left(1-\cancel{\frac{1}{2x}}+\frac{1}{3x^2}+\cancel{\frac{1}{2x}}-\frac{2}{24x^3}\right)$$

$$=\ln\left(1+\frac{8x-1}{24x^3}\right)=\frac{8x-1}{24x^3}$$

$$-\frac{(8x-1)^2}{2\cdot 24^2x^6}+\frac{(8x-1)^3}{3\cdot 24^3x^9}=$$

$$\frac{6\cdot 24^2x^6(8x-1)-3\cdot 24x^3(8x-1)^2+2(8x-1)^3}{6\cdot 24^3x^9}$$

$$\operatorname{arctg}x=x-\frac{x^3}{3}=\frac{3x-x^3}{3}$$

$$x^2\operatorname{arctg}x=x^2\cdot x\,\frac{(3-x^2)}{3}$$

$$\lim_{x\to 0}\frac{x^3(3-x^2)\left(6\cdot 24^2x^6(8x-1)-3\cdot 24x^3(8x-1)^2+2(8x-1)^3\right)}{x^{9}\,6\cdot 24^3}$$

Заметим, что нам интересны ~~...~~ в числителе только выражения, степень которых ≥ 6. (т.к. ост. $\to 0$)

Таким образом

$$\lim_{x\to\infty} \frac{3\cdot 6\cdot 242x^6(8x-1)}{x^6\ \frac{6\cdot 3x^{2^3}}{8}} - \frac{x^2(8x-1)}{24}$$

$$+\frac{(8x-1)^2}{2\cdot 242x} = \lim_{x\to\infty} -\frac{(x^2-3)(8x-1)}{24}$$

$$+\frac{8x-1)^2}{2\cdot 24^2x} =$$

$$\lim_{x\to\infty} \frac{8x-1)(8x-1-48x^3+6x)}{2\cdot 24^2x}$$

$$=\lim_{x\to\infty} \frac{(8x-1)(152x-1-48x^3)}{2\cdot 24^2x}$$

$$=\lim_{x\to\infty} \frac{\cancel{x}(8-\cancel{\frac{1}{x}})(152-\cancel{\frac{1}{x}}-48x^2)}{2\cdot 24^2\cancel{x}}$$

$$=\lim_{x\to\infty} \frac{8\cdot 152-8\cdot 48x^2}{2\cdot 24^2} = -\infty$$

$$e^{-\infty} = 0$$