1)

Let $f(x) = \cos(x)$, $x_0 = 0$, $x_1 = 0.6$, $x_2 = 0.9$

$$h_0 = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-0.6)(x-0.9)}{(0-0.6)(0-0.9)}$$

$$= \frac{50}{27}(x-0.6)(x-0.9)$$

$f(x_0) = f(0) = \cos(0) = 1$

$$h_1 = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-0)(x-0.9)}{(0.6-0)(0.6-0.9)}$$

$$= \frac{50}{-9}(x^2-0.9x)$$

$f(x_1) = f(0.6) = \cos(0.6)$

$$h_2 = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-0)(x-0.6)}{(0.9-0)(0.9-0.6)}$$

$$= \frac{100}{27}(x^2-0.6x)$$

$f(x_2) = f(0.9) = \cos(0.9)$

$$P(x) = \sum_{k=0}^{2} f(x_k) L_k(x)$$

$$= \frac{50}{27}(x-0.6)(x-0.9) +$$

$$\frac{-50}{9}(x^2-0.9x)\cos(0.6) +$$

$$\frac{100}{27}(x^2-0.6x)\cos(0.9)$$

$$= \frac{50}{27}(x^2-1.5x+0.54) +$$

$$\frac{-50}{9}(x^2-0.9x)\cos(0.6) +$$

$$\frac{100}{27}(x^2-0.6x)\cos(0.9)$$

$$= \left(\frac{50}{27} - \frac{50\cos(0.6)}{9} + \frac{100\cos(0.9)}{27}\right)x^2 +$$

$$\left(\frac{50 \cdot -1.5}{27} + \frac{50\cos(0.6)\cdot 0.9}{9} - \frac{100\cos(0.9)\cdot 0.6}{27}\right)x +$$

$$\frac{50}{27} \cdot 0.54$$

For the sake of accuracy, let us not evaluate the coefficients. Thus,

To approximate $f(0.45)$

$$f(0.45) \approx P(0.45)$$

$$= \left(\frac{50}{27} - \frac{50\cos(0.6)}{9} + \frac{100\cos(0.9)}{27}\right)(0.45)^2 +$$

$$\left(\frac{50 \cdot -1.5}{27} + \frac{50\cos(0.6)\cdot 0.9}{9} - \frac{100\cos(0.9)\cdot 0.6}{27}\right)(0.45) +$$

$$\frac{50}{27} \cdot 0.54 \approx 0.8981$$

RAD M More...

$$\left(\frac{50}{27} - \frac{50\cos(0.6)}{9} + \frac{100\cos(0.9)}{27}\right) 0.45^2 + \left(\frac{-50 \times 1.5}{27} + \frac{50\cos(0.6)\times 0.9}{9} - \frac{100\cos(0.9)\times 0.6}{27}\right)(0.45) + \frac{50 \times 0.54}{27}$$

$= 0.898\,100\,074\,705\,721\,970\,274\,892\,523\,473\,014\,666\,557\,412\,08$

CPLX

For the record $\cos(0.45) \approx 0.9004$

So, $|0.9004 - 0.8981| = \underline{0.0023}$

So we are pretty closed!!

2) To find an error bound for $p(x)$.

Since $f(x) = \cos x$

$$f'(x) = -\sin x$$

$$f''(x) = -\cos x$$

$$f'''(x) = \sin x$$

let $m = \xi$

The $p(x)$ has a corresponding error term:

$$\frac{f'''(m(x))}{3!}(x-x_0)(x-x_1)(x-x_2)$$

Input values

$$\frac{\sin(m(x))}{6}(x-0)(x-0.6)(x-0.9)$$

$$= \frac{\sin(m(x))}{6}\, x(x^2-1.5x+0.54)$$

$$= \frac{\sin(m(x))}{6}\, x^3-1.5x^2+0.54x$$

Since,

$$|\sin(m(x))| \leq 1$$

$$\frac{|\sin(m(x))|}{6} \leq \frac{1}{6}$$

let $g(x) = x^3 - 1.5x^2 + 0.54x$

$g'(x) = 3x^2 - 3x + 0.54$

$a = 3 \quad b = -3 \quad c = 0.54$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\frac{3 \pm \sqrt{2.52}}{6}$$

$$\frac{3 \pm \sqrt{9 - 4 \cdot 3 \cdot 0.54}}{2 \cdot 3} = \frac{3 \pm \sqrt{63/25}}{6}$$

$$= \frac{1}{2} \pm \frac{\frac{\sqrt{63}}{5}}{6}$$

$$= \frac{1}{2} \pm \frac{\sqrt{63}}{30}$$

$$x = \frac{1}{2} + \frac{\sqrt{63}}{30}, \frac{1}{2} - \frac{\sqrt{63}}{30}$$

$$g\left(\frac{1}{2}+\frac{\sqrt{63}}{30}\right) \approx -0.01704$$

$$g\left(\frac{1}{2}-\frac{\sqrt{63}}{30}\right) \approx 0.057040516$$

So the bound for $g(x)$ at $(0, 0.9)$ is $\left|g\left(\frac{1}{2}-\frac{\sqrt{63}}{3}\right)\right| \leq 0.057040516$

note that $\frac{1}{2}-\frac{\sqrt{63}}{5} \approx 0.2354 \in (0, 0.9)$

Therefore the error term $f(x)$.

$$\left|\frac{\sin(\ln(x))}{6} x^3 - 1.5x^2 + 0.54x\right| \leq \left|\frac{1}{6}\right| \cdot |0.057040518|$$

$$\boxed{\approx 0.0095}$$

3)

$$P_{2,3} = \frac{1}{x_3 - x_2}\left[(x - x_2)P_3 - (x - x_3)P_2\right]$$

$$2.4 = \frac{1}{0.75 - 0.5}\left[(0.4 - 0.5)6 - (0.4 - 0.75)P_2\right]$$

$$2.4 = \frac{1}{0.25}\left[-\frac{4}{5} + \frac{7}{20}P_2\right]$$

$$\frac{3}{5} = -\frac{4}{5} + \frac{7}{20}P_2$$

$$\frac{7}{5} = \frac{7}{20}P_2$$

$$\boxed{4 = P_2}$$

$$P_{1,2} = \frac{1}{x_2 - x_1}\left[(x - x_1)P_2 - (x - x_2)P_1\right]$$

$$= \frac{1}{0.5 - 0.25}\left[(0.4 - 0.25)4 - (0.4 - 0.5)2\right]$$

$$= \frac{1}{0.25}\left[\frac{3}{5} + \frac{1}{5}\right]$$

$$= \frac{1}{0.25}\left[\frac{4}{5}\right] = 4\left[\frac{4}{5}\right] = \boxed{\frac{16}{5}}$$

Checking

$$P_{123} = \frac{1}{0.75 - 0.25}\left[(0.4 - 0.25)2.4 - (0.4 - 0.75)\frac{16}{5}\right]$$

$$2.96 = 2.96$$

$$P_{0,1,2} = \frac{1}{x_2 - x_0}\left[(x-x_0)P_{1,2} - (x-x_2)P_{0,1}\right]$$

$$= \frac{1}{0.5-0}\left[(0.4-0)\frac{16}{5} - (0.4-0.5)2.6\right]$$

$$= 2\left[\frac{32}{25} + \frac{13}{50}\right] = 2\left[\frac{64}{50} + \frac{13}{50}\right]$$

$$= 2\left[\frac{77}{50}\right] = \frac{154}{50} = \boxed{\frac{77}{25}}$$

Checking

$$P_{0,1,2,3} = \frac{1}{x_3 - x_0}\left[(x-x_0)P_{1,2,3} - (x-x_3)P_{0,1,2}\right]$$

$$= \frac{1}{0.75-0}\left[(0.4-0)2.96 - (0.4-0.75)\frac{77}{25}\right]$$

$$= \frac{4}{3}\left[\frac{148}{125} + \frac{539}{500}\right] = \frac{4}{3}\left[\frac{1131}{500}\right]$$

$$= 3.016 \; //$$

$$
\begin{bmatrix}
f x_0 & & & \\
f x_1 & f(x_0,x_1) & & \\
f(x_2) & f(x_1,x_2) & f(x_0,x_1,x_2) & \\
f(x_3) & f(x_2,x_3) & f(x_1,x_2,x_3) & f(x_0,x_1,x_2,x_3)
\end{bmatrix}
$$

$$
\begin{bmatrix}
f(x_0) \\
f(x_0,x_1) \\
f(x_0,x_1,x_2) \\
f(x_0,x_1,x_2,x_3)
\end{bmatrix}
\begin{bmatrix}
1 \\
(x-x_0) \\
(x-x_0)(x-x_1) \\
(x-x_0)(x-x_1)(x-x_2)
\end{bmatrix}
$$