Operations Research

Summative Assessment 1 (Part 1)

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Case 1

HiDec produces two models of electronic gadgets that use resistors, capacitors, and chips. The following table summarizes the data of the situation:

1. Formulate the problem as a linear program, and find the optimum solution.

Linear Model

Decision Variables

x = # Model 1 produced

y = # Model 2 produced

Objective Function

Max $Z = 3x + 4y$

Constraints

$2x + 3y \leq 1200$

$3x + y \leq 1000$

$4y \leq 800$

Adding Slack Variables

let $s_1, s_2, s_3 \geq 0$

$2x + 3y + s_1 = 1200$

$3x + y + s_2 = 1000$

$4y + s_3 = 800$

Simplex Method

Basic	-Z	x	y	s_1	s_2	s_3	RHS
-Z	1	3	4	0	0	0	0

Basic	-Z	x	y	s_1	s_2	s_3	RHS
s_1	0	2	3	1	0	0	1200
s_2	0	3	1	0	1	0	1000
s_3	0	0	4	0	0	1	800

Choose y as pivot column.

Minimum Ratio Test $Min\{\frac{1200}{3}, \frac{1000}{1}, \frac{800}{4}\}$

$Min\{400, 1000, 200\} = 200$

Choose R_3 as pivot row.

First Iteration

$R_0 \rightarrow R_0 + (-1)R_1$

$R_1 \rightarrow R_1 + (\frac{-3}{4})R_1$

$R_2 \rightarrow R_2 + (\frac{-1}{4})R_1$

$R_3 \rightarrow R_3 + (\frac{-3}{4})R_1$

Basic	negZ	x	y	s_1	s_2	s_3	RHS
Obj	1	3	0	0	0	-1	-800
s_1	0	2	0	1	0	-3/4	600
s_2	0	3	0	0	1	-1/4	800
y	0	0	1	0	0	1/4	200

Choose x as pivot column.

Minimum Ratio Test $Min\{\frac{600}{2}, \frac{800}{3}, \frac{200}{0}\}$

$Min\{300, 266.67, n/a\} = 266.67$

Choose R_2 as pivot row.

Second Iteration

$R_0 \rightarrow R_0 + (-1)R_2$

$R_1 \rightarrow R_1 + (\frac{-2}{3})R_2$

$R_2 \rightarrow R_2 + (\frac{-2}{3})R_2$

$R_3 \rightarrow R_3 + (0)R_2$

Basic	negZ	x	y	s_1	s_2	s_3	RHS
Obj	1	0	0	0	-1	-3/4	-1600
s_1	0	0	0	1	-2/3	-7/12	200/3
x	0	1	0	0	1/3	-1/12	800/3
y	0	0	1	0	0	1/4	200

Since all entries in R_0 are all ≤ 0, we can now terminate the simplex method.

Therefore, the most optimal x and y values are 800/3 and 200, respectively.

Thus the max Z given the contraints is:

$$P = 3x + 4y = 3(\frac{800}{3}) + 4(200) = 1600$$

2. Determine the status of each resource. (used/unused proportions or scarcity/abundance)

For Resistor (s_1):

- $\frac{200}{3}$ units was left.
- This means that $1 - \frac{200/3}{1200} = 0.9444 \text{ or } 94.44\%$ was used.

For Capacitor (s_2):

- 0 units was left.
- This means that $1 - \frac{0}{1000} = 1 \text{ or } 100\%$ was used.

For Chips (s_3):

- 0 units was left.
- This means that $1 - \frac{0}{1000} = 1 \text{ or } 100\%$ was used.

3. In terms of the optimal revenue, determine the dual prices for the resistors, capacitors, and chips.

From the final iteration:

Basic	negZ	x	y	s_1	s_2	s_3	RHS
Obj	1	0	0	0	-1	-3/4	-1600

The dual prices can be computed by taking the negative of the coefficients of the slack variables in the objective row:

1. For Resistor (s_1):

- Coefficient in objective row: 0
- Dual price = -0 = 0
- Adding one more unit of Resistor would increase the objective value by $0. This makes sense since we already have unused units of this resource (it has slack).

2. For Capacitor (s_2):

- Coefficient in objective row: -1
- Dual price = -(-1) = $1
- Adding one more unit of Capacitor would increase the objective value by $1.

3. For Chips (s_3):

- Coefficient in objective row: -3/4
- Dual price = -(-3/4) = $0.75
- Adding one more unit of Chips would increase the objective value by $0.75.

Resource	Coefficient in Objective Row	Dual Price
Resistors	0	$0
Capacitors	-1	$1
Chips	-3/4	$0.75

4. Determine the feasibility ranges for the dual prices obtained in (3).

Basic	negZ	x	y	s_1	s_2	s_3	RHS
Obj	1	0	0	0	-1	-3/4	-1600
s_1	0	0	0	1	-2/3	-7/12	200/3
x	0	1	0	0	1/3	-1/12	800/3
y	0	0	1	0	0	1/4	200

Resistors (s_1) - Dual Price = $0

- Slack value: s_1 = 200/3
- Current usage = 1200 - 200/3 = 1133.33
- Since constraint has slack, feasibility range for resistors is [1133.33, ∞)

Capacitors (s_2) - Dual Price = $1

From final tableau equations:

- $x = 800/3 + (1/3)s_2 - (1/12)s_3$
- $s_1 = 200/3 - (2/3)s_2 - (7/12)s_3$

Lower bound:

Lower bound can be derrived by how much Δ_{s_2} we can **decrease** to the s_2 before basic variable reach infeasibility, violates non-negativity.

Let x and s_1 be equals to 0.

- $0 = 800/3 + (1/3)(-\Delta_{s_2}) - (1/12)(0)$, since s_3 is a non-basic variable, $s_3 = 0$.
- $0 = 200/3 - (2/3)(-\Delta_{s_2}) - (7/12)(0)$

Solving with x equation gives us:

$$0 = 800/3 + (1/3)(-\Delta_{s_2})$$

$$800/3 = (1/3)(\Delta_{s_2})$$

$$800 = \Delta_{s_2}$$

and with s_1 equation:

$$0 = 200/3 - (2/3)(-\Delta_{s_2})$$

$$-200/3 = (2/3)(\Delta_{s_2})$$

$$-100 = \Delta_{s_2}$$

Note that increasing the $-\Delta_{s_2}$ in the case of s_1 equation only increases the s_1 so it will never vilate non-negativity.

Therefore we can only **decrease** x by $\Delta = 100$.

Computing for the lower bound:

$$LowerBound = Available_{s_2} - \Delta_{s_2}$$

$$LowerBound = 1000 - 800 = 200$$

Upper bound:

Applying the same logic, but in **increasing** fashion.

$$0 = 200/3 - (2/3)(\Delta_{s_2})$$

$$200/3 = (2/3)(\Delta_{s_2})$$

$$100 = \Delta_{s_2}$$

$$UpperBound = Available_{s_2} + \Delta_{s_2}$$

$$UpperBound = 1000 + 100 = 1100$$

Feasibility range for Capacitors: [200, 1100]

Chips (s_3) - Dual Price = $0.75

From final tableau equations:

- $y = 200 + (1/4)s_3$

- $s_1 = 200/3 - (7/12)s_3$

Lower bound:

$$0 = 200 + (1/4)(-\Delta_{s_3})$$

$$200 = (1/4)(\Delta_{s_3})$$

$$800 = \Delta_{s_3}$$

$$Lowerbound = Available_{s_3} - \Delta_{s_3}$$

$$Lowerbound = 800 - 800 = 0$$

Upper bound:

$$0 = 200/3 - (7/12)(\Delta_{s_3})$$

$$200/3 = (7/12)(\Delta_{s_3})$$

$$2400/21 = \Delta_{s_3} \approx 114.29$$

$$Upperbound = Available_{s_3} + \Delta_{s_3}$$

$$Upperbound = 800 + 114.29 = 914.29$$

Feasibility range for Chips: [0, 914.29]

5. If the available number of resistors is increased to 1300 units, find the new optimum solution.

The optimal solution of $x = 800/3$, $y = 200$, and $Z = 1600$ remains the same given the new constraints in Resistors.

This is due to the fact that Resistors already has slack (s_3). The only change that would happen will be the increase from 200/3 to 500/3 units of unused Resistor.

6. If the available number of chips is reduced to 350 units, will you be able to determine the new optimum solution directly from the given information? Explain.

No because reducing the constraints for Chips from maximum of 700 to maximum of 350 units (450 units difference) will turn the corresponding slack, s_3 , from 0 to -450, thereby violating the $s_3 \geq 0$ constraint.

We have to do the simplex method from the scatch with contraints modified.

7. If the availability of capacitors is limited by the feasibility range computed in (4), determine the corresponding range of the optimal revenue and the corresponding ranges for the numbers of units to be produced of Models 1 and 2.

Recall that the **Feasibility range for Capacitors (c): [200, 1100]** and a dual price (DP)of 1$.

Also, the current optimal Z ($Current_Z$) is 1600.

If capacitor availability decreases to its lower bound of 200 units:

$$\Delta_{s_3} = Initial_{s_3} - LB_{s_3} = 1000 - 200 = 800$$

The Lower Bound for Z becomes:

$$LB_Z = Current_Z - DP \cdot \Delta_{s_3}$$

$$LB_Z = 1600 - 1 \cdot 800 = 800$$

If capacitor availability increases to its upper bound of 200 units:

$$\Delta_{s_3} = UB_{s_3} - Initial_{s_3} = 1100 - 1000 = 100$$

The Upper Bound for Z becomes:

$$UB_Z = Current_Z + DP \cdot \Delta s_3$$

$$UB_Z = 1600 + 1 \cdot 100 = 1700$$

Optimal Z : [800, 1700]

Recall from final tableau equations:

- $x = 800/3 + (1/3)s_2 - (1/12)s_3$
- $s_1 = 200/3 - (2/3)s_2 - (7/12)s_3$
- y is unaffected by changes by s_2

At lower bound,

- $x = 800/3 - (1/3)(800) = 0$
- $s_1 = 200/3 + (2/3)(800) = 1800/3 = 600$
- y remains the same at 200

At upper bound,

- $x = 800/3 + (1/3)(100) = 900/3 = 300$
- $s_1 = 200/3 - (2/3)(100) = 0$
- $y = 200$

Optimal Z range: [800, 1700]

x range: [0, 300]

y remains constant at 200

8. A new contractor is offering to sell HiDec additional resistors at 40 cents each, but only if HiDec would purchase at least 500 units. Should HiDec accept the offer?

No because the corresponding dual price of resistor is 0. This means that purchasing additional resistors won't add anything to the objective function. It will be an unnecessary cost for HiDec, evenmoreso given the fact that resistor already has slack of $s_1 = 200/3$ or unused resistors.

Case 2

Case 2:

Three refineries with daily capacities of 6, 5, and 8 million gallons, respectively, supply three distribution areas with daily demands of 4, 8, and 7 million gallons, respectively. Gasoline is transported to the three distribution areas through a network of pipelines. The transportation cost is 10 cents per 1000 gallons per pipeline mile. The table gives the mileage between the refineries and the distribution areas. Refinery 1 is not connected to distribution area 3.

TABLE 5.26 Mileage Chart for Problem 5-8

	Distribution area 1	2	3
Refinery 1	180	180	—
Refinery 2	300	800	900
Refinery 3	220	200	120

1. Construct the associated transportation model.
2. Determine the optimum shipping schedule in the network.
3. Suppose that the capacity of ...

Recall that $\text{cost} = \sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij} X_{ij}$ and that

transportation cost is 10 cents per 1000 gallons per pipeline mile.

can be written as $\frac{10 \text{ cents}}{1000 \text{ gal} \cdot \text{mi}}$

Applying the general formula to the prob.

$$\text{cost} = \frac{10 \text{ cents}}{1000 \text{ gal} \cdot \text{mi}} \sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij} \text{ mi } X_{ij} \text{ E6 gal}$$

$$= \frac{\text{cents}}{100} \sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij} X_{ij} \text{E}6$$

$$= \text{cents} \sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij} X_{ij} \text{E}4$$

where C_{ij} is the milleage for (i,j)
and X_{ij} is the allocated gallon for (i,j)
rounded in to the nearest millionth.

Let i be the index of Refineries (R1, R2, R3)

j be the index of destinations (1, 2, 3)

$$\begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 8 \end{bmatrix}$$ be daily capacity of refinery i.

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 7 \end{bmatrix}$$ be daily demands of destination j.

Decision Variables:

X_{ij} = quantity of gasoline to be shipped from refinery i to destination j

Obj. Function

Minimize $\text{Cost} = \text{Cents} \sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij} X_{ij} E4$

where C_{ij} is the milleage for (i, j)

and X_{ij} is the allocated gallon for (i, j)

rounded in to the nearest millionth.

Constraints

$$\sum_{j=1}^{3} X_{ij} \leq S_i \quad \text{for all } i = 1, 2, 3$$

$$X_{ij} \geq 0 \quad \text{for all } i \text{ and } j.$$

$$\sum_{i=1}^{3} X_{ij} \geq D_j \quad \text{for all } j = 1, 2, 3$$

		40	20 (4)	780 (1)	
		1	2	3	S
0	R1	180	6 / 180	—	~~6~~ 0
600 (2)	R2	4 / 300	1 / 800	900	~~5~~ ~~1~~ 0
80 (3)	R3	220	1 / 200	7 / 120	~~8~~ ~~1~~ 0
	0	~~4~~ 0	~~8~~ ~~1~~ 0	~~7~~ 0	sum 19

Using VAM

$$\text{Cost} = \text{cents} \sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij} X_{ij} E4$$

$$= \text{cents}\left[(180 \cdot 6E4) + (300 \cdot 4E4) + 900E4 + 200E4 + (120 \cdot 7E4)\right]$$

$$= 41\,200\,000 \text{ cents} = 412\,000 \,\$$$

is the cost for this solution.

$u_i \backslash v_j$		$v_1 = -320$	$v_2 = 180$	$v_3 = 100$	
		1	2	3	S
$u_1 = 0$	R1	$=500$ [180]	6 [180]	—	0
$u_2 = 620$	R2	4 [300]	1 [800]	$=180$ [900]	0
$u_3 = 20$	R3	$=520$ [220]	1 [200]	7 [120]	0
	0	0	0	0	sum 19

Since $\overline{C_{ij}}$ are nonnegative this is the optimal solution!

$C_{12} = u_1 + v_2 \iff 180 = 0 + v_2$

$C_{22} = u_2 + v_2 \iff 800 = u_2 + 180$
$620 = u_2$

$C_{32} = u_3 + v_2 \iff 200 = u_3 + 180$
$20 = u_3$

$C_{21} = u_2 + v_1 \iff 300 = 620 + v_1$
$-320 = v_1$

$C_{33} = u_3 + v_3 \iff 120 = 20 + v_3$
$100 = v_3$

$\overline{C_{11}} = C_{11} - (u_1 + v_1) = 180 - (-320) = 500$

$\overline{C_{23}} = C_{23} - (u_2 + v_3) = 900 - (720) = 180$

$\overline{C_{31}} = C_{31} - (u_3 + v_1) = 220 - (-300) = 520$

Optimal Solution

$\boxed{X_{12} = 6,\ X_{21} = 4,\ X_{22} = 1,\ X_{32} = 1,\ X_{33} = 7}$ and all other $x_{ij} = 0$

Also, the cost for optimal solution, as was computed earlier is $\boxed{41\,200\,000 \text{ cents or } 412\,000 \$}$

3. Suppose that the capacity of refinery 3 is 6 million gallons only and that distribution area 1 must receive all its demand. Additionally, any shortages at areas 2 and 3 will incur a penalty of 5 cents per gallon. Determine the optimum shipping schedule.

For this problem, we have redefine the linear model, instead of C_{ij} being in terms of milleage, let's convert it to raw cents per galloon.

$$\text{transpo cost} = \frac{10 \text{ cents}}{1000 \text{ gal} \cdot \text{mi}} = \frac{\text{cents}}{100 \text{ gal} \cdot \text{mi}}$$

Taking the C_{ij} from previous item (C_{old})

$$C_{new} = \text{transpo cost } C_{old} = \frac{\text{cents}}{100 \text{ gal} \cdot \cancel{\text{mi}}} \begin{bmatrix} 180 & 180 & - \\ 300 & 800 & 900 \\ 220 & 200 & 120 \end{bmatrix} \cancel{\text{mi}}$$

$$= \begin{bmatrix} 1.8 & 1.8 & - \\ 3.0 & 8.0 & 9.0 \\ 2.2 & 2.0 & 1.2 \end{bmatrix} \frac{\text{cents}}{\text{gal}}$$

Linear Model

Let i be the index of Refineries (R1, R2, R3)

j be the index of destinations (1, 2, 3)

$\begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 6 \end{bmatrix}$ be daily capacity of refinery i.

$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 7 \end{bmatrix}$ be daily demands of destination j.

Decision Variables.

X_{ij} = quantity of gasoline to be shipped from refinery i to destination j

Obj. Function

Minimize Cost = Cents $\sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij} X_{ij} E6$

where C_{ij} is the $\frac{\text{cent}}{\text{gallon}}$ rate for (i, j)

and X_{ij} is the allocated gallon for (i, j)

rounded in to the nearest millionth.

Constraints

$\sum_{j=1}^{3} X_{ij} \leq S_i$ for all $i = 1, 2, 3$

$X_{ij} \geq 0$ for all i and j.

$\sum_{i=1}^{3} X_{ij} \geq D_j$ for all $j = 1, 2, 3$

Since $\sum_{i=1}^{3} S_i < \sum_{j=1}^{3} D_j$ let $S_4 = \sum_{j=1}^{3} D_j - \sum_{i=1}^{m} S_i$

and $C_4 = \begin{bmatrix} C_{41} \\ C_{42} \\ C_{43} \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix}$ be the unit shortage cost at destination j.

Also, D_1 must be 0 by the end.

Using VAM

		1.4	0.2	3.4 ②	
		1	2	3	
0	R1	$\boxed{1.4}$	6 $\boxed{1.8}$	-	~~6~~ 0
① 5	R2	4 $\boxed{3}$	1 $\boxed{8}$	$\boxed{9}$	~~5~~ ~~1~~ 0
0.4	R3	$\boxed{2.2}$	6 $\boxed{2}$	$\boxed{1.2}$	~~6~~ 0 ?
	S_4	$\boxed{0}$	$\boxed{5}$	$\boxed{5}$	
		~~4~~ 0	~~8~~ ~~2~~ 1	~~2~~ 1	

V_2 \ V_2		$v_1=-3.2$ 1	$v_2=1.8$ 2	$v_3=1.8$ 3
$u_1=0$	R1	$=5$ $\boxed{1.8}$	6 $\boxed{1.8}$	$-$
$u_2=6.2$	R2	4 $\boxed{3}$	1 $\boxed{8}$	$=1$ $\boxed{9}$
$u_3=-0.6$	R3	$=6$ $\boxed{2.2}$	$=0.6$ $\boxed{2}$	6 $\boxed{1.2}$
$u_4=3.2$	S_4	$\boxed{0}$	1 $\boxed{5}$	1 $\boxed{5}$

$$\overline{C_{11}} = 1.8 - (-3.2) = 5$$

$$\overline{C_{23}} = 9 - (8) = 1$$

$$\overline{C_{31}} = 2.2 - (-3.8) = 6$$

$$\overline{C_{32}} = 2 - (1.2) = 0.6$$

$$\overline{C_{41}} = 0 - 0 = 0$$

Since all $\overline{C_{ij}}$ are non negative, this already optimal!!

Optimal solution:

$X_{12} = 6$, $X_{21} = 4$, $X_{22} = 1$, $X_{33} = 6$ and unmet demands are 1 for "2" and 1 for "3" destination.

Total cost is:

$$\text{Cost} = \text{cents}\left[(6 \cdot 1.8) + (4 \cdot 3) + 8 + (6 \cdot 1.2) + 5 + 5\right] E6$$

$$= \boxed{48\,E6 \text{ cents or } 480\,000\,\$}$$