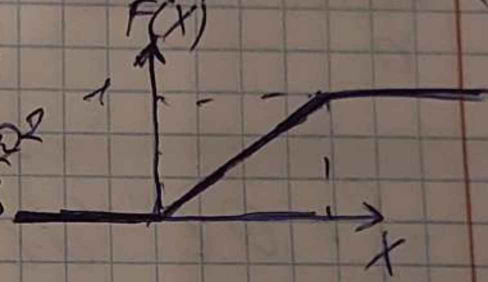
Первое задание

1. $\xi \sim R[0;\ \theta]$

$\tilde{\theta}_1 = 2\bar{x}$, $\tilde{\theta}_2 = x_{min}$ $\tilde{\theta}_3 = x_{max}$ $\theta_4 = x_1 + \dfrac{\sum\limits_{k=2}^{n} x_k}{(n-1)}$

$\vec{X}_n$ = выборка объема n



$M[\xi] = \dfrac{\theta}{2}$ $M[\xi^2] = \dfrac{\theta^2}{3}$

$p(x) = \dfrac{x}{\theta}\{0;2\}$

$D[\xi] = \dfrac{\theta^2}{3} - \dfrac{\theta^2}{4} = \dfrac{\theta^2}{12}$

$\tilde{\theta}_1 = 2\bar{X}$ $\forall \theta > 0$, $M[\tilde{\theta}_1] = \theta$

$M\left[\dfrac{2}{n}\sum\limits_{i=1}^{n} x_i\right] = \dfrac{2}{n}\sum\limits_{i=1}^{n} M[x_i] = 2M[\xi] = \theta \Rightarrow$ несмещ.

$D[\tilde{\theta}_1] = D\left[\dfrac{2}{n}\sum x_i\right] = \dfrac{4}{n^2}\sum D[x_i] = \dfrac{4}{n}D[\xi] =$

$= \dfrac{\theta^2}{3n} \xrightarrow[n\to\infty]{} 0 \Rightarrow$ состоятельная

$\tilde{\theta}_2 = \min(x_i)$ $\quad \Phi(y) = 1-(1-F(y))^n$

$\varphi(y) = n(1-F(y))^{n-1} p(y)$

$M[\tilde{\theta}_2] =$

$$= \int_0^\theta n\left(1-\frac{y}{\theta}\right)^{n-1} \frac{1}{\theta} y\, dy = \begin{Bmatrix} t = 1-\frac{y}{\theta} \\ dt = -\frac{dy}{\theta} \end{Bmatrix} =$$

$$= \int_0^1 n\theta t^{n-1} dt - \int_0^1 n\theta t^n dt =$$

$= \theta\left[1-\frac{n}{n+1}\right] = \frac{\theta}{n+1}$ смещенная

$\tilde{\theta}_2' = (n+1)\tilde{\theta}_2$ не смещ $M[\tilde{\theta}_2'] = \theta$

$$M[\tilde{\theta}_2^2] = \int_0^\theta n\left(1-\frac{y}{\theta}\right)^{n-1} \frac{1}{\theta} y^2 dy =$$

$$= \int_0^1 \theta^2 n t^{n-1} \cdot (1-\cancel{2t}+t^2)\, dt =$$

$$= \theta^2\left[1 - 2\frac{n}{n+1} + \frac{n}{n+2}\right] =$$

$$= \theta^2\left[\frac{n^2+3n+2-2n^2-4n+n^2+n}{(n+1)(n+2)}\right] =$$

$$= \frac{2\theta^2}{(n+1)(n+2)}$$

$$D[\tilde{\theta}_2] = \frac{2\theta^2}{(n+1)(n+2)} - \frac{\theta^2}{(n+1)^2} = \frac{2(n+1)-(n+2)}{(n+1)^2(n+2)}\theta^2 =$$

$$= \frac{n}{(n+1)^2(n+2)}\theta^2 \xrightarrow[n\to\infty]{} 0$$

$$D[\tilde{\theta}_2'] = \frac{n}{n+2}\theta^2 \not\to 0$$

??

$\theta_3 \in X_{max}$

$\tilde{\theta}_2'$ по определению

$$\forall \theta > 0 \ \forall \varepsilon > 0 \quad P(|\tilde{\theta}_2' - \theta| \geq \varepsilon) \xrightarrow[n\to\infty]{} 0$$

$$P(|\tilde{\theta}_2' - \theta| \geq \varepsilon) \geq P(\tilde{\theta}_2' \geq \theta + \varepsilon) =$$

$$= P((n+1)X_{min} \geq \theta+\varepsilon) = P\left(X_{min} \geq \frac{\theta+\varepsilon}{n+1}\right) =$$

$$= 1 - P\left(X_{min} < \frac{\theta+\varepsilon}{n+1}\right) = 1 - \left(1 - \left(1 - F\left(\frac{\theta+\varepsilon}{n+1}\right)\right)^n\right) =$$

$$= \left(1 - \left(\frac{\theta+\varepsilon}{\theta(n+1)}\right)\right)^n \to e^{-\frac{\theta+\varepsilon}{\theta}} > 0$$

не состоятельная

$\tilde{\Theta}_3$ по опр.

$\forall \Theta > 0 \; \forall \varepsilon > 0 \quad P(|\tilde{\Theta}_3 - \Theta| \geq \varepsilon) = P(x_{max} \leq \Theta - \varepsilon) +$

$+ P(\cancel{x_{max} \geq \Theta + \varepsilon}) = (F(\Theta - \varepsilon))^n = \begin{cases} \left(\frac{\Theta - \varepsilon}{\Theta}\right)^n \xrightarrow[n\to\infty]{} 0, & \varepsilon < \Theta \\ (0)^n \xrightarrow[n\to\infty]{} 0 & \varepsilon \geq \Theta \end{cases}$

$\tilde{\Theta}_3'$ по опр.

$\forall \Theta > 0 \; \forall \varepsilon > 0$

$P(|\tilde{\Theta}_3' - \Theta| \geq \varepsilon) = P(x_{max} \cdot \frac{n+1}{n} \leq \Theta - \varepsilon) +$

$+ P(x_{max} \frac{n+1}{n} \geq \Theta + \varepsilon) = P\left(x_{max} \leq \frac{n(\Theta - \varepsilon)}{n+1}\right) +$

$P\left(x_{max} \geq \frac{n(\Theta + \varepsilon)}{n+1}\right) = \underbrace{\left(F\left(\frac{n(\Theta - \varepsilon)}{n+1}\right)\right)^n}_{F_1} +$

$+ 1 - \underbrace{\left(F\left(\frac{n(\Theta + \varepsilon)}{n+1}\right)\right)^n}_{F_2} = F_1 + 1 - F_2$

F_1: $\quad \Theta > \varepsilon : \left(\frac{n(\Theta - \varepsilon)}{\Theta(n+1)}\right)^n = \left(\frac{(\Theta - \varepsilon)}{\Theta(1 + \frac{1}{n})}\right)^n =$

$\Rightarrow 0, \; n \to \infty, \quad \Theta < \varepsilon : (0)^n \to 0 \; n \to \infty$

$$n(\theta + \varepsilon - \theta^2) \leq \theta^2$$

$$F_2 \quad \frac{n(\theta+\varepsilon)}{n+1} > \theta \; : \quad F_2 = 1^n \xrightarrow[n\to\infty]{} 1$$

$$\cancel{n\theta + \varepsilon \leq \theta}$$

$$\frac{n(\theta+\varepsilon)}{n+1} \leq \theta \qquad n\theta + \varepsilon n \leq \theta n + \theta$$

$$\cancel{\theta} \lambda \leq \frac{\theta}{n}$$

$\Rightarrow \exists \tilde{N}(\varepsilon, \theta): \forall n \geq \tilde{N}(\varepsilon,\theta) \Leftrightarrow \frac{n(\theta+\varepsilon)}{n+1} > \theta$

$\Rightarrow F_2 \to 1,\ n \to \infty$

$\circledast$ $0 + 1 - 1 \to 0 \Rightarrow$ состоятельная

$$\tilde{\theta}_n = X_1 + \frac{\sum_{k=2}^{n} X_k}{(n-1)} \qquad M[\tilde{\theta}_n] = M[X_1] + \frac{\sum_{k=2}^{n} M[X_k]}{n-1} =$$

$$= \frac{\theta}{2} + (n-1)\frac{\theta}{2} \cdot \frac{1}{n-1} = \theta$$

$\Rightarrow$ несмещ.

$$D[\tilde{\theta}_4] = D[X_1] + \sum_{k=2}^{n} D[X_k] \cdot \frac{1}{(n-1)^2} =$$

$$= \cancel{\frac{1}{n-1} \cdot \frac{\theta^2}{12} \Rightarrow 0,\ n \to}$$

$$= \left(1 + \frac{1}{n-1}\right)\frac{\theta^2}{12} = \frac{n-1+1}{n-1}\frac{\theta^2}{12} \underset{n\to\infty}{\not\to} 0$$

$\tilde{\theta}$ ~~не состоятельная~~

$\tilde{\theta}_4$ не определенно

$$\tilde{\theta} = \underbrace{x_1}_{\xi_n} + \underbrace{\frac{1}{n+1}\sum_{i=2}^{n} x_i}_{\eta_n}$$

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$$x_1 \xrightarrow[n]{P} x_1 \qquad \frac{1}{n}\sum_{i=1}^{n} \xi_i \xrightarrow{P} M[\xi_i].$$

$$\frac{1}{n-1}\sum_{i=2}^{n} x_i \xrightarrow{P} M[\xi] = \frac{\theta}{2}$$

$$\tilde{\theta}_4 = x_1 + \frac{1}{n-1}\sum_{i=2}^{n} x_i \xrightarrow{P} x_1 + \frac{\theta}{2} \not\to \theta \Rightarrow$$

несостоятельна

θ_1, θ_3, $\tilde{\theta}_3'$
несмещ сост, смещ сост, несм сост

$D[\theta_1] = \frac{\theta^2}{3n}$ $D[\tilde{\theta}_3'] = \frac{\theta^2}{n(n+2)}$

$\forall \theta > 0 \quad \frac{\theta^2}{n(n+2)} < \frac{\theta^2}{3n} \Rightarrow \tilde{\theta}_3'$ самая эффективная

T3

$$p(x) = \begin{cases} \frac{e^{-\frac{x}{\theta}}}{\theta}, & x \geq 0 \\ 0, & x < 0 \end{cases} \quad \theta > 0$$

по выборке $n=3$

$\tilde{\theta}_1 = \bar{x}$, $\tilde{\theta}_2 = x_{(2)}$

$$M[\xi] = \int_{-\infty}^{+\infty} x\,p(x) = \int_0^{+\infty} \frac{x}{\theta} e^{-\frac{x}{\theta}} dx =$$

$x = u$ $dx = du$
$e^{-\frac{x}{\theta}}dx = dv$ $-\theta e^{-\frac{x}{\theta}} = v$

$$= \frac{1}{\theta}\left(x \cdot (-\theta e^{-\frac{x}{\theta}}) \Big|_0^{\infty} + \int_0^{\infty} \theta e^{-\frac{x}{\theta}} dx \right) =$$

$$= \frac{1}{\theta}\left(-\theta^2 e^{-\frac{x}{\theta}} \Big|_0^{\infty} \right) = \frac{1}{\theta}(+\theta^2) = \theta$$