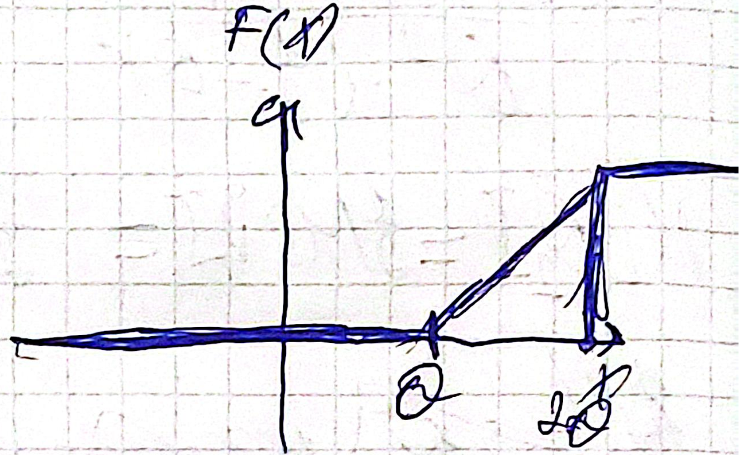
T5.

$\xi \sim R(\theta, 2\theta)$

$\vec{X_n}$ – выборка



Метод моментов

$g(x,\theta) = \frac{1}{\theta}\{\theta, 2\theta\}$

$\alpha_1 = M[\xi] = \int_{\theta}^{2\theta} x \cdot \frac{1}{\theta} d\theta = \frac{x^2}{2\theta}\Big|_{\theta}^{2\theta} = \frac{3\theta}{2}$

$\alpha_2 = \frac{x^3}{3} \cdot \frac{1}{\theta}\Big|_{\theta}^{2\theta} = \frac{7\theta^2}{3}$

$D[\xi] = \frac{7\theta^2}{3} - \frac{9\theta^2}{4} = \frac{1}{12}\theta^2$

$\alpha_1 = \tilde{\alpha}_1 = \frac{1}{n}\sum_{i=1}^{n} x_i = \overline{X}$

$\tilde{\theta}_1 = \frac{2}{3}\overline{X}$

$M[\tilde{\theta}_1] = \frac{2}{3}M[\overline{X}] = \frac{2}{3}M[\xi] = \theta$ несмещ

$D[\tilde{\theta}_1] = \frac{4}{9n} D\xi = \frac{1}{27n}\theta^2 \to 0, n \to \infty$

$\Rightarrow$ состоятельная

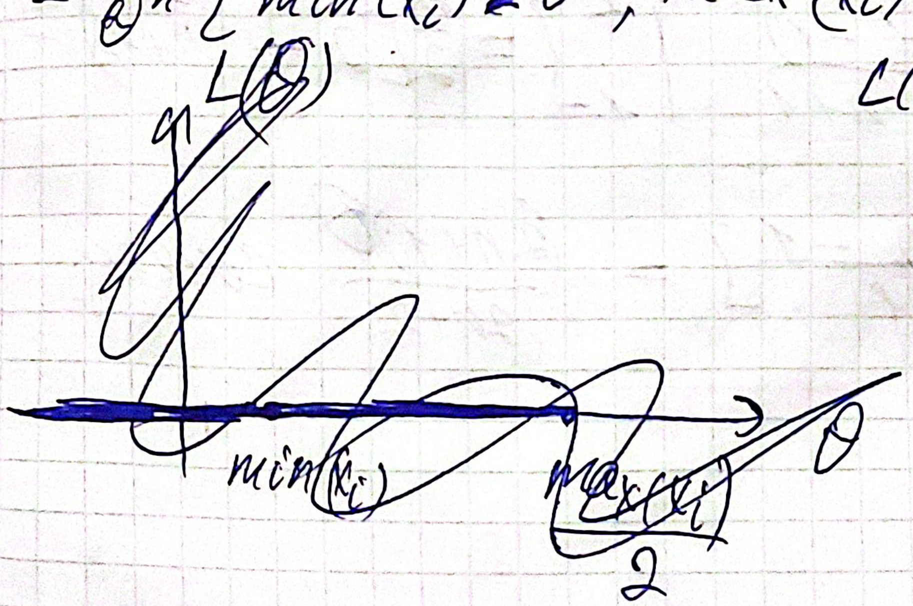
~~эффективность~~

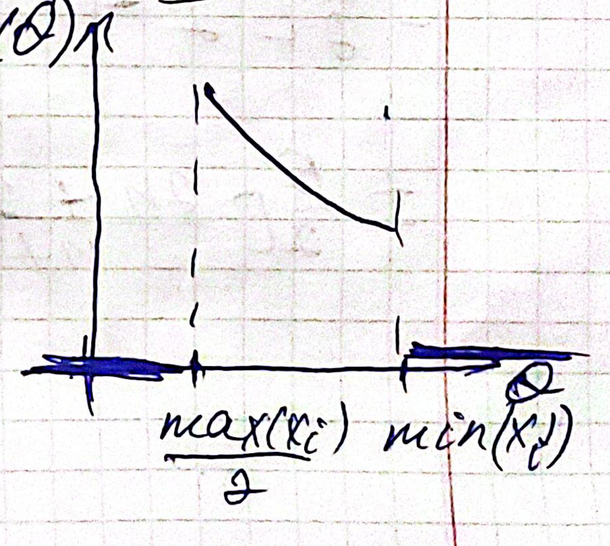
~~as регулярность~~

Метод максимального правдоподобия

$L(\theta) = \prod_{i=1}^{n} p(x_i, \theta) = \frac{1}{\theta^n}\{\forall i, \ \theta < x_i < 2\theta\}$

$= \frac{1}{\theta^n}\{\min(x_i) > \theta, \ \max(x_i) < 2\theta\}$





$$\Rightarrow \tilde{\theta}_2 = \frac{x_{max}}{2}$$

$$M[\tilde{\theta}_2] = \frac{1}{2} M[X_{max}] =$$

$$\Phi(z) = (F(z))^n \quad \varphi(z) = n(F(z))^{n-1} \cdot p(z) =$$

$$= \cancel{n\left(\frac{z}{\theta}\right)^{n-1} \cdot \frac{1}{\theta}} \quad n\left(\frac{z}{\theta} - 1\right)^{n-1} \cdot \frac{1}{\theta}$$

$$= \frac{1}{2}\int_{\theta}^{2\theta} zn\left(\frac{z}{\theta} - 1\right)^{n-1} \cdot \frac{1}{\theta}\, dz =$$

$$= \frac{\theta}{2}\int_{1}^{2} nt(t-1)^{n-1}\, dt = \quad t = u \quad dt = du \quad n(t-1)^{n-1}dt = dv \quad (t-1)^n = v$$

$$= \frac{\theta}{2}\left[t(t-1)^n \Big|_1^2 - \int_1^2 (t-1)^n\, dt\right] =$$

$$= \frac{\theta}{2}\left[2 - \frac{(t-1)^{n+1}}{n+1}\Big|_1^2\right] = \frac{\theta}{2}\left[2 - \frac{1}{n+1}\right] =$$

$$= \frac{\theta}{2}\left[\frac{2n+2-1}{n+1}\right] = \frac{(2n+1)\theta}{2n+2}$$ смещ.

$\tilde{\theta}_2' = \frac{2n+2}{2n+1}\tilde{\theta}_2$ несмещ.

$\cancel{D[\tilde{\theta}_2']}$

$$M[\tilde{\theta}_2^2] = \int_{-\infty}^{+\infty} z^2 \varphi(z)\,dz = \frac{1}{2}\int_{\theta}^{2\theta} n z^2 \left(\frac{z}{\theta}-1\right)^{n-1}\cdot\frac{1}{\theta}\,dz$$

$$= \frac{1}{4}\int_{\theta}^{2\theta} n\,\frac{z^2}{\theta^2}\,\theta\left(\frac{z}{\theta}-1\right)^{n-1}\cdot\frac{1}{\theta}\,dz = \frac{\theta^2}{4}\int_1^2 n t^2 (t-1)^{n-1}\,dt$$

$$= \frac{\theta^2}{4}\left[t^2(t-1)^n\Big|_1^2 - 2\int_1^2 t(t-1)^n\,dt\right] =$$

$$= \frac{\theta^2}{4}\left[4 - 2\left[t\,\frac{(t-1)^{n+1}}{n+1}\Big|_1^2 - \int_1^2 \frac{(t-1)^{n+1}}{n+1}\,dt\right]\right] =$$

$$= \theta^2 - \frac{\theta^2}{2}\left[\frac{2}{n+1} - \frac{(t-1)^{n+2}}{(n+1)(n+2)}\Big|_1^2\right] =$$

$$= \theta^2 - \frac{\theta^2}{2}\left[\frac{2(n+2)-1}{(n+1)(n+2)}\right] = \theta^2 - \frac{\theta^2}{2}\,\frac{2n+3}{(n+1)(n+2)}$$

$$= \frac{\theta^2\, 2(n^2+3n+2) - \theta^2(2n+3)}{2(n+1)(n+2)} = \frac{2n^2\theta^2 + 4\theta^2 n + 1\theta^2}{2(n+1)(n+2)}$$

$$= \cancel{\frac{\theta^2\, 4n-1}{2(n+1)(n+2)}}\ \theta^2 \quad \frac{(2n^2+4n+1)\,\theta^2}{2(n+1)(n+2)}$$

$$D[\tilde{\theta}_2] = \left(\frac{2n^2+4n+1}{2(n+1)(n+2)} - \frac{(2n+1)^2}{4(n+1)^2} \right) \theta^2 =$$

$4n^2+4n+1$

$$= \frac{2(n+1)(2n^2+4n+1) - (2n+1)^2(n+2)}{4(n+1)^2(n+2)} \theta^2 =$$

$$= \left(\frac{4n^3 + 8n^2 + 2n + 4n^2 + \cancel{8n+2} - 4n^3 - 4n^2 - n}{4(n+1)^2(n+2)} - \right.$$

$$\left. - \frac{8n^2 - \cancel{8n} - \cancel{2}}{4(n+1)^2(n+2)} \right) \theta^2 = \frac{n}{4(n+1)^2(n+2)} \theta^2$$

$$D[\tilde{\theta}_2'] = \left(\frac{2n+2}{2n+1} \right)^2 \cdot \frac{n}{4(n+1)^2(n+2)} \theta^2 =$$

$$= \frac{n}{(2n+1)^2(n+2)} \theta^2 \to 0, n \to \infty \text{ состоят.}$$

$$D[\tilde{\theta}_1] = \frac{\theta^2}{27n} \qquad D[\tilde{\theta}_2'] = \frac{n}{(2n+1)^2(n+2)}$$

$D[\tilde{\theta}_2'] < D[\tilde{\theta}_1] \ \forall n \geq 3$

$\Rightarrow \tilde{\theta}_2'$ эффективнее $\tilde{\theta}_1$

$$F(t) = \begin{cases} 0, & t < \theta \\ \frac{t}{\theta} - 1, & \theta \le t \le 2\theta \\ 1, & t > 2\theta \end{cases}$$

d) Точность интервал.

$$\frac{\max(X_1, \dots X_n)}{\theta} - 1$$

$$q(t) = P(\xi < t) = P(\max(X_1, \dots X_n) < \theta t + \theta) =$$

$$= P(X_1 < \theta t + \theta, \dots, X_n < \theta t + \theta) =$$

$$= P((X_1 < \theta t + \theta)^n) = (F(\theta t + \theta))^n$$

$$q(t) = \begin{cases} 0, & t \le 0 \\ t^n, & 0 \le t \le 1 \\ 1, & t > 1 \end{cases}$$

$$q(X_{\frac{\alpha}{2}}) = \frac{\alpha}{2} \qquad q(X_{\beta + \frac{\alpha}{2}}) = \beta + \frac{\alpha}{2}$$

$$t_1 = X_{\frac{\alpha}{2}} = \sqrt[n]{\frac{\alpha}{2}} = \sqrt[n]{\frac{1-\beta}{2}}$$

$$t_2 = X_{\frac{1+\beta}{2}} = \sqrt[n]{\frac{1+\beta}{2}}$$

$$t_1 < \frac{\max(x_i)}{\theta} - 1 < t_2$$

$$\cancel{\left(\frac{1+\beta}{2}\right)^{\frac{1}{n}} < \frac{\theta}{\max(x_i)} < \left(\frac{1-\beta}{2}\right)^{\frac{1}{n}}}$$

$$\frac{\max(x_i)}{\left(\frac{1+\beta}{2}\right)^{\frac{1}{n}}+1} < \theta < \frac{\max(x_i)}{\left(\frac{1-\beta}{2}\right)^{\frac{1}{n}}+1}$$

e) асимптотический

ОММ: $\tilde{\theta} = \frac{2}{3}\overline{x} = \frac{2}{3}\tilde{\alpha}_1 = g(\tilde{\alpha}_1)$

$$\frac{\sqrt{n}\,(g(\tilde{\alpha}_1) - g(\alpha_1))}{\sqrt{\nabla^T g\; K\; \nabla g}} \rightarrow N(0,1)$$

$\nabla g = \frac{2}{3}$ $\quad K_{11} = \alpha_2 - \alpha_1^2$

$$\frac{\sqrt{n}\,(\tilde{\theta} - \theta)}{\frac{2}{3}\sqrt{\tilde{\alpha}_2 - \tilde{\alpha}_1^2}} \rightarrow N(0,1)$$

$$\frac{(\tilde{\theta}_1 - \theta)}{\frac{2}{3} S \cdot \sqrt{n-1}} \to N(0,1)$$

$$P\left(t_1 < \frac{3n(\tilde{\theta}_1 - \theta)}{2S\sqrt{n-1}} < t_2\right) = \beta$$

$$\tilde{\theta}_1 - \frac{t_2 \cdot 2S\sqrt{n-1}}{3n} < \theta_4 < \tilde{\theta}_1 - \frac{t_1 \cdot 2S\sqrt{n-1}}{3n}$$