θ_1, θ_3, $\tilde{\theta}_3'$
несмещ сост, смещ сост, несм сост

$D[\theta_1] = \frac{\theta^2}{3n}$ $\quad D[\tilde{\theta}_3'] = \frac{\theta^2}{n(n+2)}$

$\forall \theta > 0 \quad \frac{\theta^2}{n(n+2)} < \frac{\theta^2}{3n} \Rightarrow \tilde{\theta}_3'$ самая эффективная

Т3

$$p(x) = \begin{cases} \frac{e^{-\frac{x}{\theta}}}{\theta}, & x \geq 0 \\ 0, & x < 0 \end{cases} \quad \theta > 0$$

по выборке $n=3$

$\tilde{\theta}_1 = \bar{x}$, $\tilde{\theta}_2 = x_{(2)}$

$$M[\xi] = \int_{-\infty}^{+\infty} x\, p(x) = \int_0^{+\infty} \frac{x}{\theta} e^{-\frac{x}{\theta}} dx = \begin{matrix} x = u & dx = du \\ e^{-\frac{x}{\theta}} dx = dv & -\theta e^{-\frac{x}{\theta}} = v \end{matrix}$$

$$= \frac{1}{\theta}\left(x \cdot (-\theta e^{-\frac{x}{\theta}}) \Big|_0^{\infty} + \int_0^{\infty} \theta e^{-\frac{x}{\theta}} dx \right) =$$

$$= \frac{1}{\theta}\left(-\theta^2 e^{-\frac{x}{\theta}} \Big|_0^{\infty} \right) = \frac{1}{\theta}(+\theta^2) = \theta$$

$$M[\xi^2] = \int_{-\infty}^{+\infty} x^2 p(x)\,dx = \int_0^{\infty} \frac{x^2}{\theta} e^{-\frac{x}{\theta}} dx =$$

$$= \frac{1}{\theta}\left(x^2 \left(e^{-\frac{x}{\theta}}(-\theta)\right)\Big| + 2\theta \int_0^{+\infty} x e^{-\frac{x}{\theta}} dx \right) =$$

$$= \frac{1}{\theta} \cdot 2\theta^3 = 2\theta^2$$

$$D[\xi] = 2\theta^2 - \theta^2 = \theta^2$$

$$\tilde{\theta}_1 = \overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$M[\tilde{\theta}_1] = M[\frac{1}{n}\sum_{i=1}^{n} x_i] = M[\xi] = \theta$$

$\Rightarrow \tilde{\theta}_1$ несмещ.

$$D[\tilde{\theta}_1] = D[\frac{1}{n}\sum_{i=1}^{n} x_i] = \frac{1}{n^2}\sum D[\xi] =$$

$$= \frac{\theta^2}{n}$$

$$\tilde{\theta}_2 = x_{(2)}$$

$$\varkappa(x) = n\frac{e^{-\frac{x}{\theta}}}{\theta}(n-1) e^{-\frac{(n-2)x}{\theta}}\left(1 - e^{-\frac{x}{\theta}}\right)$$

$$M[x_{(2)}] = n(n-1)\int_0^{+\infty} \frac{x}{\theta}\left(e^{-\frac{(n-1)x}{\theta}} - e^{-\frac{nx}{\theta}}\right)dx$$

$$= n(n-1)\int_0^{+\infty} \theta t \left(e^{-(n-1)t} - e^{-nt}\right) dt =$$

$$= n(n-1)\,\theta\left(\frac{1}{n-1}\int_0^{+\infty} e^{-(n-1)t}\, dt - \frac{1}{n}\int_0^{+\infty} e^{-(n-1)t}\, dt =\right.$$

$$= \left(\frac{n(n-1)}{(n-1)^2} - \frac{n(n-1)}{n^2}\right)\theta = \theta\left(\frac{n}{n-1} - \frac{n-1}{n}\right)$$

$$= \frac{2n-1}{n(n-1)}\theta$$ смещ.

$$\tilde{\theta}_2' = \frac{n(n-1)}{2n-1}\tilde{\theta}_2$$ - несмещ

$$M[\tilde{\theta}_2^2] = n(n-1)\theta^2\int_0^{+\infty} t^2 \left(e^{-(n-1)t} - e^{-nt}\right) dt =$$

$$= n(n-1)\,\theta^2\left(\frac{2}{n-1}\int_0^{+\infty} t e^{-(n-1)t}\, dt - \frac{1}{n}\int_0^{+\infty} t e^{-nt}\, dt =\right.$$

$$= n(n-1)\,\theta^2\left(\frac{2}{(n-1)^2}\int_0^{+\infty} e^{-(n-1)t}\, dt - \frac{2}{n^2}\int_0^{+\infty} e^{-nt}\, dt =\right.$$

$$= \left(\frac{2n}{(n-1)^2} - \frac{2(n-1)}{n^2}\right)\theta^2 = 2\theta^2 \cdot \frac{3n^2-3n+1}{n^2(n+1)^2}$$

$$D[\tilde{\theta}_2] = \frac{3n^2-3n+1}{n^2(n+1)^2}\theta^2 - \frac{4n^2-4n+1}{n^2(n+1)^2}$$

$$= \frac{2n^2-2n+1}{n^2(n+1)^2}\theta^2.$$

$$D[\tilde{\theta}_2'] = \frac{2n^2-2n+1}{4n^2-4n+1}\theta^2 \#$$

б) $D[\tilde{\Theta}_1] = \frac{\Theta^2}{n}$ $\quad D[\tilde{\Theta}_2'] = \frac{\Theta^2 (2 - \frac{2}{n} + \frac{1}{n^2})}{(4 - \frac{4}{n} + \frac{1}{n^2})}$

$\Rightarrow \exists n_0 : \forall n \geq n_0 \ \forall \Theta \hookrightarrow D[\tilde{\Theta}_1] < D[\tilde{\Theta}_2']$

$\Rightarrow \tilde{\Theta}_1$ эффективнее $\tilde{\Theta}_2'$

в) Проверим модель на регулярность

1) $p(x)$ непр. на $\Theta > 0$

2) $\frac{\partial}{\partial\Theta} \int\limits_{-\infty}^{+\infty} p(x)\,dx = \frac{\partial}{\partial\Theta}\left(\frac{1}{\Theta}\int\limits_{0}^{+\infty} e^{-\frac{x}{\Theta}}dx\right) =$

$= \frac{\partial}{\partial\Theta}\left(-\frac{1}{\Theta}\cdot\Theta\, e^{-\frac{x}{\Theta}}\right)\Big|_0^{+\infty} = \frac{\partial}{\partial\Theta}(1) = 0$

$\int\limits_{-\infty}^{+\infty} \frac{\partial}{\partial\Theta} p(x)\,dx = \int\limits_{0}^{+\infty} \frac{\partial}{\partial\Theta} \frac{e^{-\frac{x}{\Theta}}}{\Theta}\,dx =$

$= \int\limits_{0}^{+\infty} \frac{\frac{x}{\Theta^2} e^{-\frac{x}{\Theta}} \cdot \Theta - e^{-\frac{x}{\Theta}}}{\Theta^2}\,dx = \int\limits_{0}^{+\infty} \frac{x e^{-\frac{x}{\Theta}} - \Theta e^{-\frac{x}{\Theta}}}{\Theta^3}\,dx$

$= \frac{1}{\Theta^2}\left(\int\limits_{0}^{+\infty} \frac{x}{\Theta} e^{-\frac{x}{\Theta}} - \int\limits_{0}^{+\infty} e^{-\frac{x}{\Theta}}dx\right) = 0$

3) $\ln p(x) = \ln \frac{e^{-\frac{x}{\theta}}}{\theta} = -\frac{x}{\theta} - \ln\theta$

$\left(\frac{\partial \ln p(x)}{\partial \theta}\right)^2 = \left(+\frac{x}{\theta^2} - \frac{1}{\theta}\right)^2$

$I(\theta) = M\left[\left(\frac{\partial \ln p(x)}{\partial \theta}\right)^2\right] = \int_0^{+\infty} \frac{(x-\theta)^2}{\theta^4} \frac{e^{-\frac{x}{\theta}}}{\theta} dx =$

$= \frac{1}{\theta^2}$ непр на $\theta > 0$ и $I(\theta) > 0$

$\Rightarrow$ модель регулярна

регулярность оценки

$\tilde{\theta}_1 = \overline{x}$ модель регулярна,

$\tilde{\theta}_1$ = несмещ. $D[\tilde{\theta}_1] = \frac{\theta^2}{n}$ огр.

$\Rightarrow$ оценка на $\forall$ компакт

$\tilde{\theta}_2' = \frac{n(n-1)}{2n-1} x_{(2)}$ - регулярна по дост. усл.

несмещ.

$D[\tilde{\theta}_2'] = \theta^2\left[\frac{2n^2-2n+1}{4n^2-4n+1}\right]$ - огр на $\forall$ компакте

$\Rightarrow \tilde{\theta}_2'$ - регулярна

выполнены все условия
нер-ва Крамера-Рао

$$D[\tilde{\theta}_1] \geq \frac{1}{3\frac{1}{\theta^2}} = \frac{\theta^2}{3}$$

По т. о единственности эф. оценки
$\tilde{\theta}_1$ эффективна