№6

$$p(x) = \begin{cases} \dfrac{\theta - 1}{x^{\theta}}, & x \geq 1 \\ 0, & x < 1 \end{cases}, \quad \theta > 1$$

$$L(\theta) = \prod_{i=1}^{n} p(x_i; \theta) = \{x_{min} \geq 1\} = \frac{(\theta - 1)^n}{\left(\prod_{i=1}^{n} x_i\right)^{\theta}}$$

$$\ln L(\theta) = n \ln(\theta - 1) - \theta \left(\ln \prod_{i=1}^{n} x_i\right)$$

$$\ln L(\theta) = n\ln(\theta-1) - \theta(\ln \prod x_i)$$

$$(\ln L)' = \frac{n}{\theta-1} - \ln \prod x_i$$

$$\theta = 1 + \frac{n}{\ln \prod x_i}$$

$$\frac{\partial^2 \ln L}{\partial \theta^2} = \frac{-n}{(\theta-1)^2} < 0 \Rightarrow \text{лок. max}$$

$$\tilde{\theta} = 1 + \frac{n}{\sum_{i=1}^{n} \ln x_i} = 1 + \frac{1}{\ln x}$$

c) Построить асимптотический доверительный интервал для θ

$I(\theta) =$

$$\ln p(x) = \ln(\theta - 1) - \theta \ln x$$

$$\left(\frac{\partial \ln p}{\partial \theta}\right)^2 = \left(\frac{1}{\theta - 1} - \ln x\right)^2$$

$$I(\theta) = \int_1^{+\infty} \left(\frac{1}{(\theta-1)^2} - \frac{2\ln x}{\theta - 1} + \ln^2 x\right) \frac{\theta - 1}{x^{\theta}} dx$$

$$= \int_1^{+\infty} \frac{x^{-\theta}}{\theta - 1} dx - \int_1^{+\infty} \frac{2\ln x}{x^{\theta}} dx + \int_1^{+\infty} \frac{\ln^2 x(\theta-1)}{x^{\theta}} dx$$

$= I_1 - I_2 + I_3$

$$I_1 = \frac{x^{-\theta+1}}{(1-\theta)(\theta-1)} \cdot (\ldots) = \frac{-x^{1-\theta}}{(\theta-1)^2}\Bigg|_1^{+\infty} = \frac{1}{(\theta-1)^2}$$

$$I_2 = 2\int_1^{+\infty} \frac{\ln x\,dx}{x^{\theta}} = \quad \ln x = u \quad \frac{1}{x}dx = du$$

$$x^{-\theta}dx = dv \quad v = \frac{x^{1-\theta}}{1-\theta}$$

$$= 2\left(\cancel{\frac{x^{1-\theta}\ln x}{1-\theta}}\Bigg|_1^{+\infty} - \int_1^{+\infty} \frac{x^{-\theta}}{1-\theta} \right) = -2\frac{x^{1-\theta}}{(1-\theta)^2}\Bigg|_1^{+\infty} =$$

$$= \frac{2}{(1-\theta)^2}$$

$$I_3 = (\theta-1)\int_1^{+\infty} \frac{\ln^2 x}{x^{\theta}}dx = \quad \ln^2 x = u \quad \frac{2\ln x}{x}dx = du$$

$$x^{-\theta}dx = dv \quad v = \frac{x^{1-\theta}}{1-\theta}$$

$$= \left(\cancel{\frac{x^{1-\theta}\ln^2 x}{1-\theta}}\Bigg|_1^{+\infty} - \int_1^{+\infty} \frac{2\ln x\,dx}{x^{\theta}(1-\theta)} \right)\theta - 1 =$$

$$= 2\int_1^{+\infty} \frac{\ln x}{x^\theta}\,dx = \frac{1}{(1-\theta)^2}$$

$$I(\theta) = \frac{1}{(\theta-1)^2}$$

$$\frac{\sqrt{n}(\tilde{\theta}-\theta)}{\sqrt{I^{-1}(\theta)}} \to N(0,1)$$

$$\frac{\sqrt{n}(\tilde{\theta}-\theta)}{\tilde{\theta}-1} \to N(0,1)$$

$$t_1 < \frac{\sqrt{n}(\tilde{\theta}-\theta)}{\tilde{\theta}-1} < t_2$$

$$\frac{(\tilde{\theta}-1)t_1}{\sqrt{n}} - \tilde{\theta} < -\theta < \frac{(\tilde{\theta}-1)}{\sqrt{n}}t_2 - \tilde{\theta}$$

$$\tilde{\theta} - \frac{(\tilde{\theta}-1)t_1}{\sqrt{n}} < \theta < \tilde{\theta} - \frac{(\tilde{\theta}-1)}{\sqrt{n}}t_1$$

в) дов. интервал для медианы

сильная регулярность.

$r=2$:

$$\int_1^{+\infty} \frac{\partial}{\partial\theta} f(x,\theta)\,dx = \int_1^{+\infty} \frac{x^{\theta} - (\theta-1)x^{\theta}\ln x}{x^{2\theta}}\,dx$$

$$= \int_1^{+\infty} \left(\frac{1}{x^{\theta}} - \frac{(\theta-1)\ln x}{x^{\theta}}\right) dx =$$

$$\frac{x^{1-\theta}}{1-\theta}\Bigg|_1^{+\infty} - \frac{1}{1-\theta} = 0$$

)

$$\int_1^{+\infty} \frac{-2\ln x + \cancel{\ldots} (\theta-1)\ln^2 x}{x^\theta}\,dx =$$

см. пункт б.

$\int = 0$

$\Rightarrow$ модель регулярна

$$\int_1^{\hat{x}} \frac{\theta-1}{x^\theta}\,dx = \frac{1}{2} \Rightarrow \hat{x} = (2)^{\frac{1}{\theta-1}}$$

$$g(\hat{\theta}) = 2^{\frac{1}{\hat{\theta}-1}}$$

$$\frac{\sqrt{n}\,(g(\hat{\theta}) - g(\theta))}{\sqrt{\nabla^T g(\hat{\theta})\, I^{-1}(\hat{\theta})\, \nabla g(\hat{\theta})}} \to N(0,1)$$

$$\nabla g = -\frac{2^{\frac{1}{\hat{\theta}-1}} \cdot \ln 2}{(\hat{\theta}-1)^2}$$

$$\frac{\sqrt{n}\,(g(\hat{\theta})-g(\theta))(\hat{\theta}-1)}{\ln 2\cdot 2^{\frac{1}{\hat{\theta}-1}}} \to N(0,1)$$

$$-1,96 < \frac{\sqrt{n}\,(g(\hat{\theta})-g(\theta))(\hat{\theta}-1)}{\ln 2\cdot 2^{\frac{1}{\hat{\theta}-1}}} < 1,96$$

$$-\frac{\cancel{(\hat{\theta}-1)}\ln 2\cdot 2^{\frac{1}{\hat{\theta}-1}}}{\sqrt{n}\,(\hat{\theta}-1)}\,1,96 < g(\hat{\theta})-g(\theta) < 1,96\cdot\cancel{(\hat{\theta}-1)}\ln 2\cdot 2^{\frac{1}{\hat{\theta}-1}}\cdot\frac{1}{\sqrt{n}}\cdot\frac{1}{\hat{\theta}-1}$$

$$g(\hat{\theta})-1,96\cdot\frac{\ln 2\cdot 2^{\frac{1}{\hat{\theta}-1}}}{(\hat{\theta}-1)\sqrt{n}} < g(\theta) < g(\hat{\theta})+\frac{1,96\cdot\ln 2\cdot 2^{\frac{1}{\hat{\theta}-1}}}{(\hat{\theta}-1)\sqrt{n}}$$