1: Vectors and Vector Spaces

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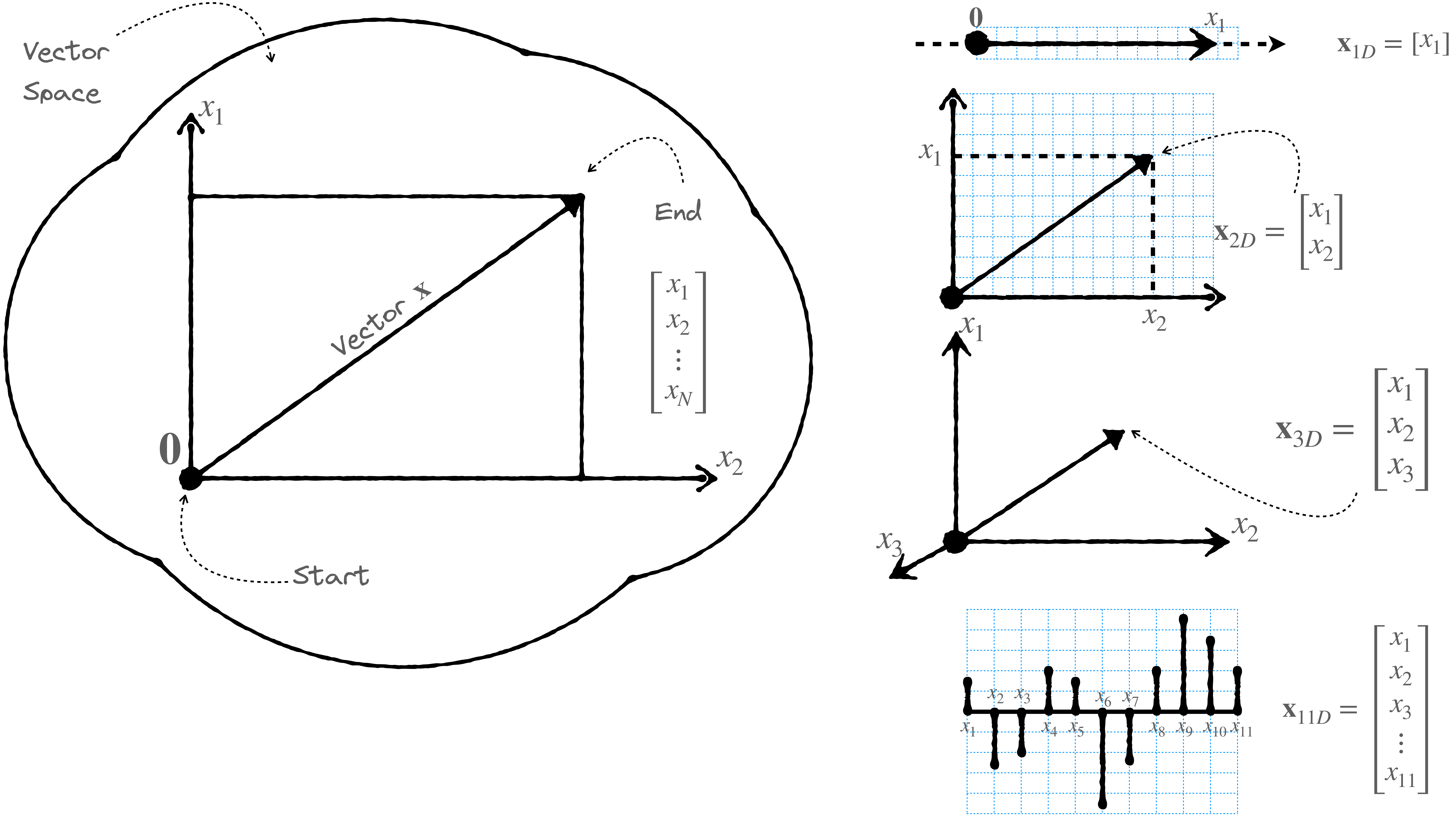
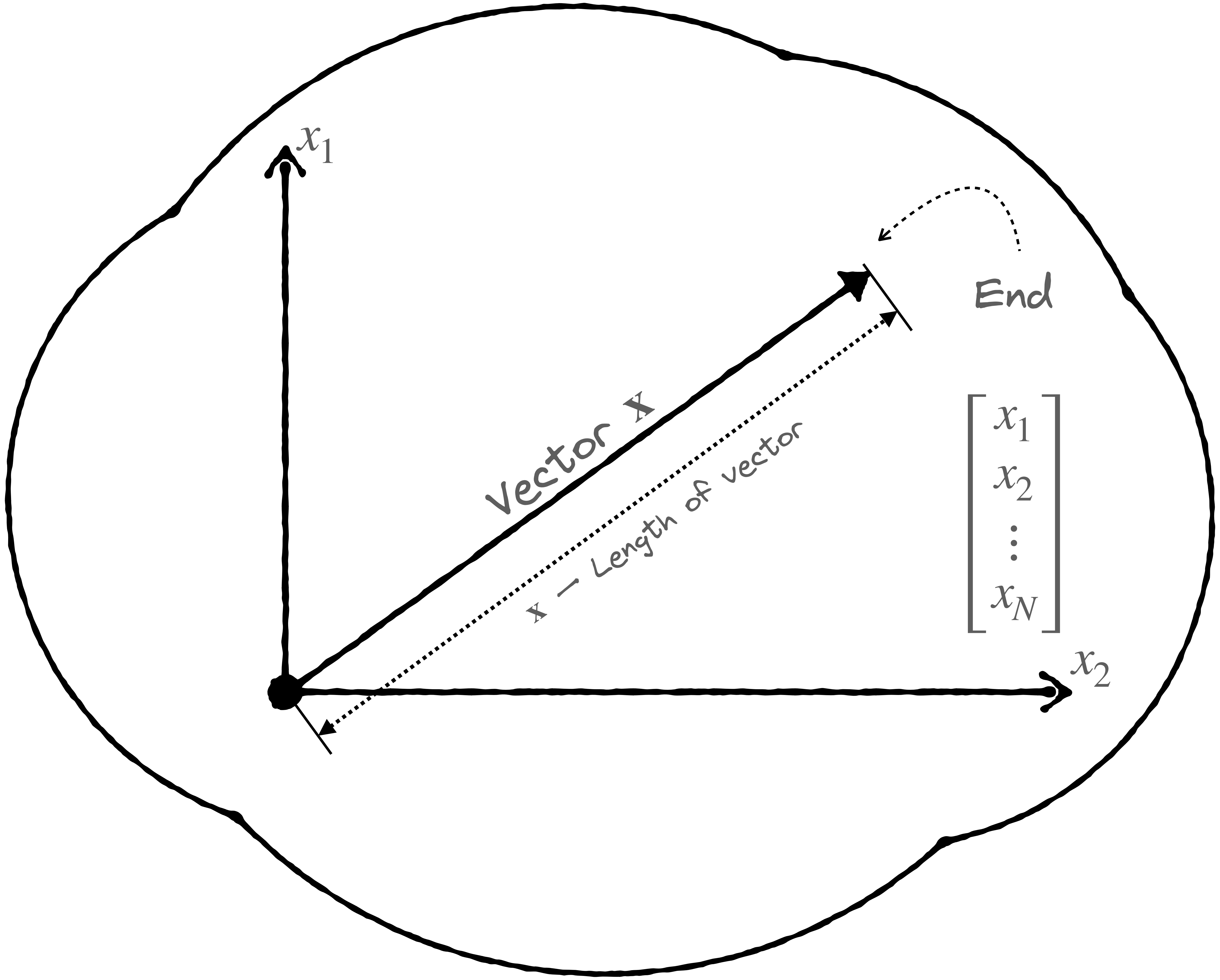
Vectors, Lengths and Directions

Vectors and their Representations

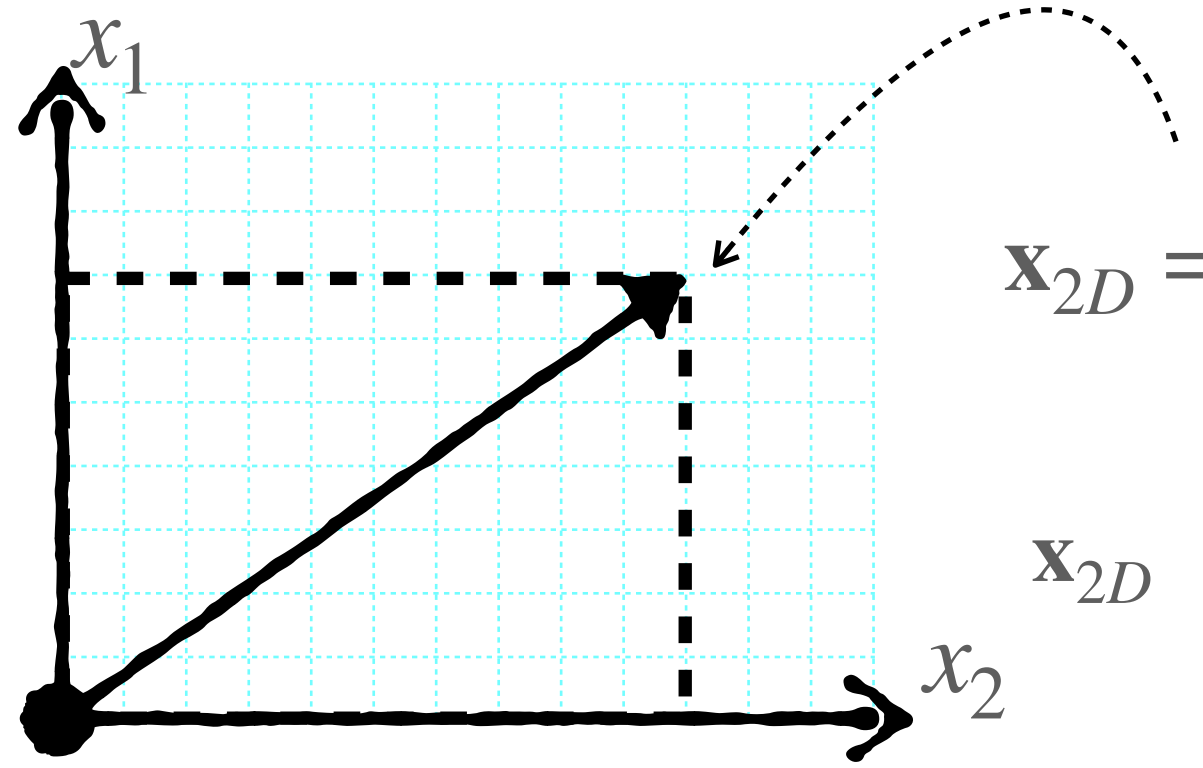


Length of Vector



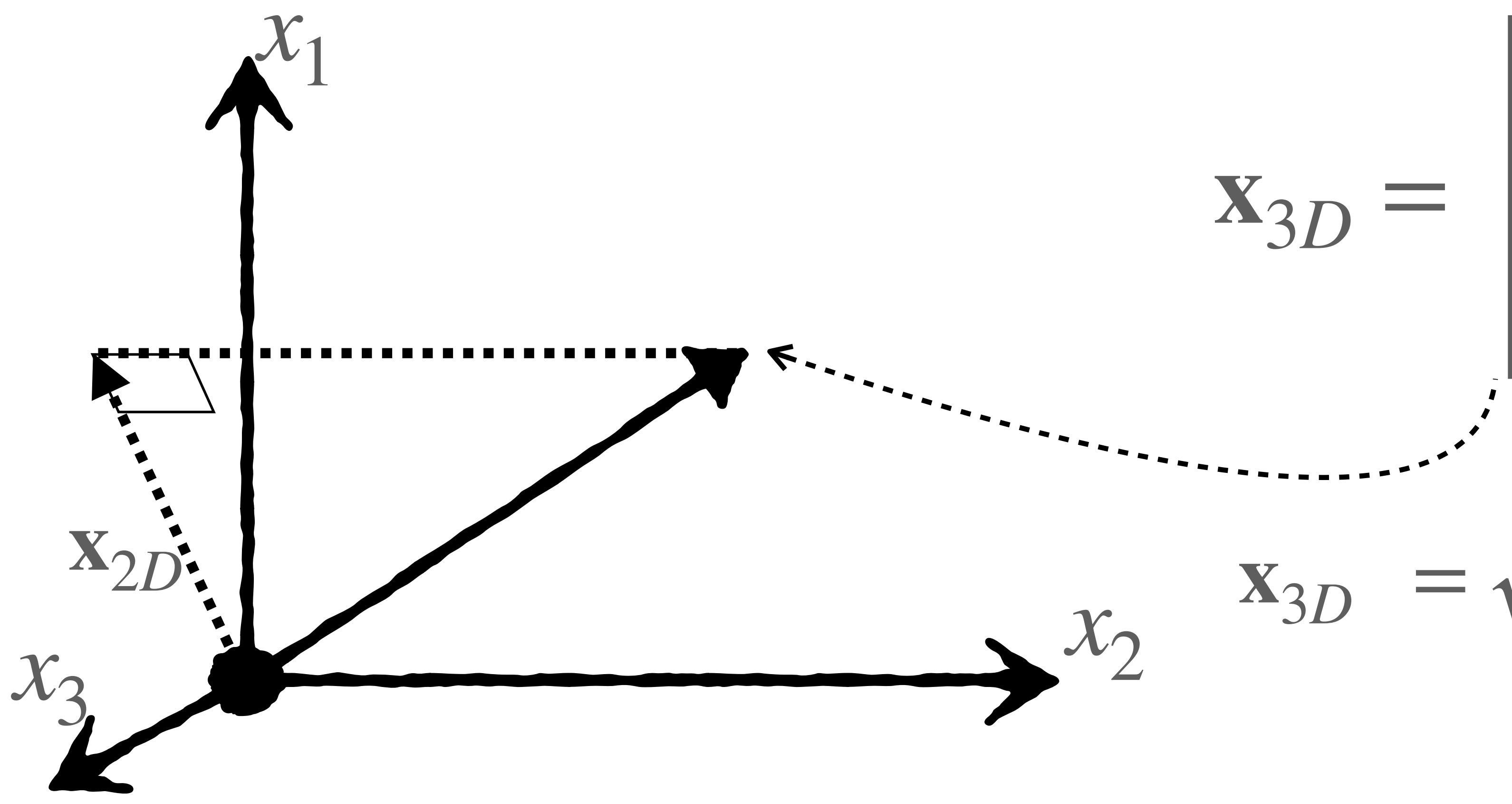
$$\mathbf{x}_{1D} = [x_1]$$

$$\mathbf{x}_{1D} = x_1$$



$$\mathbf{x}_{2D} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

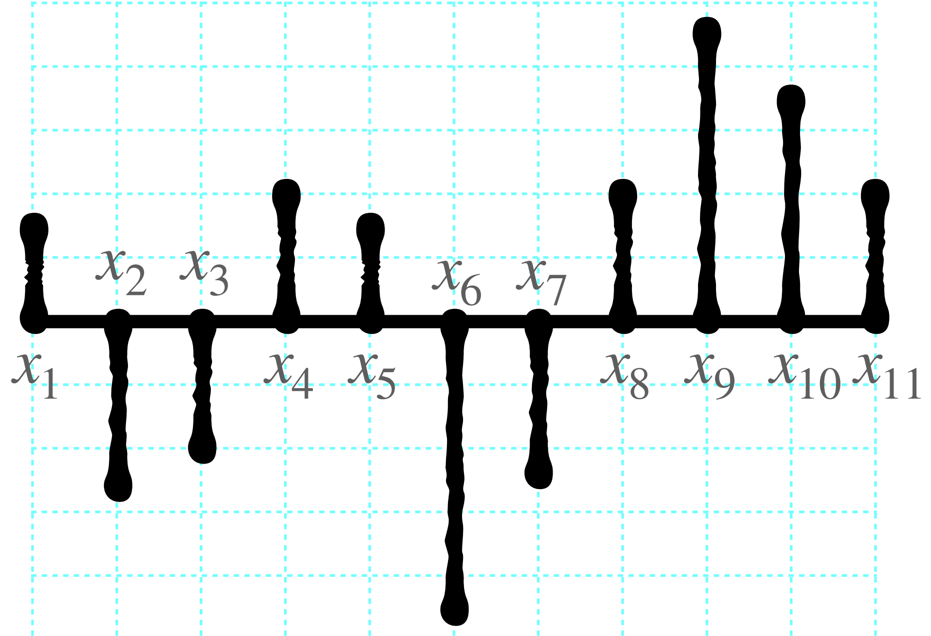
$$\mathbf{x}_{2D} = \sqrt{x_1^2 + x_2^2}$$



$$\mathbf{x}_{3D} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

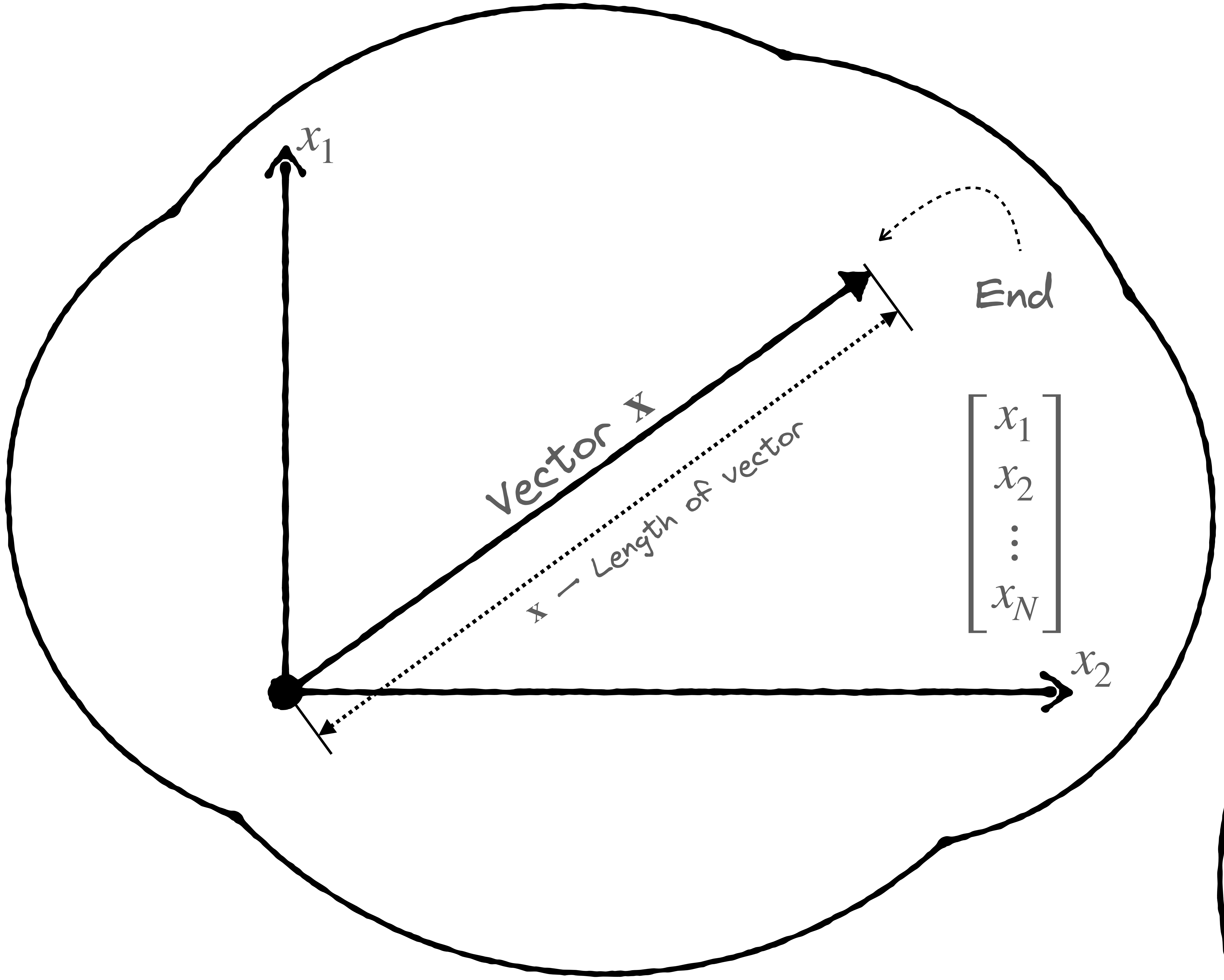
$$\mathbf{x}_{3D} = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

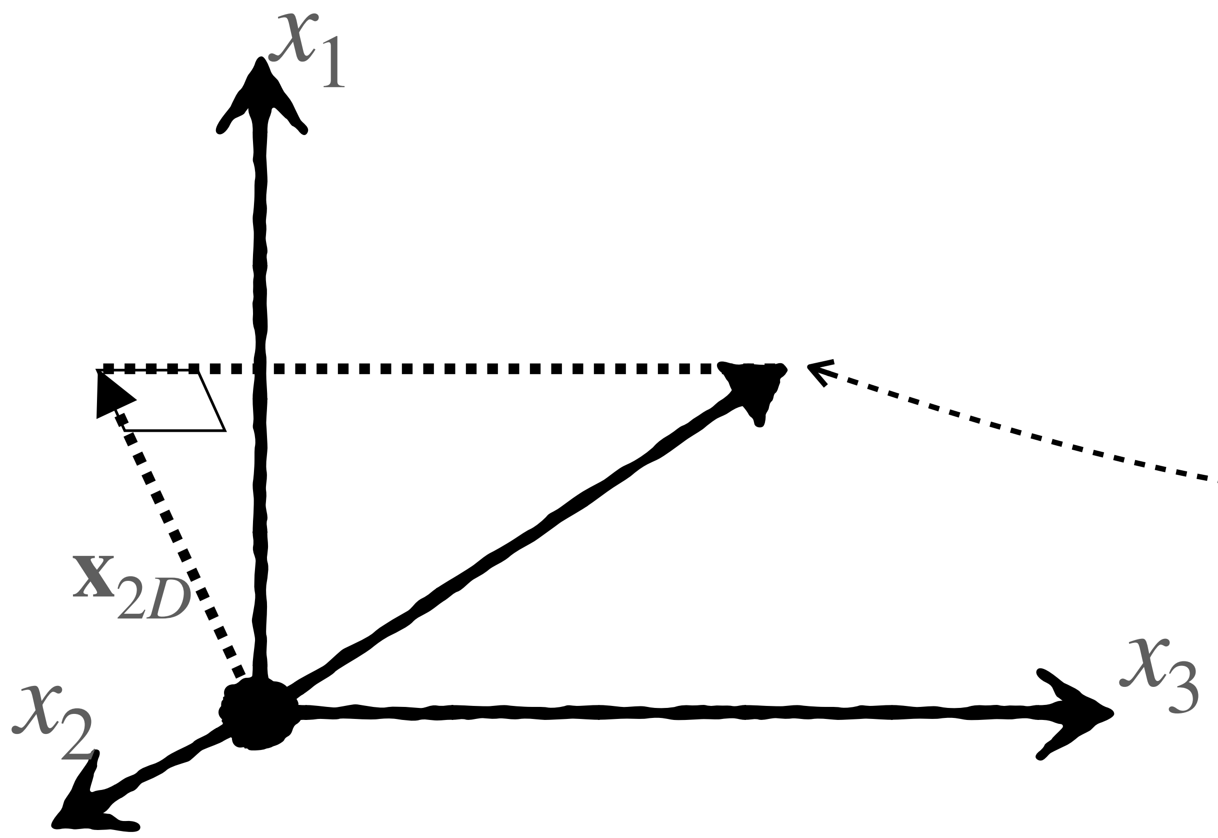
$$\mathbf{x}_{11D} = \sqrt{\sum_{k=1}^{11} x_k^2}$$



$$\mathbf{x}_{11D} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{11} \end{bmatrix}$$

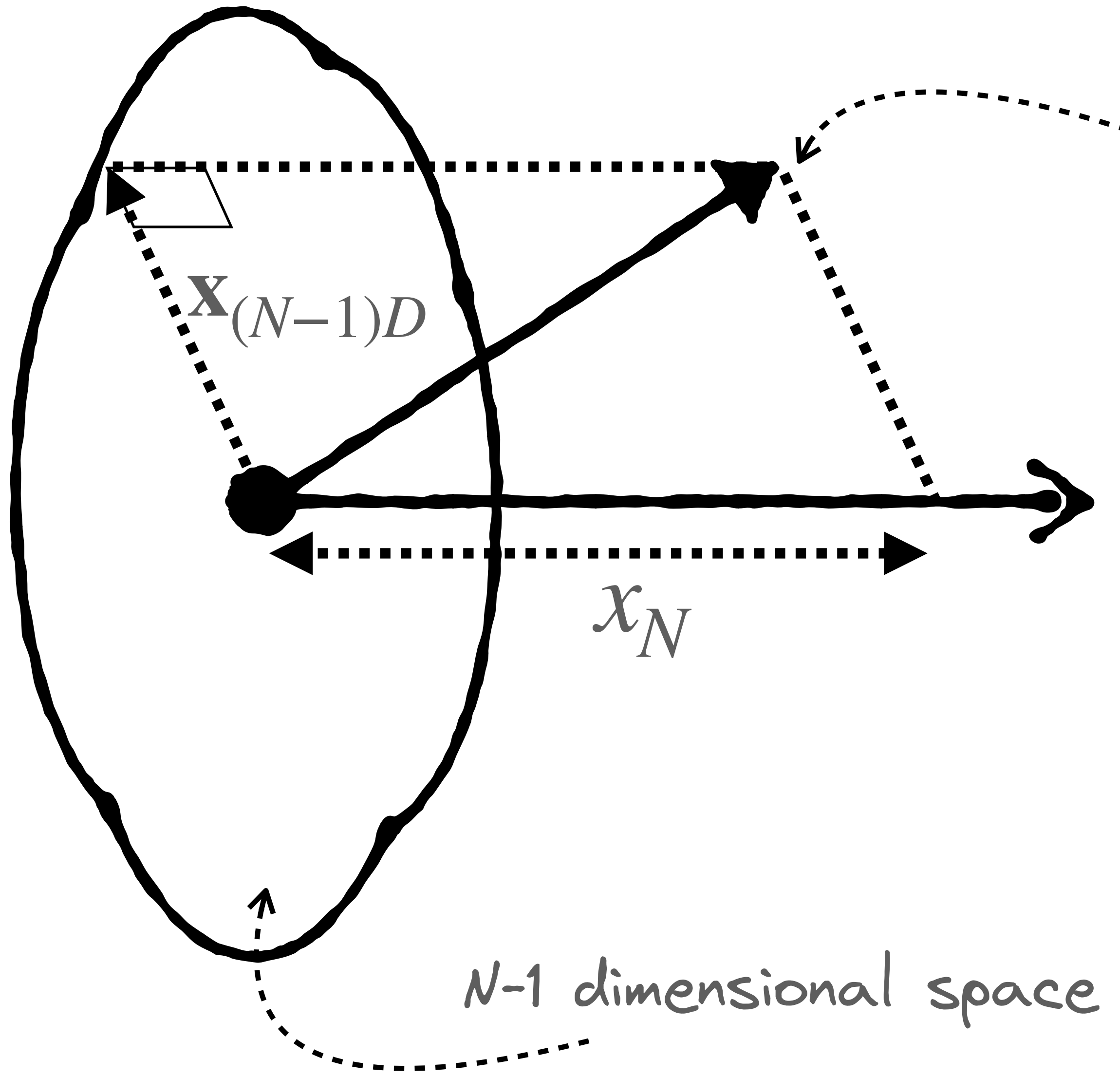
Length of Vector





$$\mathbf{x}_{3D} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

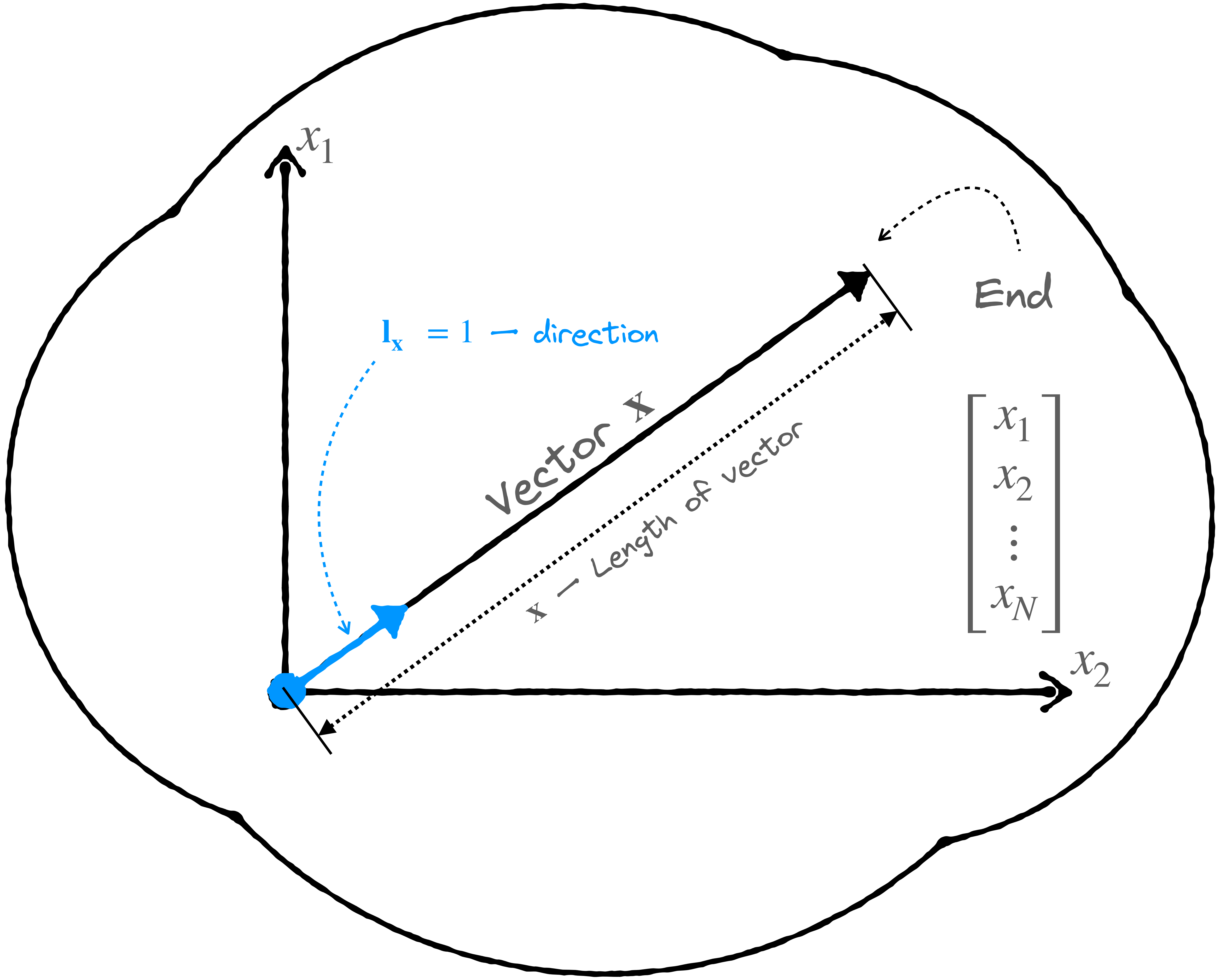
$$\mathbf{x}_{3D} = \sqrt{x_3^2 + \mathbf{x}_{2D}{}^2}$$



$$\mathbf{x}_{ND} = \sqrt{\mathbf{x}_{(N-1)D}{}^2 + x_N^2}$$

$$\mathbf{x}_{ND} = \sqrt{\sum_{k=1}^{N} x_k^2}$$

Direction of Vector



$$\mathbf{l_x} = \begin{bmatrix} \frac{x_1}{\mathbf{x}} \\ \frac{x_2}{\mathbf{x}} \\ \vdots \\ \frac{x_N}{\mathbf{x}} \end{bmatrix}$$

Direction is a unit vector that is collinear with the original vector

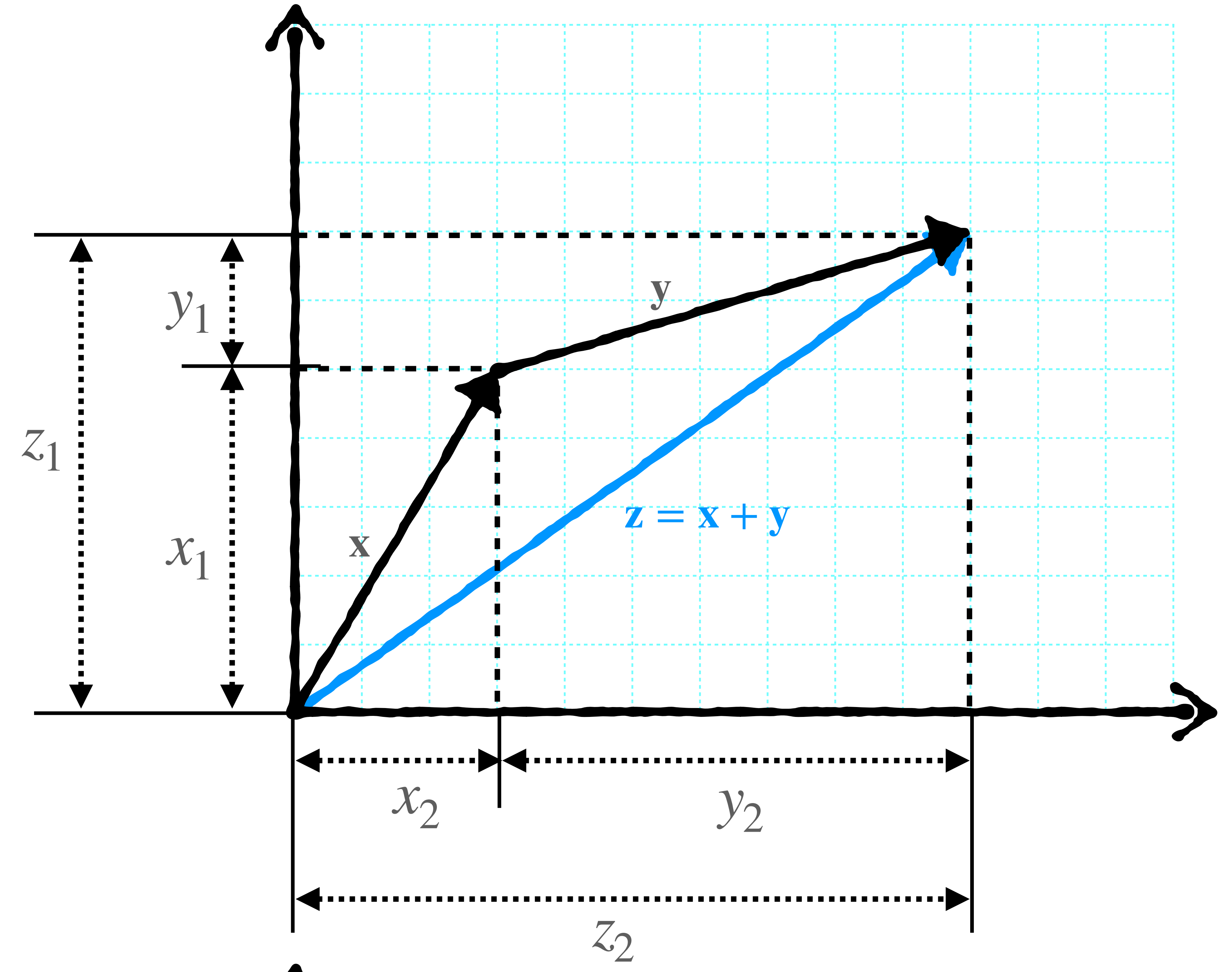
Collinear — points in the same direction, parallel

Operations with Vectors
Scalar Product

Vector Operations

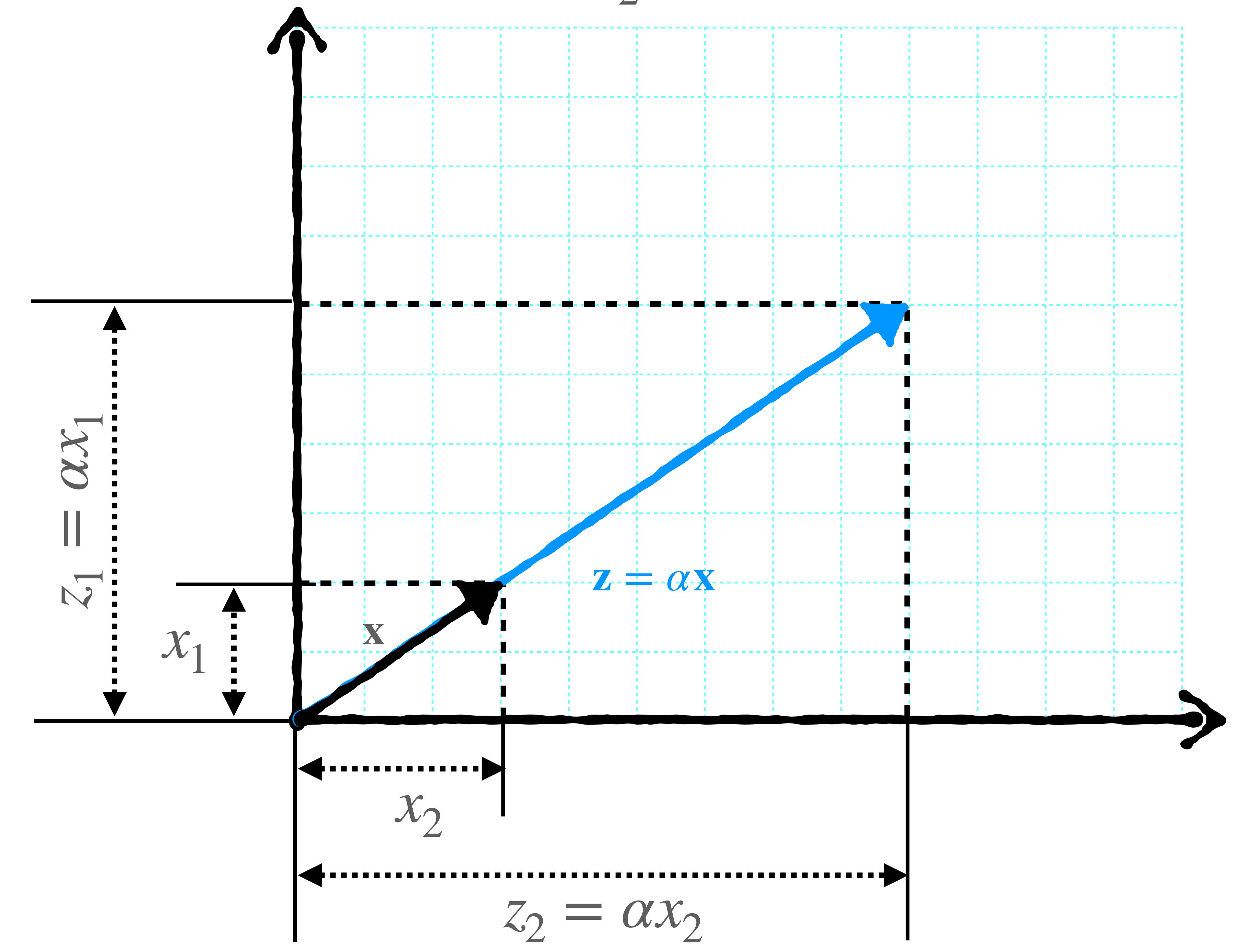
Sum of vectors:

$$\mathbf{z} = \mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_N + y_N \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{bmatrix}$$



Scaling:

$$\mathbf{z} = \alpha \mathbf{x} = \alpha \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_N \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{bmatrix}$$



Elementwise Operations

Elementwisely you can do with vectors everything that you can do with normal numbers:

Sum $\mathbf{z} = \mathbf{x} + \mathbf{y}$

Difference $\mathbf{z} = \mathbf{x} - \mathbf{y}$

Product $\mathbf{z} = \mathbf{x} \cdot \mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} x_1 y_1 \\ x_2 y_2 \\ \vdots \\ x_N y_N \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{bmatrix}$

Division $\mathbf{z} = \mathbf{x}/\mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} / \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} \frac{x_1}{y_1} \\ \frac{x_2}{y_2} \\ \vdots \\ \frac{x_N}{y_N} \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{bmatrix}$

Remainder Extraction $\mathbf{z} = \mathbf{x} \,\%\, \mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} \% \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} x_1 \,\%\, y_1 \\ x_2 \,\%\, y_2 \\ \vdots \\ x_N \,\%\, y_N \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{bmatrix}$

And even you can calculate functions:

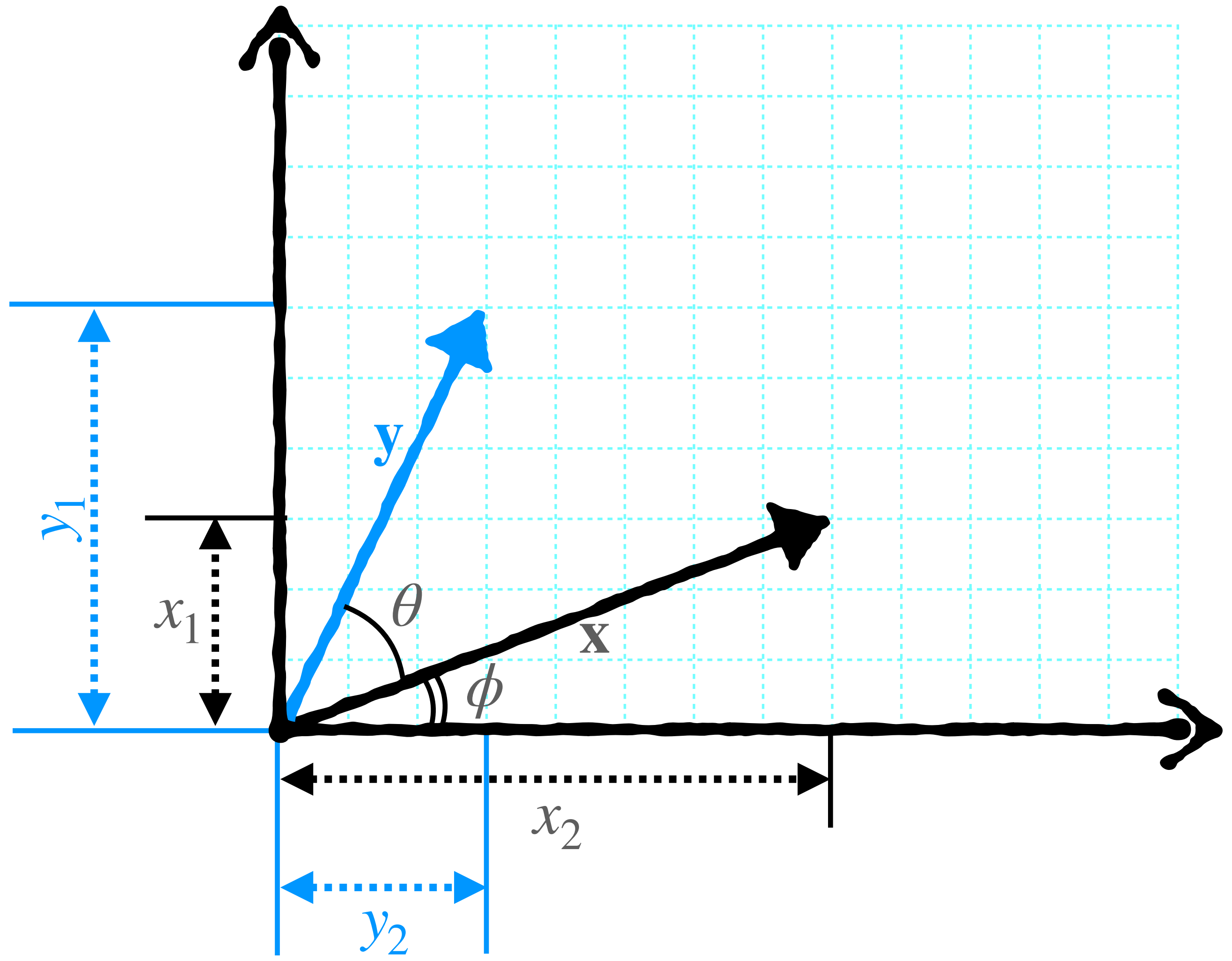
Square root: $\mathbf{z} = \sqrt{\mathbf{x}} = \begin{bmatrix} \sqrt{x_1} \\ \sqrt{x_2} \\ \vdots \\ \sqrt{x_N} \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{bmatrix}$

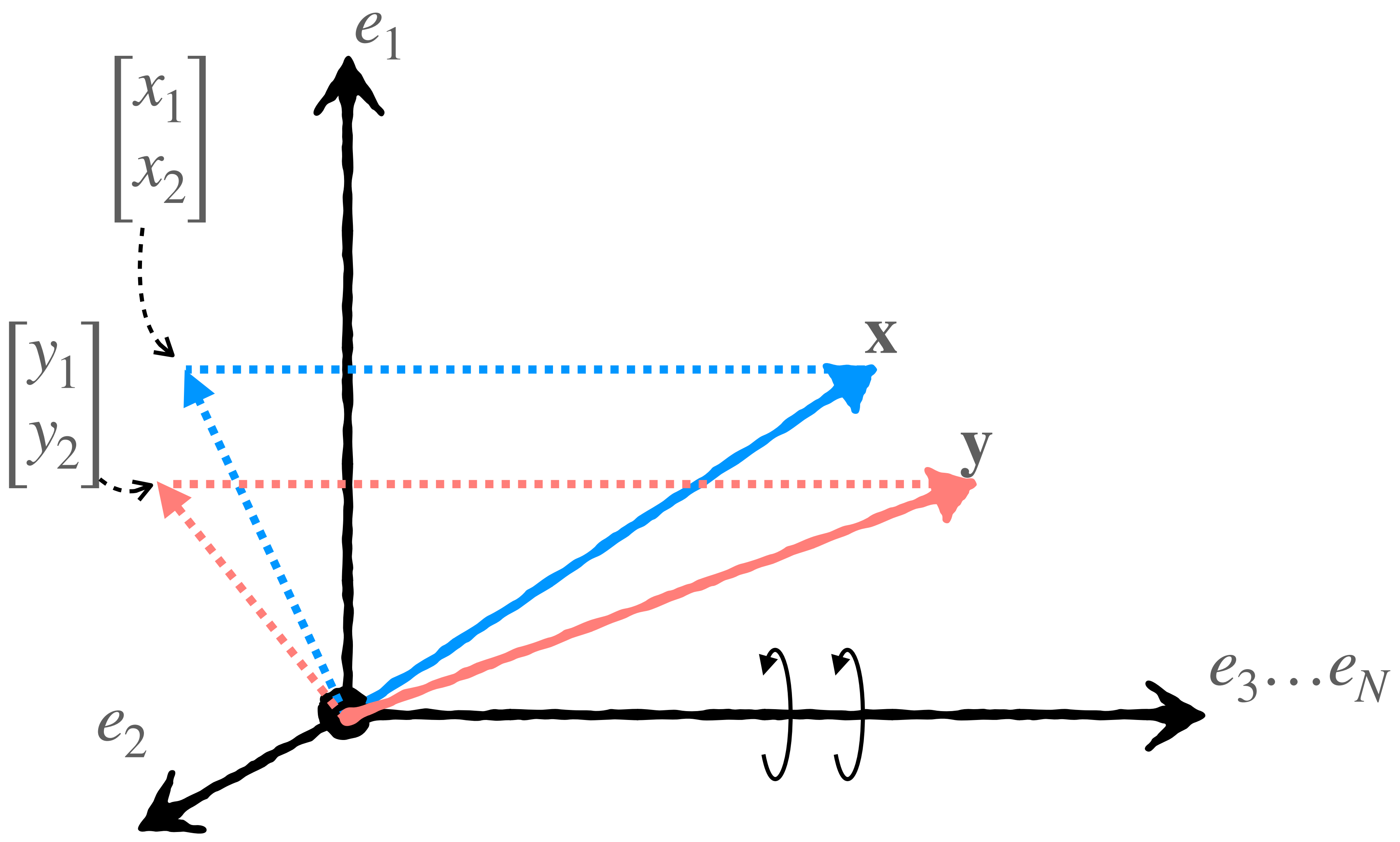
Sinus: $\mathbf{z} = \sin(\mathbf{x}) = \begin{bmatrix} \sin x_1 \\ \sin x_2 \\ \vdots \\ \sin x_N \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{bmatrix}$

And many others …

Dot Product

Scalar (Dot) Product: Equivalence



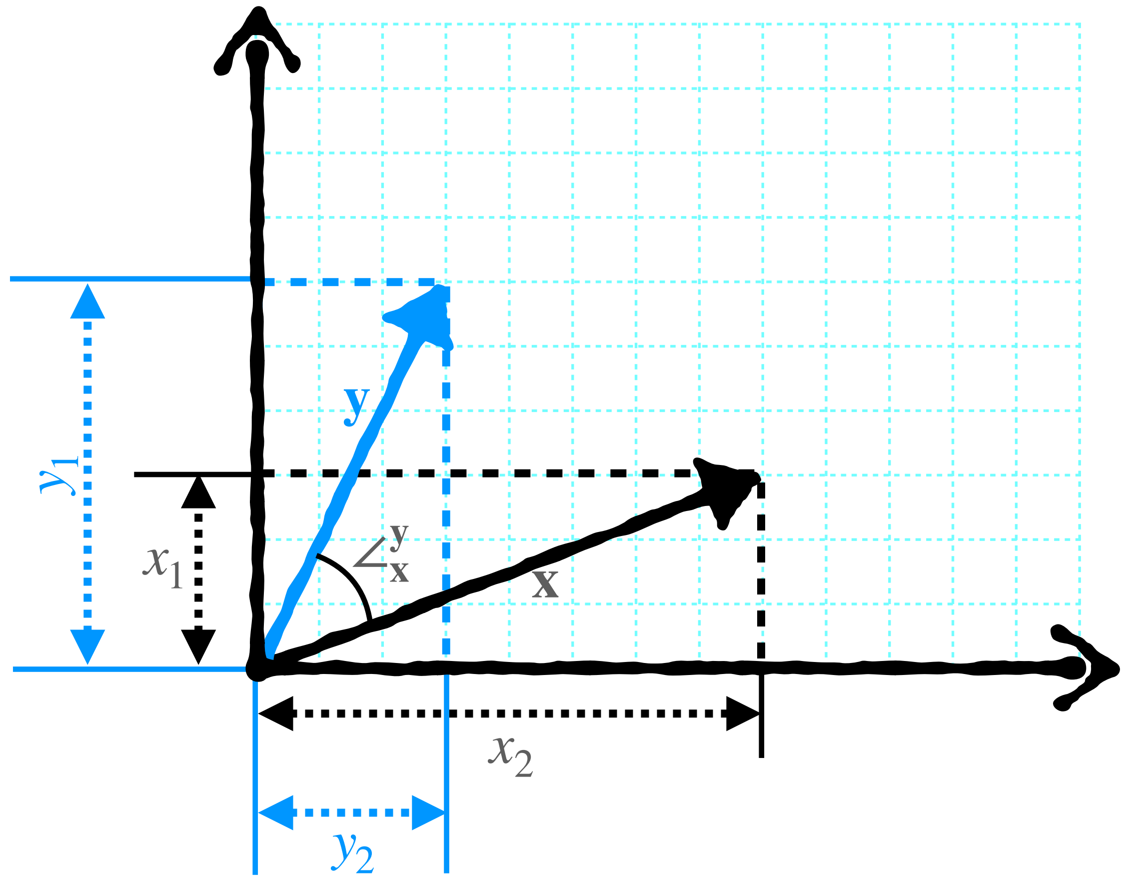


Definition #1

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{k=1}^{N} x_k y_k$$

Definition #2

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \quad \mathbf{y} \ \cos\left(\angle_{\mathbf{x}}^{\mathbf{y}}\right)$$



Dot Product:
Length, Cosine, Projection

Definition #1

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{k=1}^{N} x_k y_k$$

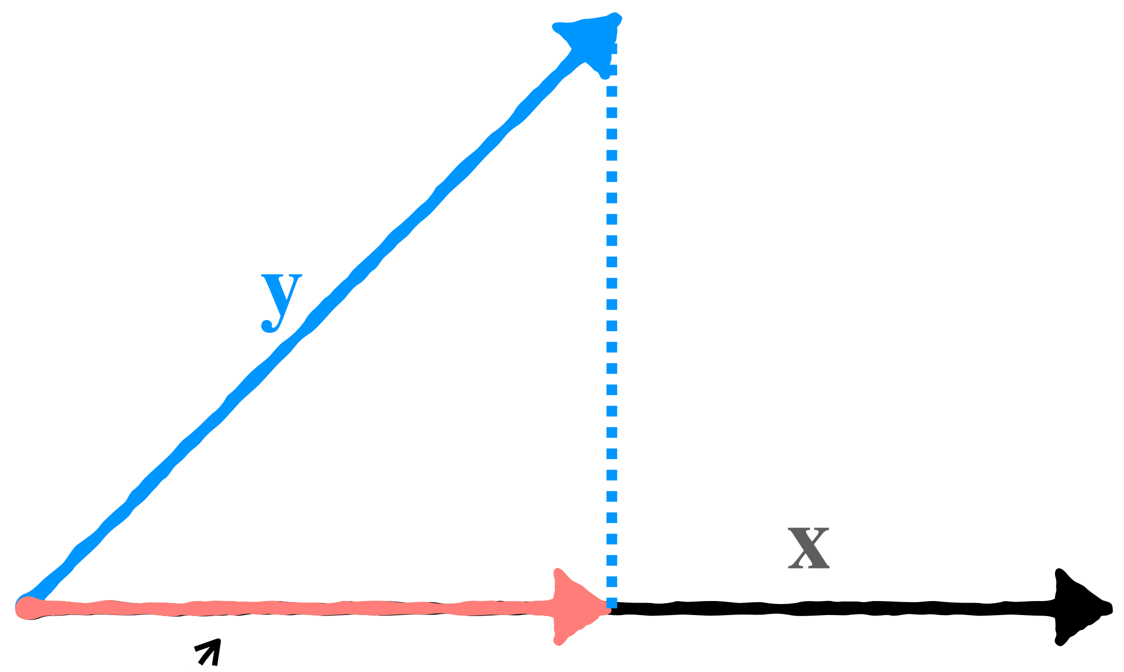
Definition #2

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \quad \mathbf{y} \ \cos\left(\angle_{\mathbf{x}}^{\mathbf{y}}\right)$$

Using scalar product one can calculate:

Length of vector: $\mathbf{x}\ ^{2} = \langle \mathbf{x}, \mathbf{x} \rangle$

Cosine between vectors: $\cos(\angle_{\mathbf{x}}^{\mathbf{y}}) = \dfrac{\langle \mathbf{x}, \mathbf{y} \rangle}{\mathbf{x} \quad \mathbf{y}}$



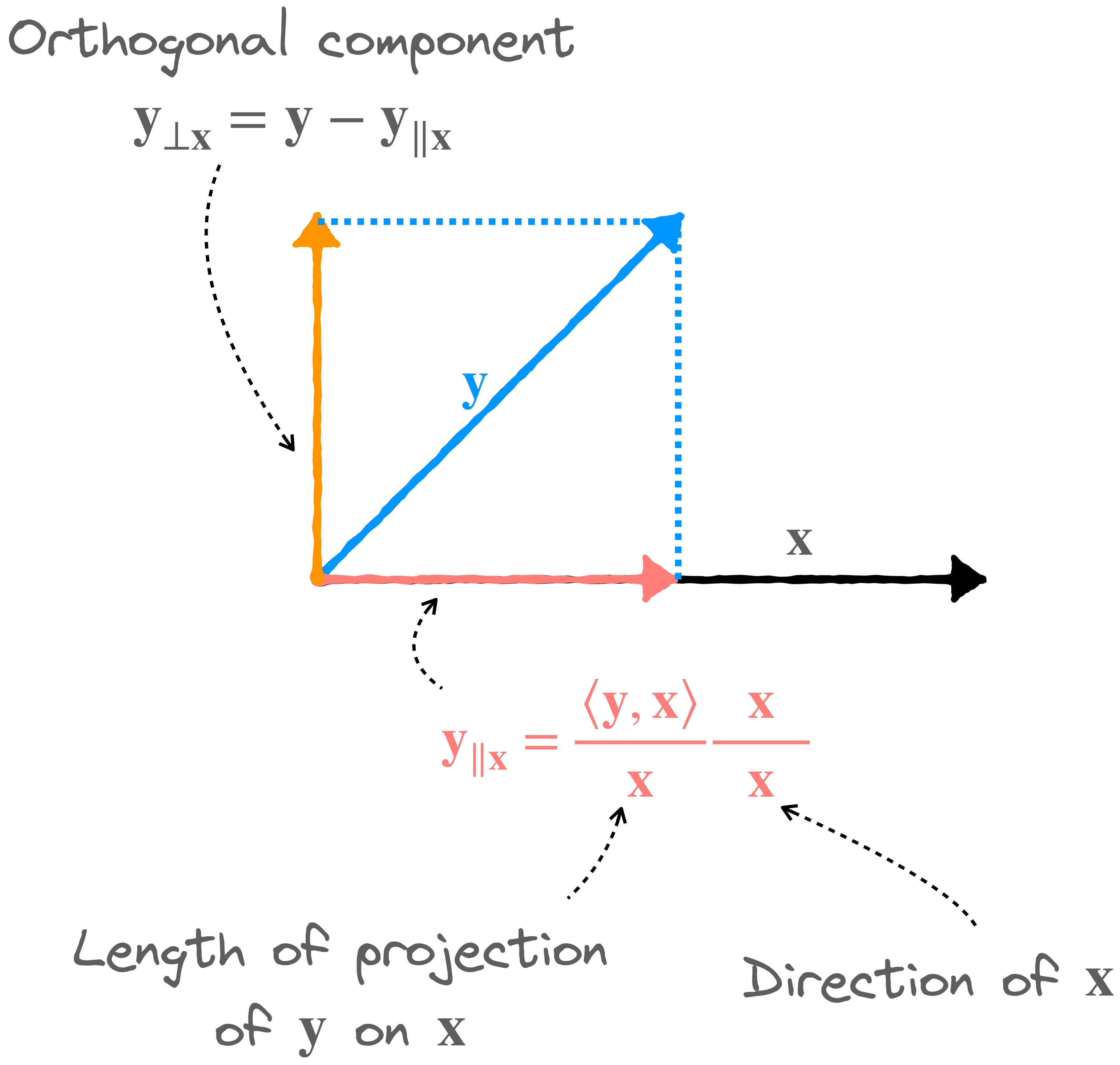
$$\mathbf{y}_{\|\mathbf{x}} = \frac{\langle \mathbf{y}, \mathbf{x} \rangle}{\mathbf{x}} \frac{\mathbf{x}}{\mathbf{x}}$$

Direction of $\mathbf{x}$

Length of projection
of $\mathbf{y}$ on $\mathbf{x}$:

$$\mathbf{y}_{\|\mathbf{x}} = \mathbf{y} \ \cos\angle_{\mathbf{y}}^{\mathbf{x}} = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\mathbf{x}}$$

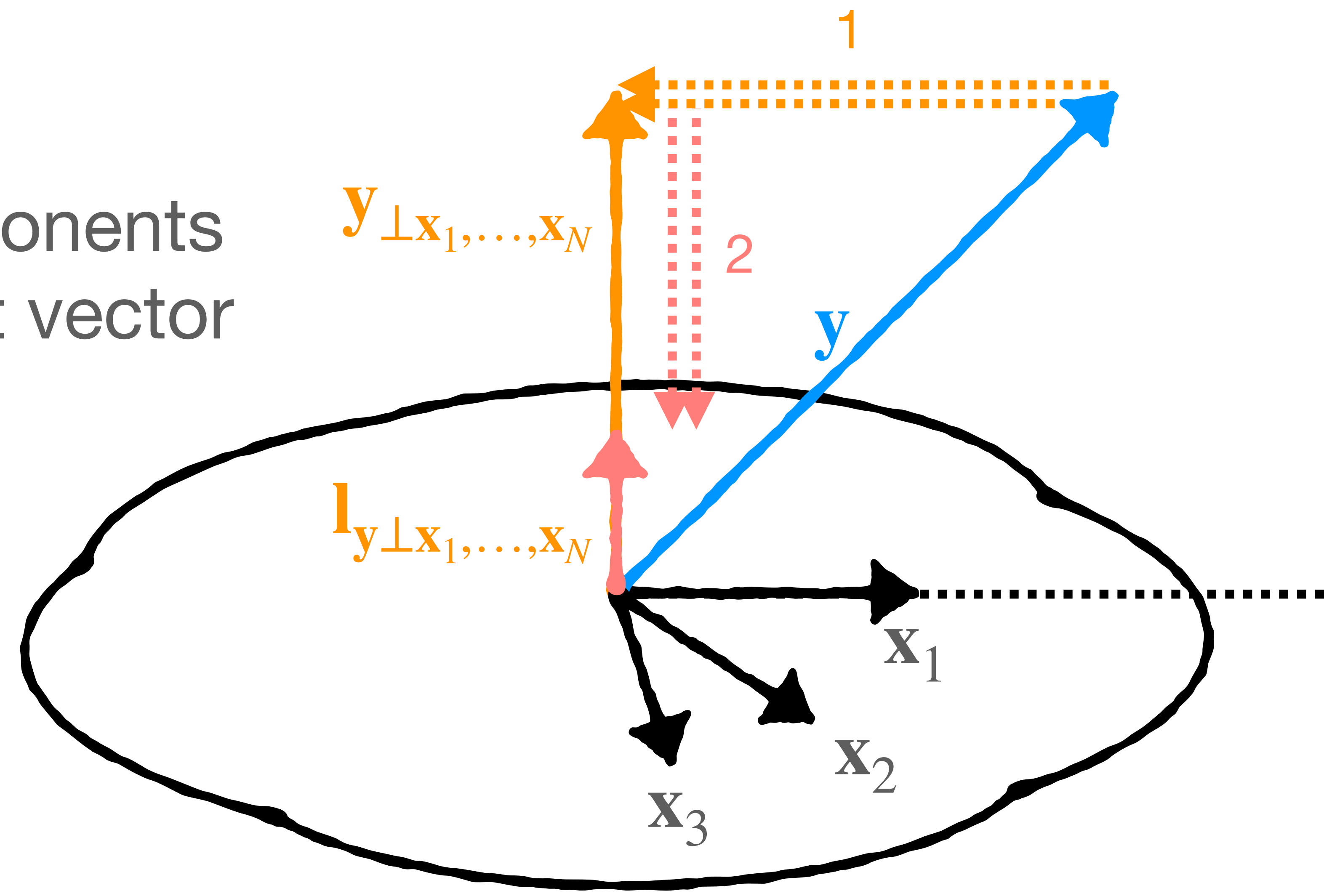
Collinear and Orthogonal Components



Orthonormalisation Procedure

Two principal steps:

1. Remove collinear components
2. Shrink a vector to a unit vector




```
res = []

for y in input_vectors:
    for x in res:
        # Remove all collinearities
        y = y - projection(y, x)
    # Make vector unit
    y = y / length(y)
    # Add vector to new system
    res.append(y)
```

Linear (In)Dependency
Basis and Coordinates

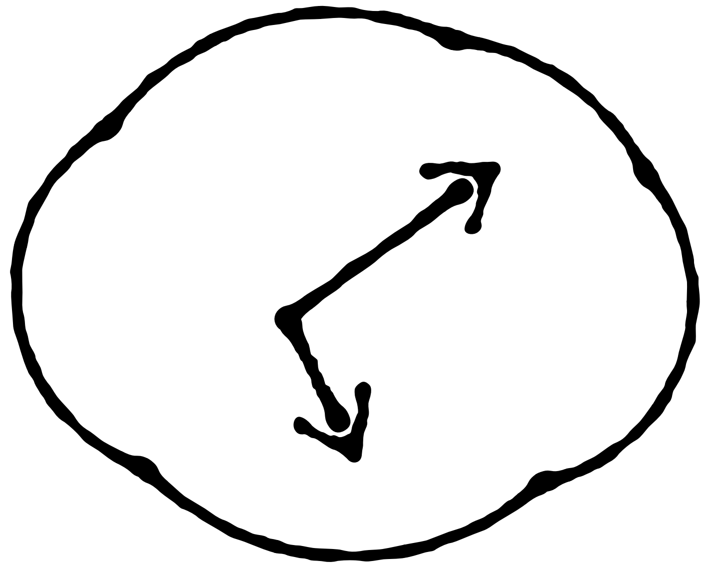
Linear (in)Dependency

System of vectors:

$$\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\}$$

Linearly dependent if:

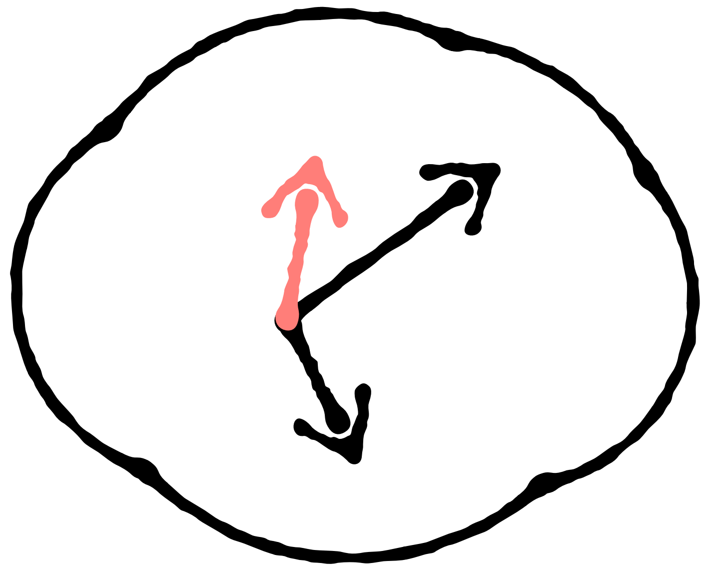
$$\exists \mathbf{a} \neq \mathbf{0} : \sum_{k=1}^{N} a_k \mathbf{x}_k = \mathbf{0}$$



None of the vectors can be represented as weighted sum of the others

Linearly independent if:

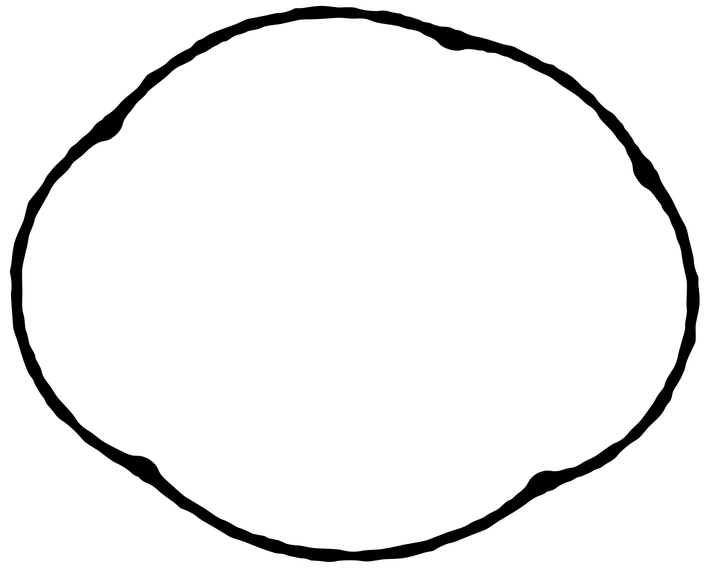
$$\forall \mathbf{a} \neq \mathbf{0} : \sum_{k=1}^{N} a_k \mathbf{x}_k \neq \mathbf{0}$$



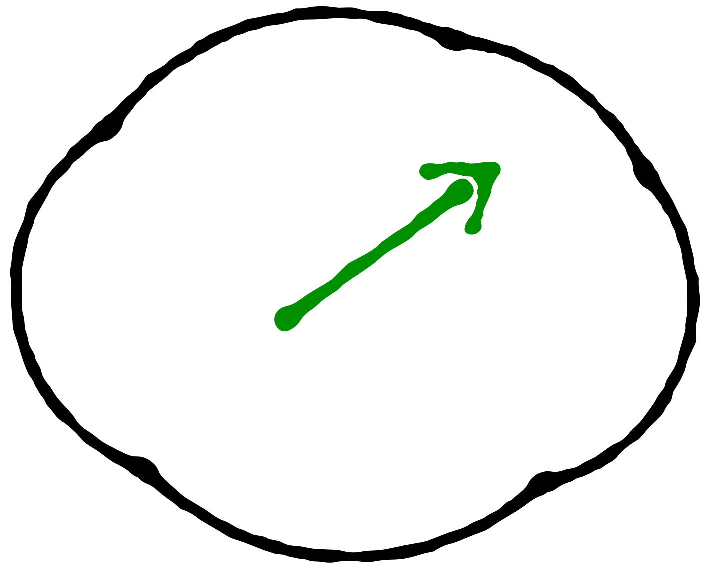
Some vector can be represented as weighted sum of the others

$$\mathbf{x}_1 = -\frac{a_2}{a_1}\mathbf{x}_2 - \frac{a_3}{a_1}\mathbf{x}_3 - \ldots - \frac{a_N}{a_1}\mathbf{x}_N$$

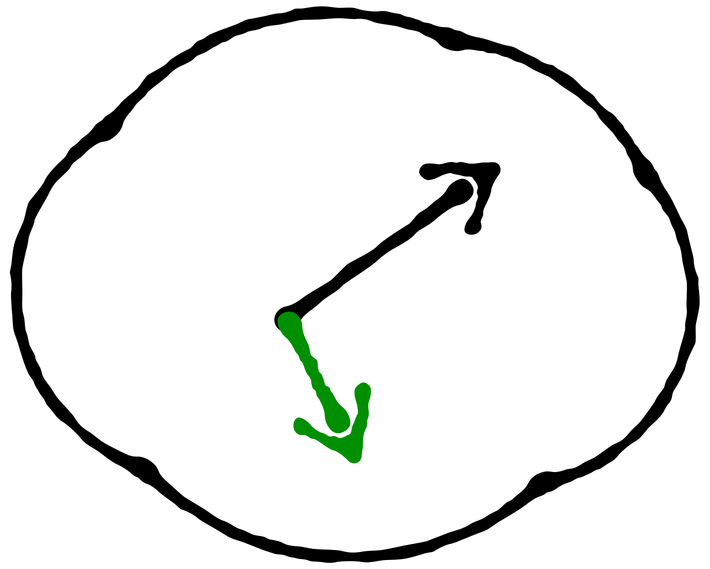
Basis



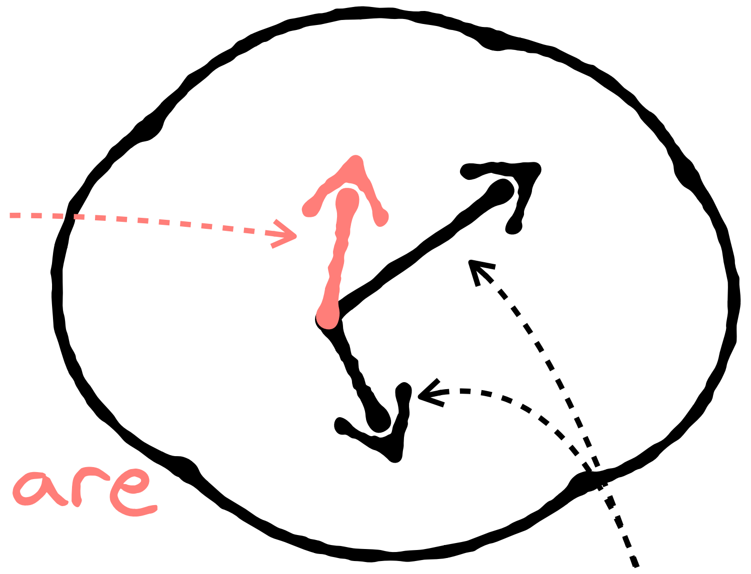
Some vector space



Select some vector from it



And another one linearly independent from previous ones

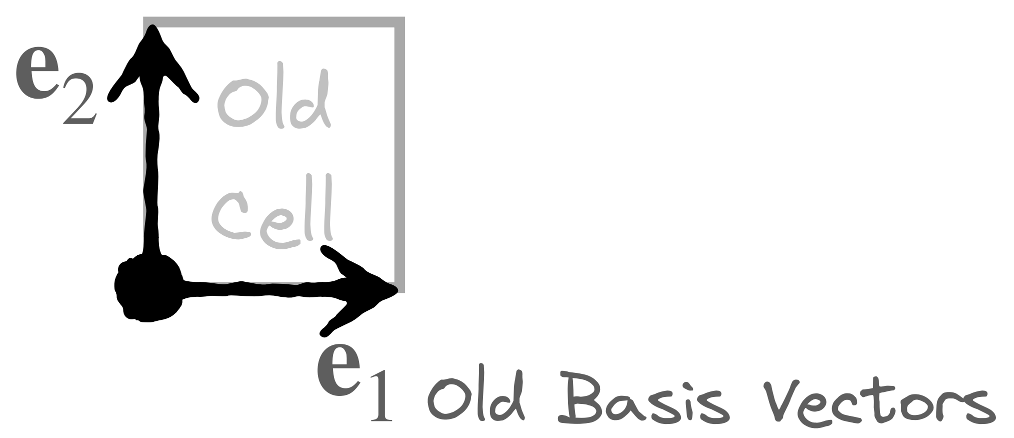


Other vectors are linear combinations of basis

Basis

When you cannot select linearly independent vector — you've found basis!

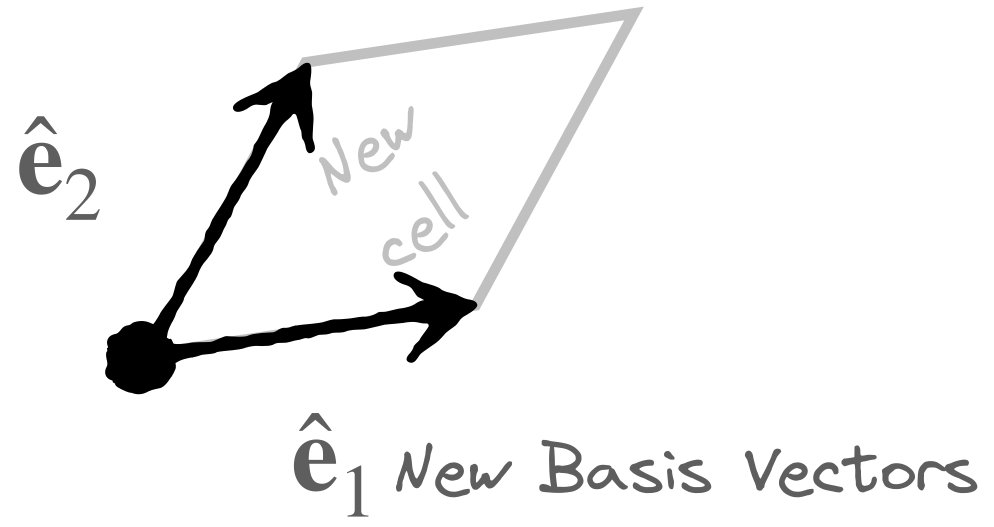
Basis and Coordinates

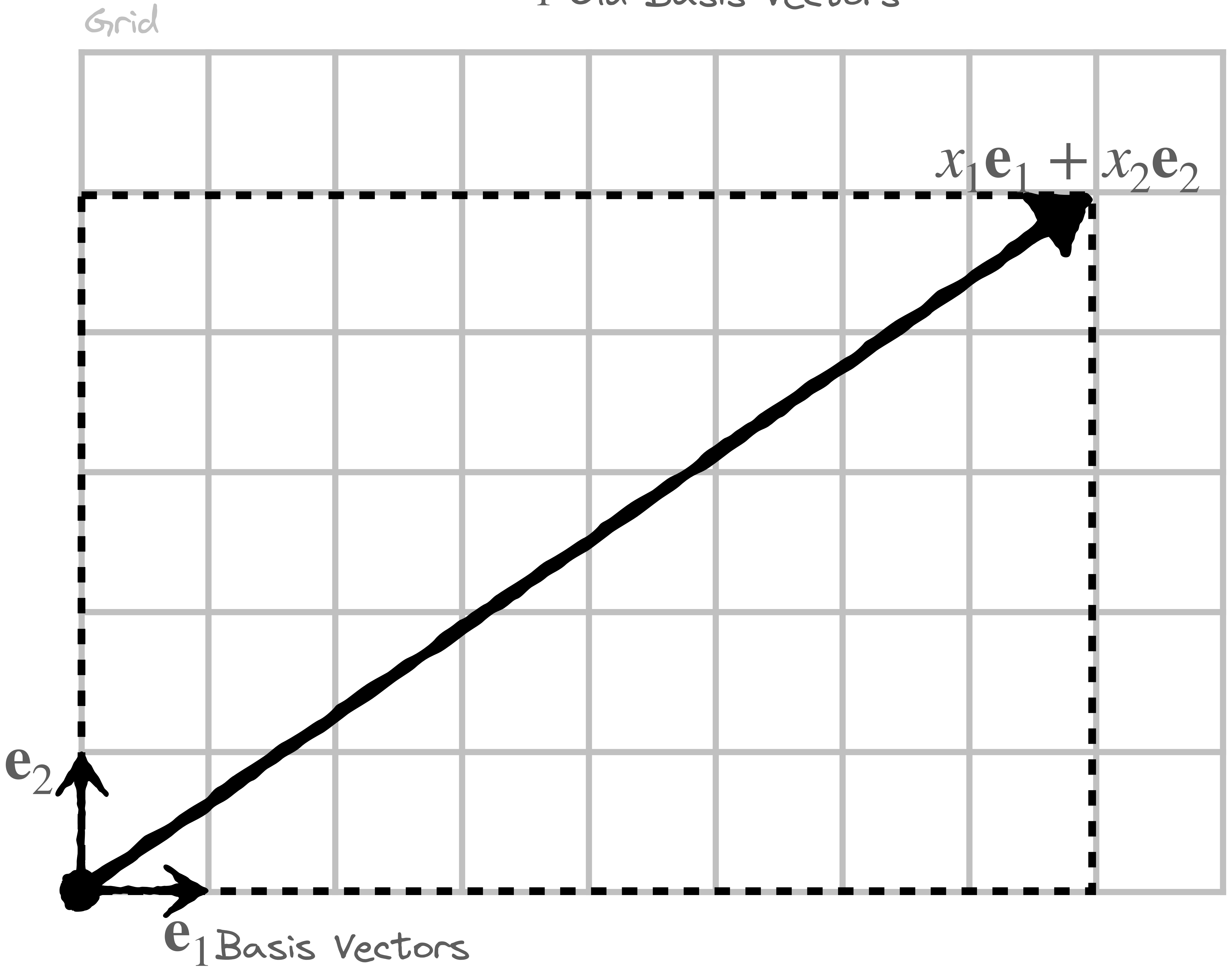


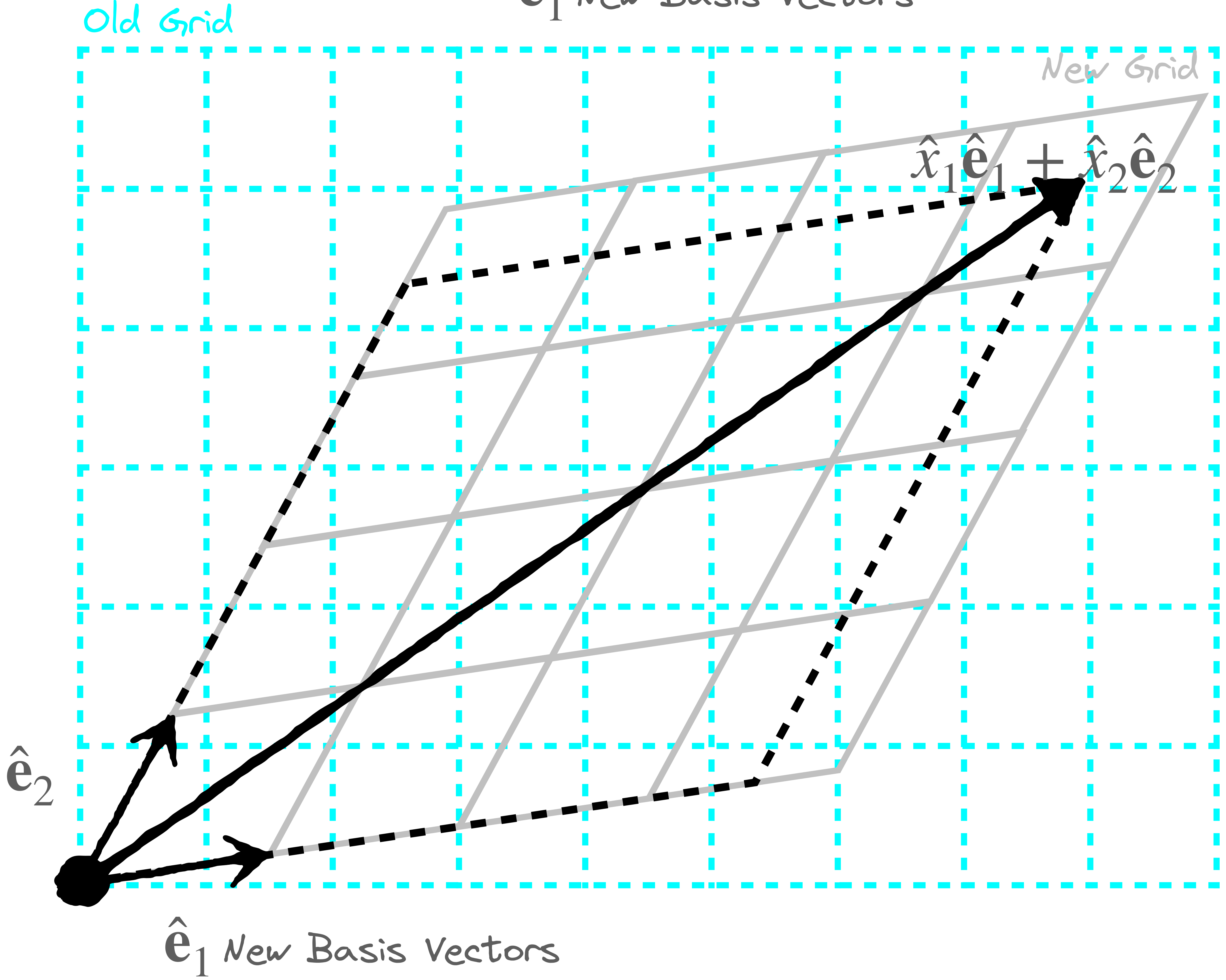
Basis Vectors

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \ldots + x_N\mathbf{e}_N$$

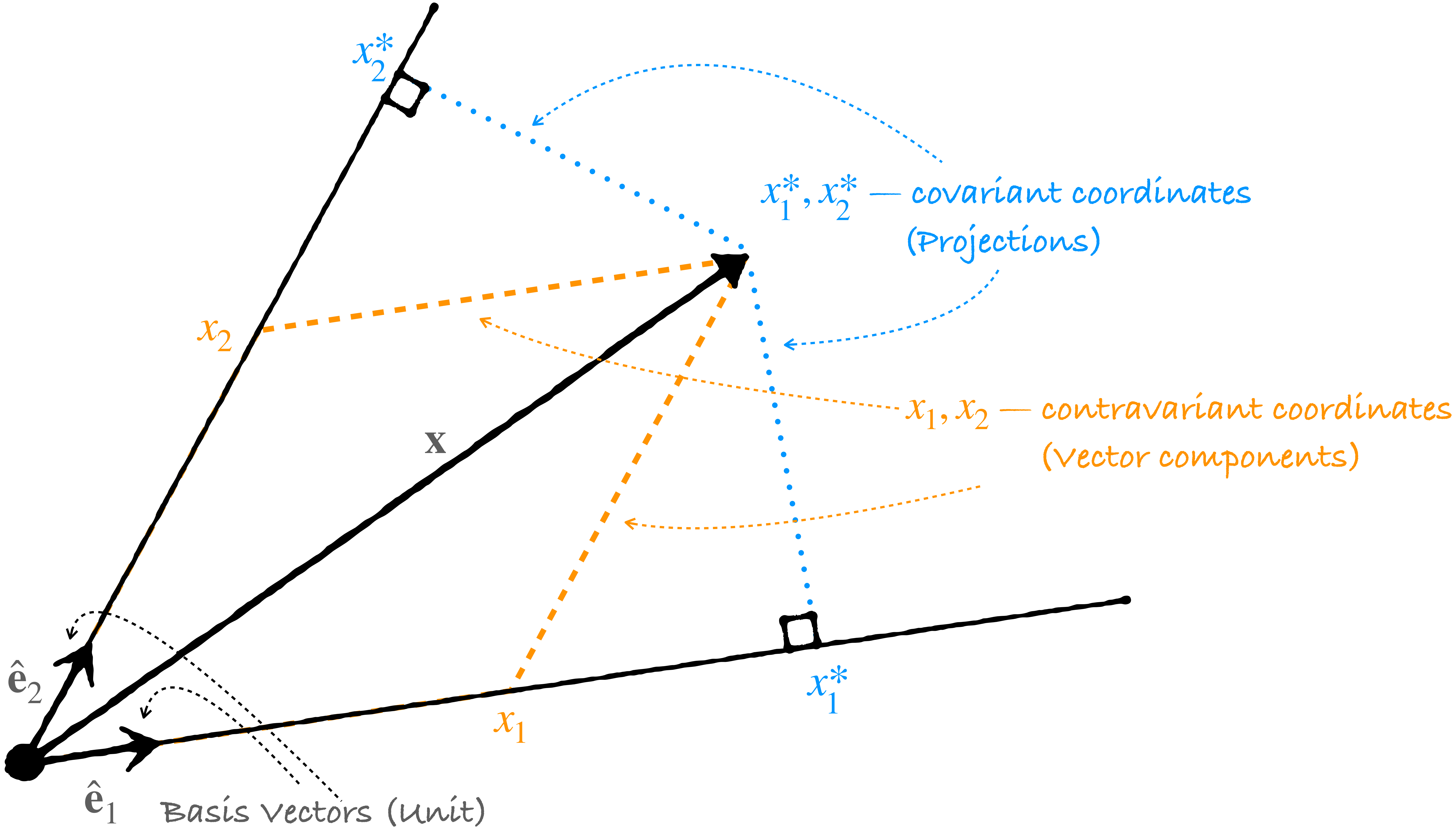
Coordinates







Contravriant and Covariant Coordinates



Contravariant coordinates:

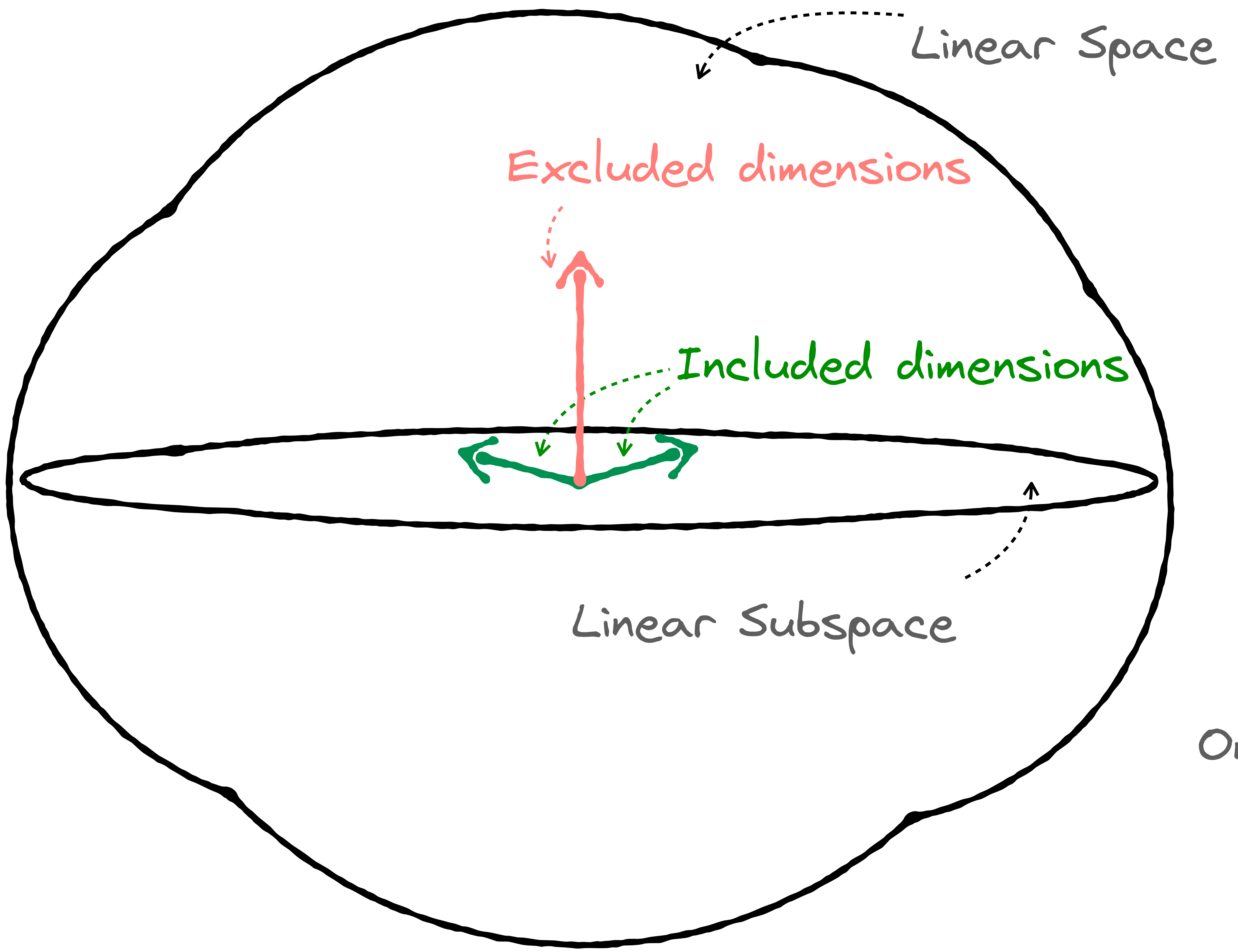
$$\mathbf{x} = x_1\mathbf{e}_1 + \ldots + x_N\mathbf{e}_N$$

Covariant coordinates:

$$x_k^* = \langle \mathbf{x}, \mathbf{e}_k \rangle$$

Linear Subspaces

Vector Spaces: Planes and HyperSurfaces

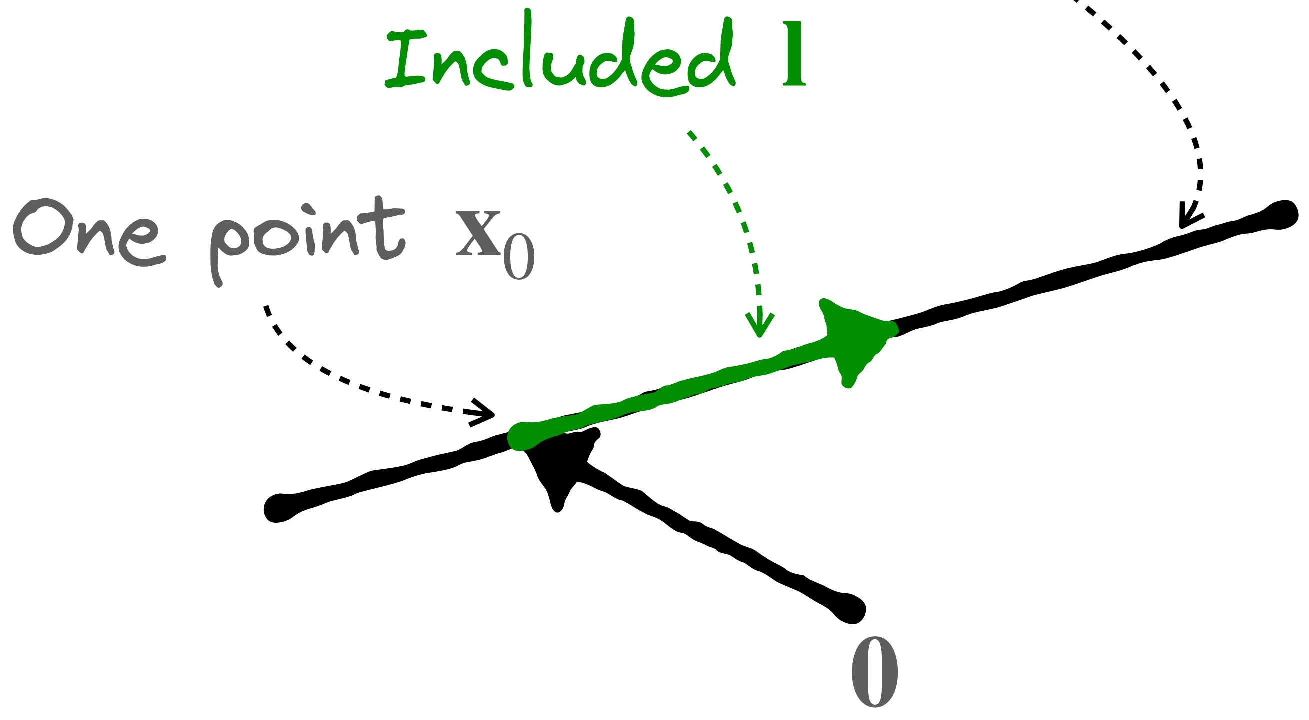


Linear Subspace

Inclusion:

All the vectors that are linear combination of the given ones:

$$\forall \mathbf{x} : \mathbf{x} = a\mathbf{l} + \mathbf{x}_0, \quad a \in \mathbb{R}$$



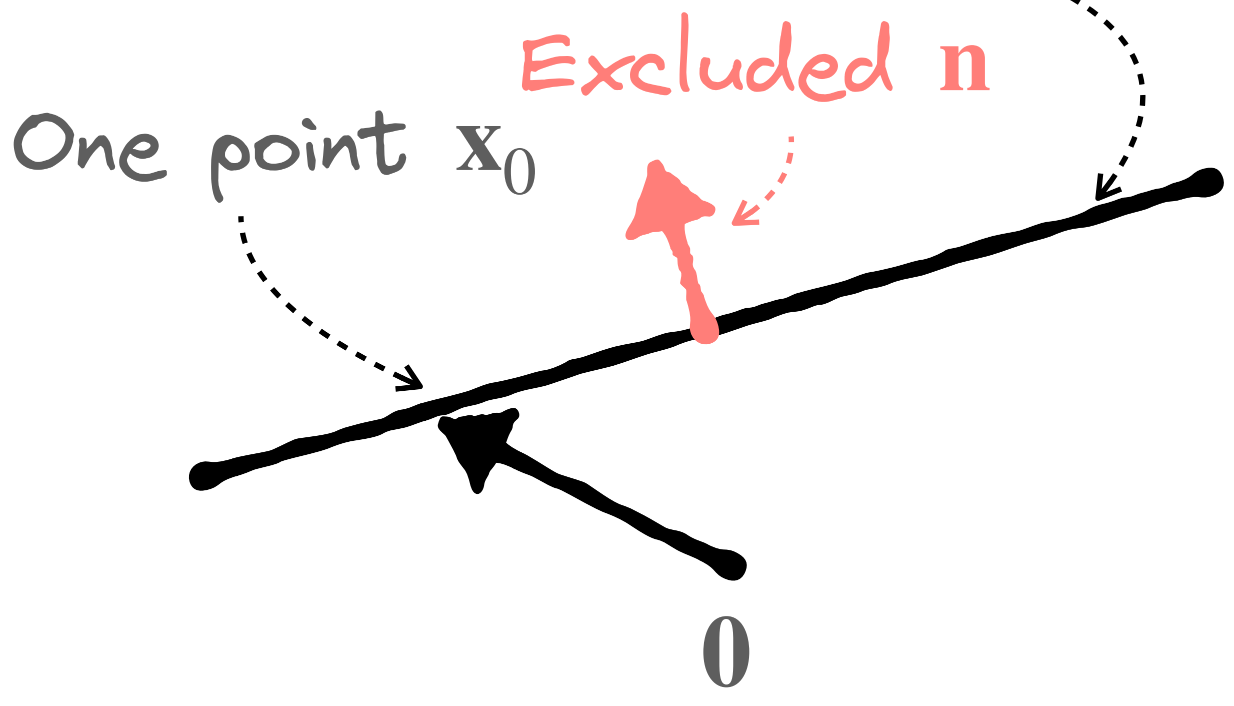
To select more than one dimension:

$$\forall \mathbf{x} : \mathbf{x} = \sum_{k=1}^{M} a_k \mathbf{l}_k + \mathbf{x}_0, \quad a \in \mathbb{R}$$

Exclusion

All the vectors that are orthogonal to the given ones

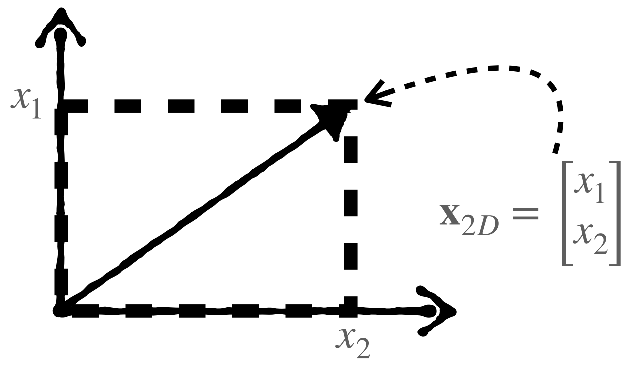
$$\forall \mathbf{x} : \langle \mathbf{x}, \mathbf{n} \rangle + b = 0$$



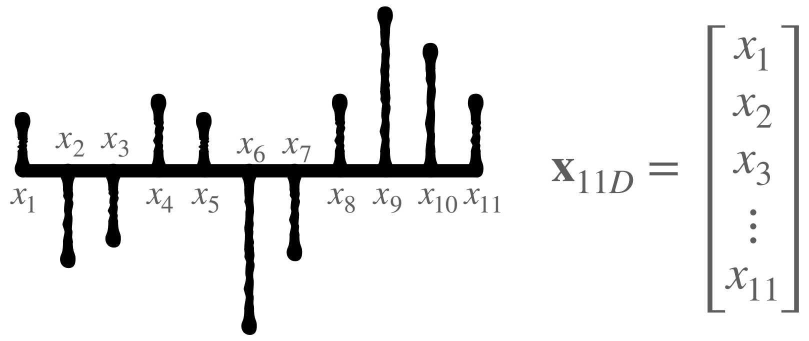
Cannot exclude more than one dimension

Takeaways

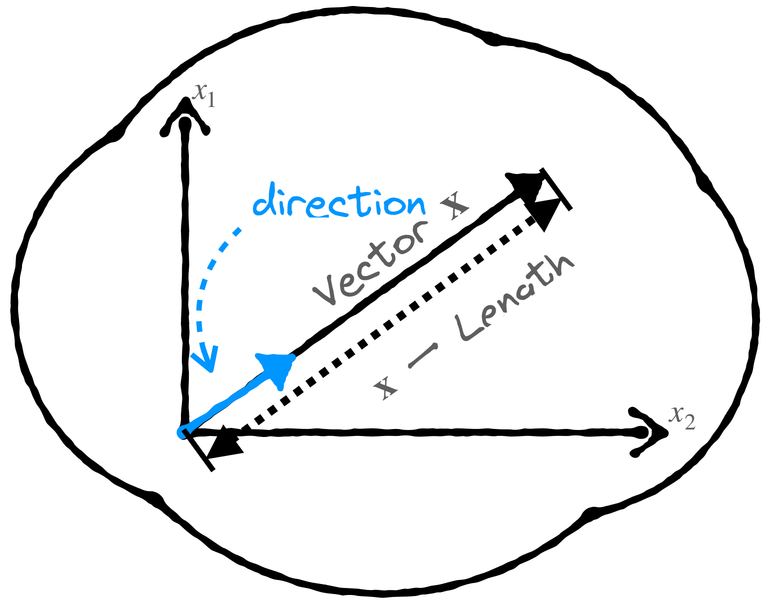
- Vector and its representation:
  - Arrow with coordinates



  - Bars

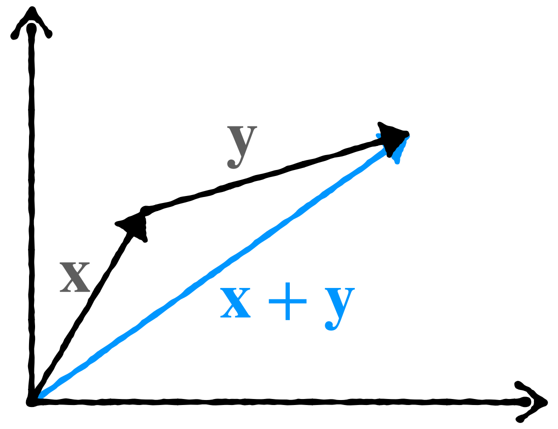


- Length and direction of vector



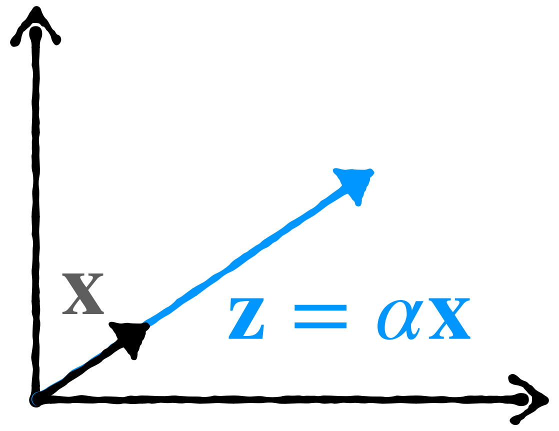
- Operations with vectors:
  - Summation

$$\mathbf{z} = \mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_N + y_N \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{bmatrix}$$



  - Scaling

$$\mathbf{z} = \alpha\mathbf{x} = \alpha \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_N \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{bmatrix}$$



- Dot product

Definition #1
$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{k=1}^{N} x_k y_k$$

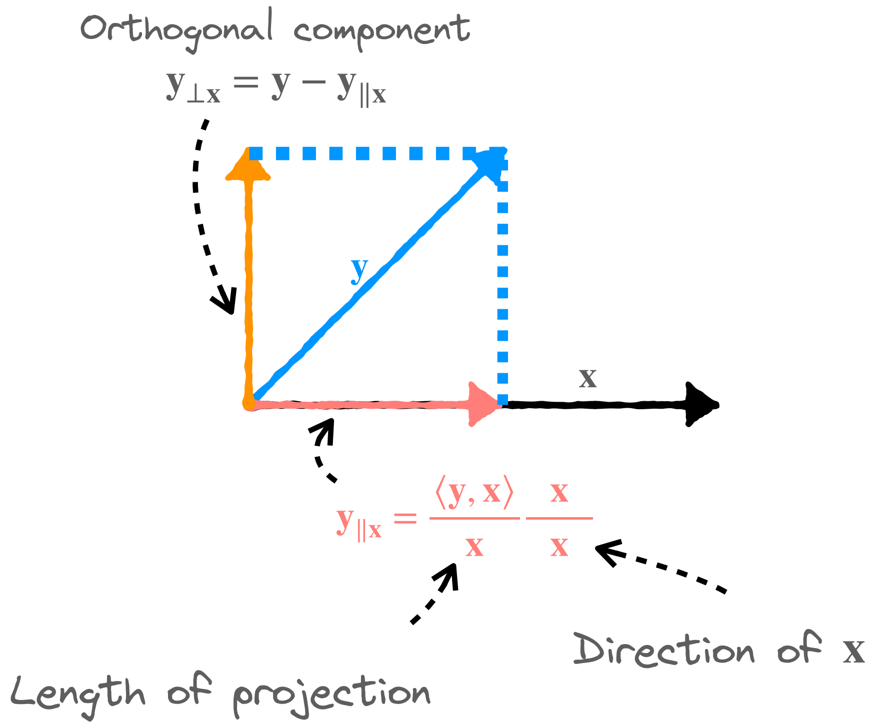
Definition #2
$$\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\| \, \|\mathbf{y}\| \cos\left(\angle_{\mathbf{x}}^{\mathbf{y}}\right)$$

- Length and cosine through dot product

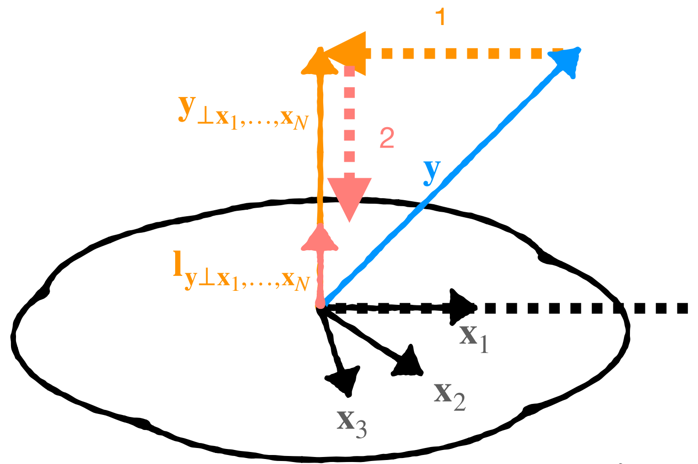
$$\|\mathbf{x}\|^2 = \langle \mathbf{x}, \mathbf{x} \rangle$$

$$\cos(\angle_{\mathbf{x}}^{\mathbf{y}}) = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\| \, \|\mathbf{y}\|}$$

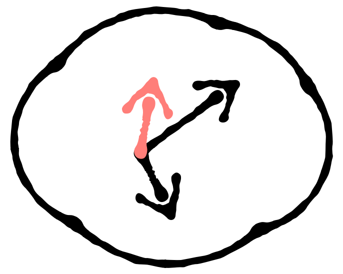
- Normal and collinear components



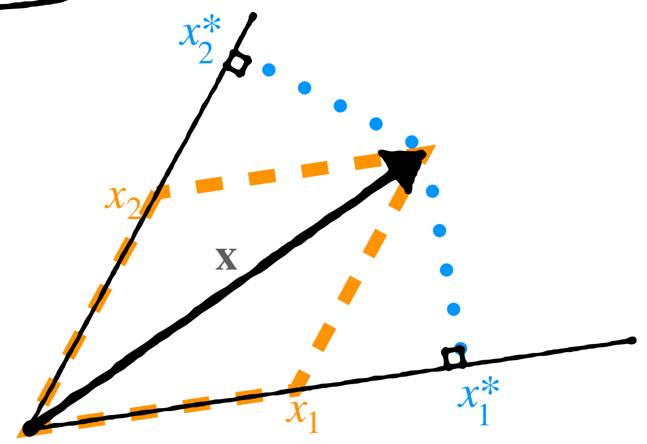
- Orthogonalisation



- Basis

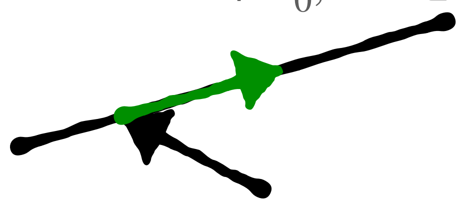


- Covariant and contravariant coordinates



- Linear subspaces

$\forall \mathbf{x} : \mathbf{x} = a\mathbf{l} + \mathbf{x}_0, \quad a \in \mathbb{R}$



$\forall \mathbf{x} : \langle \mathbf{x}, \mathbf{n} \rangle + b = 0$