3: SLEs (with square matrix)

Systems of Linear Equations with Square Matrix

!!! Gaussian Elimination !!!

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Solving Systems of Linear Equations

System of Linear Equations:

$$\begin{cases} ax_1 + bx_2 = y_1 \\ cx_1 + dx_2 = y_2 \end{cases}$$

You are asked to find x_1, x_2

a, b, c, d, y_1, y_2 are numbers

In matrix form

$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\mathbf{y}}$$

$$A\mathbf{x} = \mathbf{y}$$

How do we usually do that:

$$\begin{cases} ax_1 + bx_2 = y_1 \\ x_2 = \frac{1}{d}y_2 - \frac{c}{d}x_1 \end{cases}$$

$$\begin{cases} ax_1 + b\left(\frac{1}{d}y_2 - \frac{c}{d}x_1\right) = y_1 \\ x_2 = \frac{1}{d}y_2 - \frac{c}{d}x_1 \end{cases}$$

$$\begin{cases} x_1 = -\frac{b}{a}\left(\frac{1}{d}y_2 - \frac{c}{d}x_1\right) + \frac{1}{a}y_1 \\ x_2 = \frac{1}{d}y_2 - \frac{c}{d}x_1 \end{cases}$$

Another way:

$$\begin{cases} x_1 + \frac{b}{a}x_2 = \frac{y_1}{a} \\ x_1 + \frac{d}{c}x_2 = \frac{y_2}{c} \end{cases}$$

$[2] - [1]$

$$\begin{cases} x_1 + \frac{b}{a}x_2 = \frac{y_1}{a} \\ \left(\frac{d}{c} - \frac{b}{a}\right)x_2 = \frac{y_2}{c} - \frac{y_1}{a} \end{cases}$$

$$\begin{bmatrix} a* & b* \\ 0 & d* \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1^* \\ y_2^* \end{bmatrix}$$

But if we know the inverse A^{-1}:

$$A\mathbf{x} = \mathbf{y} \qquad A^{-1} \times \cdot$$

$$\underbrace{A^{-1}A}_{I}\,\mathbf{x} = A^{-1}\mathbf{y}$$

$$I\mathbf{x} = A^{-1}\mathbf{y}$$

$$\mathbf{x} = A^{-1}\mathbf{y}$$

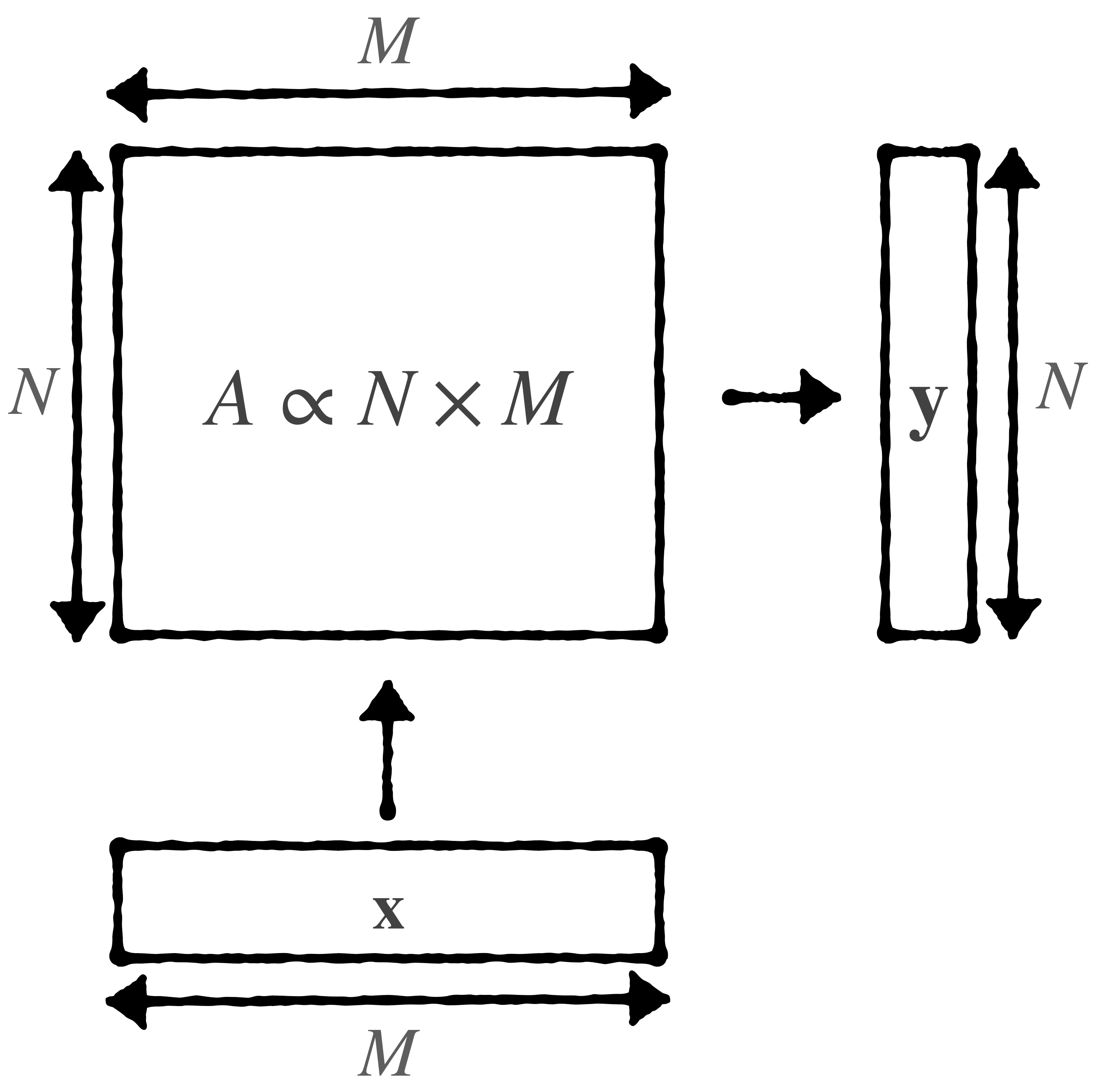
And we solved it!

Understanding SLE

When we are solving SLE, we answer the question:

Which $\mathbf{x}$ being transformed by A yields $\mathbf{y}$?

$A\mathbf{x} = \mathbf{y}$



If N = M:

Number of variables equal to number of data (usually only solution)

If N > M:

More data than variables (often inconsistent)

If N < M:

More variables than data (often infinite solutions)

Sometimes there are no solutions of SLE

Example:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

No solution as x_1 cannot be simultaneously equal to 2 and 1

Sometimes there is only one solution of the SLE.

Example:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

the only solution is:

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Sometimes there are infinite number of solutions.

Example:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solution:

$$\mathbf{x} = \begin{bmatrix} t \\ 1 \\ -t \end{bmatrix}, \quad t \in \mathbb{R}$$

Solving SLE: Gaussian Elimination

Stage 1: Make the matrix upper-triangular

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \dots & a_{NN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}$$

$$\begin{bmatrix} 1 & a_{12} & \dots & a_{1N} \\ 0 & a_{22} & \dots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{N2} & \dots & a_{NN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}$$

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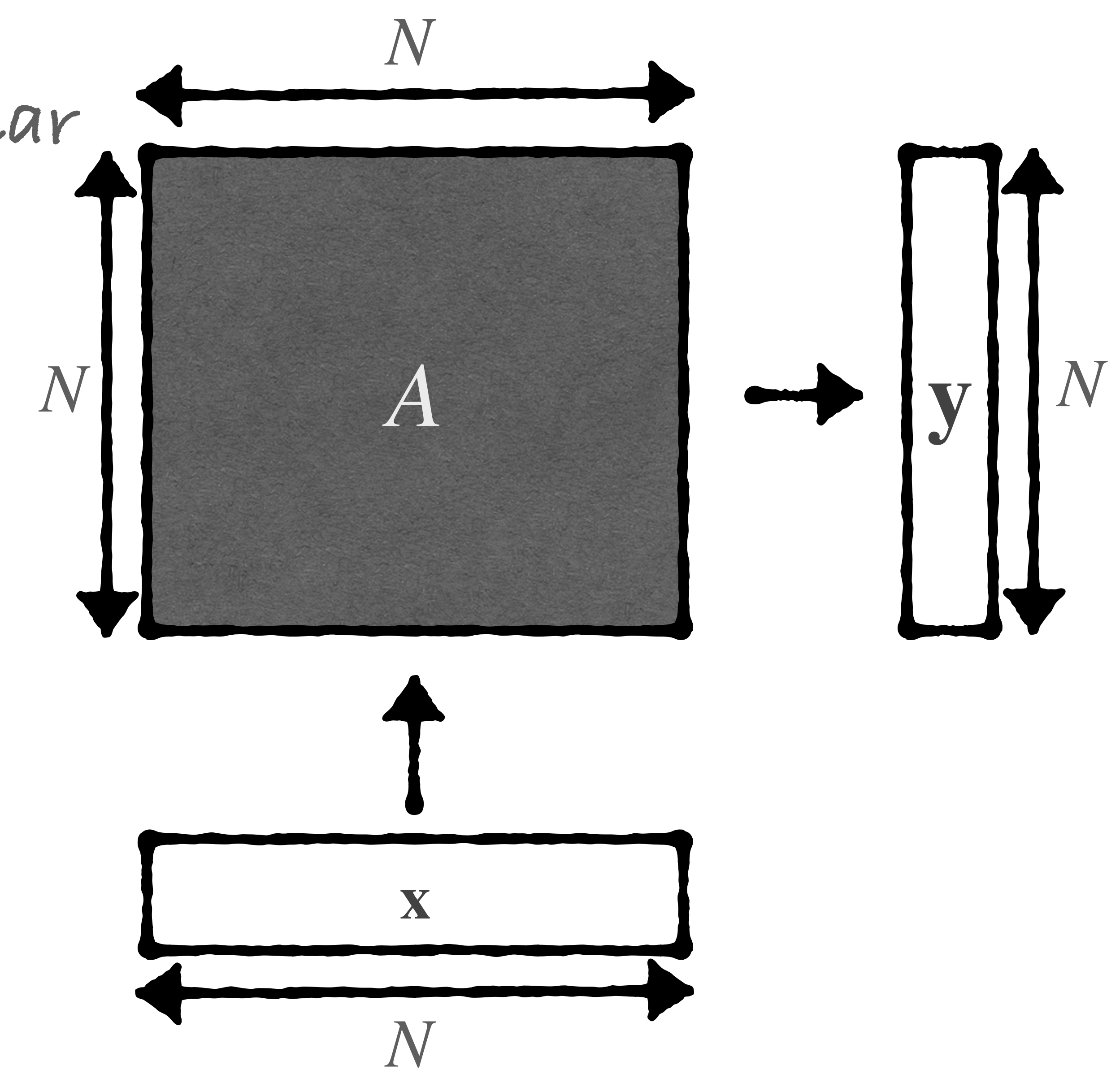
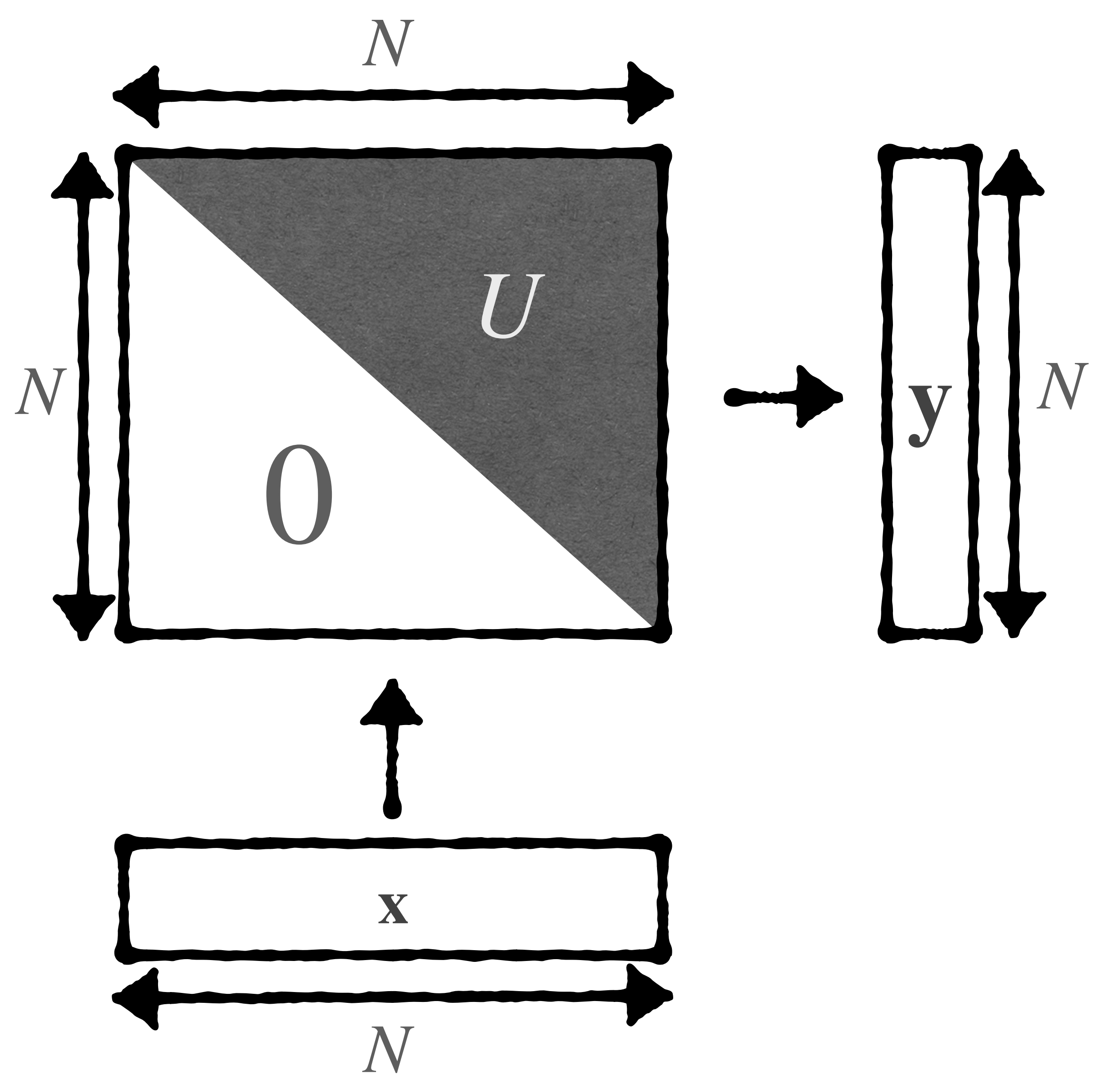




What could go wrong?

Any of $a_{11}, a_{22}, \dots, a_{NN}$ may happen to be zero

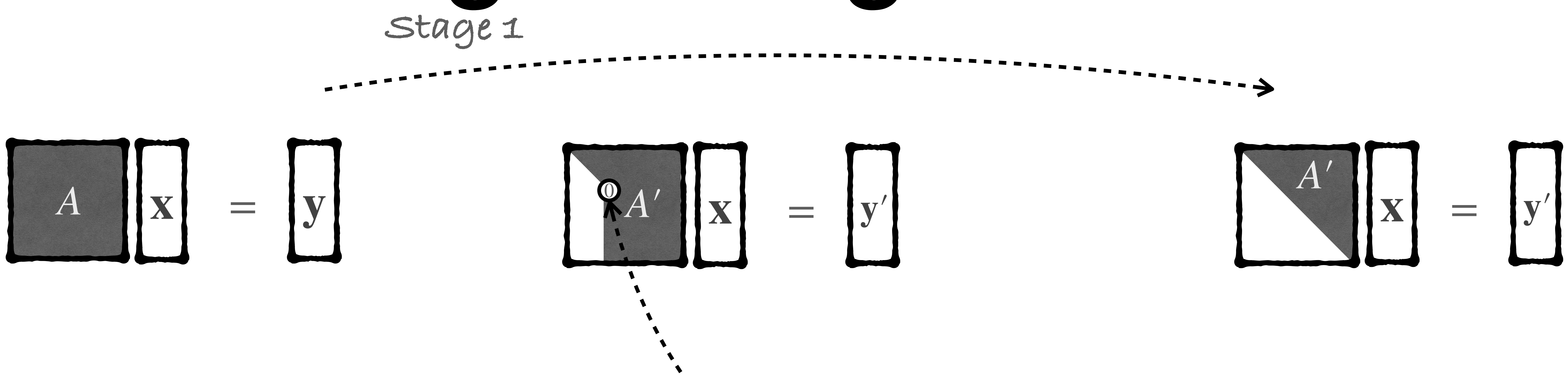
Then we cannot perform elimination of a column

In this case change order of equations so that the required item in matrix is non-zero.

In case that is not possible — too bad, we do not know what to do with it at the moment

* Note: coefficients $a_{..}$ and $y_.$ change throughout the process

What could go wrong?



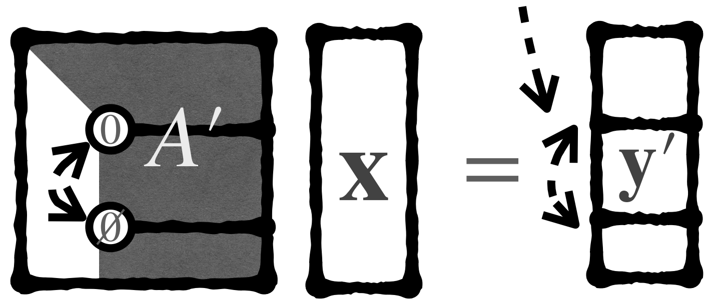
Emergencies may occur only at stage #1

Throughout the process we encountered zero on the main diagonal

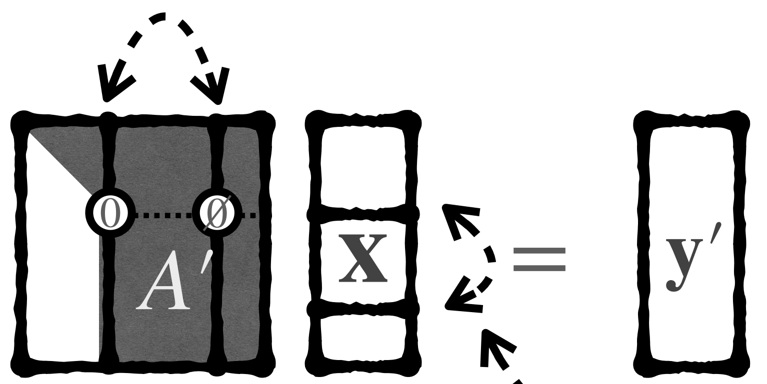
How to resolve:

Step 1: try to swap equations with the one that has non-zero in that place.

Note that coordinates in $\mathbf{y}$ swap then!



Step 2: try to swap the variables with the one that has non-zero there

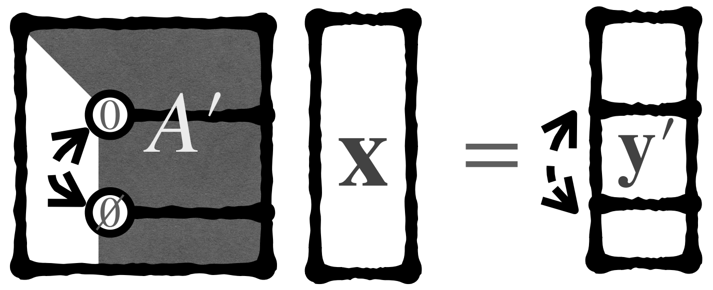


Note that coordinates in $\mathbf{x}$ swap then!

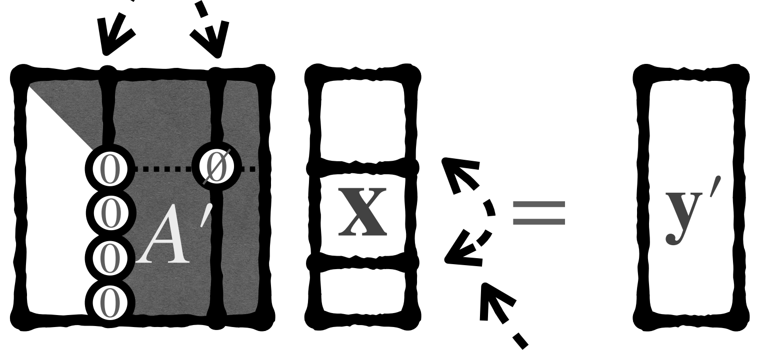
No success?

- Left part is strictly 0
- If corresponding $y = 0$, this equation Is duplicate
- Otherwise, the system is inconsistent (no solution)

What could go completely wrong?

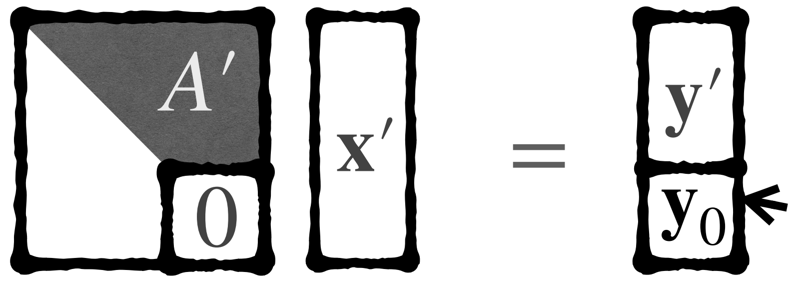


In this case we successfully fix the issue



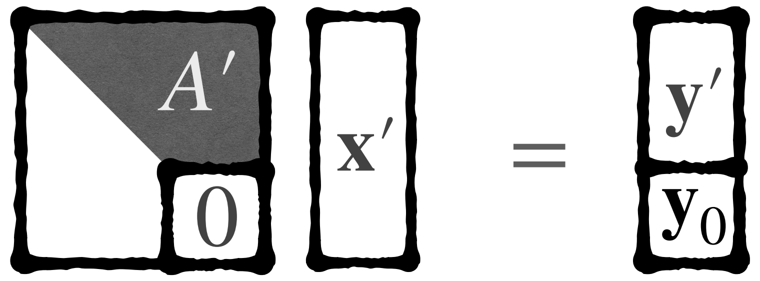
In this case we postpone the issue

Note that coordinates in $\mathbf{x}$ swap then!



* If $\mathbf{y}_0 = \mathbf{0}$, system has infinitely many solutions

* Otherwise system is <u>inconsistent</u>



How did we come to this?

* We performed linear combinations of the rows
* We also changed order of the coordinates and equations sometimes
* In the end some row turned into a zero vector, $\mathbf{0}$

$$\begin{bmatrix} - & \mathbf{a}_1 & - \\ - & \mathbf{a}_2 & - \\ & \vdots & \\ - & \mathbf{a}_N & - \end{bmatrix} \underbrace{\rightarrow}_{P} \begin{bmatrix} - & \mathbf{a}'_1 & - \\ - & \mathbf{a}'_2 & - \\ & \vdots & \\ - & \mathbf{a}'_N & - \end{bmatrix} \underbrace{\rightarrow}_{L} \begin{bmatrix} - & \mathbf{a}''_1 & - \\ - & \mathbf{a}''_2 & - \\ & \vdots & \\ - & \mathbf{0} & - \end{bmatrix}$$

After linear combinations

Initial matrix

Permutation

Matrix with permuted rows

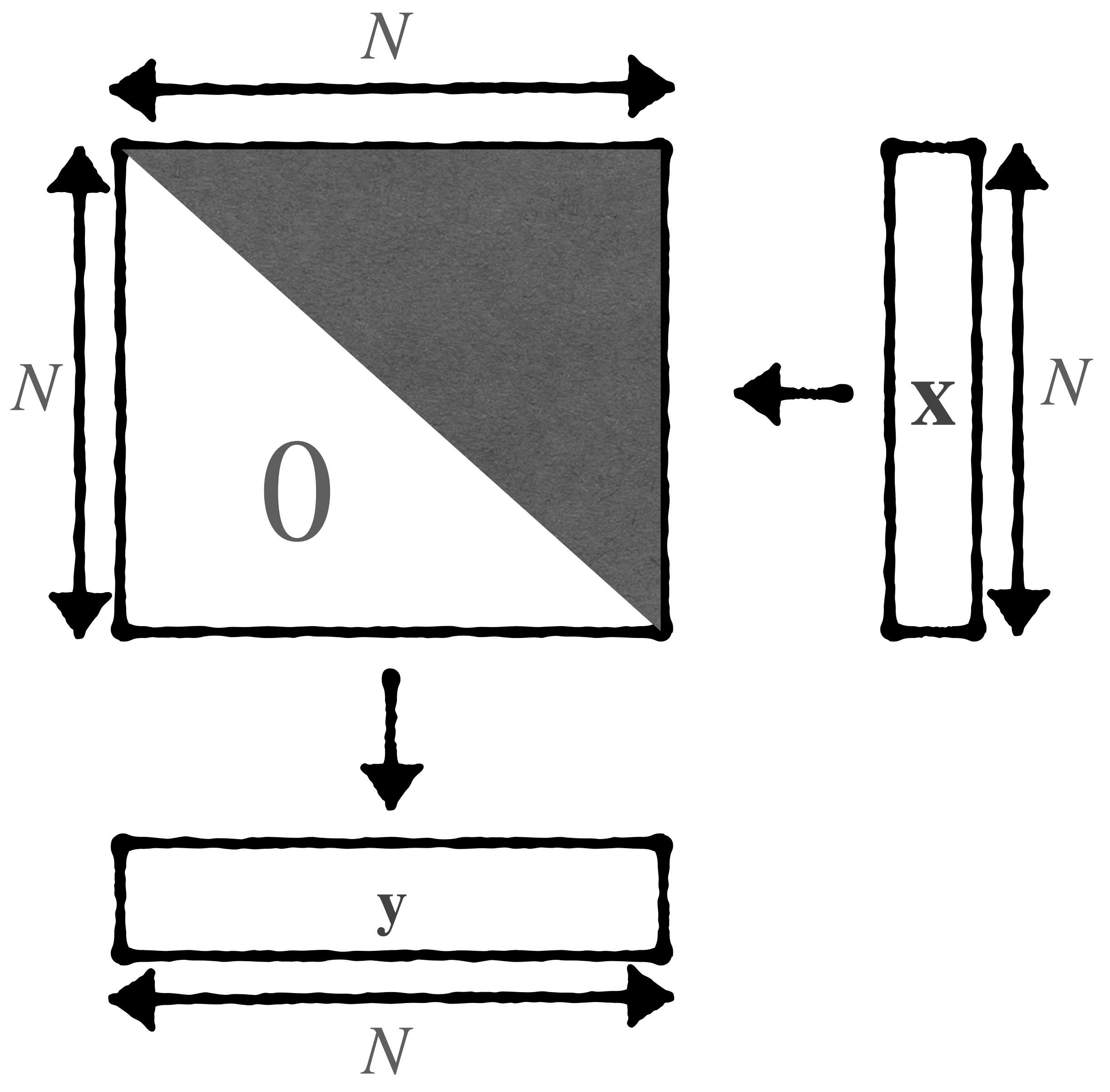
Permutation

Hence, the system $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_N\}$ is linearly dependent!

Solving SLE: Gaussian Elimination, Step 2

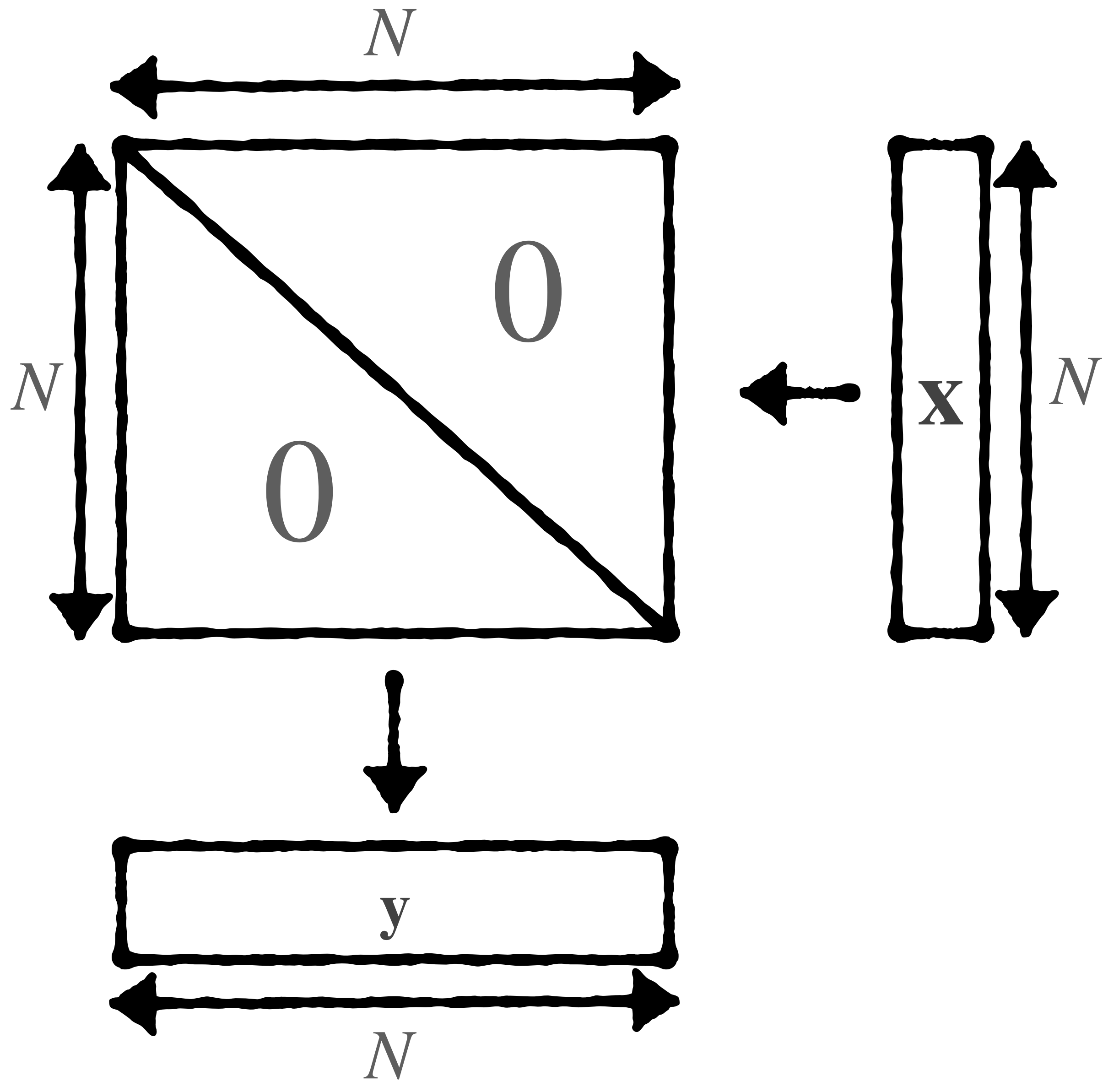
Stage 2: Make the matrix diagonal

$$\begin{bmatrix} 1 & a_{12} & \dots & a_{1N} \\ 0 & 1 & \dots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}$$



$$\begin{bmatrix} 1 & a_{12} & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}$$



What could go wrong?

In this stage nothing can go wrong!

Result:

Here we have solution in $\mathbf{y}$ vector

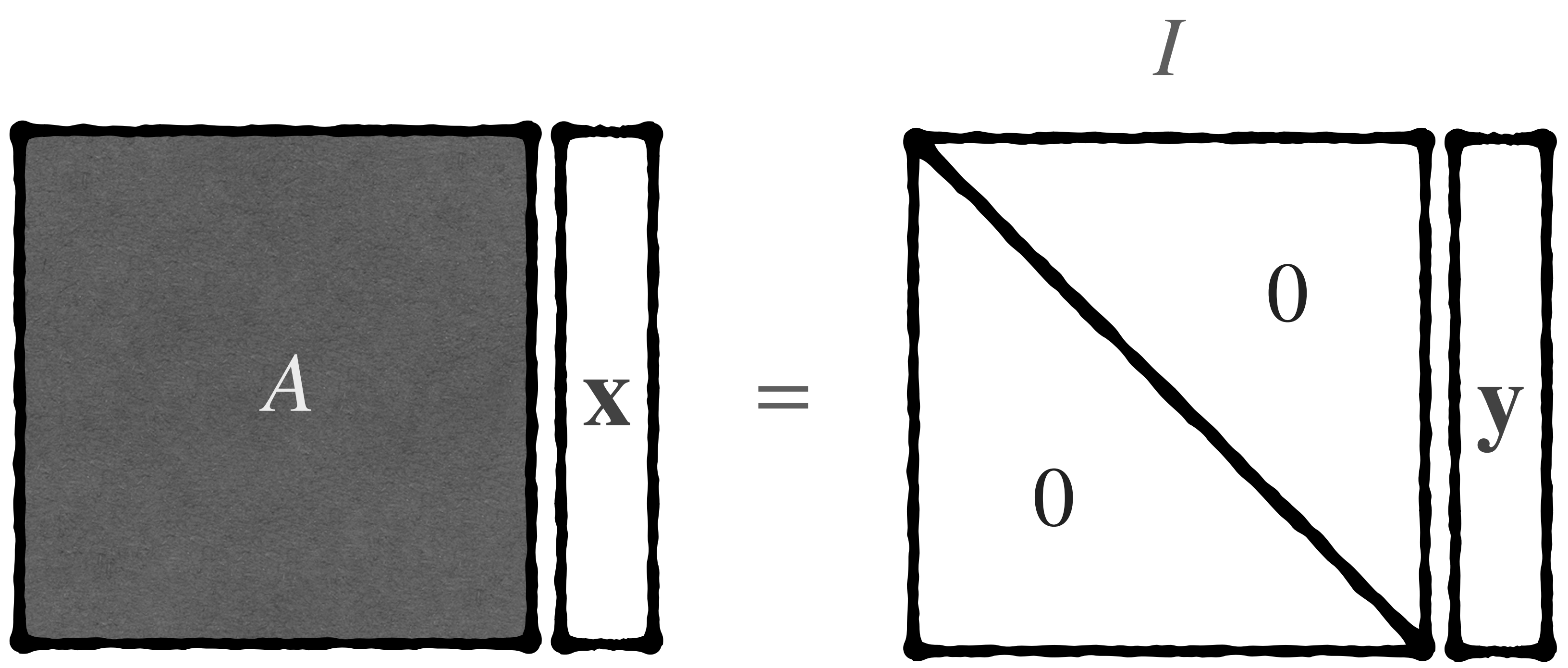
Solved the SLE!

Gaussian Elimination for Matrix Inversion 1

Stage 1: Make the matrix upper-triangular

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}$$

$$A\mathbf{x} = I\mathbf{y}$$



$$\begin{bmatrix} 1 & a_{12} & \cdots & a_{1N} \\ 0 & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{N2} & \cdots & a_{NN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ b_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{N1} & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ 0 & 1 & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{NN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ b_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{N1} & b_{N2} & \cdots & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}$$

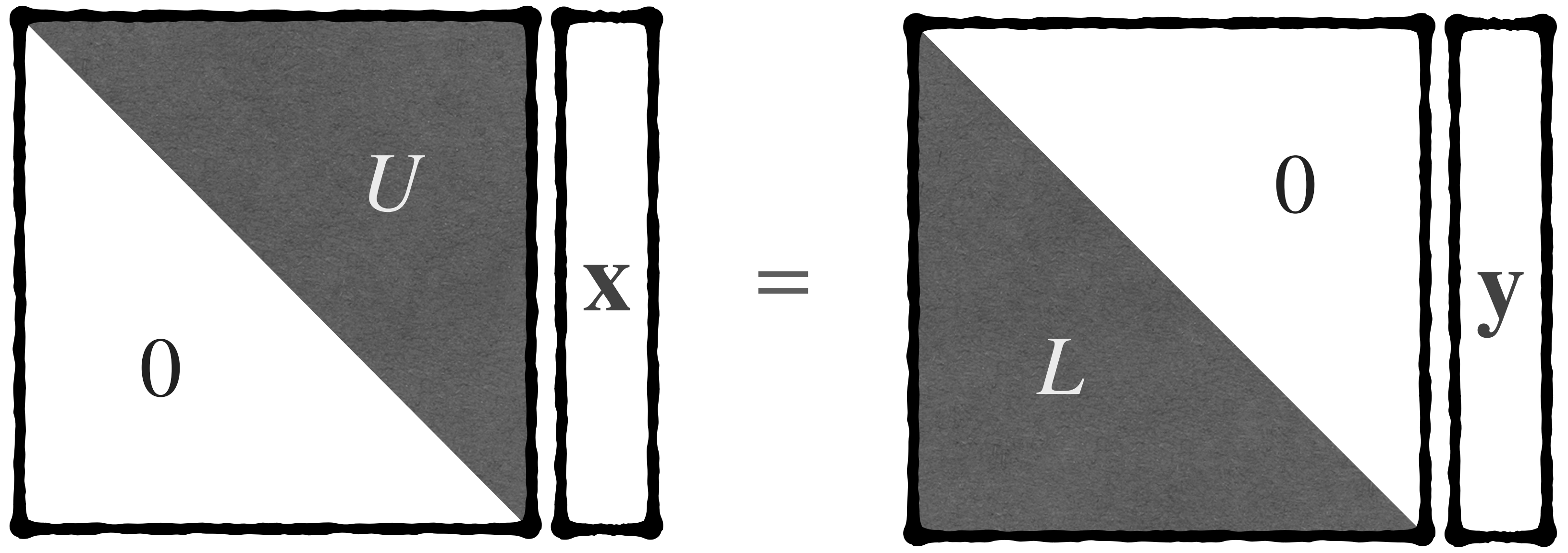
$$\begin{bmatrix} 1 & a_{12} & \cdots & a_{1N} \\ 0 & 1 & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ b_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{N1} & b_{N2} & \cdots & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}$$

Upper triangular matrix

Lower triangular matrix

$$U\mathbf{x} = L\mathbf{y}$$

* Note: here $\mathbf{y}$ does not change throughout the process

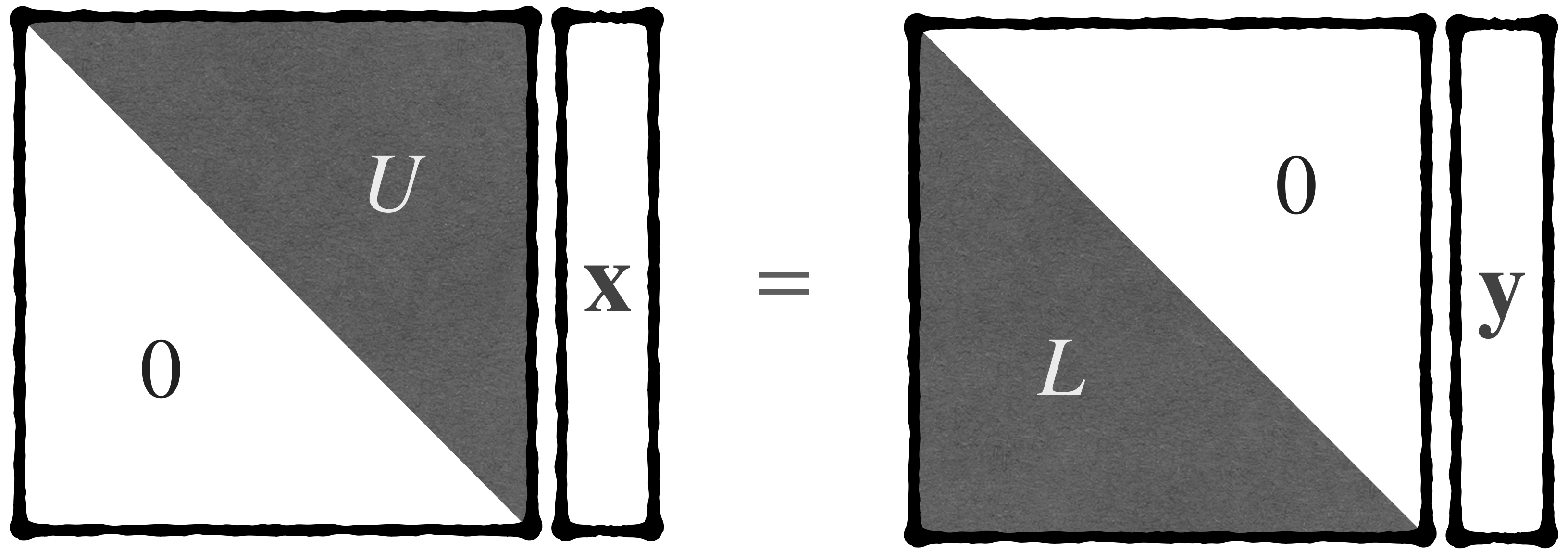


Gaussian Elimination for Matrix Inversion 2

Stage 2: Make the matrix diagonal

$$\begin{bmatrix} 1 & a_{12} & \dots & a_{1N} \\ 0 & 1 & \dots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}$$

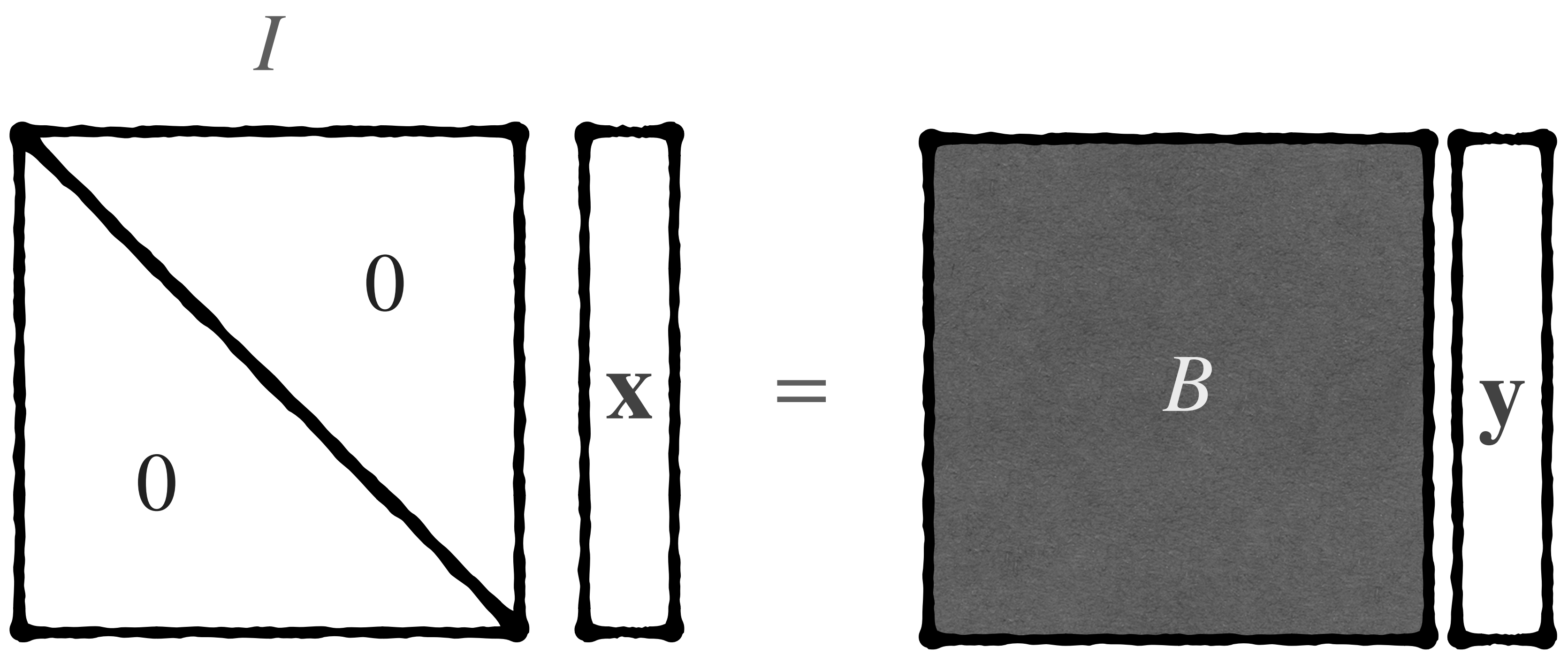
$$U\mathbf{x} = L\mathbf{y}$$



$$\begin{bmatrix} 1 & a_{12} & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}$$

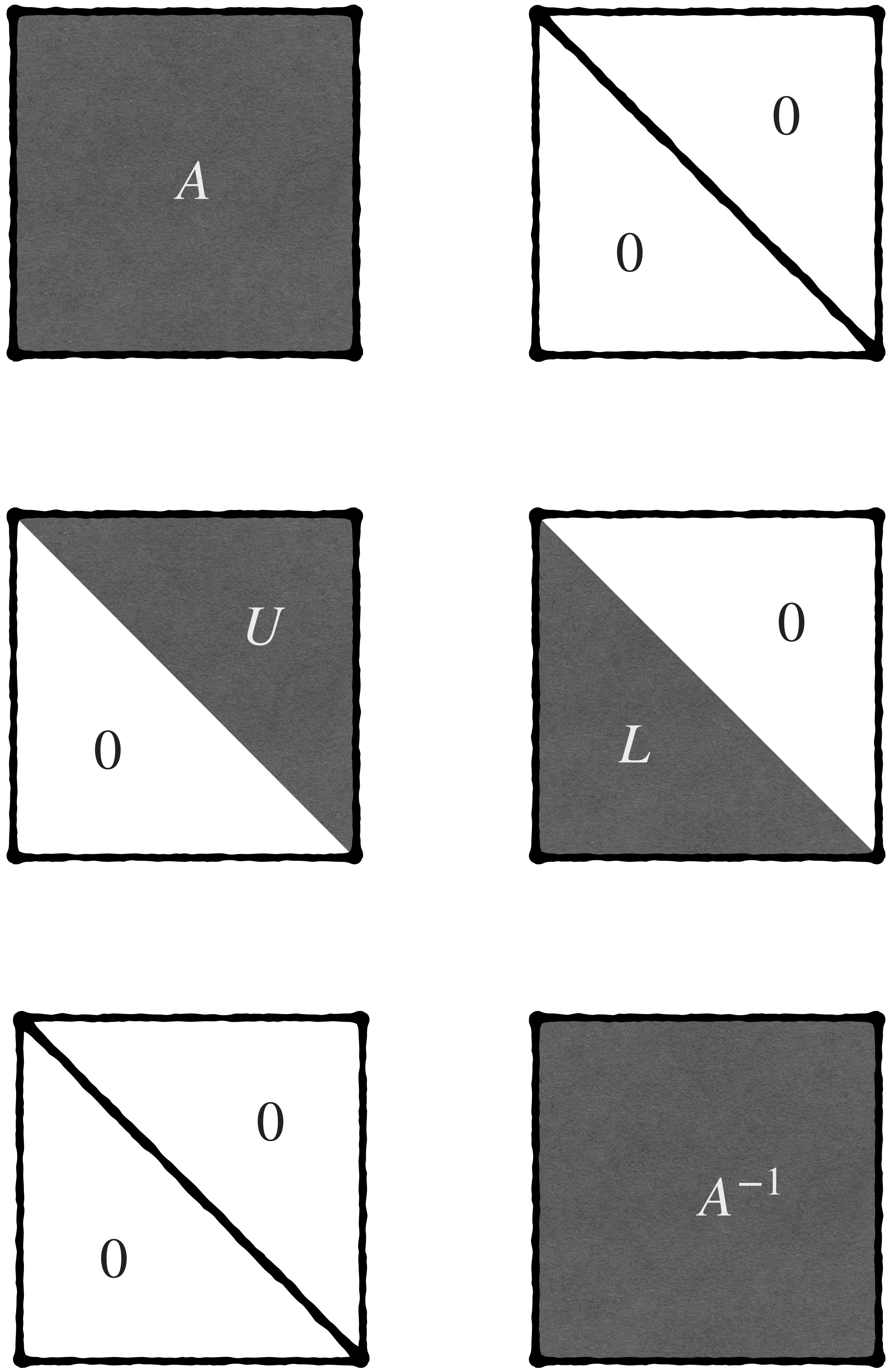
$$I\mathbf{x} = B\mathbf{y}$$



Very easy to see that:

$$B = A^{-1}$$

Gauss Algorithm: LU decomposition



Start

Stage 1:

Column by column eliminate lower triangle of matrix

Stage 2:

Column by column eliminate upper triangle of matrix

* Inverse of lower-triangle matrix is a lower-triangle matrix

* as to inverse it, you need only Stage 1 Gauss Elimination

$$U\mathbf{x} = \tilde{L}\mathbf{y}$$

$$\tilde{L}^{-1}U\mathbf{x} = \tilde{L}^{-1}\tilde{L}\mathbf{y}$$

$$\tilde{L}^{-1} = L$$

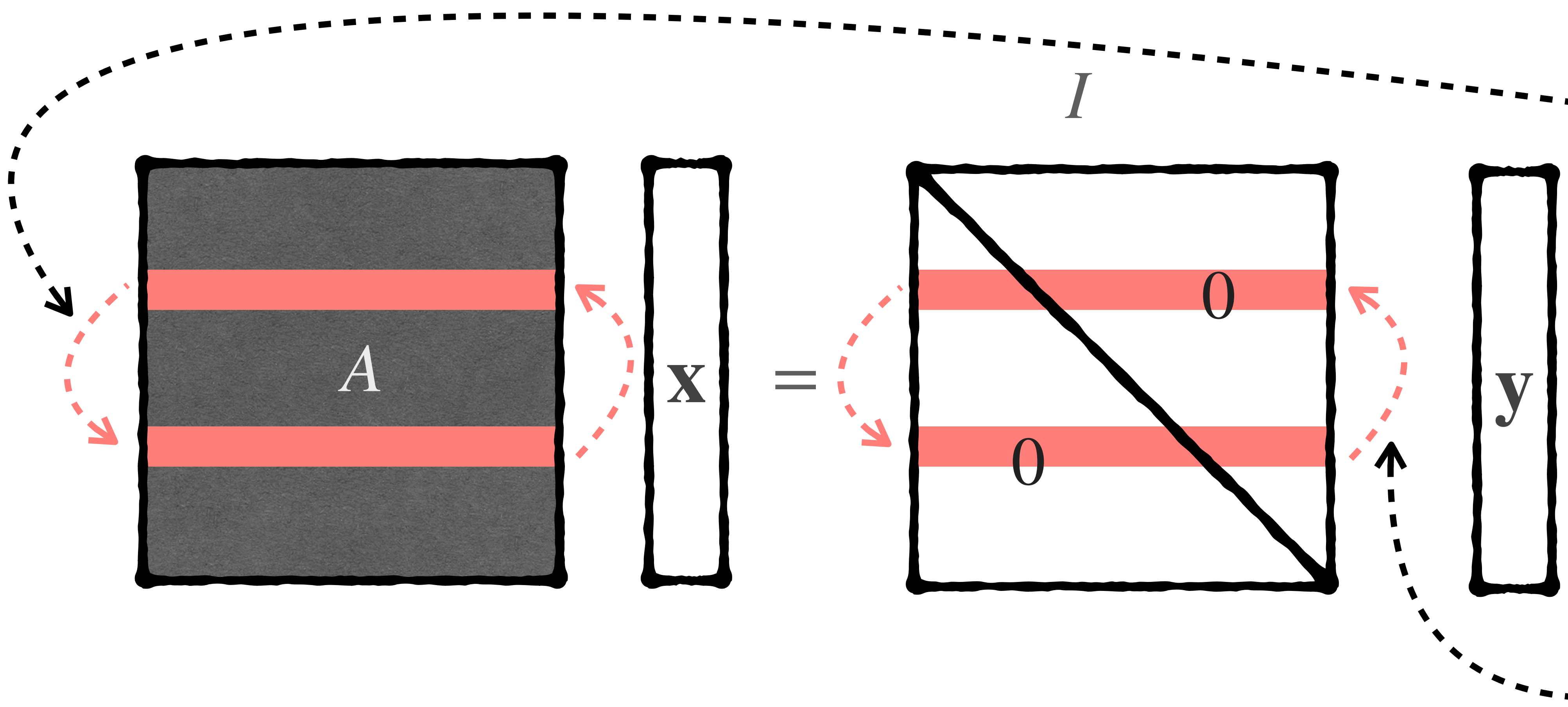
$$LU\mathbf{x} = I\mathbf{y}$$

$$A = LU$$

LU decomposition

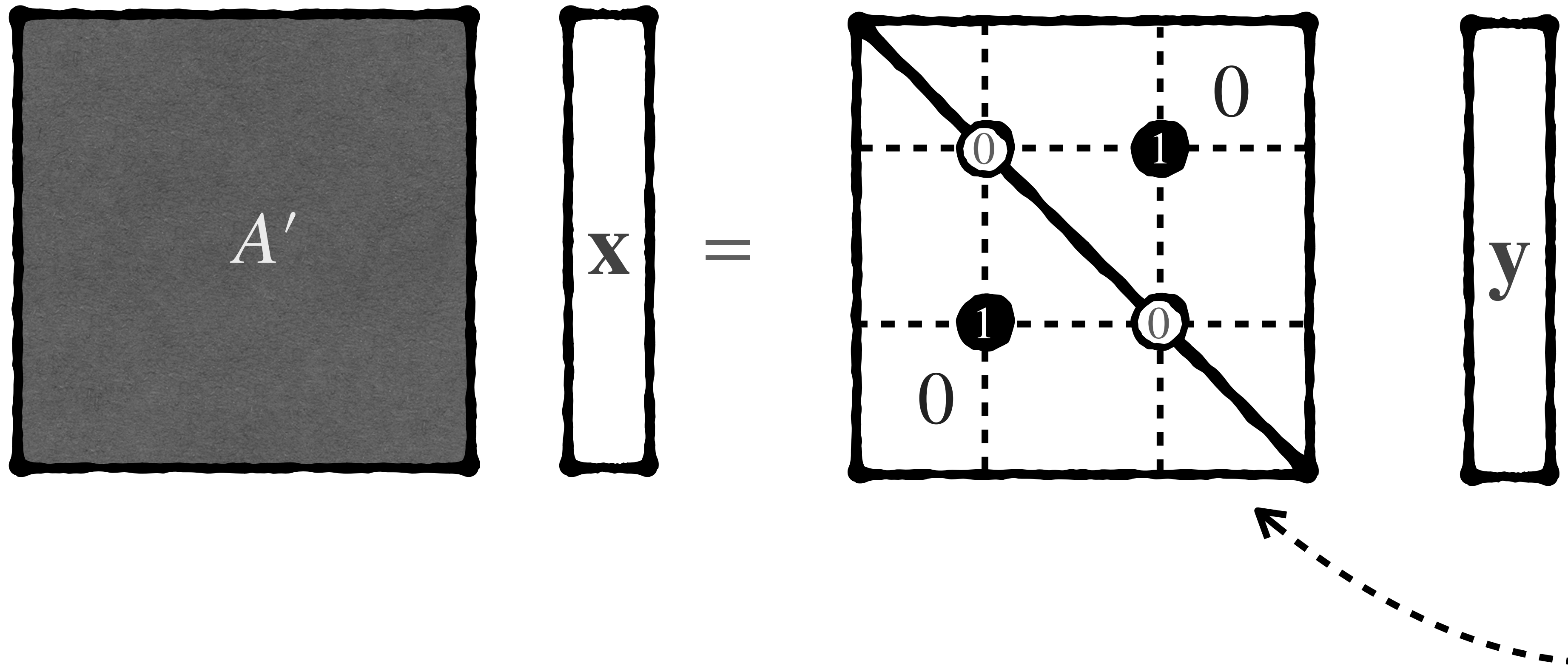
* Thus, matrix A can be represented as
* product LU of lower-triangle matrix L and upper-triangle matrix U
* if you can perform Stage 1 of Gauss Elimination without swapping the rows (or equations)

Permutation matrix



Swapping two equations, we are swapping both left and right parts

$$PA\mathbf{x} = PI\mathbf{y}$$



$$\underbrace{PA}_{A'}\,\mathbf{x} = \underbrace{P}\,\mathbf{y}$$

This is the matrix using which you may permute the rows of any matrix

<u>Example:</u>

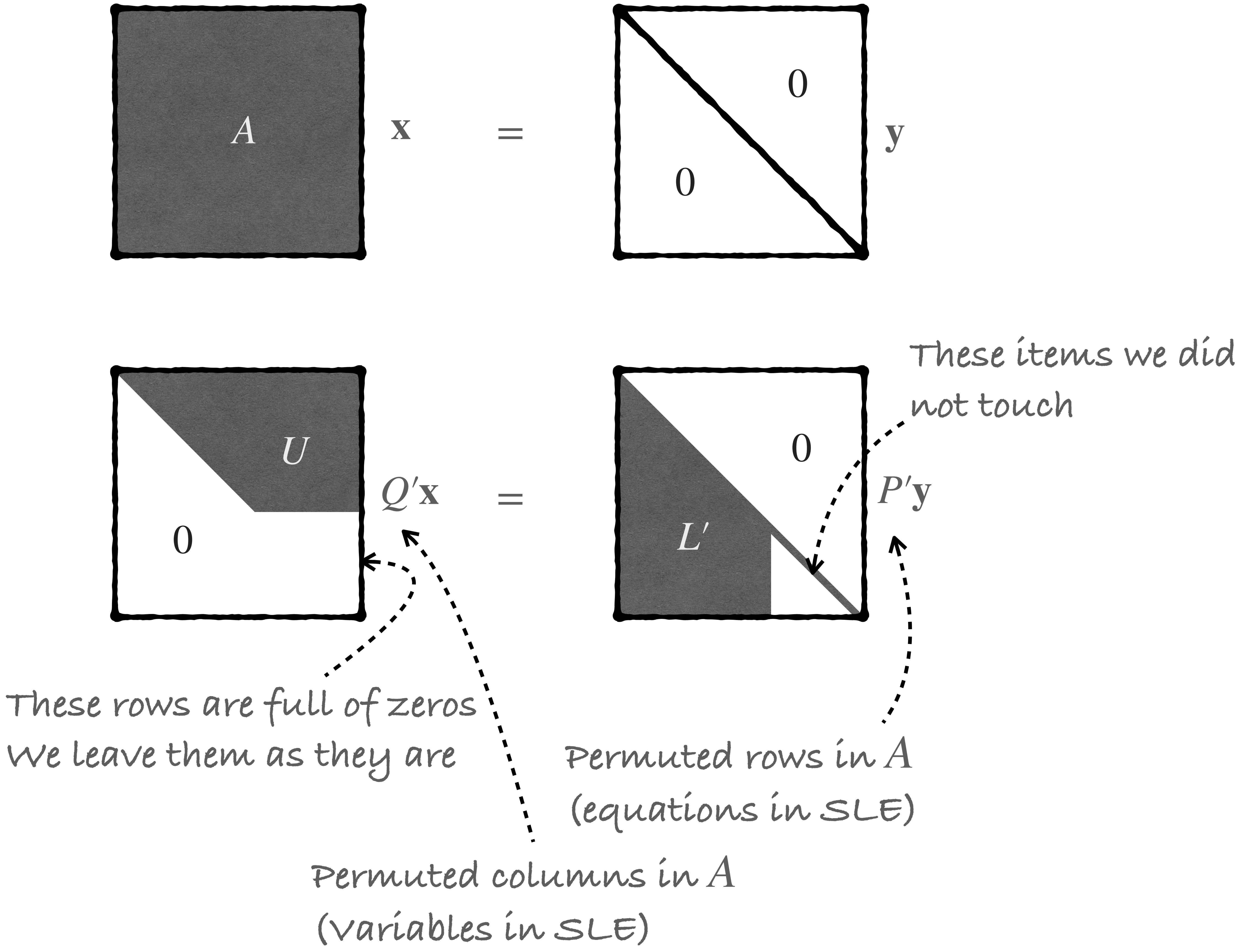
Permutes 2 and 4th rows:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Any matrix A can be represented as product of permutation matrix, low-triangle matrix and upper-triangle matrix:

$$A = PLU$$

PLU decomposition



$$U\mathbf{x} = L'P'\mathbf{y}$$

As inverse of L' is also a lower triangular matrix

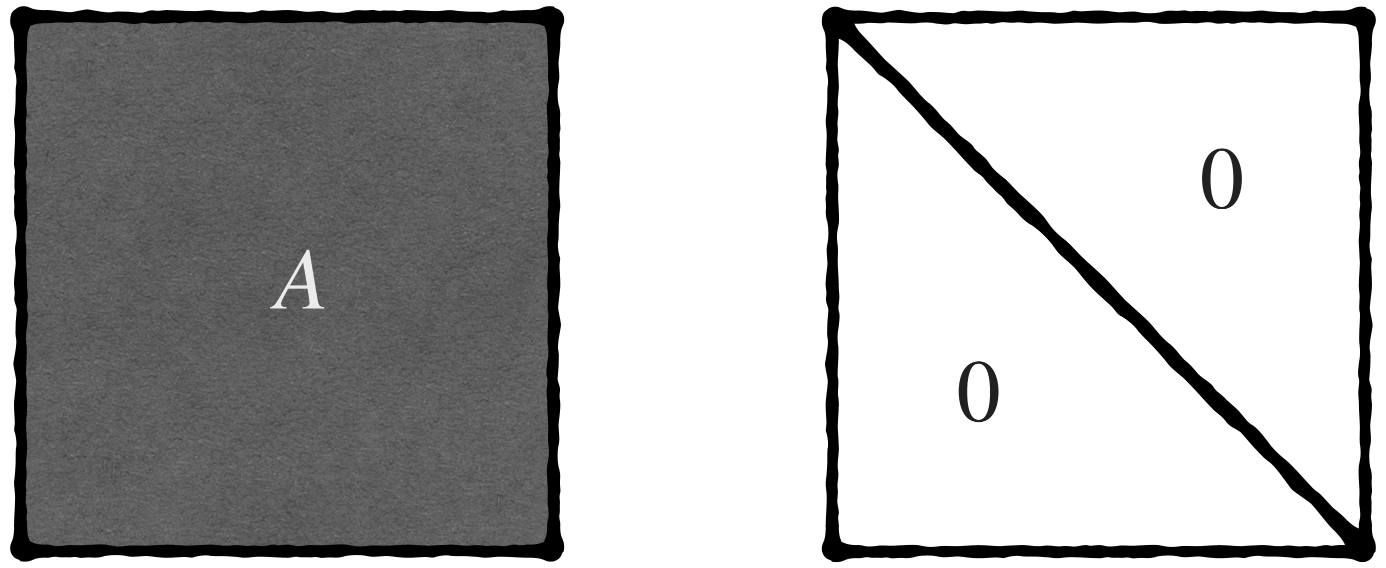
$$LU\mathbf{x} = P'\mathbf{y}$$

As inverse of P' is also a permutation matrix

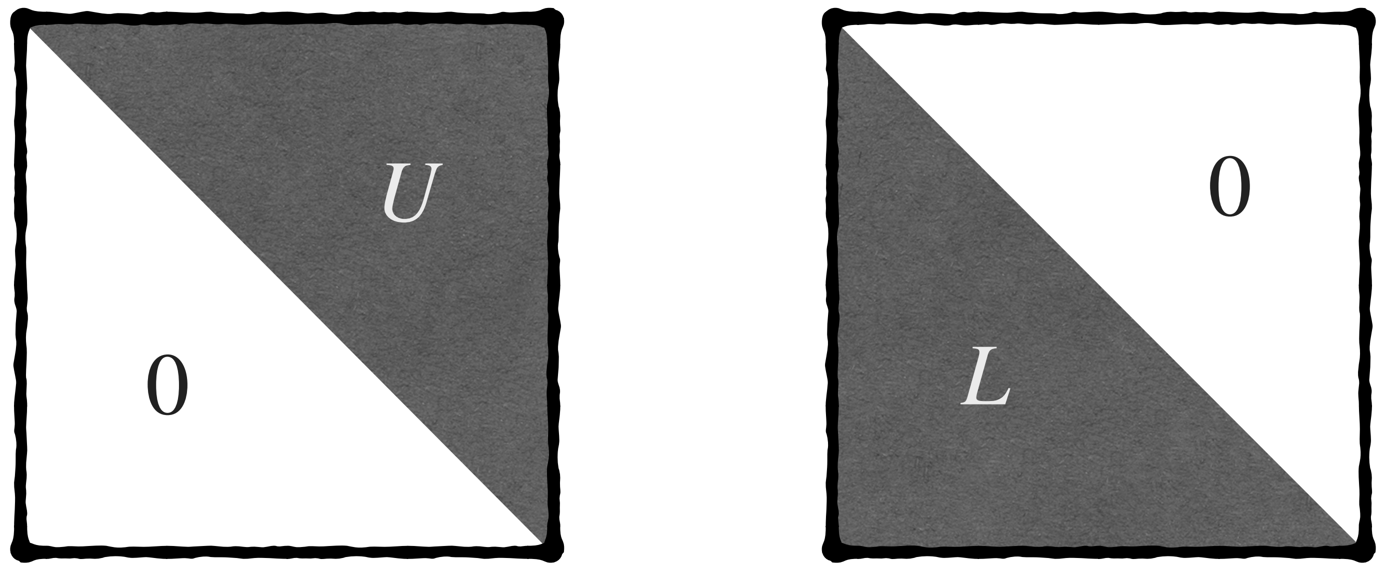
$$PLU\mathbf{x} = \mathbf{y}$$

Every square matrix has a PLU decomposition

Gauss Elimination: Summary

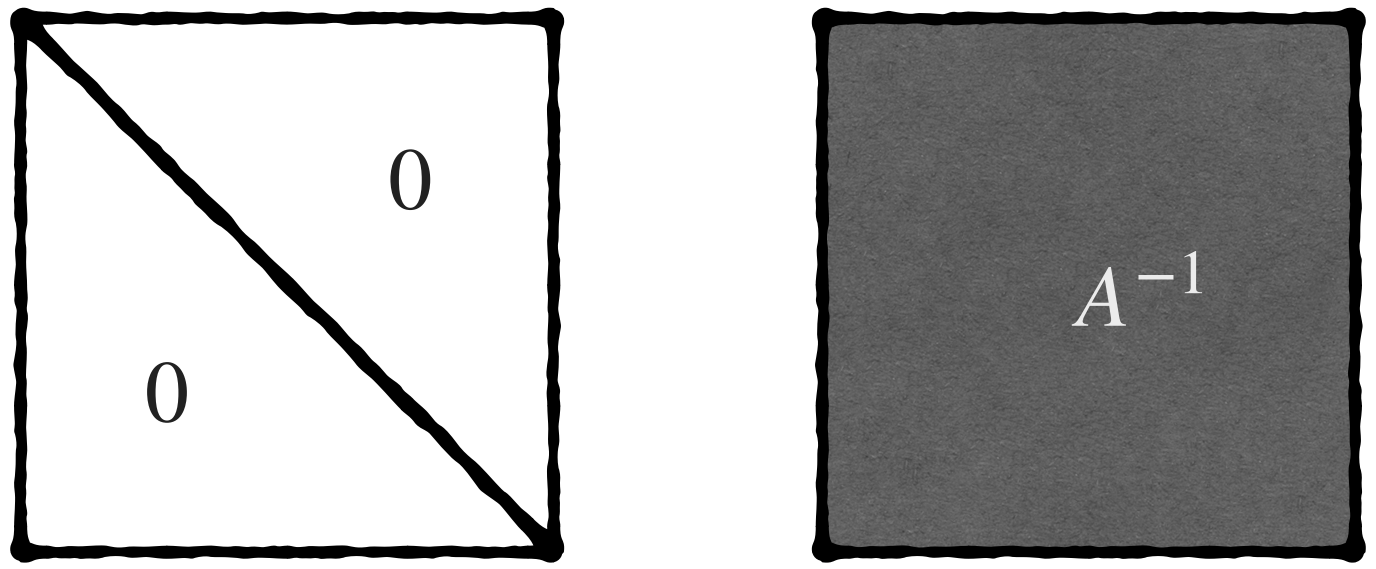


Start



Stage 1:

Column by column
eliminate lower triangle
of matrix



Stage 2:

Column by column
eliminate upper triangle
of matrix

What can be done using it:

* Solving SLE
* Matrix inversion
* Performing (P)LU decomposition
* Many other things that are yet to come

Takeaways

- System of Linear Equations

$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\mathbf{y}}$$

$$A\mathbf{x} = \mathbf{y}$$

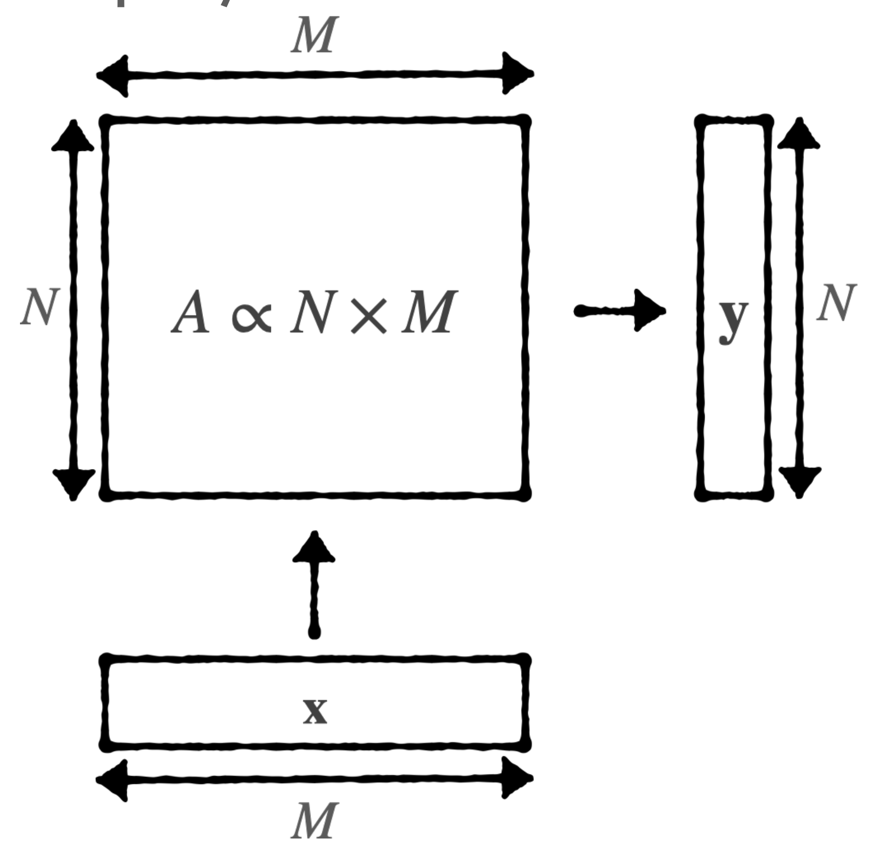
- Ways to solve it
  - Gaussian Elimination

$$\begin{bmatrix} a^* & b^* \\ 0 & d^* \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1^* \\ y_2^* \end{bmatrix}$$

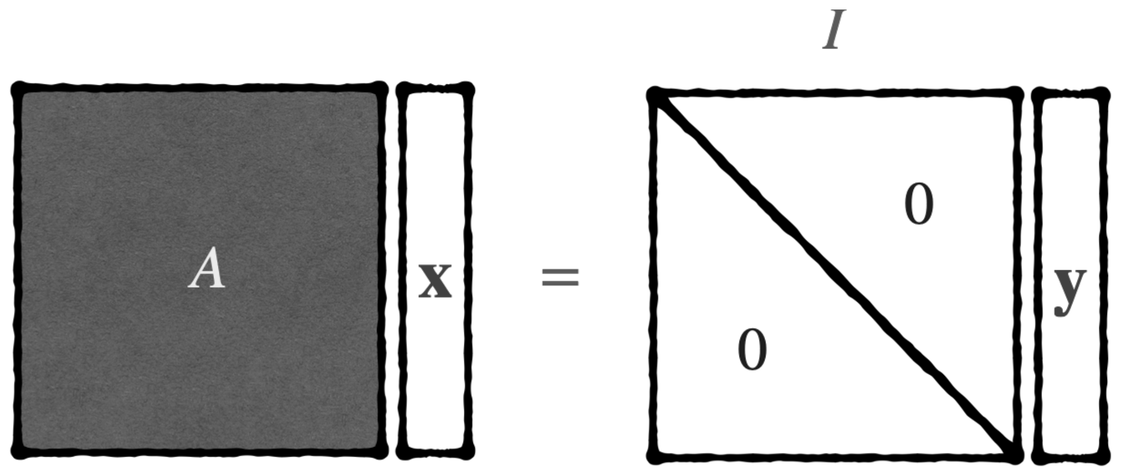
  - With Inverse Matrix

$$\mathbf{x} = A^{-1}\mathbf{y}$$

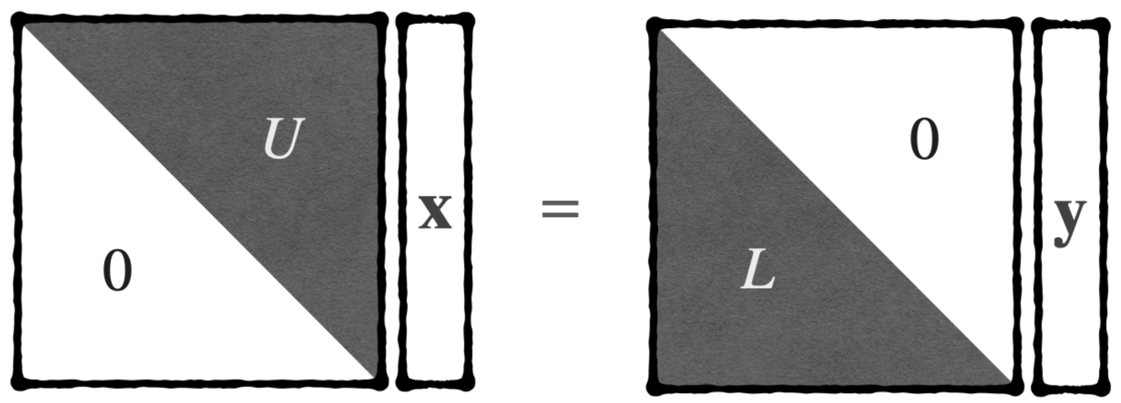
- Philosophy of SLEs



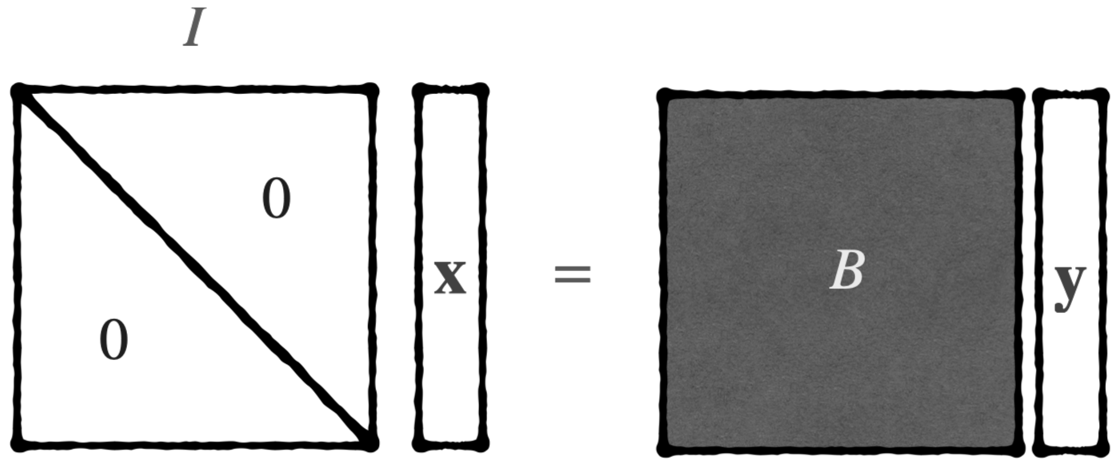
- Gaussian Elimination



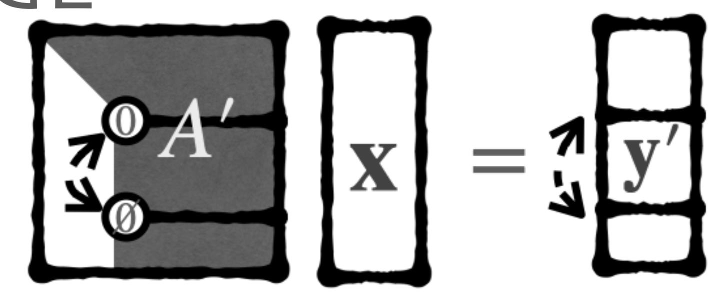
- Stage I



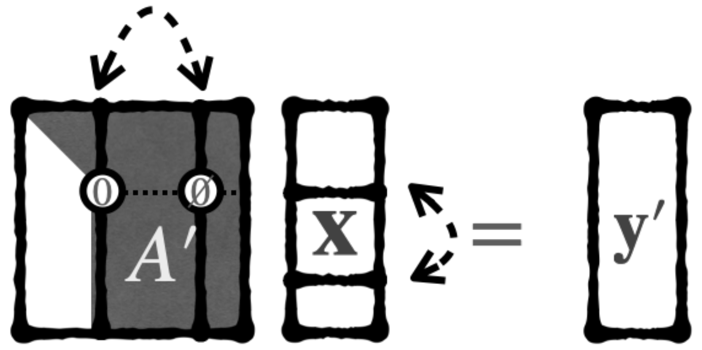
- Stage II ($B = A^{-1}$)



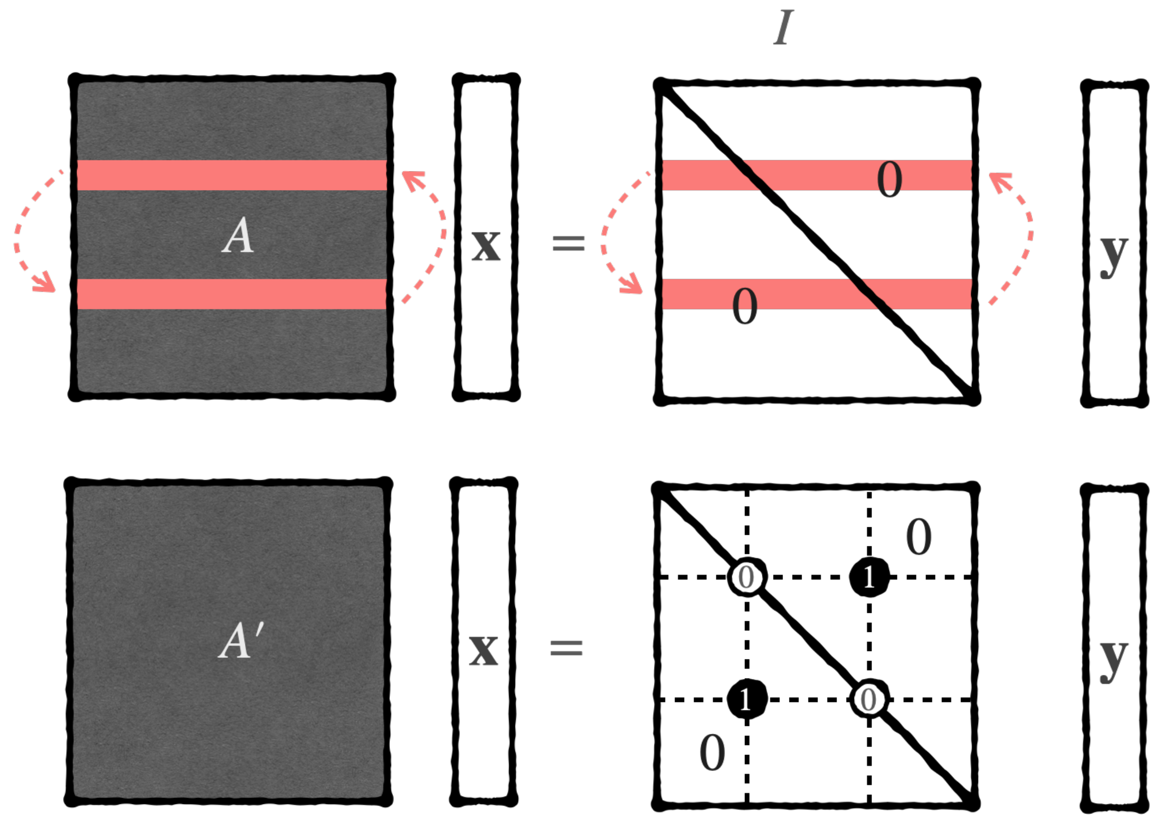
- Fixing issues in GE
  - Method I (resolves)



  - Method II (postpones)



- Permutation matrices



- PLU decomposition
  - Start: $A\mathbf{x} = \mathbf{y}$
  - Stage I GE: $U\mathbf{x} = L'P'\mathbf{y}$
  - Invert L': $LU\mathbf{x} = P'\mathbf{y}$
  - Invert P': $PLU\mathbf{x} = \mathbf{y}$
  - The procedure can be completed for any square matrix
  - Any A can be represented as PLU