2: Matrices and Operations

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Matrix

Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{M1} & a_{M2} & \dots & a_{MN} \end{bmatrix}$$

Table of elements

$$A = [a_{kl}]$$

Elements indexed by:

* row index k
* Column index l

$$A = \begin{bmatrix} \\ \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_N \\ \\ \end{bmatrix}$$

Or list of column-vectors

$$A \in \mathbb{R}^{N \times M}$$

A contains $N \times M$

$\mathbb{R}$ (real numbers)

$$A = \begin{bmatrix} - & \mathbf{a}_1 & - \\ - & \mathbf{a}_2 & - \\ & \vdots & \\ - & \mathbf{a}_M & - \end{bmatrix}$$

Or list of row-vectors

Matrix Operations

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{M1} & a_{M2} & \dots & a_{MN} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1N} \\ b_{21} & b_{22} & \dots & b_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ b_{M1} & b_{M2} & \dots & b_{MN} \end{bmatrix}$$

Sin*: $$\sin A = \begin{bmatrix} \sin a_{11} & \sin a_{12} & \dots & \sin a_{1N} \\ \sin a_{21} & \sin a_{22} & \dots & \sin a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \sin a_{M1} & \sin a_{M2} & \dots & \sin a_{MN} \end{bmatrix}$$

Root*: $$\sqrt{A} = \begin{bmatrix} \sqrt{a}_{11} & \sqrt{a}_{12} & \dots & \sqrt{a}_{1N} \\ \sqrt{a}_{21} & \sqrt{a}_{22} & \dots & \sqrt{a}_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{a}_{M1} & \sqrt{a}_{M2} & \dots & \sqrt{a}_{MN} \end{bmatrix}$$

And many others

Sum: $$A + B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1N}+b_{1N} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2N}+b_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{M1}+b_{M1} & a_{M2}+b_{M2} & \dots & a_{MN}+b_{MN} \end{bmatrix}$$

Difference: $$A - B = \begin{bmatrix} a_{11}-b_{11} & a_{12}-b_{12} & \dots & a_{1N}-b_{1N} \\ a_{21}-b_{21} & a_{22}-b_{22} & \dots & a_{2N}-b_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{M1}-b_{M1} & a_{M2}-b_{M2} & \dots & a_{MN}-b_{MN} \end{bmatrix}$$

Product*: $$A \cdot B = \begin{bmatrix} a_{11}b_{11} & a_{12}b_{12} & \dots & a_{1N}b_{1N} \\ a_{21}b_{21} & a_{22}b_{22} & \dots & a_{2N}b_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{M1}b_{M1} & a_{M2}b_{M2} & \dots & a_{MN}b_{MN} \end{bmatrix}$$

Division*: $$A \div B = \begin{bmatrix} a_{11}\div b_{11} & a_{12}\div b_{12} & \dots & a_{1N}\div b_{1N} \\ a_{21}\div b_{21} & a_{22}\div b_{22} & \dots & a_{2N}\div b_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{M1}\div b_{M1} & a_{M2}\div b_{M2} & \dots & a_{MN}\div b_{MN} \end{bmatrix}$$

* Non-orthodox operations. Use with caution! Some mathematicians get insulted.

Matrix Transposal

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ \vdots & \vdots & & \vdots \\ a_{M1} & a_{M2} & \dots & a_{MN} \end{bmatrix}$$

Transposition operation

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{N1} \\ a_{12} & a_{22} & \dots & a_{N2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1M} & a_{2M} & \dots & a_{NM} \end{bmatrix}$$

$$A = [a_{kl}]$$

$$A^T = [a_{lk}]$$

Matrix Product

$$AB = \begin{bmatrix} - & \mathbf{a}_1 & - \\ - & \mathbf{a}_2 & - \\ & \vdots & \\ - & \mathbf{a}_N & - \end{bmatrix} \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_M \end{bmatrix} = \begin{bmatrix} \langle \mathbf{a}_1, \mathbf{b}_1 \rangle & \langle \mathbf{a}_1, \mathbf{b}_2 \rangle & \dots & \langle \mathbf{a}_1, \mathbf{b}_M \rangle \\ \langle \mathbf{a}_2, \mathbf{b}_1 \rangle & \langle \mathbf{a}_2, \mathbf{b}_2 \rangle & \dots & \langle \mathbf{a}_2, \mathbf{b}_M \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \mathbf{a}_N, \mathbf{b}_1 \rangle & \langle \mathbf{a}_N, \mathbf{b}_2 \rangle & \dots & \langle \mathbf{a}_N, \mathbf{b}_M \rangle \end{bmatrix}$$

$A \propto N \times K$

$B \propto K \times M$

$$A\mathbf{x} = \begin{bmatrix} - & \mathbf{a}_1 & - \\ - & \mathbf{a}_2 & - \\ & \vdots & \\ - & \mathbf{a}_N & - \end{bmatrix} \begin{bmatrix} \mathbf{x} \end{bmatrix} = \begin{bmatrix} \langle \mathbf{a}_1, \mathbf{x} \rangle \\ \langle \mathbf{a}_2, \mathbf{x} \rangle \\ \vdots \\ \langle \mathbf{a}_N, \mathbf{x} \rangle \end{bmatrix} = \mathbf{y}$$

$A \propto M \times N$

$\mathbf{x} \in \mathbb{R}^N$

$\mathbf{y} \in \mathbb{R}^M$

Matrix A transforms $\mathbf{x}$ to $\mathbf{y}$:

$\mathbf{x} \to A \to \mathbf{y}$

Several multiplications:

$ABC\mathbf{x} : \mathbf{x} \to C \to B \to A \to \mathbf{y}$

Properties

Matrix-vector product is linear:

$$A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y}$$
$$(A + B)\mathbf{x} = A\mathbf{x} + B\mathbf{x}$$

Comes from linearity of
$\langle \mathbf{a} + \mathbf{b}, \mathbf{y} \rangle = \langle \mathbf{a}, \mathbf{y} \rangle + \langle \mathbf{b}, \mathbf{y} \rangle$

Matrices in product do not commute:

$$AB \neq BA$$

Result does not depend on the order

$$ABC = (AB)C = A(BC)$$

$$ABC = \sum_{m=1}^{M} a_{nm} \left(\sum_{k=1}^{K} b_{mk} c_{kl} \right) = \sum_{k=1}^{K} \left(\sum_{m=1}^{M} a_{nm} b_{mk} \right) c_{kl}$$

Identity Matrix

$\mathbf{x} \to A? \to \mathbf{x}$ Which matrix works like that?

That is what we want

$$A\mathbf{x} = \begin{bmatrix} - & \mathbf{a}_1 & - \\ - & \mathbf{a}_2 & - \\ & \vdots & \\ - & \mathbf{a}_N & - \end{bmatrix} \begin{bmatrix} \mathbf{x} \end{bmatrix} = \begin{bmatrix} \langle \mathbf{a}_1, \mathbf{x} \rangle \\ \langle \mathbf{a}_2, \mathbf{x} \rangle \\ \vdots \\ \langle \mathbf{a}_N, \mathbf{x} \rangle \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$

Identity matrix

$$A = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} = I$$

$IA = A$
$AI = A$

Inverse Matrix

Inverse matrix of A

A^{-1}

It reverts the result of original matrix

$\mathbf{x} \to A \to \mathbf{y} \to A^{-1} \to \mathbf{x}$

$\mathbf{x} = A^{-1}(A\mathbf{x})$

$\mathbf{x}$ goes to A and the result then to A^{-1}

$A^{-1}A = I$

$A^{-1}\mathbf{x} = A^{-1} \underbrace{AA^{-1}} \mathbf{x}$

Take A^{-1} (because we can) And feed it to A and then to A^{-1}

$AA^{-1} = I$

Functions as Transforms

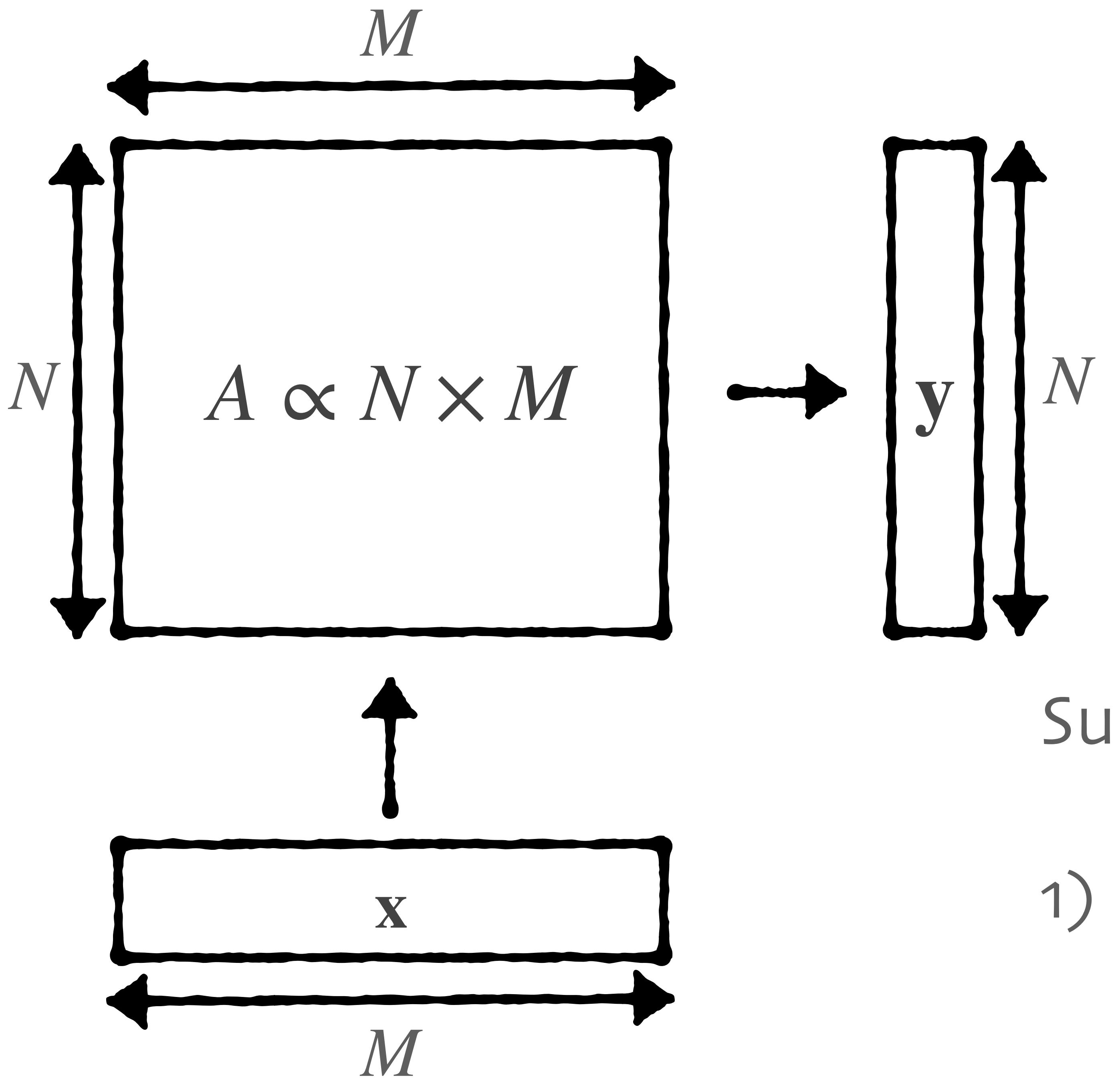
How projections work

What's the point?

The point is: NOT ALL THE MATRICES HAVE INVERSE!

Matrix A has A^{-1} if it works like case #3!

Otherwise — NO INVERSE (as we cannot resolve the conflicts)

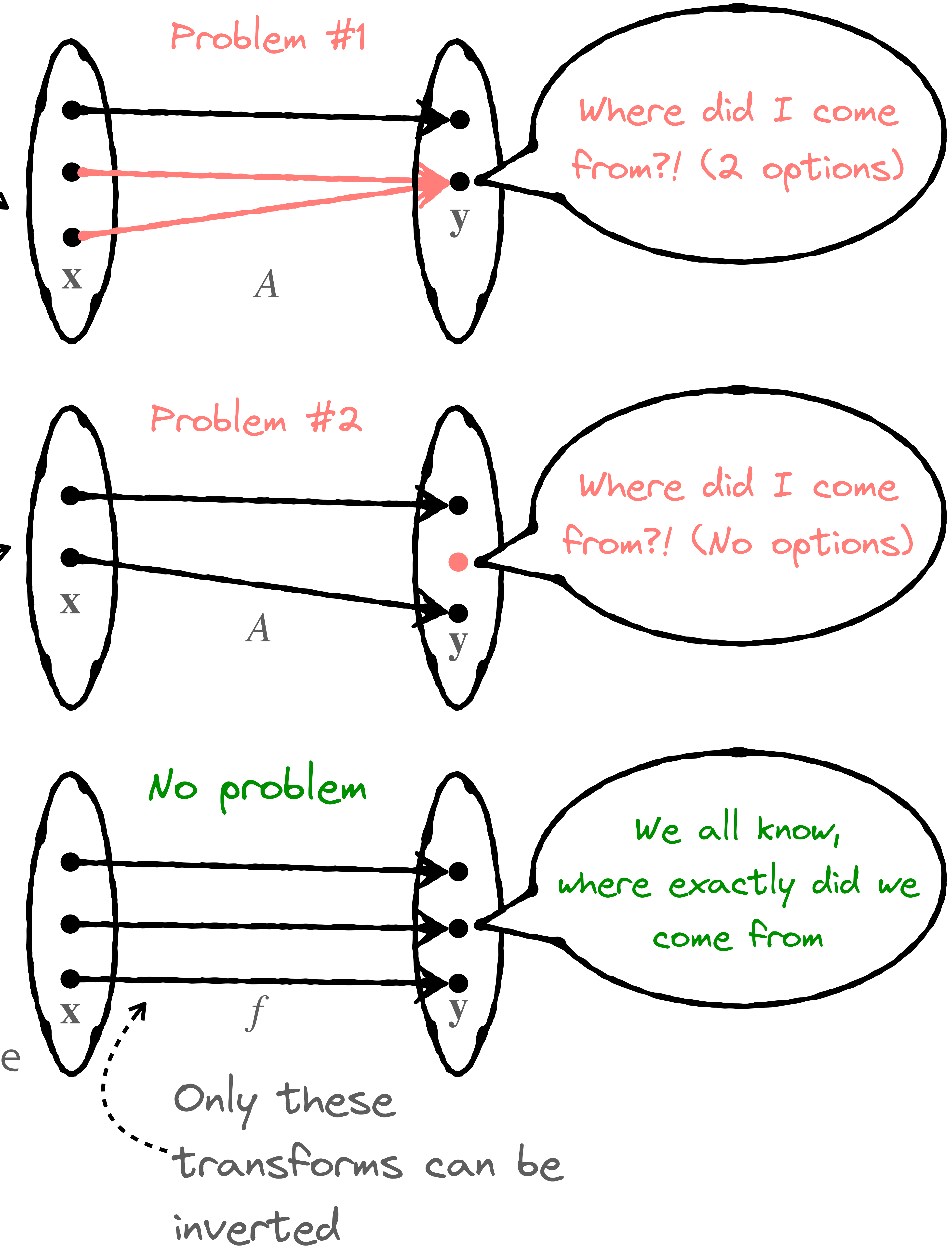


Suppose $A \propto N \times M$

1) If $M \neq N$
    1) If $M > N$, problem #1 is inevitable
    2) If $M < N$, now problem #2 is inevitable

2) Only when $M = N$, A^{-1} exists

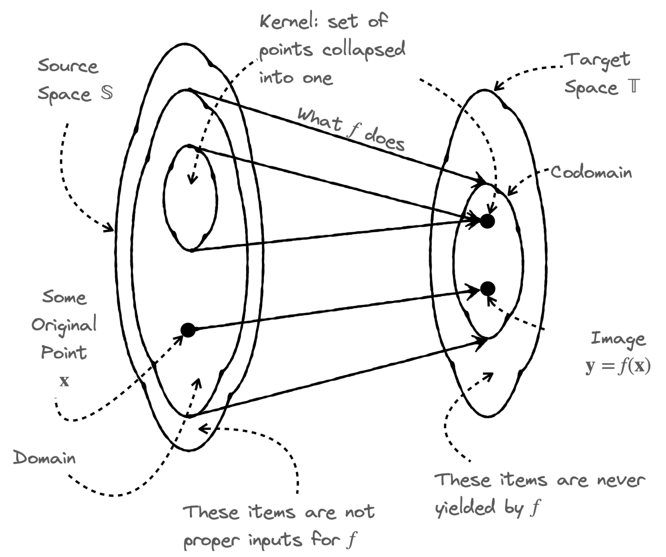
3) But not always!!
    Not always … :'(



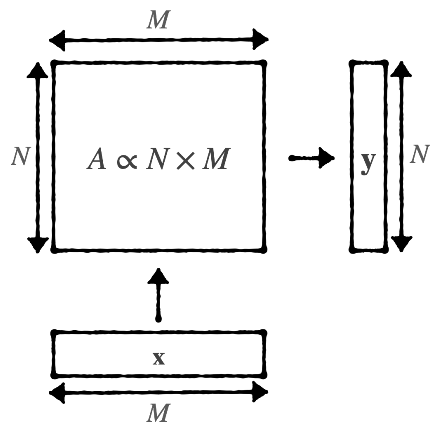
Takeaways

- Matrix as table of numbers
  - Table of values
    $$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{M1} & a_{M2} & \dots & a_{MN} \end{bmatrix} \qquad A = [a_{kl}] \qquad A \in \mathbb{R}^{N \times M}$$
  - List of column vectors
    $$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_N \end{bmatrix}$$
  - List of row vectors
    $$A = \begin{bmatrix} - & \mathbf{a}_1 & - \\ - & \mathbf{a}_2 & - \\ & \vdots & \\ - & \mathbf{a}_M & - \end{bmatrix}$$
- Elementwise operations
  - Unary
    $$f(A) = \begin{bmatrix} f(a_{11}) & \dots & f(a_{1N}) \\ \vdots & \ddots & \vdots \\ f(a_{M1}) & \dots & f(a_{MN}) \end{bmatrix}$$
  - Binary
    $$A \odot B = \begin{bmatrix} a_{11} \odot b_{11} & \dots & a_{1N} \odot b_{1N} \\ \vdots & \ddots & \vdots \\ a_{M1} \odot b_{M1} & \dots & a_{MN} \odot b_{MN} \end{bmatrix}$$

- Matrix Transposition
  $$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ \vdots & \vdots & & \vdots \\ a_{M1} & a_{M2} & \dots & a_{MN} \end{bmatrix}$$
  Transposition operation
- Matrix product
  $$AB = \begin{bmatrix} - & \mathbf{a}_1 & - \\ - & \mathbf{a}_2 & - \\ & \vdots & \\ - & \mathbf{a}_N & - \end{bmatrix} \begin{bmatrix} | & | & & | \\ \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_M \\ | & | & & | \end{bmatrix} = \begin{bmatrix} \langle \mathbf{a}_1, \mathbf{b}_1 \rangle & \langle \mathbf{a}_1, \mathbf{b}_2 \rangle & \dots & \langle \mathbf{a}_1, \mathbf{b}_M \rangle \\ \langle \mathbf{a}_2, \mathbf{b}_1 \rangle & \langle \mathbf{a}_2, \mathbf{b}_2 \rangle & \dots & \langle \mathbf{a}_2, \mathbf{b}_M \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \mathbf{a}_N, \mathbf{b}_1 \rangle & \langle \mathbf{a}_N, \mathbf{b}_2 \rangle & \dots & \langle \mathbf{a}_N, \mathbf{b}_M \rangle \end{bmatrix}$$
- Identity matrix
  $$I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$
- Inverse Matrix $A^{-1}A = AA^{-1} = I$
- Transforms



- Matrix as transform



- Invertible Transforms