

1 Ordinary points and singular points

We are considering general second order linear homogeneous equations

$$P(x)y'' + Q(x)y' + R(x)y = 0, \quad (1)$$

and looking for series solutions

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

We assume that P, Q and R all have power series expansions around x_0 . (In most cases that we will consider, P, Q and R are polynomials. Any polynomial has a Taylor series around any point, and since eventually the derivatives of the function are zero, the Taylor series is a finite sum. This is the expansion we are referring to.

Example: $P(x) = x^2 + x + 1$. Find the Taylor series around $x = 2$.

Method 1

$$\begin{aligned} P'(x) &= 2x + 1 \\ P''(x) &= 2 \\ P^{(j)}(x) &= 0 \text{ for } j > 1. \end{aligned}$$

Hence,

$$\begin{aligned} P(2) &= 7 \\ P'(2) &= 5 \\ P''(2) &= 2 \end{aligned}$$

so

$$P(x) = 7 + 5(x - 2) + \frac{2}{2!}(x - 2)^2 + 0x^3 + \dots$$

with all the remaining terms 0.

Method 2

$$\begin{aligned}
 x^2 + x + 1 &= \{(x-2)^2 + 4x - 4\} + \{(x-2) + 2\} + 1 \\
 &= (x-2)^2 + \{4(x-2) + 8\} - 4 + (x-2) + 3 \\
 &= (x-2)^2 + 5(x-2) + 7
 \end{aligned}$$

Many other ways of algebraic manipulation could be used. The Taylor series is the most straightforward, but not necessarily the quickest.

Definition 1 Assume that $P, Q,$ and R have no common factor. Then a point x_0 is a “ordinary point” for (1) if $P(x_0) \neq 0$. Otherwise x_0 is a “singular” point for (1).

There are several examples given in the text. Be sure to reach Theorem 5.3.1. It was illustrated at the end of the last section of notes.

2 Hermite's equation

Hermite's equation (problem 21, pg. 260) is

$$y'' - 2xy' + \lambda y = 0,$$

where y is a function of x . Notice that there are no singular points. We can find a series solution by the usual way:

$$\begin{aligned}
 y &= \sum_{n=0}^{\infty} a_n x^n \\
 y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
 y'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.
 \end{aligned}$$

Substituting this into Hermite's equation gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 2 \sum_{n=1}^{\infty} n a_n x^n + \lambda \sum_{n=0}^{\infty} a_n x^n = 0$$

Rearranging the indices, we let $m = n - 2$ in the first sum and then change m back to n , to get

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - 2\sum_{n=1}^{\infty} na_nx^n + \lambda\sum_{n=0}^{\infty} a_nx^n = 0$$

The coefficient of x^0 then gives us

$$\begin{aligned} 2a_2 + \lambda a_0 &= 0 \\ a_2 &= -\frac{\lambda a_0}{2} \end{aligned}$$

and the coefficient of x^n for $n > 0$ gives

$$a_{n+2} = -\frac{(\lambda - 2n)}{(n+2)(n+1)}a_n$$

Hence, the even terms are

$$\begin{aligned} a_4 &= -\frac{\lambda - 4}{(4)(3)}a_2 = \frac{\lambda(\lambda - 4)}{4!}a_0 \\ a_6 &= -\frac{8 - \lambda}{(6)(5)}a_4 = -\frac{\lambda(\lambda - 4)(\lambda - 8)}{6!}a_0 \end{aligned}$$

and so forth, while the odd numbered terms are

$$\begin{aligned} a_3 &= -\frac{\lambda - 2}{3!}a_1 \\ a_5 &= \frac{(\lambda - 6)(\lambda - 2)}{5!}a_1 \\ a_7 &= -\frac{(\lambda - 10)(\lambda - 6)(\lambda - 2)}{7!}a_1 \end{aligned}$$

and so forth. We get two linearly independent solutions by first setting $a_0 = 1$, $a_1 = 0$, and then setting $a_0 = 0$, $a_1 = 1$. Notice that $y(0) = a_0$ and $y'(0) = a_1$. The solutions are

$$y_1(x) = 1 - \frac{\lambda}{2!}x^2 + \frac{\lambda(\lambda - 4)}{4!}x^4 - \frac{\lambda(\lambda - 4)(\lambda - 8)}{6!}x^6 + \dots$$

and

$$y_2(x) = x - \frac{\lambda - 2}{3!}x^3 + \frac{(\lambda - 2)(\lambda - 6)}{5!}x^5 - \frac{(\lambda - 2)(\lambda - 6)(\lambda - 10)}{7!}x^7 + \dots$$

Since Hermite's equation has no singular points, Theorem 5.3.1 tells us that the series converges for all x . But the most interesting cases are for $\lambda = 2, 4, 6, 8$, etc., that is, λ any even positive integer. In these cases we can see that one or the other of these functions is not an infinite sum, but only a finite sum.

For example, if $\lambda = 10$, then

$$y_2(x) = x - \frac{8}{3!}x^3 + \frac{32}{5!}x^5$$

and all the later terms are zero. The functions we get then are the so-called "Hermite polynomials", or multiples thereof. For a reason we will give below, the Hermite polynomial H_n , where the highest order term is a multiple of x^n , is chosen so the coefficient of x^n is 2^n . For example, H_5 is supposed to start off with $32x^5$, so we get

$$H_5(x) = 32x^5 - \frac{8}{3!}5!x^3 + 5!x = 32x^5 - 160x^3 + 120x.$$

The first four H_n are (as given in the text), pg. 260)

$$\begin{aligned} H_0(x) &= 1 \\ H_1(x) &= 2x \\ H_2(x) &= 4x^2 - 2 \\ H_3(x) &= 8x^3 - 12x \\ H_4(x) &= 16x^4 - 48x^2 + 12 \end{aligned}$$

I will now list some of the properties of the Hermite polynomials. We will not have time to prove these in general, but I will try to give some examples. You can find these properties on Wikipedia, or on "MathWorld", another good math website.

1. They satisfy a "recurrence relation"

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

This is a quick way of figuring them out.

2. They have a "generating function", obtained from the power series for e^x . The relevant formula is

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n.$$

This is seen from

$$\begin{aligned} e^{2xt-t^2} &= 1 + \frac{1}{1!} (2xt - t^2) + \frac{1}{2!} (2xt - t^2)^2 + \dots \\ &= 1 + 2xt - t^2 + \frac{1}{2!} (4x^2t^2 + t^4 - 4xt^3) + \dots \\ &= 1 + \frac{2x}{1!}t + \frac{4x^2 - 2}{2!}t^2 + \dots \end{aligned}$$

Notice that the first three terms agree with what we got earlier, and this formula gives a reason for choosing H_n as we did, with 2^n as the leading coefficient.

3. They satisfy an “orthogonality relation”:

$$\int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-x^2} dx = 0.$$

Recall from math 1180 that polynomials can be orthogonal with respect to an inner product. Here the inner product is weighted by the factor e^{-x^2} , which is necessary for the indefinite integrals to converge.

Many more properties are listed on the websites.

The Hermite polynomials are examples of what are called “special functions”, which are functions, usually defined by either polynomials or infinite series, which are important in physics and other applications. Doing a Google search for "Hermite polynomials" yielded about 117,000 links.

We shall run into several more special functions in this chapter. The next comes from:

3 Chebyshev’s equation

Chebyshev’s equation (problem 10, pg. 265) is

$$(1 - x^2) y'' - xy' + \alpha^2 y = 0,$$

where y is a function of x . Now there are singular points at $x = \pm 1$. However $x_0 = 0$ is an ordinary point.

We can find these solutions in the usual way:

$$y = \sum_{n=0}^{\infty} a_n x^n, \text{ etc. for the derivatives.}$$

Substituting this into Chebyshev's equation results eventually in

$$\begin{aligned} 2a_2 &= -\alpha^2 a_0 \\ a_2 &= -\frac{\alpha^2 a_0}{2} \end{aligned}$$

The coefficient of x^1 gives

$$\begin{aligned} 6a_3 + (\alpha^2 - 1)a_1 &= 0 \\ a_3 &= \frac{1 - \alpha^2}{3!} a_1. \end{aligned}$$

and the coefficient of x^n for $n > 0$ gives

$$a_{n+2} = \frac{n(n-1) + n - \alpha^2}{(n+2)(n+1)} a_n = \frac{n^2 - \alpha^2}{(n+2)(n+1)} a_n.$$

Hence, the even terms are

$$\begin{aligned} a_4 &= \frac{4 - \alpha^2}{(4)(3)} a_2 = -\frac{\alpha^2(4 - \alpha^2)}{4!} a_0 \\ a_6 &= \frac{16 - \alpha^2}{(6)(5)} a_4 = -\frac{(16 - \alpha^2)(4 - \alpha^2)\alpha^2}{6!} a_0 \end{aligned}$$

and so forth, while the odd numbered terms are

$$\begin{aligned} a_3 &= \frac{1 - \alpha^2}{3!} a_1 \\ a_5 &= \frac{(3^2 - \alpha^2)(1^2 - \alpha^2)}{5!} a_1 \\ a_7 &= \frac{(5^2 - \alpha^2)(3^2 - \alpha^2)(1^2 - \alpha^2)}{7!} a_1 \end{aligned}$$

and so forth. We get two linearly independent solutions as usual:

$$y_1(x) = 1 - \frac{\alpha^2}{2!} x^2 - \frac{\alpha^2(4 - \alpha^2)}{2!} x^2 - \frac{\alpha^2(4 - \alpha^2)(16 - \alpha^2)}{4!} x^4 - \dots$$

and

$$y_2(x) = x + \frac{1 - \alpha^2}{3!}x^3 + \frac{(3^2 - \alpha^2)(1^2 - \alpha^2)}{5!}x^5 + \frac{(5^2 - \alpha^2)(3^2 - \alpha^2)(1^2 - \alpha^2)}{7!}x^7 + \dots$$

From Theorem 5.3.1 it follows that the radius of convergence in each case is at least 1, because the series for $\frac{1}{1-x^2}$ has radius of convergence equal to 1.

But we can also see that if α is an integer, then one of the series stops after a certain point, and we get a polynomial, as before, with Hermite's equation. In this case, there is no issue of convergence. We can say that $r = \infty$. This emphasizes that Theorem 5.3.1 gives a minimum value for r . It could be larger, as it is in this example if α is an integer.

The Chebyshev polynomials are denoted by $T_n(x)$. (Perhaps someone who has studied Russian can explain why.) They are defined to be the polynomial solution of Chebyshev's equation, with $\alpha = n$, normalized so that $T_n(1) = 1$. They give another class of special functions, and have the same sorts of properties as the Hermite polynomials:

1. There is a "generating function", with

$$\frac{1 - t^2}{1 - 2xt + t^2} = T_0(x) + \sum_{n=1}^{\infty} 2T_n(x)t^n$$

2. They have an orthogonality property

$$\int_{-1}^1 T_n(x) T_m(x) \frac{1}{\sqrt{1-x^2}} dx = 0 \text{ if } n \neq m.$$

Note the weight function.

3. There is a recurrence relation:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

4. Here is a direct formula for $T_n(x)$:

$$T_n(x) = \frac{1}{2} \left[\left(x + \sqrt{x^2 - 1} \right)^n + \left(x - \sqrt{x^2 - 1} \right)^n \right]$$

This seems to involve complex numbers for $|x| < 1$, but the complex parts will cancel.

5. Here is another neat formula: If $x = \cos \theta$, then

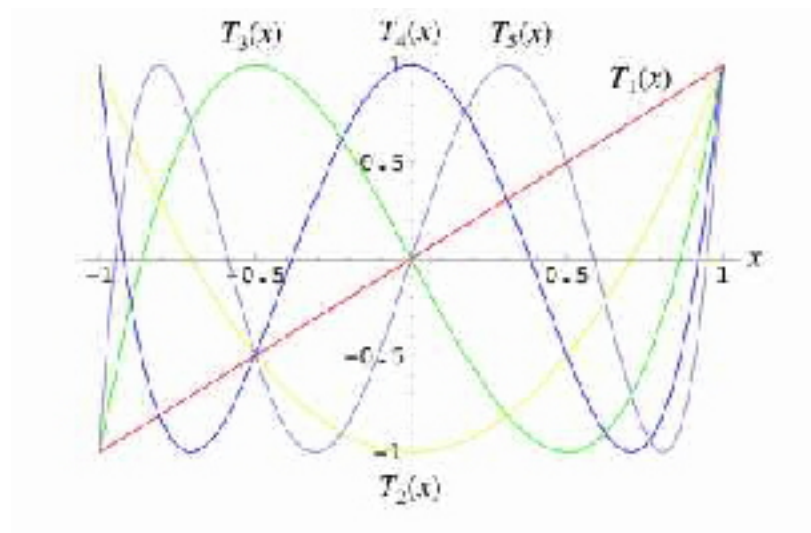
$$T_n(x) = \cos(n\theta).$$

In other words,

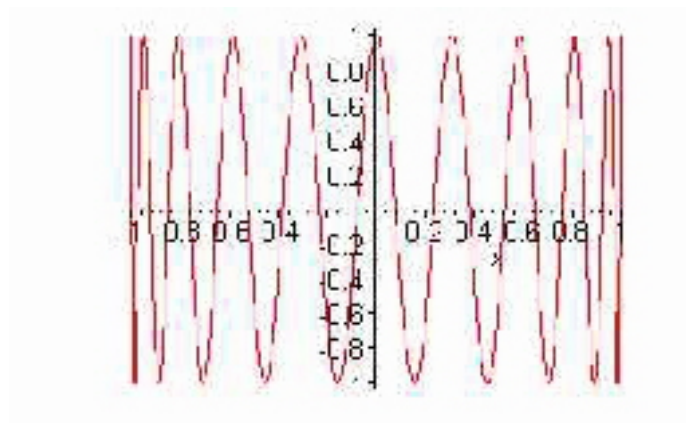
$$T_n(x) = \cos(n \arccos(x))$$

Notice that this formula cannot be used if $|x| > 1$, since then we can't have $x = \cos \theta$. (Or can we? It turns out that we can if we allow θ to be complex, but we are not considering that here.)

The Chebyshev polynomials turn out to be important in numerical analysis, because when they are used to approximate non-polynomial functions, they are efficient in giving a good approximation. We can see that there is something unusual about them from the two formulas involving trig functions. Their values lie in $[-1, 1]$ if $|x| \leq 1$. (“Obviously”, this cannot be true for all x . Why not?)



In this sense, they resemble trigonometric functions, over this interval, and yet they are polynomials, not infinite sums. Here is the plot of $T_{20}(x)$;



Another quite surprising property is that T_n has all of its maxima and minima in $[-1, 1]$, and the maximum and minimum values are ± 1 . It is perhaps surprising that any polynomial could have such a property. Plus, as property (2) above states, they are orthogonal with respect to a particular weight function.

These properties make the Chebyshev polynomials perhaps even more important than those of Hermite. A Google search for "Chebyshev polynomials" gives about 130,000 links.

4 Homework

Section 5.2, # 2, 7, 15

Section 5.3, # 6, 23 (you can use the result from 22).