4C8 - Tutorial 1 - Guide to Question 1

Image and Video Processing

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1. Let's consider for now the 1 dimensional case for a simple example. Recall that the 1 dimensional z transform of a sequence x_n is noted $X(z) = \mathcal{Z}\{x_n\}$ and defined as follows:

$$X(z) = \mathcal{Z}\{x_n\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Say the transfer function is expressed in the z domain as:

$$H(z) = 1 - z^{-3}$$

Y, H and x are related by $Y(z) = H(z)X(z)$. Your first step should be to substitute $H(z)$ by its expression and expand the complex polynomial:

$$Y(z) = H(z)X(z) = (1 - z^{-3})X(z) = X(z) - z^{-3}X(z)$$

We then need to apply the inverse z-transform. As the z-transform and its inverse are transforms, it can be applied independently on each of the polynomial terms:

$$\mathcal{Z}^{-1}\{Y(z)\} = \mathcal{Z}^{-1}\{X(z)\} - \mathcal{Z}^{-1}\{z^{-3}X(z)\}$$

We can then find the inverse z transform for each term using the fact that (see proof below):

$$z^{-k}X(z) = \mathcal{Z}\{x_{n-k}\}$$

We thus end up with the following difference equation:

$$y[n] = x[n] - x[n-3]$$

Proof that $z^{-k}X(z) = \mathcal{Z}\{x_{n-k}\}$:

$$\begin{aligned}
z^{-k}X(z) &= z^{-k}\mathcal{Z}\{x_n\} \\
&= z^{-k}\sum_{n=-\infty}^{\infty} x[n]z^{-n} \\
&= \sum_{n=-\infty}^{\infty} x[n]z^{-(n+k)} \\
&= \sum_{n=-\infty}^{\infty} x[n-k]z^{-n} \qquad \text{(changing } n \to n+k\text{)} \\
&= \mathcal{Z}\{x_{n-k}\}
\end{aligned}$$

IIR filters

For FIR (finite impulse response) filters, the transfer function is a simple polynomial, eg. $H(z) = 1 + z^{-1} + z$. For IIR (infinite impulse response) filters, the transfer function is a rational fraction, eg. $H(z) = \frac{1+z^{-1}+z}{1+z^{-3}}$.

To solve the difference equations for IIR filters, we start by substituting the expression for $H(z)$ and then multiplying both sides of the equation by the denominator:

$$Y(z) = H(z)X(z) = \frac{P(z)}{Q(z)}X(z)$$

$$Y(z)Q(z) = P(z)X(z)$$

For instance, if $H(z) = \frac{0.1}{1-0.9z^{-1}}$, we then have:

$$\begin{aligned} Y(z) = H(z)X(z) &= \frac{0.1}{1-0.9z^{-1}}X(z) \\ Y(z)(1-0.9z^{-1}) &= 0.1X(z) \\ Y(z) - 0.9z^{-1}Y(z) &= 0.1X(z) \\ \Rightarrow \quad y[n] - 0.9y[n-1] &= 0.1x[n] \\ y[n] &= 0.1x[n] + 0.9y[n-1] \end{aligned}$$

We can see why this is called a infinite impulse response filter. If the input is a pulse followed by zeros $x_n = [1, 0, 0, \cdots]$, then the output response will be of the form: $y[n] = 0.1 \times 0.9^n$: a decaying but never ending response.

Coming back to the original questions for 2D transfer functions and only covering example 1:

$$H_1(z_1, z_2) = \frac{1}{1 - 0.99z_1^{-1} - 0.9z_2^{-1} + 0.891z_1^{-1}z_2^{-1}}$$

The same arguments as for 1D give us the following 2D difference equations:

$$y[h, k] = 0.99y[h-1, k] + 0.9y[h, k-1] - 0.891y[h-1, k-1] + x[h, k]$$

This is a IIR filter (rational fraction leads to a difference equation requiring other output values).

It is causal because we only require sample values that are in the past, for both directions.

It is also separable because we can factorise the transfer function as follows:

$$H_1(z_1, z_2) = A(z_1)B(z_2) = \frac{1}{1-0.99z_1^{-1}}\frac{1}{1-0.9z_2^{-1}}$$

The other examples are left for exercise.