Numerical Methods I

Assignment Sheet 6. Due: March 18, 2020

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Exercise 26 [3 x 5 Points]: Consider the measurement values $p_0 = 0$, $p_1 = 2$, and $p_2 = 1$ that have been obtained at the nodes $u_0 = 0$, $u_1 = 1$, and $u_2 = 2$. Assume that a linear function $p(u) = au + b$ is supposed to approximate these values in the least squares sense.

a) Formulate the normal equations as linear system of equations.

b) Solve the normal equations.

c) Compute the error in the ℓ_2 sense that is minimized in b).

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Exercise 27 [3 x 5 Points]:

a) Consider the backward differencing scheme for computing the first order derivative of a function f. Apply the idea of Richardson extrapolation to derive an estimate with quadratic error term in the step-size h. Compute the exact (not just asymptotic) error.

b) Apply Richardson extrapolation to the estimate from a) in order to derive an estimate with even better asymptotic error. Compute the exact (not just asymptotic) error.

c) Apply the estimates from a) and b) and the backward differencing scheme itself to compute $f'(0)$ for a function $f(x) = 100x^2 + 1$ and step size $h = 0.2$. For which of the three estimates can we assume that the error is zero? Why?

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Exercise 28 [not graded, w/o Points]: We want to evaluate the integral

$$\int_0^1 2x^2 - 4x + 8\,dx$$

numerically with an error less than $\frac{1}{12}$.

a) How many intervals n do we need to satisfy this error bound when using the composite trapezoid rule?

b) Apply the composite trapezoid rule with the number of intervals that you found in a) to obtain an approximation for the value of the integral.

c) Apply the recursive trapezoid rule to obtain the integral iteratively until the difference between two consecutive estimates is below the given error threshold. How does your computation compare to what you did in b)?

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Exercise 29 [not graded, w/o Points]: The composite Simpson rule for computing a definite integral $I = \int_0^1 f(x)\,dx$ is given by

$$I \approx \frac{h}{3}\left(f(0) + f(1) + 4\sum_{i=1}^{\frac{n}{2}} f\big((2i-1)h\big) + 2\sum_{i=1}^{\frac{n}{2}-1} f\big(2ih\big)\right)$$

for an even number of intervals n and a step size $h = \frac{1}{n}$. Its asymptotic error is $\mathcal{O}(h^4)$.

a) Derive a recursive scheme from the composite Simpson rule assuming that n is a power of 2.

b) Use Richardson extrapolation to the recursive scheme from a) in order to derive an estimate that has an improved asymptotic error.

c) Apply your estimates from a) and b) to compute the integral $\int_0^1 x^2\,dx$ for $n = 4$ intervals.

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Exercise 30 [not graded, w/o Points]: Again, consider the integral $I = \int_0^1 f(x), dx$.

a) Assuming the normalization $q(1) = 1$ for the polynomial q of the Gaussian quadrature theorem, derive the two Gaussian nodes x_0 and x_1 for Gaussian quadrature of I.

b) Derive the optimal weights A_i for the Gaussian nodes from a) according to the Gaussian quadrature scheme.

c) For which polynomials f is the Gaussian quadrature scheme with nodes from a) and weights from b) exact? *Note that the maximal degree of these polynomials is called* degree of exactness *of the quadrature scheme.*

d) Apply the quadrature scheme from a) and b) plus an interval transformation (substitution) in order to compute the integral $\int_0^4 3x^3 + x + 3\,dx$.