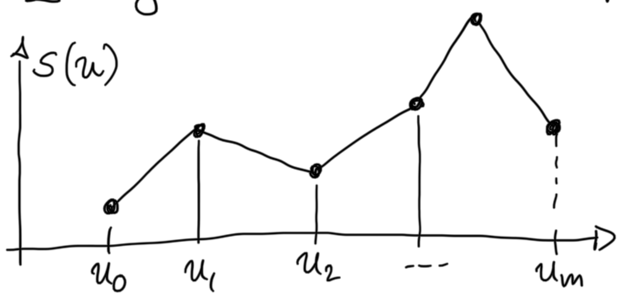
4.6 Spline interpolation:

Def. 38: A function $s(u)$ is called a spline of degree k on the domain $[a,b]$ if

- $s \in C^{k-1}([a,b])$, i.e. $k-1$-times cont. differentiable
- there exist nodes $a = u_0 < \dots < u_m = b$ such that $s|_{[u_i, u_{i+1}]}$ is a polynomial of degree k for $i = 0, \dots, m-1$.

Spline of degree 1: $s \in C^0$, $s|_{[u_i, u_{i+1}]}$ linear



Def 39: A spline in B-spline representation is of the form

$$s(u) = \sum_{i=0}^{m} c_i N_i^n(u)$$

where the $N_i^n(u)$ are the basis spline functions of degree n with minimal support.

We define the basis splines (B-splines) $N_i^n(u)$ by a recursion: Let u_i be set of nodes

$$u_0 < u_1 < \dots u_i < u_{i+1} < \dots$$

Then:

$$N_i^0(u) = \begin{cases} 1 & \text{for } u_i \le u < u_{i+1} \\ 0 & \text{else} \end{cases}$$

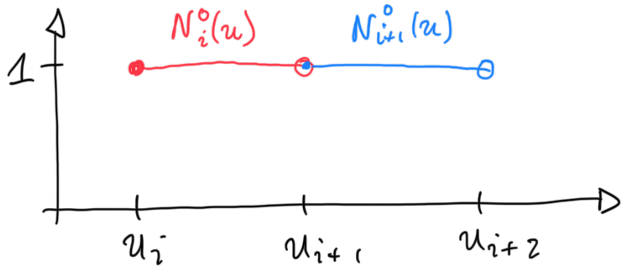
and further

$$N_i^n(u) = \alpha_i^{n-1}(u)\, N_i^{n-1}(u) + \left(1 - \alpha_{i+1}^{n-1}(u)\right) N_{i+1}^{n-1}(u)$$

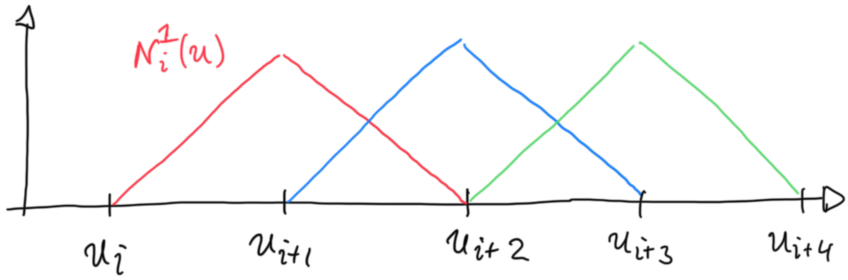
where $\alpha_i^{n-1}(u) = \dfrac{u - u_i}{u_{i+n} - u_i}$ is a local parameter

Example 40:

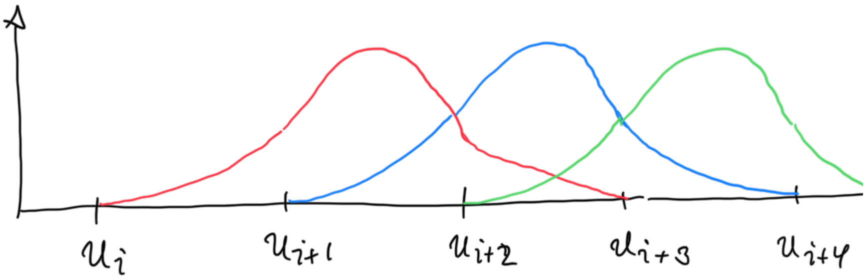
- $n = 0$:



- $n = 1$:



- $n = 2$:



We find that:

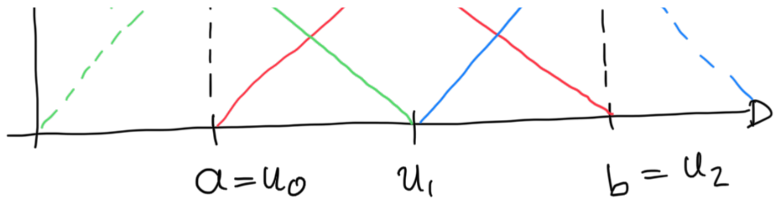
(1) $\operatorname{supp} N_i^n = [u_i, u_{i+n+1}]$

(2) N_i^n is a piecewise poly of degree $n-1$

(3) N_i^n is positive on (u_i, u_{i+n+1}), zero outsi

(4) N_i^n have a partition of unity property,

i.e. $\displaystyle\sum_{i=-n}^{m} N_i^n(u) = 1$



Now, let us do interpolation with splines. Given nodes $u_0 < \dots < u_m$ and values $p_0, \dots, p_m$. Find

$$s(u) = \sum_i c_i N_i^n(u)$$

such that $s(u_i) = p_i$ for $i = 0, \dots, m$.

Note that degree n and number m can be independent.

Next step: collocation matrix

$$\Phi = \begin{bmatrix} N_0^n(u_0) & N_1^n(u_0) & \cdots & N_m^n(u_0) \\ N_0^n(u_1) & \cdots & & N_m^n(u_1) \\ \vdots & & & \\ N_0^n(u_m) & \cdots & & N_m^n(u_m) \end{bmatrix} \in \mathbb{R}^{(m+1)\times(m}$$

and we need to solve $\Phi \vec{c} = \vec{p}$, $\vec{c} = (c_0, \dots, c_m) \in \mathbb{R}^{m+}$ $\vec{p} \in \mathbb{R}^{m+1}$.

Example 41: Cubic spline interpolation, $n=3$.

- Assume equidistant nodes $u_i = i+2$