Jacobs University Bremen
Prof. Danilov

Spring Semester 2019

Numerical Methods I

A brief summary of problems for Final Exam on March 20, 2019

Problem 1
Solution of nonlinear equations. Bisection method.

(a) What are the requirements on function $f(x)$ allowing one to use the bisection method to find a root of $f(x)$?

(b) How to compute accuracy of this method. Why we say that it converges linearly? How to determine the number of iterations needed to reach a specified accuracy?

(c) Take, e.g., $f(u) = u^2 - 6$. Execute 2 steps of the bisetion method on the interval $[0, 3]$. Step 1: $a = 0$, $b = 3$; $f(a) = -6$, $f(b) = 3$, signs are opposite, we can use the method. Take $c = (a + b)/2 = 3/2$. Step 2: $f(c) = -3/2$, $f(a)f(c) > 0$, hence we have to use another interval. We redefine $a = c = 3/2$, leave b and compute new $c = (a+b)/2 = (3/2+3)/2 = 9/4$. Return c. (Continue if better accuracy is needed).

Problem 2
Consider the following interpolation problem with points p_i at knots u_i

i	0	1	2
u_i	0	3	10
p_i	1	4	15

Consider the Newton polynomials as basis polynomials.

(a) How to compute the collocation matrix for the given interpolation problem? How will it look for the Lagrange polynomials?

(b) How to determine the coefficients of interpolation (nodes) if we have the collocation matrix?

(c) In the case above, a polynomial $q_2(u)$ passing through points p_0 and p_1 is $q_2(u) = c_0P_0(u)+c_1P_1(u)$ where $P_0 = 1$ and $P_1 = (u-u_0)$ are the Newton polynomials. (We sometimes denote c_0 as $p[u_0]$ and c_1 as $p[u_0u_1]$). We require that $c_0 \cdot 1 = p_0$, $c_0 \cdot 1+c_1(u_1 - u_0) = p_1$, and use them to find coefficients. These equations will also follow from consideration of the collocation matrix. If we are willing to compute $q_3(u)$ passing through all three points, $q_3(u) = c_0P_0(u) + c_1P_1(u) + c_2P_2(u)$. We will only need to add one more equation to solve for c_2: $c_0 \cdot 1 + c_1(u_2 - u_0) + c_2(u_2 - u_0)(u_2 - u_1) = p_2$.

Problem 3

(a) How to derive forward, backward and centered differencing formulas? How to estimate error terms using Taylor's theorem?.

(b) How to apply Richardson's extrapolation to forward, backward and centered differencing schemes? How to determine their errors? (Use Taylor's theorem). Note that applying Richardson's extrapolation to forward and backward scheme will make them second order accurate.

Problem 4

On interval $[a, b]$ consider the integral $I = \int_b^a f(x)\, dx$ and a partition P with $n = 2^m$ subintervals $[x_i, x_{i+1}]$, $i = 0, \ldots, n-1$ of length $h = (b-a)/2^m$. The composite trapezoid rule (the name of composite trapezoid rule is used to indicate that trapezoid rule is applied at each subinterval of $[a, b]$) to approximate the integral for partition with $n = 2^m$ gives

$$T_m(f) = \frac{h}{2}(f(a) + f(b)) + h \sum_{i=1}^{2^m-1} f(x_i).$$

(a) What is the error term in the case of trapezoid rule?

(b) How to derive a recursive scheme of the composite trapezoidal rule connecting T_{m+1} with T_m. What will be $T_0(f; P)$ in this case?

(c) Take, e.g., $[a, b] = [0, 3]$ and $f(x) = x^3 + x$ compute T_2 and compare it with the exact answer.