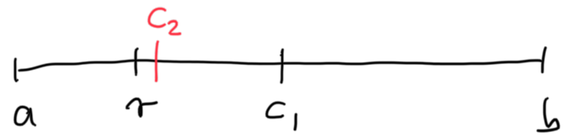
Bisection method:

Producing a sequence of intervals $[a_i, b_i]$ such that the root r is inside of these interval.
Given is a starting interval $[a,b] = [a_0, b_0]$.

Theorem 22: The bisection method, when applied to an interval $[a,b]$ and $f \in C^0([a,b])$ with $f(a)\,f(b) < 0$ after n steps has computed an approximation c_n of the root r with an error

$$|r - c_n| < \frac{b-a}{2^n}.$$

Proof:



we have

$$|r - c_1| < |a - c_1| = \frac{b-a}{2}$$

and then

$$|r - c_2| < |a - c_2| = \frac{b-a}{4}$$

so iteratively, we get

$$|r - c_n| < \frac{b-a}{2^n} \qquad \blacksquare$$

Example 23: Let $[a,b] = [0,1]$. How many iterations are needed to decrease the error below 2

Find $n \in \mathbb{N}$ such that

$$\frac{b-a}{2^n} < 2^{-20}$$

$$\Leftrightarrow \frac{b-a}{2^{-20}} < 2^n$$

$$\Rightarrow \log_2\left(\frac{b-a}{2^{-20}}\right) < n$$

because $\log_2$ strictly increasing

Here: $\log_2\left(\frac{1}{2^{-20}}\right) < n$

$$\Leftrightarrow \log_2(1) - \log_2(2^{-20}) < n$$

$$\Leftrightarrow \boxed{n > 20}$$

This is slow convergence: For every iteration step we get one binary digit as accuracy increase. Recall, for the approximation of cos(x) by its Taylor poly, we got 2 decimal digits per term of the poly.

<u>Def 24</u>: Suppose the sequence x_n converges to r as $n \to \infty$. Then, the sequence

(a) <u>converges linearly</u> to r, if $\exists M \in (0,1)$ such that

$$\lim_{n\to\infty} \frac{|x_{n+1} - r|}{|x_n - r|} = M$$

For bisection method, $M = 1/2$

(b) <u>converges super-linearly</u> to r if

$$\lim_{n\to\infty} \frac{|x_{n+1} - r|}{|x_n - r|} = 0$$

(c) <u>converges sub-linearly</u> to r if

$$\lim_{n\to\infty} \frac{|x_{n+1} - r|}{|x_n - r|} = 1$$

(d) converges with order $q > 1$ if $\exists M > 0$ s.th

$$\lim_{n\to\infty} \frac{|x_{n+1} - r|}{|x_n - r|^q} < M$$

if $q = 2$ we call the convergence quadratic
$q = 3$ ——— " ——— cubic

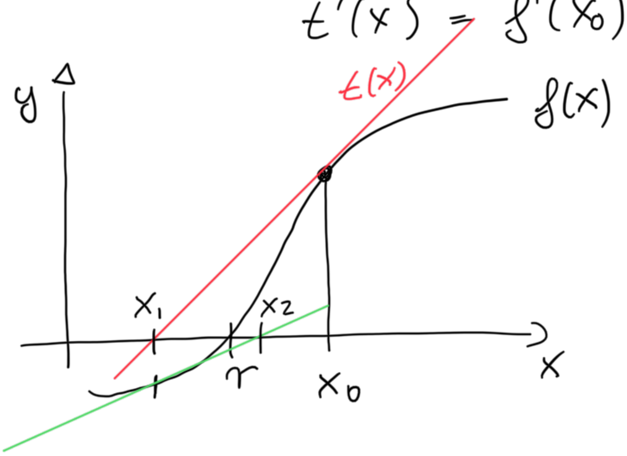
3.2 Newton's method:

Let $f \in C^1([a,b])$, then at every $x_0 \in (a,b)$ there exists a tangent to the graph of f:

tangent: $t(x) = f(x_0) + f'(x_0)(x - x_0)$

the tangent at x_0 is the first order Taylor poly

obviously: $t(x_0) = f(x_0)$
$t'(x) = f'(x_0)$



Use $t(x)$ as an approximation to f whose root can be easily found:

$0 = t(x_1) = f(x_0) + f'(x_0)(x_1 - x_0)$

$\iff \boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$

Iterating this yields the Newton sequence (Newton–Raphson iteration)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad x_0 = \text{initial guess}$$

Theorem 25: (cf. Calculus I/II notes)

When the Newton sequence converges, it converges to a root of f.

Theorem 26: (cf. Calculus I/II)

Let $f \in C^1([a,b])$ fulfill:

(1) $f(a)f(b) < 0$

(2) f has no critical pt in (a,b)

(3) f'' exists, is continuous and either $f'' > 0$ OR $f'' < 0$ in whole (a,b)

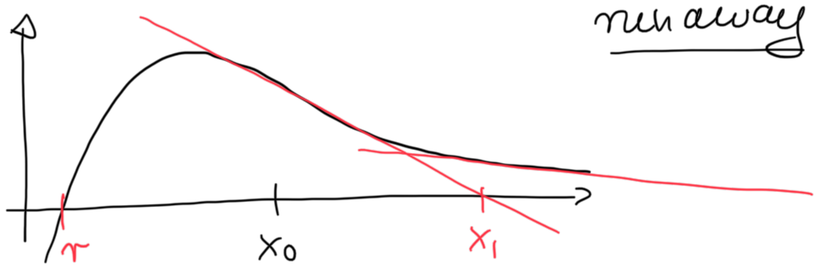
Then $f(x) = 0$ has exactly one solution r. The Newton sequence always converges to r as $n \to \infty$, when the initial guess is chosen accord to:

- if $f(a) < 0$, $f'' < 0$ OR $f(a) > 0$, $f'' > 0$ then $x_0 \in [a, r]$ e.g. $x_0 = a$.
- if $f(a) < 0$, $f'' > 0$ OR $f(a) > 0$, $f'' < 0$ then $x_0 \in [r, b]$, e.g. $x_0 = b$

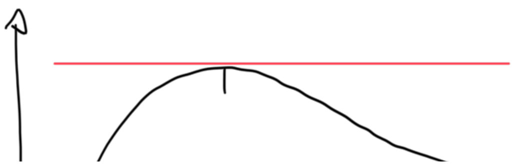
In any case we have the estimate

$$|x_n - r| < \frac{f(x_n)}{\min_{[a,b]} |f'(x)|}$$

Problems / violations of assumptions:

(1) 

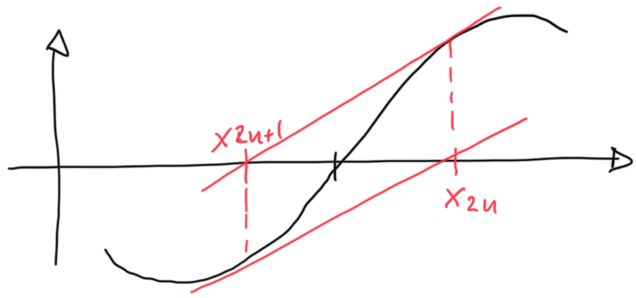
runaway

(2) 

flat point

cycle

(3)



Theorem 27: Convergence of Newton's method

Let $f:[a,b]\to\mathbb{R}$ fulfill

(1) $f(a)f(b)<0$

(2) f has no critical pt in (a,b)

(3) f'' exists and is continuous

and x_0 needs to be "close enough" to the true roo[t] r. Then, the Newton sequence converges quadratically.

Proof: To show:

$$|x_{n+1}-r| \leq M\,|x_n-r|^2 \quad \text{as } n\to \infty$$

By Taylor series approximation:

$$f(x)=f(x_n)+f'(x_n)(x-x_n)+\frac{1}{2}f''(\xi_n)(x-x_n)^2$$

Since r is a root we have

$$0=f(r)=f(x_n)+f'(x_n)(r-x_n)+\frac{f''(\xi_n)}{2}(r-x_n)^2$$

rearranging terms:

$$\underbrace{\frac{f(x_n)}{f'(x_n)}+(r-x_n)}_{-x_{n+1}}=-\frac{f''(\xi_n)}{2f'(x_n)}(r-x_n)^2$$

$$\Leftrightarrow \quad (r - x_{n+1}) = -\frac{f''(\xi_n)}{2f'(x_n)} (r - x_n)^2$$

$$\Rightarrow \quad |r - x_{n+1}| = \frac{f''(\xi_n)}{2f'(x_n)} |r - x_n|^2$$

let $M := \sup_{x_n, \xi_n} \frac{f''(\xi_n)}{2f'(x_n)}$ then

$$|r - x_{n+1}| \leq M \, |r - x_n|^2$$

which is what we had to show.

3.3 Secant method:

When the conditions for Newton's method are not fulfilled but something more sophisticated than bisection shall be done, we approximate derivatives by secants / difference quotient.

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$