Theorem 20: Let $A \in \mathbb{R}^{n \times n}$ be invertible. Then there exists a decomposition $A = LU$ with L being lower triangular and U being upper triangular, $L, U \in \mathbb{R}^{n \times n}$. And we have

$$L = M_1^{-1} M_2^{-1} \cdots M_{n-1}^{-1}$$

where M_i is the matrix describing step i of the forward elimination in GE. And where U is upper triangular (echelon form) that results from

$$U = M_{n-1} \cdots M_2 M_1 A$$

Note that LU decomposition may also be done with pivoting, then it is called LUP decomposition.

Example 19: (cf. Example 13)

$$A = \begin{bmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{bmatrix}$$

yields:

$$U = \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

and

$$L = M_1^{-1} M_2^{-1} M_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ \frac{1}{2} & 3 & 1 & 0 \\ -1 & -\frac{1}{2} & 2 & 1 \end{bmatrix}$$

Now, for solving $Ax = b$ observe that

$$Ax = b \iff LUx = b$$

Do a substitution, let $y := Ux$. Then, we can solve for y the equation $Ly = b$. This is not complicated, because L is triangular. We do forward substitution (from top to bottom). We have found y and can then solve

$$Ux = y$$

for x. Again, this is easy and involves backward substitution.

2.4 Cholesky decomposition

Cholesky (1875–1918)

When $A \in \mathbb{R}^{n\times n}$, symmetric and pos. def (SPD)

> When $B \in \mathbb{R}^{n\times n}$ is pos. def. then
> (1) all e-values > 0
> (2) $v \cdot Bv > 0$ when $v \neq 0$
> $v \cdot Bv = 0$ iff $v = 0$
> etc.

> When B is symmetric then all e-values λ are real numbers. When v is e-vector then $\lambda v = Av$, then
> $$\langle \lambda v, v\rangle = \langle Av, v\rangle$$
> $$\lambda\langle v,v\rangle = \langle v, A^t v\rangle = \langle v, Av\rangle$$
> $$= \langle v, \lambda v\rangle = \bar{\lambda}\langle v, v\rangle$$
> so $\lambda = \bar{\lambda}$, i.e. e-values are real, $\in$

When A SPD then A can be decomposed as

$$A = LDL^t$$

where L is lower triangular, L has ones on the diagonal, and D is a diagonal matrix with positive entries

Because D has positive entries we can "take a square root", i.e we find $D^{1/2}$ such that $D = D^{1/2} D^{1/2}$

Then $A = LDL^t = (LD^{1/2})(D^{1/2}L^t)$

$$= \tilde{L}\tilde{L}^t$$

When such decomposition is available, solving $Ax = b$ means solving $\tilde{L}\tilde{L}^t x = b$ and this will be even simpler, since twice back-substitution of the same kind will be done. Also, just half of the entries need to be computed.

→ effort is still $O(n^3)$, but just half of the operations.

How can we compute $\tilde{L}$?

Assume that $\tilde{L} = \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & 0 \\ \vdots & & \ddots & 0 \\ l_{n1} & \cdots & \cdots & l_{nn} \end{bmatrix}$

and

$\tilde{L}^t$!

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ \vdots & \ddots & 0 & \vdots \\ \boxed{l_{n1}} & \cdots & \cdots & l_{nn} \end{bmatrix} \begin{bmatrix} l_{11} & \cdots & \cdots & \boxed{l_{n1}} \\ & \ddots & & \vdots \\ 0 & & & l_{nn} \end{bmatrix}$$

row i col j

Then:

$$\boxed{a_{ij} = \sum_{k=1}^{n} l_{ik} l_{\boxed{jk}}}$$

note that $\tilde{L}^t$ is used here!

of course, in this sum, several term will vanish.

Now, for $i = j$:

$$a_{ii} = \sum_{k=1}^{n} l_{ik} l_{ik} = \sum_{k=1}^{i} (l_{ik})^2$$

(terms $k = i+1, \ldots, n$ vanish)

$$\Leftrightarrow \boxed{l_{ii} = \sqrt{\underbrace{a_{ii} - \sum_{k=1}^{i-1} l_{ik}^2}_{\text{this is always } > 0 \text{ because } A \text{ pos def}}}}$$

Now, for $i > j$:

we get $$a_{ij} = \sum_{k=1}^{n} l_{ik} l_{jk}$$

$$= \sum_{k=1}^{j} l_{ik} l_{jk}$$

and so: $$\boxed{l_{ij} = \frac{1}{l_{jj}} \left(a_{ij} - \sum_{k=1}^{j-1} l_{ik} l_{jk} \right)}$$

Finally, $i < j$: $l_{ij} = 0$.

This all means that we can compute the entries of $\tilde{L}$ column-wise.

Example 21: $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 5 & 5 \\ 1 & 5 & 14 \end{bmatrix}$

Compute Cholesky decomposition $A = \tilde{L} \tilde{L}^t$:

column-wise computation:

1st col, 1st row: $l_{11} = \sqrt{a_{11} - 0} = \sqrt{1} = \underline{\underline{1}}$

$$l_{21} = \frac{1}{l_{11}} \left(a_{21} - 0 \right) = \frac{1}{1} = \underline{\underline{1}}$$

$$l_{31} = \frac{1}{l_{11}} \left(a_{31} - 0 \right) = \frac{1}{1} = \underline{\underline{1}}$$

$l_{11} \quad (a_{31}$

<u>2nd row:</u>

$$l_{22} = \sqrt{a_{22} - \sum_{k=1}^{1} l_{2k}^2}$$

$$= \sqrt{5 - 1} = \underline{\underline{2}}$$

$$l_{32} = \frac{1}{l_{22}}\left(a_{32} - l_{31}\, l_{21}\right)$$

$$= \frac{1}{2}\left(5 - 1\right) = \underline{\underline{2}}$$

<u>3rd row:</u>

$$l_{33} = \sqrt{a_{33} - \sum_{k=1}^{2} l_{3k}^2} = \underline{\underline{3}}$$

<u>Alltogether:</u>

$$\tilde{L} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}$$

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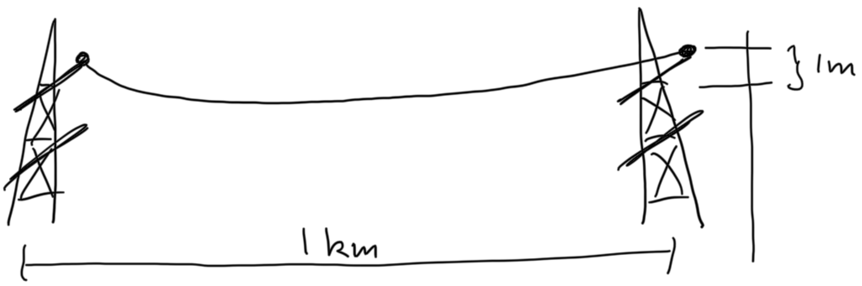
3 Nonlinear equations:

Finding the roots (solving non-linear equations) appears in many application problems.

E.g.



How long must this cable be?

The cable is described by the hyperbolic cosine cosh and the equation

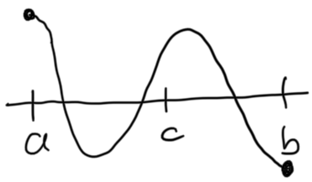
500 meters

$$\lambda \cosh\left(\frac{500}{\lambda}\right) - \lambda - 1 = 0$$

Goal: Solve this nonlinear equation for λ.
1 meter
In other words: let $g(\lambda) := \lambda \cosh\left(\frac{500}{\lambda}\right) - \lambda - 1$
we need to find a root λ so that $g(\lambda) = 0$.

3.1 Bisection method:

Recall the intermediate value theorem: For $f \in C^0([a,b])$ with $f(a)\,f(b) < 0$ then there exists a $r \in (a,b)$ at which $f(r) = 0$.



Idea of bisection method:

1) Bisect $[a,b]$ into $[a,c] \cup [c,b]$, i.e. into two subintervals, where $a < c < b$.
2) if $f(c) = 0$, then $r = c \Rightarrow$ done.
3) if $f(c)\,f(a) < 0$ then investigate $[a,c]$ further
4) if $f(c)\,f(b) < 0$ then investigate $[c,b]$ further.

Note that even when $f(c)\,f(a) > 0$ we cannot conclude that there are no roots in $[a,c]$. In fact, when $f(a)\,f(c) > 0$ there are $2m$ roots for some $m \in \mathbb{N}_0$.
