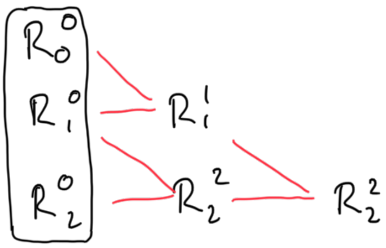
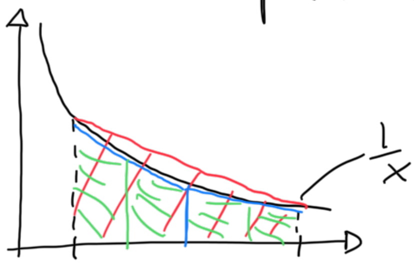
- Example on Romberg Algorithm
- Example on theorem 44

Example 46: Apply Romberg Algorithm to find $\boxed{R_2^2}$ for the integral $\int_1^3 \frac{1}{x}\,dx$.

First: $\int_1^3 \frac{1}{x}\,dx = \ln(3) - \ln(1) = \ln(3) \approx 1{,}098$

To start with Romberg we need the trapezoidal rule:





trapezoidal rule on $2^0, 2^1, 2^2$ sub-intervals

Now:

$$R_0^0 = (3-1)\cdot\frac{1}{2}\left[\frac{1}{1}+\frac{1}{3}\right] = 2\cdot\frac{1}{2}\cdot\frac{4}{3} = \boxed{\frac{4}{3}}$$

$$R_1^0 = 1\cdot\Big(\underbrace{\frac{1}{2}\left[\frac{1}{1}+\frac{1}{2}\right]}_{\text{first trapezoid}} + \underbrace{\frac{1}{2}\left[\frac{1}{2}+\frac{1}{3}\right]}_{\text{second trapezoid}}\Big)$$

$$= \frac{1}{2}\left(\frac{3}{2} + \frac{3}{6}+\frac{2}{6}\right) = \boxed{\frac{7}{6}}$$

$$R_2^0 = \frac{1}{2}\cdot\frac{1}{2}\left[\frac{1}{1} + \boxed{\frac{1}{3/2}+\frac{1}{3/2}} + \boxed{\frac{1}{2}+\frac{1}{2}} + \boxed{\frac{1}{5/2}} + \boxed{\frac{1}{5/2}} + \frac{1}{3}\right]$$

$$= \frac{1}{4}\left[1 + \frac{4}{3} + 1 + \frac{4}{5} + \frac{1}{3}\right] = \boxed{\frac{67}{60}}$$

First colum of Romberg array:

$$\frac{4}{3}$$

$$\frac{7}{6} \qquad \frac{7}{6} + \frac{1}{3}\left(\frac{7}{6} - \frac{4}{3}\right) = \frac{10}{9}$$

$$\frac{67}{60} \qquad \frac{67}{60} + \frac{1}{3}\left(\frac{67}{60} - \frac{7}{6}\right) = \frac{66}{60} \qquad \frac{66}{60} + \frac{1}{15}\left(\frac{66}{60} - \frac{10}{9}\right) = \frac{7}{6}$$

$$\approx 1.09$$

Formula for getting better approximations

$$\boxed{R_i^k = R_i^{k-1} + \frac{1}{4^k - 1}\left(R_i^{k-1} - R_{i-1}^{k-1}\right)}$$

Here

$$R_i^1 = R_i^0 + \frac{1}{3}\left(R_i^0 - R_{i-1}^0\right)$$

So we get as the desired value

$$\boxed{R_2^2 = \frac{742}{675} \approx 1.099}$$

and the real/analytical value is $\boxed{1.0986}$

Question: What is the asymptotic error of R_2^2?

$$\begin{array}{cccc}
R_0^0 & & & \\
R_1^0 & R_1^1 & & \\
R_2^0 & R_2^1 & R_2^2 & \\
R_3^0 & R_3^1 & R_3^2 & R_3^3
\end{array}$$

$R_0^0 \xrightarrow{\frac{\text{error}}{4}} R_1^0$, $R_1^0 \xrightarrow{\frac{\text{error}}{4}} R_2^0$, $R_1^1 \xrightarrow{\frac{\text{error}}{16}} R_2^1$, $R_2^2 \xrightarrow{\frac{\text{error}}{64}} R_3^2$

trapezoidal rule $\mathcal{O}(h^2)$ $\qquad \mathcal{O}(h^4)$ $\qquad \mathcal{O}(h^6)$ $\qquad \mathcal{O}(h^8)$

asymptotic error

Example 47: Approximate $\int_1^3 \frac{1}{x}\,dx$ by the trapezoi rule. How many sub-intervals (n) do you need to guarantee an error $< 10^{-2}$?

We need to use theorem 44:

$$\left| \int_1^3 \frac{1}{x}\,dx - T\left(\tfrac{1}{x}; P\right) \right| = \frac{1}{12}\cdot 2\, h^2\, |f''(\xi)| \overset{!}{<} 10$$

for $\xi \in (1,3)$.

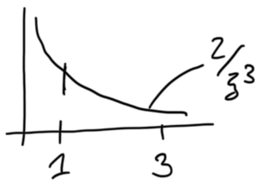
Find h such that $\frac{1}{6} h^2 |f''(\xi)| < 10^{-2}$

Difficult: We do not know where ξ is.

Remedy: Estimate how large $|f''(\xi)|$ may be in the worst case.

$$f(x) = \frac{1}{x}, \quad f'(x) = -\frac{1}{x^2}, \quad f''(x) = \frac{2}{x^3}$$

so $$\max_{\xi\in(1,3)} |f''(\xi)| = \max_{\xi\in(1,3)} \left|\frac{2}{\xi^3}\right| = 2$$



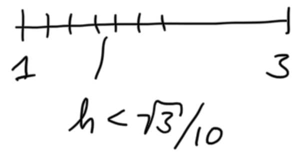
This means that

$$\text{error} = \frac{1}{6} h^2 |f''(\xi)| \le \frac{1}{6} h^2 \cdot 2 < 10^{-2}$$

solve $\frac{1}{3} h^2 < 10^{-2}$ for h:

$$\iff h^2 < 3 \cdot 10^{-2}$$

$$\Rightarrow \boxed{h < \sqrt{3} \cdot 10^{-1}}$$



For an equidistant partition we have that

$$h = \frac{b-a}{n} = \frac{2}{n}$$

to get: $\frac{2}{n} < \frac{\sqrt{3}}{10} \iff n > \frac{20}{\sqrt{3}} \approx 11{,}54$

So letting $\boxed{n = 12}$, i.e. 12 sub-intervals, will yield an error of $< 10^{-2}$.