NUMERICAL METHODS I:

I. Introduction:

Numerical methods are algorithmic approaches to solving mathematical problems / equations, which are hard to solve algebraically / analytically.

Goal of this: Study efficient numerical methods and understand them:

- How do they work?
- When do they work? Limitations?
- What is the error introduced?

Example 1: Given a function $f: \mathbb{R} \longrightarrow \mathbb{R}$ that is hard to evaluate for some $x \in \mathbb{R}$.
But the values of f and its derivatives are known for a value c that is close to x.
Can we use this information to approximate $f(x)$
How accurate is this approximation?

For example $f(x) = \cos(x)$ let $x = 0.1$
We know the values of $\cos^{(k)}(0)$:

$$\left.\begin{aligned} f(c) &= \cos(0) = 1 \\ f'(c) &= -\sin(0) = 0 \\ f''(c) &= -\cos(0) = -1 \end{aligned}\right\} \text{ for } c = 0.$$

$$\vdots$$

Can we get $\cos(0.1)$ from these values?

Def 2. (Taylor series) Let $f: \mathbb{R} \longrightarrow \mathbb{R}$ be differentiable at $c \in \mathbb{R}$. Then, the Taylor series of f at c is given by:

$$f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{6}(x-c)^3 + \dots$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

This is a power series!

For $c=0$ this is known as Maclaurin series.

Remember: A power series has a radius of convergence / interval of convergence. If $x \in$ interval of convergence, then

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

Example 3: Taylor series for $f(x) = e^x$ at $c=0$

We have $f^{(k)}(x) = e^x$, so $f^{(k)}(c) = e^0 = 1$.

Thus $$\sum_{k=0}^{\infty} \frac{1}{k!} x^k$$

and the radius of convergence is ∞, i.e. for any $x \in \mathbb{R}$, $e^x = \sum_{k=0}^{\infty} x^k/k!$.

For a numerical algorithm we need to stop the summation after a finite number of terms:

E.g. $$e^x \approx \frac{1}{0!}x^0 + \frac{1}{1!}x^1 + \frac{1}{2!}x^2$$

$$= 1 + x + \frac{1}{2}x^2$$

This is a polynomial!

<u>Example 4:</u> $f(x) = 4x^2 + 5x + 7, \quad c = 2$

Taylor series of f at c?

$f(2) = 33, \quad f'(x) = 8x + 5, \quad f''(x) = 8, \quad f'''(x) = 0$

$f'(2) = 21, \quad f''(2) = 8$

Taylor series:

$$33 + 21(x-2) + \frac{8}{2}(x-2)^2$$
$$= 4x^2 + 5x + 7 = f(x)$$

The Taylor series of a polynomial is the polynomial itself!

<u>Theorem 5:</u> (<u>Taylor theorem</u>)

Let $f \in C^{n+1}([a,b])$ i.e. f is $(n+1)$-times continuously differentiable over $[a,b]$. Then for any $c, x \in [a,b]$ we have that

$$f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k}_{\text{truncated Taylor series}} + \underbrace{\frac{f^{(n+1)}(\xi_x)}{(n+1)!}(x-c)^{n+1}}_{\text{remainder term}}$$

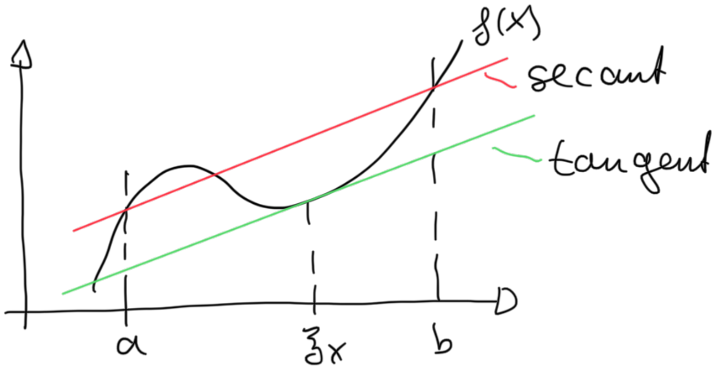
where ξ_x is a point that depends on x and which lies between c and x.

For $n = 0$: $\quad f(x) = f^{(0)}(c) + f'(\xi_x)(x-c)$

choose $c = a$, $x = b$:

$$f(b) = f(a) + f'(\xi_x)(b-a)$$

$$\Leftrightarrow \quad \frac{f(b) - f(a)}{b-a} = f'(\xi_x)$$



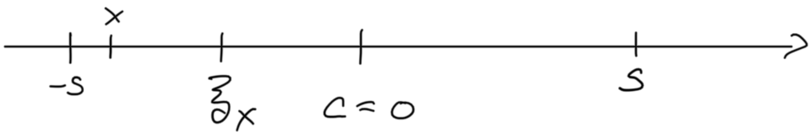
Def 6: We say that a Taylor series represents the function f at x, iff the Taylor series converges at that point, i.e. the remainder tends to zero as $n \to \infty$.

Back to Example 3: $f(x) = e^x$, $c=0$

By Taylor theorem

$$e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + \frac{e^{\xi_x}}{(n+1)!} x^{n+1}$$

For any $x \in \mathbb{R}$ we find $s \in \mathbb{R}_0^+$ so that $|x| \le s$, and $|\xi_x| \le s$ because ξ_x is between c and x.



Because e^x is monotone increasing we have $e^{\xi_x} \le e^s$.

Thus:

$$\lim_{n\to\infty} \left| \frac{e^{\xi_x}}{(n+1)!} x^{n+1} \right|$$

$$\le \lim_{n\to\infty} \frac{e^s}{(n+1)!} s^{n+1}$$

$$= e^s \lim \frac{s^{n+1}}{\dots} = 0$$

$$= e \lim_{n \to \infty} (n+1)! = 0$$

because $(n+1)!$ will grow faster than any power of s.

Thus, e^x is represented by its Taylor series.

Example 7: $f(x) = \ln(1+x)$, $c = 0$

then $f'(x) = \dfrac{1}{1+x}$

$f''(x) = \dfrac{-1}{(1+x)^2}$

$\vdots$

$f^{(k)}(x) = (-1)^{k-1}(k-1)!\,\dfrac{1}{(1+x)^k}$

so $f^{(k)}(0) = (-1)^{k-1}(k-1)!$

So, the Taylor series is:

$$f(x) = \sum_{k=0}^{n} \frac{(-1)^{k-1}}{k} x^k + (-1)^n \frac{1}{n+1} \frac{1}{(1+\xi_x)^{n+1}} x^{n+1}$$