Numerical Methods I

Assignment Sheet 4. Due: March 9, 2020

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Exercise 16 [3 + 3 + 3 + 2 + 9 Points]: We want to estimate the value $f(1/2)$ of a function f, of which we know the function values $f(-1) = 2$, $f(0) = 0$, and $f(1) = 1$. We take the approach of polynomial interpolation with respect to the polynomial basis of Legendre polynomials. Legendre polynomials are defined recursively by

$$P_0(x) = 1, P_1(x) = x, \text{ and } (n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \text{ for } n \geq 1.$$

a) Show that the Legendre polynomials form a basis.

b) List the interpolation conditions that a polynomial $p(x)$ in Legendre form has to fulfill for the given interpolation problem and derive the respective collocation matrix with respect to the Legendre basis.

c) Solve the interpolation problem by deriving polynomial $p(x)$ in Legendre form using the collocation matrix derived in b).

d) Evaluate the polynomial $p(x)$ in Legendre form derived in b) at $x = 1/2$ to obtain the desired estimate for the value $f(1/2)$.

e) Now, we want to do the same interpolation with respect to the polynomial basis of Lagrange polynomials. I.e., starting from the interpolation conditions, derive the collocation matrix, compute the Lagrange polynomials, provide the interpolating polynomial $p_L(x)$ in Lagrange form, and evaluate the polynomial $p_L(x)$ in Lagrange form at $x = 1$. How does the interpolated value compare to the one in d) when using Legendre polynomials? Explain your observation.

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Exercise 17 [3 + 3 + 4 Points]:

a) Show that the Lagrange polynomials have the property of partition of unity, i.e. $\sum_{i=0}^{n} L_i^n(u) = 1$.

b) Check whether the Newton polynomials have the property of partition of unity. Give reason/proof for your answer.

c) Given an interpolation problem with points p_i at nodes u_i, $i = 0, ..., k$, show that the weights in the Newton interpolation scheme can be expressed by

$$p[u_0, u_1, \ldots, u_k] = \sum_{i=0}^{k} \frac{p_i}{\prod\limits_{j=0,\ldots,k; j\neq i} (u_i - u_j)}.$$

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Exercise 18 [not graded, w/o Points]: Determine the Newton interpolating polynomial for this table:

u_i	0	1	2	7
p_i	51	3	1	201

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Exercise 19 [not graded, w/o Points]: Consider polynomial interpolation with $n+1$ points $u_0, \ldots, u_n$. The Vandermonde matrix

$$M = \begin{bmatrix} 1 & u_1 & u_1^2 & \ldots & u_1^n \\ 1 & u_2 & u_2^2 & \ldots & u_2^n \\ 1 & u_3 & u_3^2 & \ldots & u_3^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & u_n & u_n^2 & \ldots & u_n^n \end{bmatrix}$$

results as the collocation matrix in interpolation on these points with the standard basis $1, x, x^2, \ldots, x^n$. Prove by induction that the determinant of M is

$$\det(M) = \prod_{0 \le i < j \le n} (u_j - u_i).$$

As a consequence of this we find that the interpolation problem is solvable iff the points u_i are pairwise different.

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Exercise 20 [not graded, w/o Points]: In *Hermite* interpolation, a function as well as its derivative shall be interpolated. Using Hermite interpolation...

a) ... find a polynomial p that assumes the values $p(0) = 0, p(1) = 1, p'(\frac{1}{2}) = 2$.

b) ... find a polynomial q that satisfies the conditions $q(x_i) = f(x_i)$, $q'(x_i) = f'(x_i)$, for known values $f(x_i)$ and $f'(x_i)$ and where $i = 0, 1$.