Lecture 14:

Transformations

Contents

1. Homogeneous Coordinates
2. Basic transformations in homogeneous coordinates
3. Concatenation of transformations
4. Projective transformations

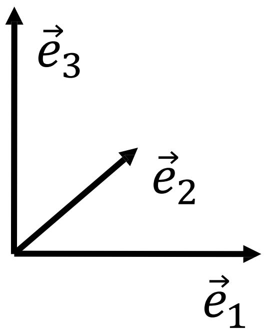
Math recap

- The **vector space** V in 3D over the real numbers

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in V^3 = \mathbb{R}^3$$

- Vectors written as $n \times 1$ matrices
- Vectors describe directions – **not positions**!
  - All vectors conceptually start from the origin of the coordinate system
- 3 linear independent vectors create a basis

$$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$



- Any 3D vector can be represented uniquely with coordinates

$$\vec{v} = v_1\vec{e}_1 + v_1\vec{e}_1 + v_1\vec{e}_1 \qquad v_1, v_2, v_3 \in \mathbb{R}$$

Standard scalar product a.k.a. dot or inner product

- Measure lengths

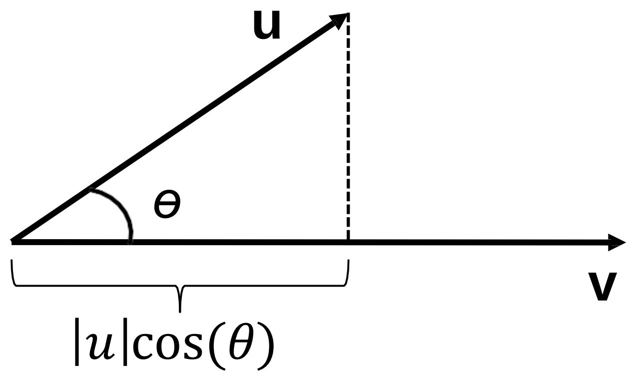
$$|\vec{v}|^2 = \vec{v} \cdot \vec{v} = v_1^2 + v_2^2 + v_3^2$$

- Compute angles

$$\vec{v} \cdot \vec{u} = |\vec{v}||\vec{u}| \cos(u, v)$$

- Projection of vectors onto other vectors

$$|\vec{u}| \cos\theta = \frac{\vec{v} \cdot \vec{u}}{|\vec{v}|} = \frac{\vec{v} \cdot \vec{u}}{\sqrt{\vec{v} \cdot \vec{v}}}$$



Orthonormal basis

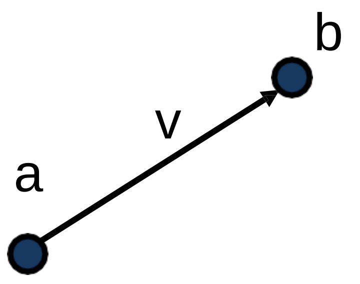
- Unit length vectors
  - $|\vec{e}_1| = |\vec{e}_2| = |\vec{e}_3| = 1$
- Orthogonal to each other
  - $\vec{e}_i \cdot \vec{e}_j = \delta_{ij}$

Handedness of the coordinate system

- Two options: $\vec{e}_1 \times \vec{e}_2 = \vec{e}_3$
  - Positive: Right-handed (RHS)
  - Negative: Left-handed (LHS)
- Example: Screen Space
  - Typical: X goes right, Y goes up (thumb & index finger, respectively)
  - In a RHS: Z goes out of the screen (middle finger)
- Be careful:
  - Most systems nowadays use a right handed coordinate system
  - But some are not (*e.g.* RenderMan) → can cause lots of confusion

Basic mathematical concepts

- The **affine space** A
  - In contrast to vector space, affine space operates with objects of 2 types:
    - Vectors and
    - Points
- Defined via its associated vector space V
  - $a, b \in A \iff \exists \vec{v} \in V : \vec{v} = b - a$
  - $\rightarrow$: unique, $\leftarrow$: ambiguous
- Operations on affine space A
  - Subtraction of two points yields a vector
  - No addition of points
    - Its not clear what the some of two points would mean
  - But: Addition of points and vectors:
    - $a + \vec{v} = b \in A^3$
  - Distance
    - $dist(a, b) = |a - b|$

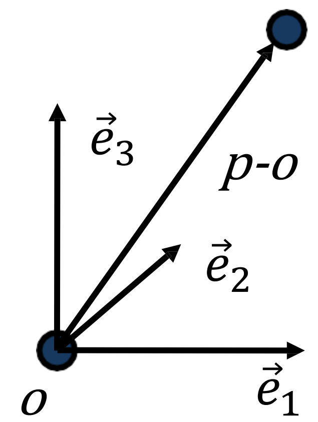


Affine Basis

- Given by its origin point o and the basis of an associated vector space
  - $\{\vec{e}_1, \vec{e}_2, \vec{e}_3, o\}: \vec{e}_1, \vec{e}_2, \vec{e}_3 \in V^3; o \in A^3$

Position vector of point p

- $(p - o)$ is in V^3



Affine Combination

- The fundamental operation on the points of an affine space
- Linear combination of $(n+1)$ points $p_0, \dots, p_n \in A$, which uniquely defines a new point:

$$p = \alpha_0 p_0 + \alpha_1 p_1 + \cdots + \alpha_n p_n$$

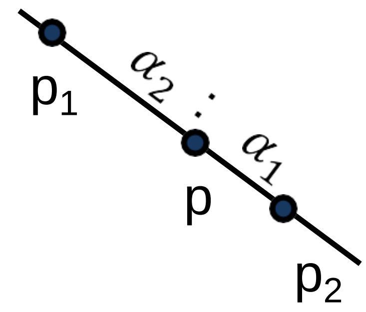
- With weights forming a partition of unity $\alpha_0, \dots, \alpha_n \in \mathbb{R}$ with $\sum_i \alpha_i = 1$
- $p = \sum_{i=0}^{n} \alpha_i p_i = p_0 + \sum_{i=0}^{n} \alpha_i (p_i - p_0) = o + \sum_{i=0}^{n} \alpha_i \vec{v}_i$

Basis

- $(n + 1)$ points form an **affine basis**
  - If none of these point can be expressed as an affine combination of the other points
  - Any point in A can then be uniquely represented as an affine combination of the affine basis $p_0, \dots, p_n \in A$
  - Any vector in another basis can be expressed as a linear combination of the p_i, yielding a matrix for the basis

Affine Coordinates

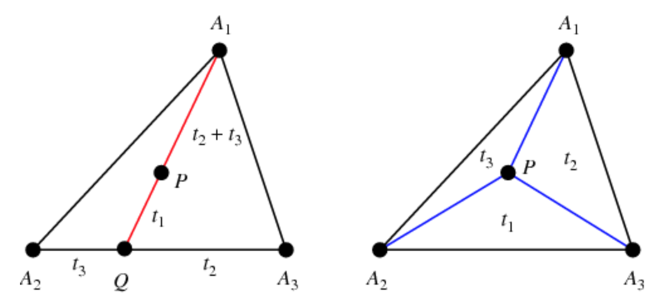
In 1D



- Point is defined by the splitting ratio $\alpha_1 : \alpha_2$
  - $p = \alpha_1 p_1 + \alpha_2 p_2 = \frac{|p - p_2|}{|p_2 - p_1|} p_1 + \frac{|p - p_1|}{|p_2 - p_1|} p_2$

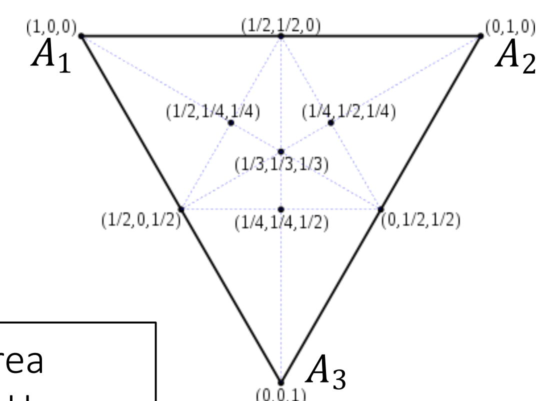
In 2D

- Weights are the relative areas in $\Delta(A_1, A_2, A_3)$
  - $t_i = \alpha_i = \frac{\Delta\left(P, A_{(i+1)\%3}, A_{(i+2)\%3}\right)}{\Delta(A_1, A_2, A_3)}$
  - $p = \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3$



Closely related to “Barycentric Coordinates”

- Center of mass of $(n + 1)$ points with arbitrary masses (weights) m_i is given as
- $p = \frac{\sum m_i p_i}{\sum m_i} = \sum \frac{m_i}{\sum m_i} p_i = \sum \alpha_i p_i$



Convex / Affine Hull

- If all α_i are non-negative than p is in the **convex hull** of the other points

Note: Length and area measures are signed here

Properties

- Affine mapping (continuous, bijective, invertible)
  - $T: A^3 \rightarrow A^3$
- Defined by two non-degenerated simplicies
  - 2D: Triangle, 3D: Tetrahedron, ...
- Invariants under affine transformations:
  - Barycentric / affine coordinates
  - Straight lines, parallelism, splitting ratios, surface/volume ratios
- Characterization via fixed points and lines
  - Given as eigenvalues and eigenvectors of the mapping

Representation

- Matrix product and a translation vector:
  - $\boldsymbol{T}p = \boldsymbol{A}p + \boldsymbol{t}$ with $\boldsymbol{A} \in \mathbb{R}^{n \times n},\ \boldsymbol{t} \in \mathbb{R}^n$
- Invariance of affine coordinates
  - $\boldsymbol{T}p = \boldsymbol{T}(\sum \alpha_i p_i) = \boldsymbol{A}(\sum \alpha_i p_i) + \boldsymbol{t} = \sum \alpha_i(\boldsymbol{A}p_i) + \sum \alpha_i \boldsymbol{t} = \sum \alpha_i(\boldsymbol{T}p_i)$

Homogeneous embedding of $\mathbb{R}^3$ into the projective 4D space $P(\mathbb{R}^4)$

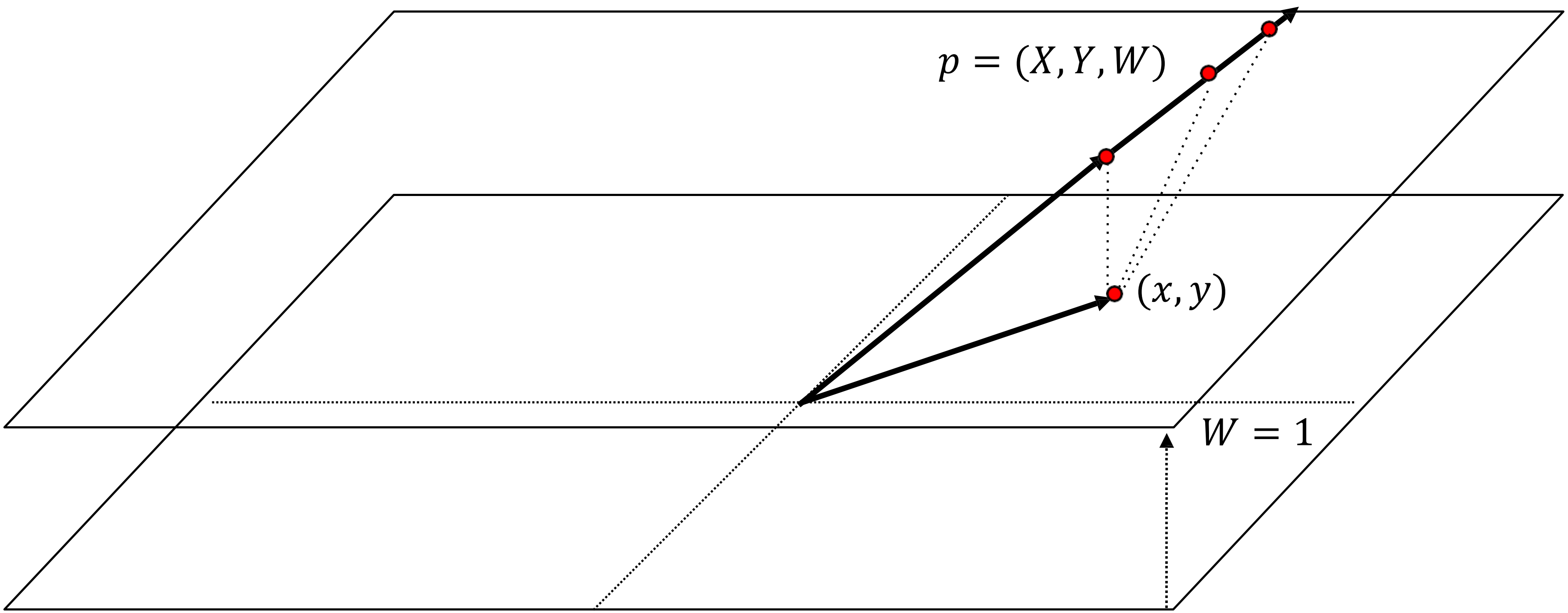
- Mapping into homogeneous space
  - $\mathbb{R}^3 \ni \begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} \in P(\mathbb{R}^4)$
- Mapping back by dividing through fourth component
  - $\begin{pmatrix} X \\ Y \\ Z \\ W \end{pmatrix} \to \begin{pmatrix} X/W \\ Y/W \\ Z/W \end{pmatrix}$

Consequence

- This allows to represent affine transformations as 4x4 matrices
- Mathematical trick
  - Convenient representation to express rotations and translations as matrix multiplications
  - Easy to find line through points, point-line / line-line intersections
- Also important for projections (later)

Point in homogeneous coordinates

- All points along a line through the origin map to the same point in 2D



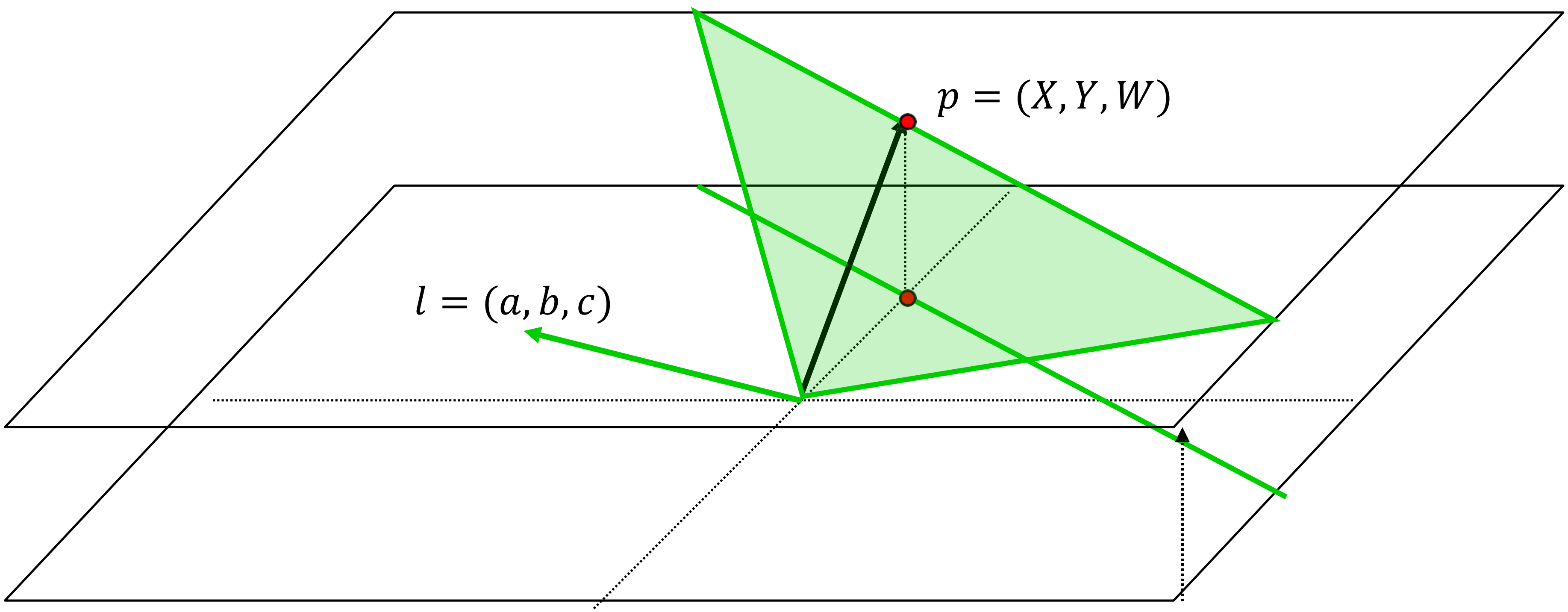
$$x = \frac{X}{W} \qquad y = \frac{Y}{W}$$

Some tricks (work only in $P(\mathbb{R}^3)$, *i.e.* only in 2D)

- Point representation
  - $(X) = \begin{pmatrix} X \\ Y \\ W \end{pmatrix} \in P(\mathbb{R}^3)$, $\quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X/W \\ Y/W \end{pmatrix}$
- Representation of a line $l \in \mathbb{R}^2$
  - Dot product of l vector with point in plane must be zero:
    - $l = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| ax + by + c \cdot 1 = 0 \right\} = \left\{ X \in P(\mathbb{R}^3) \middle| X \cdot l = 0,\ l = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\}$
  - Line l is normal vector of the plane through origin and points on line

Definition of a 2D Line in $P(\mathbb{R}^3)$

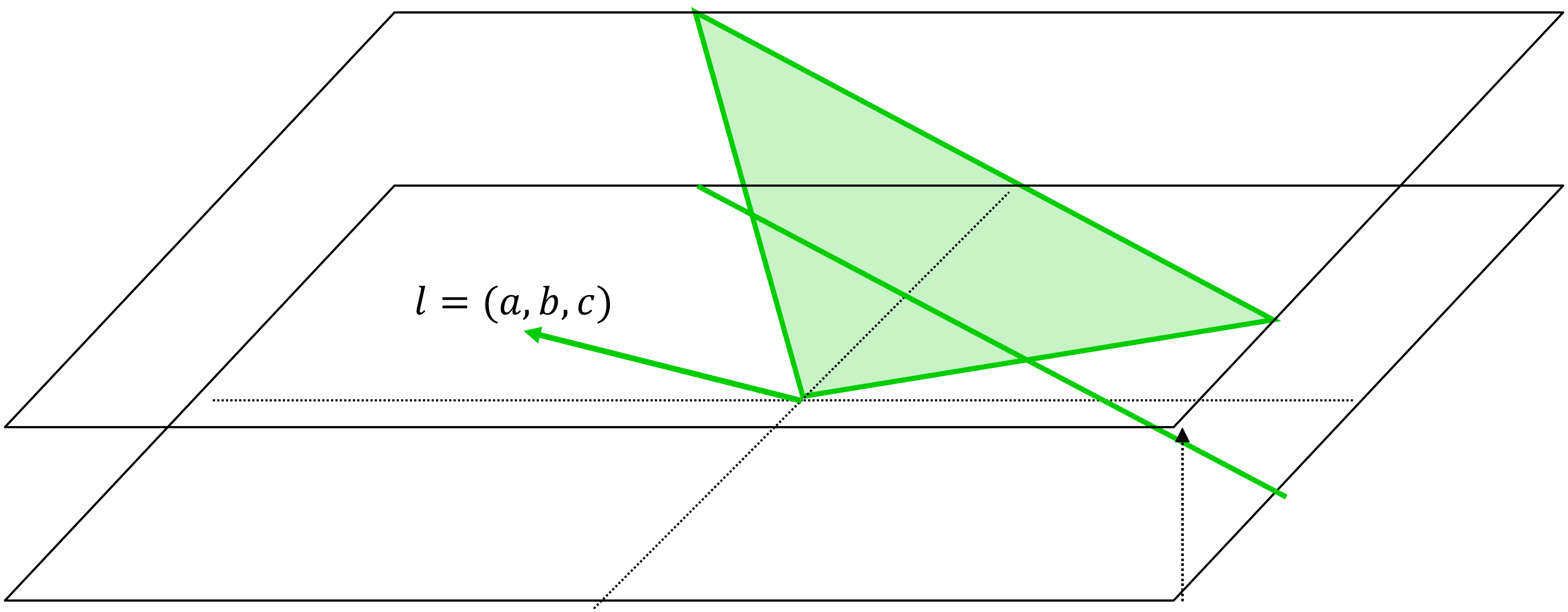
- Set of all point P where the dot product with l is zero



$$p \cdot l = 0$$

Line

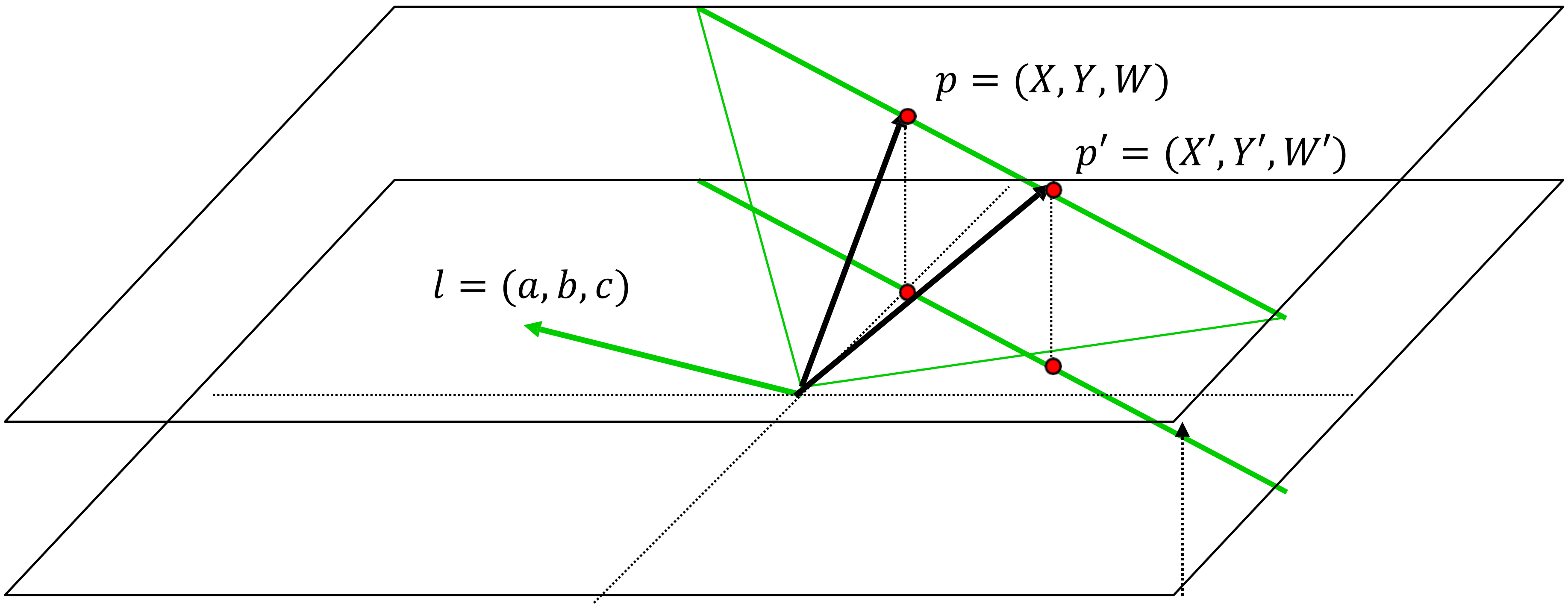
- Represented by normal vector to plane through line and origin



$$ax + by + c \cdot 1 = 0$$

Construct line through two points

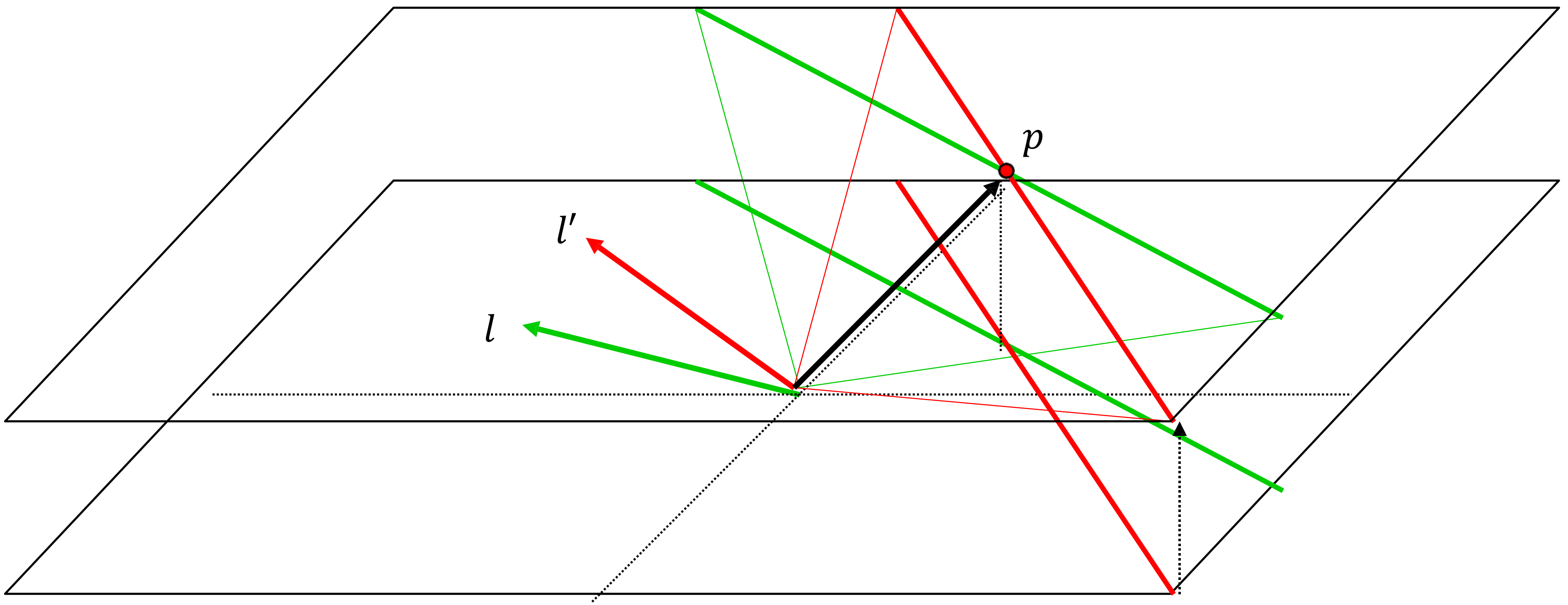
- Line vector must be orthogonal to both points
- Compute through cross product of point coordinates



$$l = p \times p'$$

Construct intersection of two lines

- A point that is on both lines and thus orthogonal to both lines
  - Computed by cross product of both line vectors



$$p = l \times l'$$

Some tricks (work only in $P(\mathbb{R}^3)$, *i.e.* only in 2D)

- Point representation
  - $(X) = \begin{pmatrix} X \\ Y \\ W \end{pmatrix} \in P(\mathbb{R}^3)$, $\quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X/W \\ Y/W \end{pmatrix}$
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  - Line l is normal vector of the plane through origin and points on line
- Line trough 2 points p and p'
  - Line must be orthogonal to both points
  - $p \in l \wedge p' \in l \Leftrightarrow l = p \times p'$
- Intersection of lines l and l':
  - Point on both lines → point must be orthogonal to both line vectors
  - $X \in l \cap l' \Leftrightarrow X = l \times l'$

Columns are orthogonal vectors of unit length

- An example:
  - $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

- Directly derived from the definition of the matrix product:
  - $M^\intercal M = 1$

- In this case the transpose must be identical to the inverse:
  - $M^{-1} := M^\intercal$

Transformations in a Vector space: Multiplication by a Matrix

- Action of a linear transformation on a vector
  - Multiplication of matrix with column vectors (*e.g.* in 3D)

$$p' = \begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = \boldsymbol{T}p = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

Composition of transformations

- Simple matrix multiplication ($\boldsymbol{T_1}$, then $\boldsymbol{T_2}$)
  - $\boldsymbol{T_2}\boldsymbol{T_1}p = \boldsymbol{T_2}(\boldsymbol{T_1}p) = \boldsymbol{T_2}\boldsymbol{T_1}\,p = \boldsymbol{T}p$
- Note: matrix multiplication is associative but not commutative!
  - $\boldsymbol{T_2}\boldsymbol{T_1}$ is not the same as $\boldsymbol{T_1}\boldsymbol{T_2}$ (in general)

Remember:

- Affine map: Linear mapping and a translation
  - $\boldsymbol{T}p = \boldsymbol{A}p + \boldsymbol{t}$

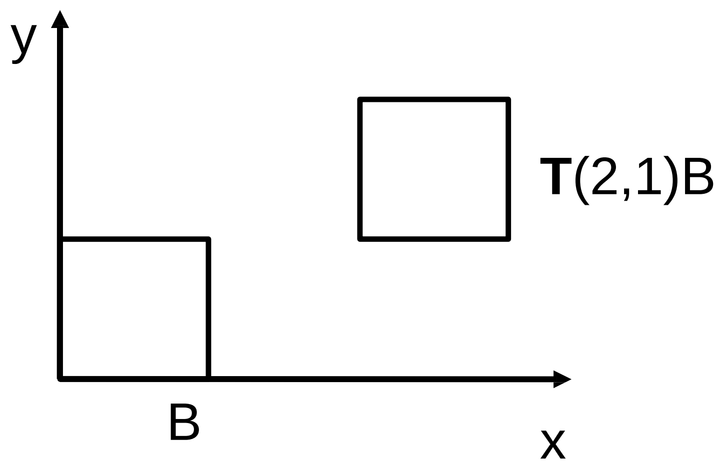
For 3D: Combining it into one matrix

- Using homogeneous 4D coordinates
- Multiplication by 4x4 matrix in $P(\mathbb{R}^4)$ space
- $$p' = \begin{pmatrix} X' \\ Y' \\ Z' \\ W' \end{pmatrix} = Tp = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} & T_{xw} \\ T_{yx} & T_{yy} & T_{yz} & T_{yw} \\ T_{zx} & T_{zy} & T_{zz} & T_{zw} \\ T_{wx} & T_{wy} & T_{wz} & T_{ww} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ W \end{pmatrix}$$
- Allows for combining (concatenating) multiple transforms into one using normal (4x4) matrix products

Let's go through the different transforms we need!

Translation ($\boldsymbol{T}$)

- $$\boldsymbol{T}(t_x, t_y, t_z)p = \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} x + t_x \\ y + t_y \\ z + t_z \\ 1 \end{pmatrix}$$



So far: only translated points

Vectors: Difference between 2 points

$$v = p - q = \begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix} - \begin{pmatrix} q_x \\ q_y \\ q_z \\ 1 \end{pmatrix} = \begin{pmatrix} p_x - q_x \\ p_y - q_y \\ p_z - q_z \\ 0 \end{pmatrix}$$

- Fourth component is zero

Consequently: Translations do not affect vectors!

- $\boldsymbol{T}(t_x, t_y, t_z)v = \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ v_z \\ 0 \end{pmatrix} = \begin{pmatrix} v_x \\ v_y \\ v_z \\ 0 \end{pmatrix}$

Properties

- Identity
  - $\boldsymbol{T}(0,0,0) \;=\; \mathbf{1}$ (Identity Matrix)

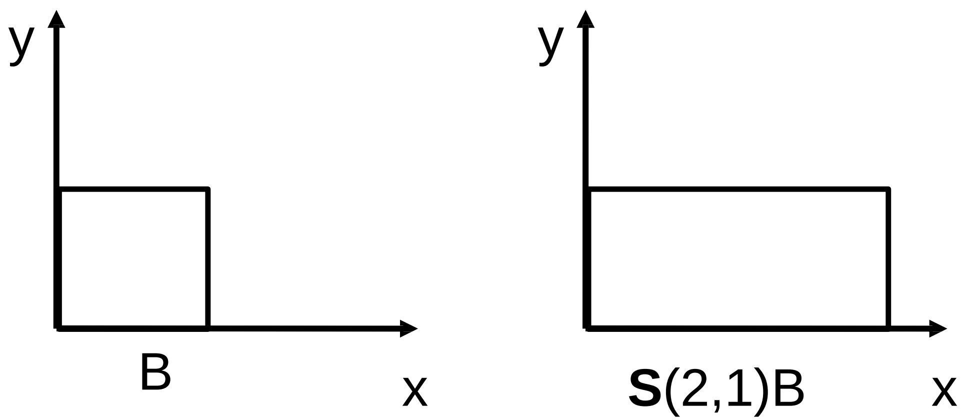
- Commutative (special case)
  - $\boldsymbol{T}(t_x, t_y, t_z)\boldsymbol{T}(t'_x, t'_y, t'_z) = \boldsymbol{T}(t'_x, t'_y, t'_z)\boldsymbol{T}(t_x, t_y, t_z) = \boldsymbol{T}(t_x + t'_x, t_y + t'_y, t_z + t'_z)$

- Inverse
  - $\boldsymbol{T}^{-1}(t_x, t_y, t_z) = \boldsymbol{T}(-t_x, -t_y, -t_z)$

Scaling (S)

$$\mathbf{S}(s_x, s_y, s_z) = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

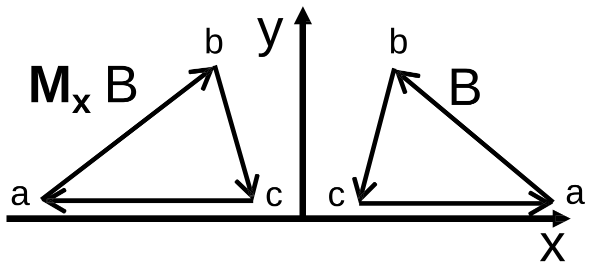
- Note: $s_x, s_y, s_z \geq 0$ (otherwise see mirror transformation)
- Uniform Scaling s: $s = s_x, s_y, s_z$



Reflection / Mirror Transformation ($\boldsymbol{M}$)

- Reflection at plane ($x = 0$)

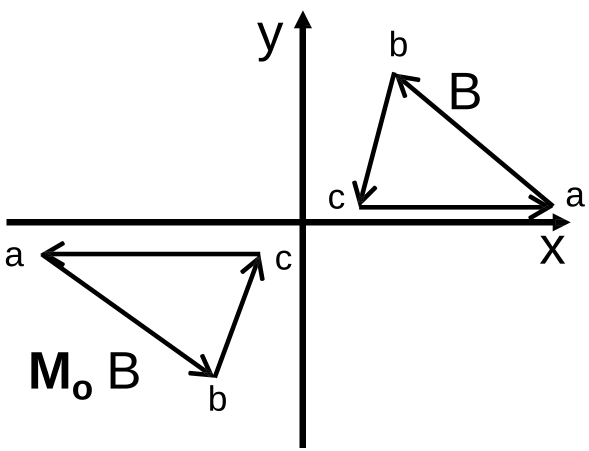
$$\boldsymbol{M_x} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} -x \\ y \\ z \\ 1 \end{pmatrix}$$



  - Analogously for other axis
  - Note: changes orientation
    - Right-handed rotation becomes left-handed and *vice versa*
    - Indicated by $\det(M_i) < 0$

- Reflection at origin

$$\boldsymbol{M_o} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} -x \\ -y \\ -z \\ 1 \end{pmatrix}$$



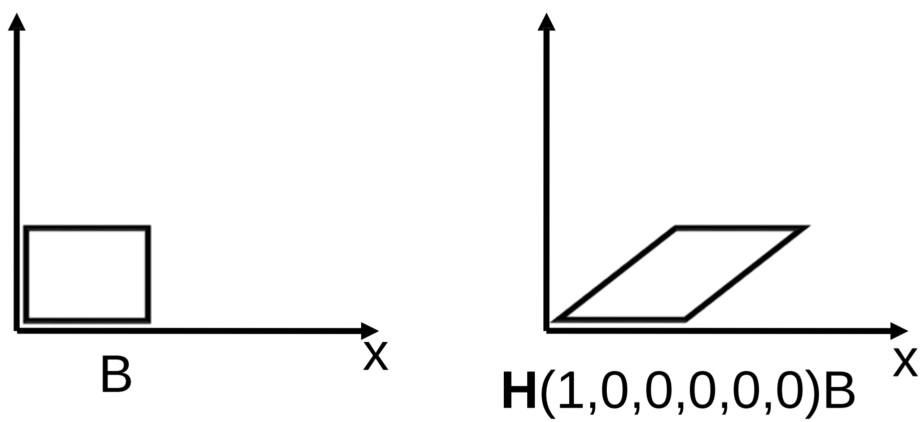
  - Note: changes orientation in 3D
    - But not in 2D (!!!): Just two scale factors
    - Each scale factor reverses orientation once

Basic Transformations (4)

Shear ($\boldsymbol{H}$)

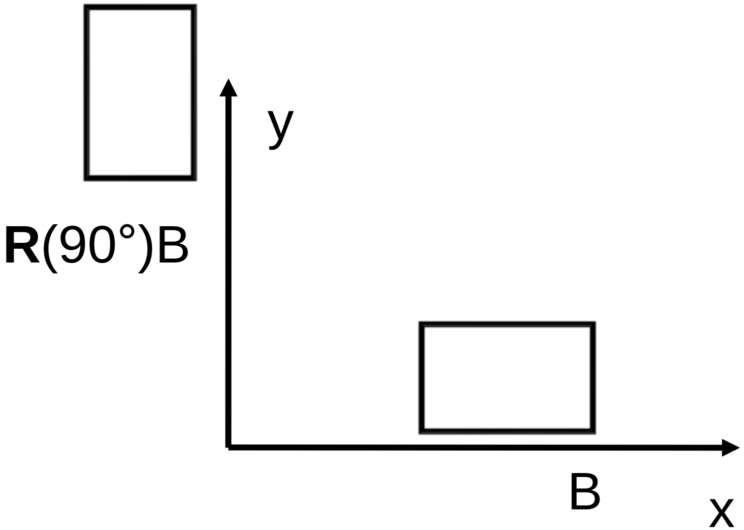
- $\boldsymbol{H}(h_{xy}, h_{xz}, h_{yx}, h_{yz}, h_{zx}, h_{zy}) = \begin{pmatrix} 1 & h_{xy} & h_{xz} & 0 \\ h_{yx} & 1 & h_{yz} & 0 \\ h_{zx} & h_{zy} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} x + h_{xy}y + h_{xz}z \\ y + h_{yx}x + h_{yz}z \\ z + h_{zx}x + h_{zy} \\ 1 \end{pmatrix}$

- Determinant is 1
  - Volume preserving (as volume is just shifted in some direction) y



In 2D: Rotation around origin

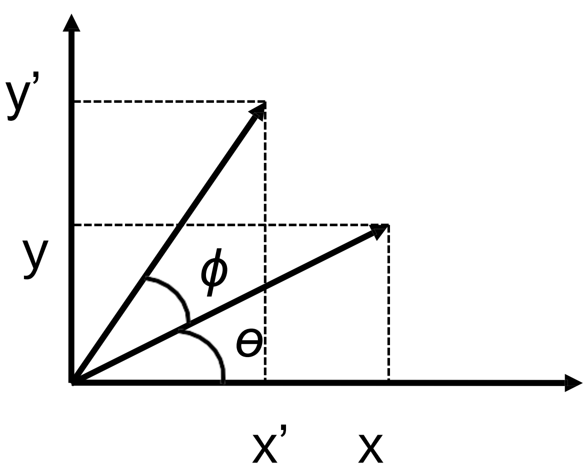
- Representation in spherical coordinates
  - $x = r\cos\theta \longrightarrow x' = r\cos(\theta+\phi)$
  - $y = r\sin\theta \longrightarrow y' = r\sin(\theta+\phi)$



- Well know property
  - $\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$
  - $sin(\theta+\phi) = \cos\theta\sin\phi - \sin\theta\cos\phi$

- Gives
  - $x' = (r\cos\theta)\cos\phi - (r\sin\theta)\sin\phi = x\cos\phi - y\sin\phi$
  - $y' = (r\cos\theta)\sin\phi - (r\sin\theta)\cos\phi = x\sin\phi + y\cos\phi$

- Or in matrix form
  - $R_z(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$



Rotation around major axes

- $$\boldsymbol{R}_x(\phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
- $$\boldsymbol{R}_y(\phi) = \begin{pmatrix} \cos\phi & 0 & \sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
- $$\boldsymbol{R}_z(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

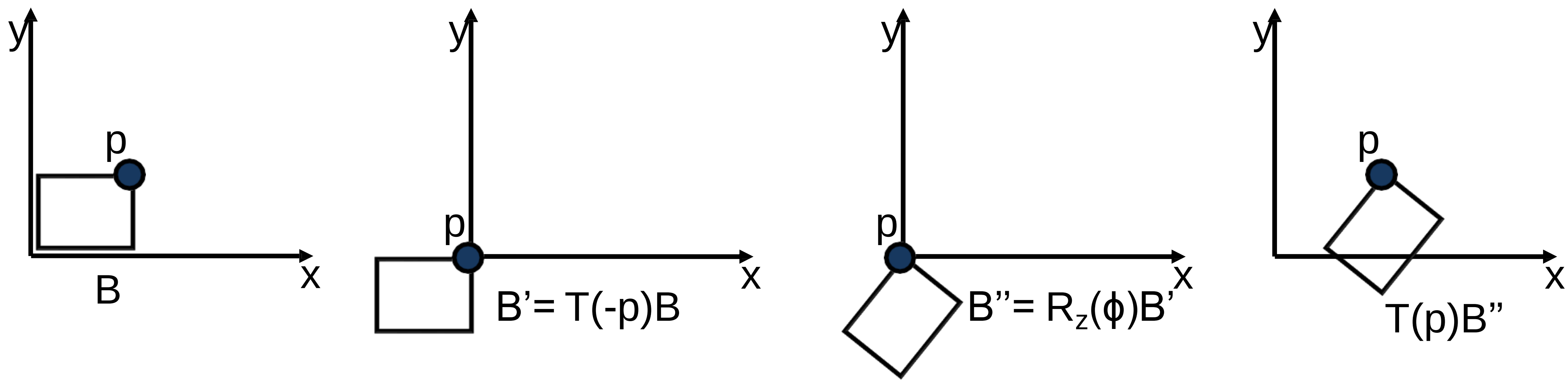
- 2D rotation around the respective axis
  - Assumes right-handed system, mathematically positive direction
- Be aware of change in sign on sines in $\boldsymbol{R}_y$
  - Due to relative orientation of other axis

Properties

- $\boldsymbol{R_a}(0) = \mathbf{1}$
- $\boldsymbol{R_a}(\theta)\boldsymbol{R_a}(\phi) = \boldsymbol{R_a}(\theta + \phi) = \boldsymbol{R_a}(\phi)\boldsymbol{R_a}(\theta)$
  - Rotations around the same axis are commutative (special case)
- In general: Not commutative
  - $\boldsymbol{R_a}(\theta)\boldsymbol{R_b}(\phi) \neq \boldsymbol{R_b}(\phi)\boldsymbol{R_a}(\theta)$
  - Order does matter for rotations around different axes
- $R_a^{-1}(\theta) = \boldsymbol{R_a}(-\theta) = \boldsymbol{R_a^{\mathsf{T}}}(\theta)$
  - Orthonormal matrix: Inverse is equal to the transpose
- Determinant is 1
  - Volume preserving

Rotate object around a point p and axis a

- Translate p to origin, rotate around axis a, translate back to p
  - $\boldsymbol{R_a}(p, \phi) = \boldsymbol{T}(p)\boldsymbol{R_a}(\phi)\boldsymbol{T}(-p)$



Rotate around a given point p and vector r ($|r| = 1$)

- Translate so that p is in the origin
- Transform with rotation $R = M^{\top}$
  - M given by orthonormal basis (r, s, t) such that r becomes the x axis
  - Requires construction of a orthonormal basis (r, s, t), see next slide
- Rotate around x axis
- Transform back with R^{-1}
- Translate back to point p

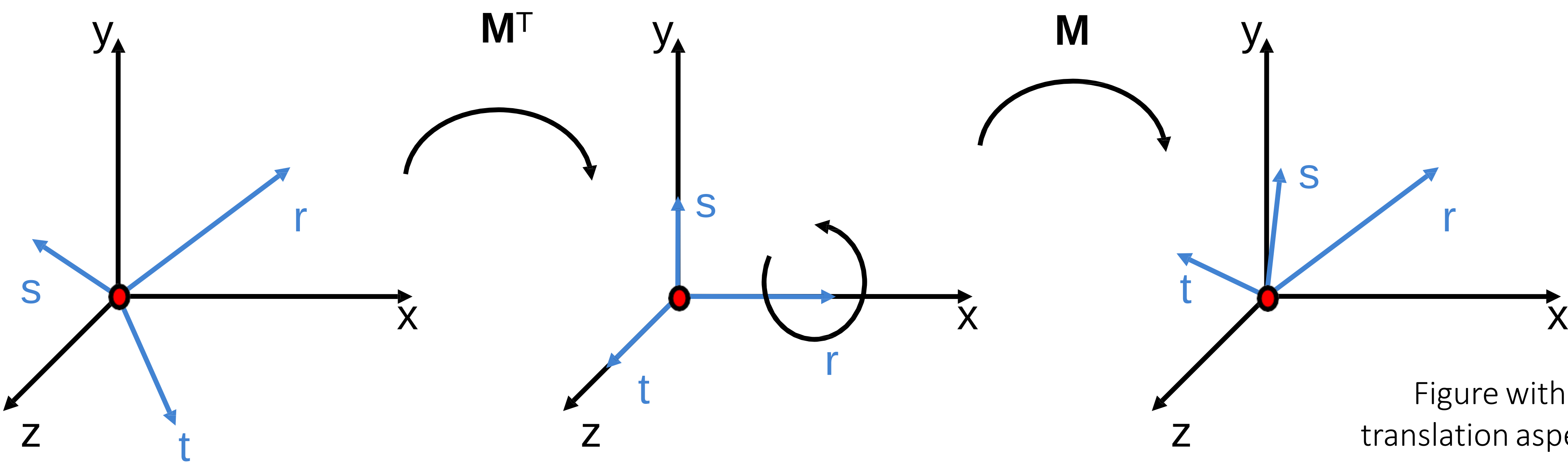


Figure without translation aspect

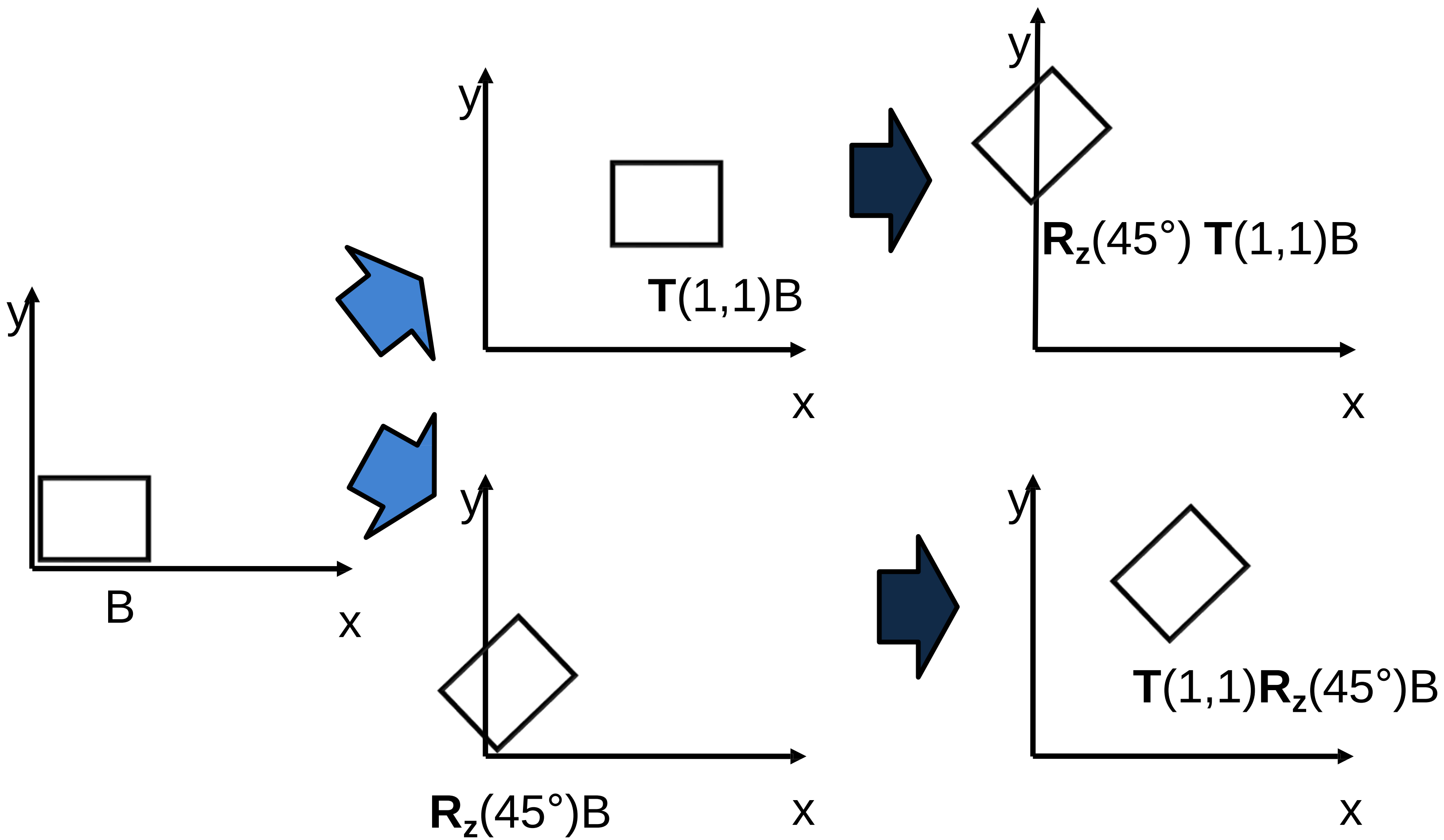
$$R(p, r, \phi) = T(p)M(r)R_x(\phi)M^{\top}(r)T(-p)$$

Compute orthonormal basis given a vector r

- Using a numerically stable method
- Construct s such that it is normal to r (verify with dot product)
  - Use fact that in 2D, orthogonal vector to (x, y) is $(-y, x)$
    - Do this in coordinate plane that has largest components
  - $s' = \begin{cases} (0, -r_z, r_y), & if\ x = \underset{x,y,z}{\operatorname{argmin}}\{|r_x|, |r_y|, |r_z|\} \\ (-r_z, 0, r_x), & if\ y = \underset{x,y,z}{\operatorname{argmin}}\{|r_x|, |r_y|, |r_z|\} \\ (-r_y, r_x, 0), & if\ z = \underset{x,y,z}{\operatorname{argmin}}\{|r_x|, |r_y|, |r_z|\} \end{cases}$
- Normalize
  - $s = {}^{s'}/_{|s'|}$
- Compute t as cross product
  - $t = r \times s$
- r, s, t forms orthonormal basis, thus M transforms into this basis
  - $M(r) = \begin{pmatrix} r_x & s_x & t_x & 0 \\ r_y & s_y & t_y & 0 \\ r_z & s_z & t_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, inverse is given as its transpose: $M^{-1} = M^{\intercal}$

Multiply matrices to concatenate

- Matrix-matrix multiplication is not commutative (in general)
- Order of transformations matters!



Submission deadline: Friday, 1. November 2019 9:45 (before the lecture)
Written solutions have to be submitted in the lecture room before the lecture. Every assignment sheets counts 100 points (theory and practice)

4.* Convolution vs Multiplication (30 Points) (voluntary / bonus points)

The convolution of a function $f(t)$ with a second function $g(t)$ is defined as:

$$(f \otimes g)(t) = \int_{-\infty}^{\infty} f(\tau) \cdot g(t-\tau) d\tau$$

The multiplication of two function is defined as the pointwise multiplication:

$$(f \cdot g)(t) = f(t) \cdot g(t)$$

The transformation of a signal $f(x)$ to Fourier space is given by:

$$F(k) = \int_{-\infty}^{\infty} f(x) \cdot e^{-2\pi i k x} dx$$

We call $\mathcal{F}$ the operator mapping f to Fourier space: $\mathcal{F}f = F$. Show that convolving in signal space is the same as multiplication in Fourier space:

$$\mathcal{F}[f \otimes g] = \mathcal{F}[f] \cdot \mathcal{F}[g]$$