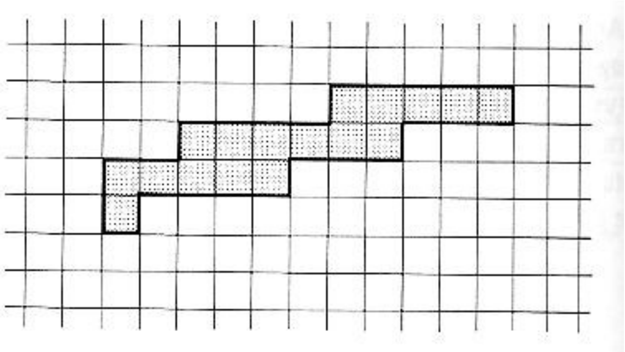
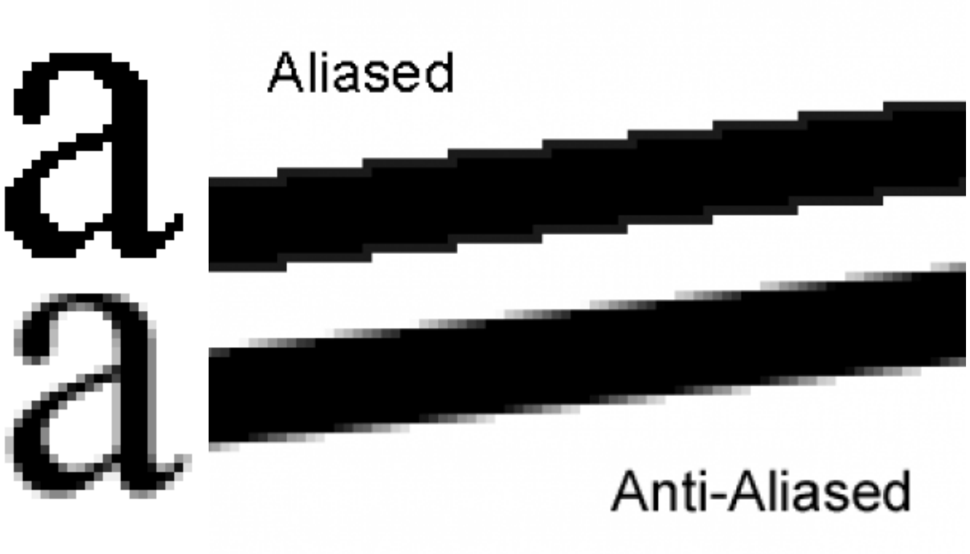
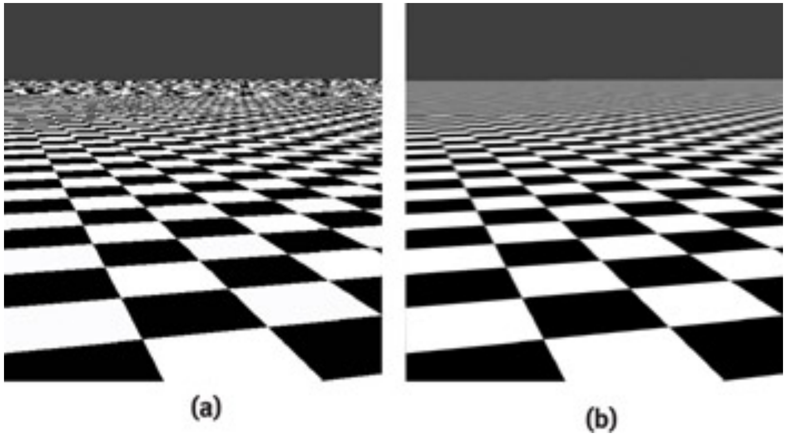
Lecture 9:

Signal Processing

Contents




Aliased
Anti-Aliased


(a)
(b)

Nature

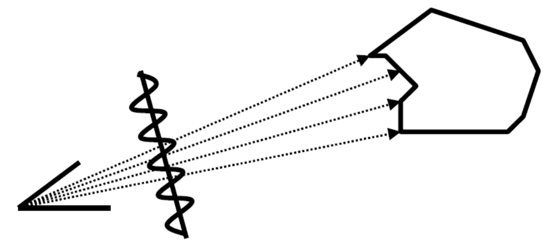
Continuous signal (2D / 3D / 4D with time)

- Defined at all points

Acquisition

Sampling

- Rays, pixel / texel, spectral values, frames, *etc*.



Representation

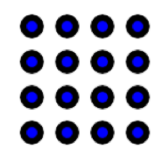
Discrete data

- Discrete points, discretized values

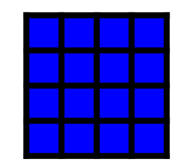
Reconstruction

Interpolation

- Mimic continuous signal



, not



Impression

Natural

- Hopefully similar to the original signal, no artifacts

Sampling of signals

- Conversion of a continuous signal to discrete samples by integrating over the sensor field
- Required by physical processes

$$R(i,j) = \int_{A_{ij}} E(x,y)P_{ij}(x,y)dxdy$$

Examples

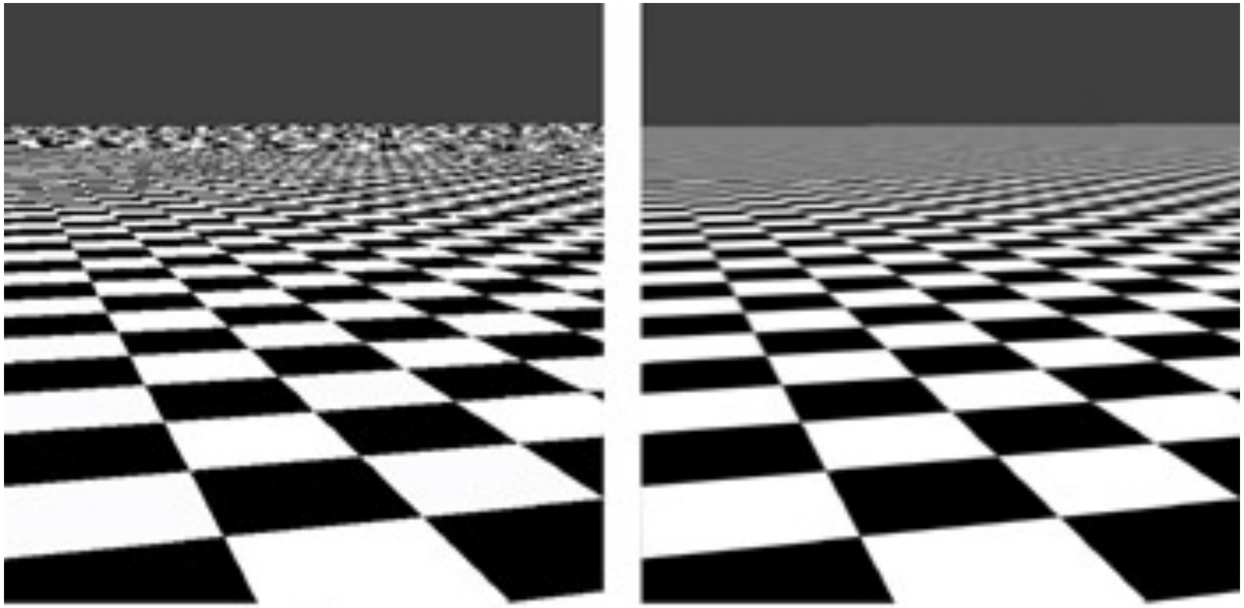
- Photo receptors in the retina
- CCD or CMOS cells in a digital camera

Virtual cameras in computer graphics

- Integration usually avoided
    - Too expensive
- Ray tracing: mathematically ideal point samples
    - Origin of alias!

Ray tracing

- Textured plane
- Checkerboard period becomes smaller than two pixels
  - Nyquist limit
- One ray for each pixel (say, at pixel center)
- Hits textured plane at only one point, black or white by “chance”
- (correct: integrate over pixel pre-image: texture mapping lecture)



(a) (b)

Frequency: period length of some structure in an image

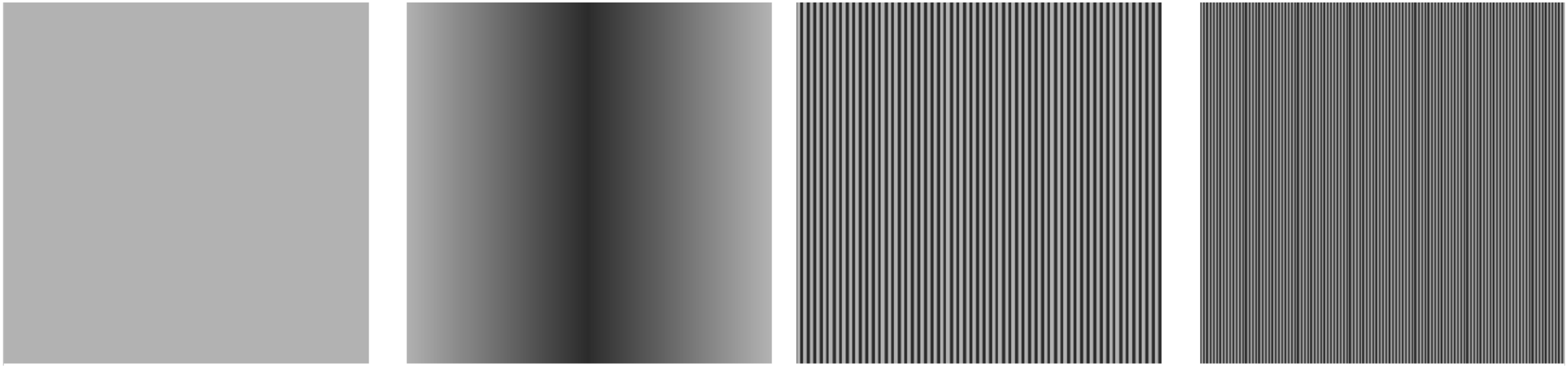
- Unit [1/pixel]
- Range: $-0.5 \cdots 0.5$ $(-\pi \cdots \pi)$

Lowest frequency

- Image average

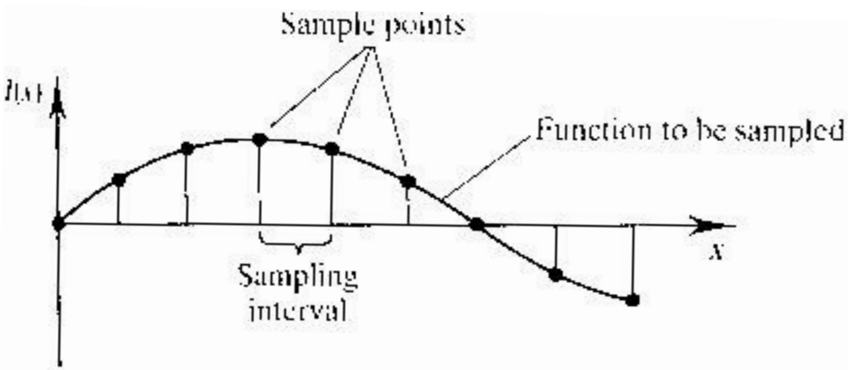
Highest frequency: Nyquist

- **In nature**: light wavelength
- **In graphics**: image resolution

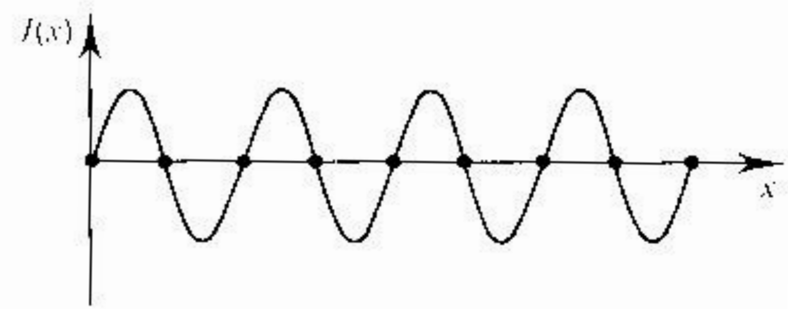


Definition:

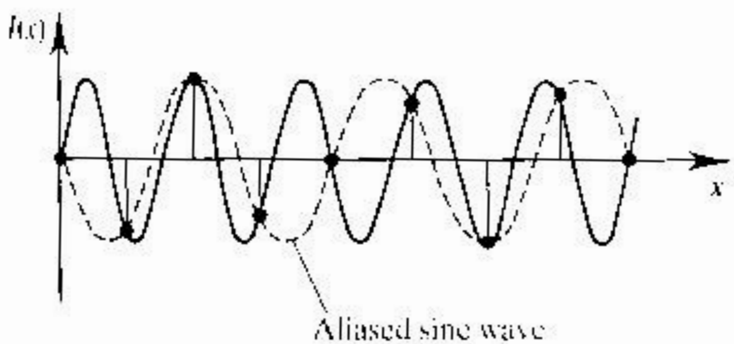
- **Nyquist Frequency** is the highest (spatial) frequency that can be represented
- Determined by image resolution (pixel size):



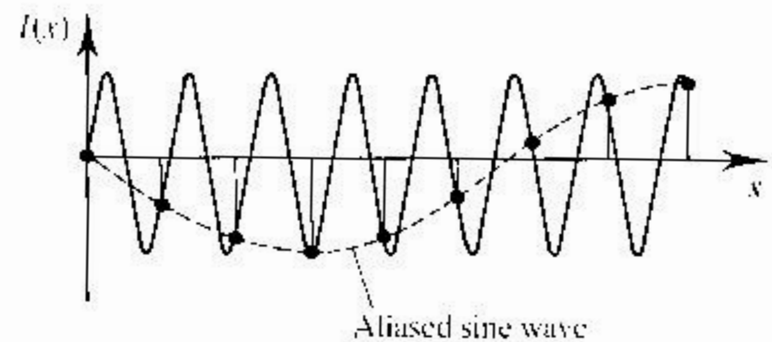
Spatial frequency < Nyquist



Spatial frequency = Nyquist
2 samples / period



Spatial frequency > Nyquist

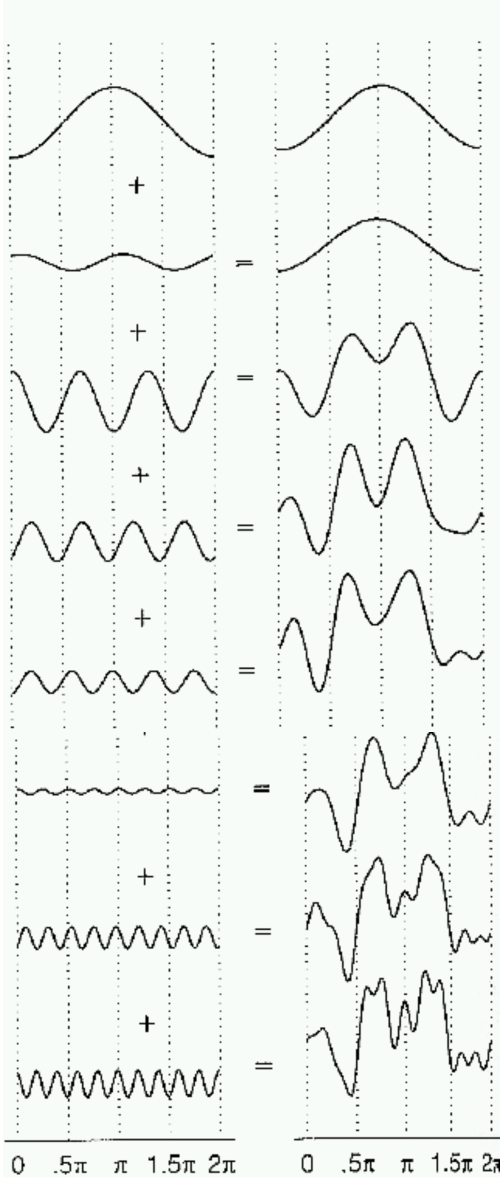


Spatial frequency >> Nyquist

Any continuous function $f(x)$ can be expressed as an integral over sine and cosine waves:

$$f(x) = \int_{-\infty}^{\infty} a(\omega)\sin(2\pi\omega x) + b(\omega)\cos(2\pi\omega x)\, d\omega =$$

$$= \int_{-\infty}^{\infty} F(\omega) e^{i2\pi\omega x} d\omega$$



- $a(\omega) = \int_{-\infty}^{\infty} \sin(2\pi\omega x)\, f(x) dx$
- $b(\omega) = \int_{-\infty}^{\infty} \cos(2\pi\omega x)\, f(x) dx$
- $\mathrm{F}(\omega) = \int_{-\infty}^{\infty} e^{-i2\pi\omega x} f(x) dx$
- Sine & cosine: orthonormal basis functions
- $a(\omega)$, $b(\omega)$: “weighting” functions
- $\mathrm{F}(\omega)$: (complex-valued) Fourier transform

Any periodic, continuous function can be expressed as the sum of an (infinite) number of sine or cosine waves:

$$f(x) = \sum_k a_k\, sin(2\pi kx) + b_k\, cos(2\pi kx)$$

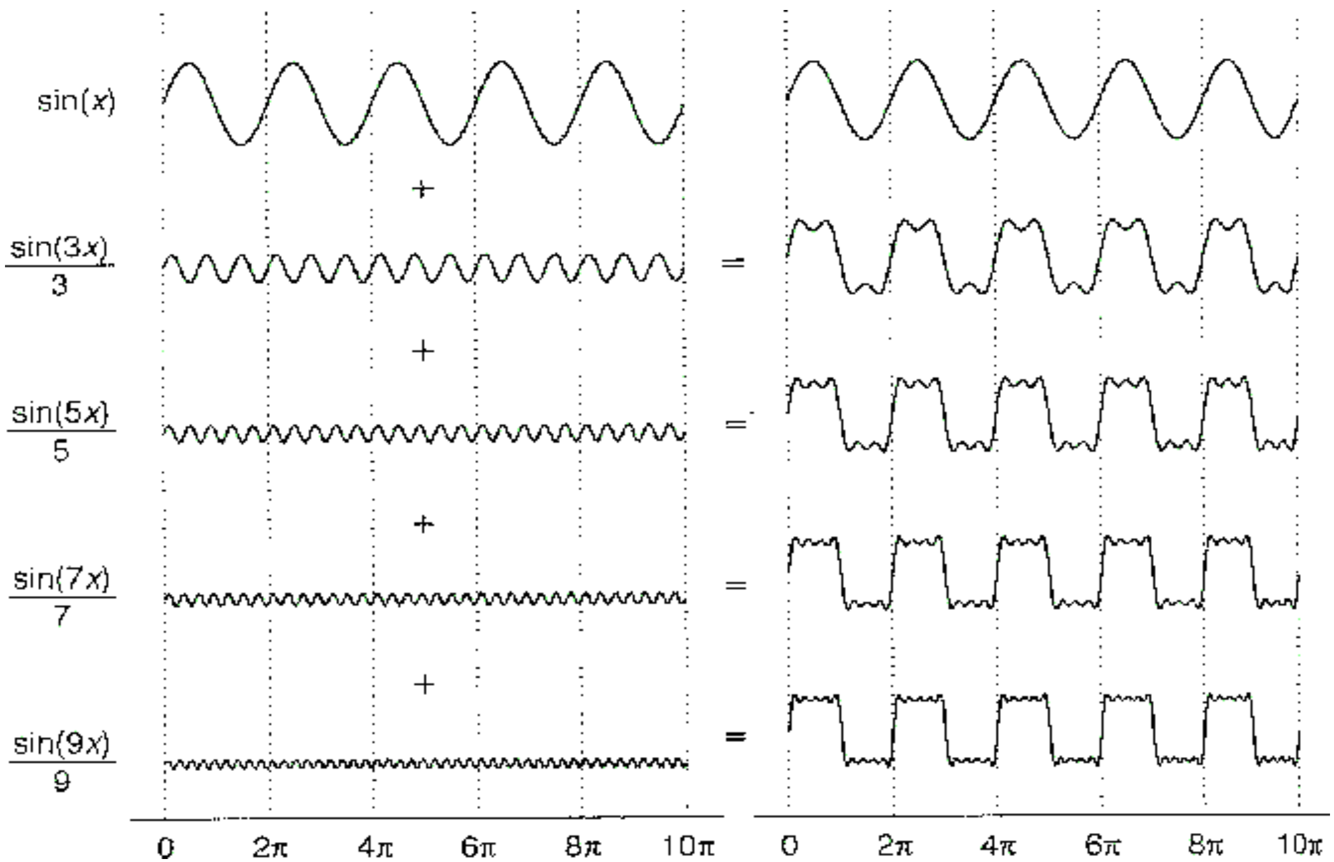
- Decomposition of signal into different frequency bands
  - Spectral analysis
- k: frequency band
  - $k = 0$ mean value
  - $k = 1$ function period, lowest possible frequency
  - $k = 1{,}5?$ not possible, periodic function $f(x) = f(x+1)$
  - $k_{max}?$ band limit, no higher frequency present in signal
- a_k, b_k : (real-valued) Fourier coefficients
- Even function $f(x) = f(-x)$: $a_k = 0$
- Odd function $f(x) = -f(-x)$: $b_k = 0$

Periodic, uneven function: square wave

$$f(x) = \begin{cases} 0.5 & \forall \quad 0 < (x \bmod 2\pi) < \pi \\ -0.5 & \forall \quad \pi < (x \bmod 2\pi) < 2\pi \end{cases}$$

$$a_k = \int \sin(kx)\, f(x)\, dx \qquad\qquad f(x) = \sum_k a_k \sin(kx)$$

- $a_0 = 0$
- $a_1 = 1$
- $a_2 = 0$
- $a_3 = {}^1/_3$
- $a_4 = 0$
- $a_5 = {}^1/_5$
- $a_6 = 0$
- $a_7 = {}^1/_7$
- $a_8 = 0$
- $a_9 = {}^1/_9$
- …



Equally-spaced function samples

- Function values known only at discrete points
  - Physical measurements
  - Pixel positions in an image!

Fourier Analysis

$$a_k = \frac{1}{N}\sum_i \sin(2\pi k \frac{i}{N}) f_i \qquad\qquad b_k = \frac{1}{N}\sum_i \cos(2\pi k \frac{i}{N}) f_i$$

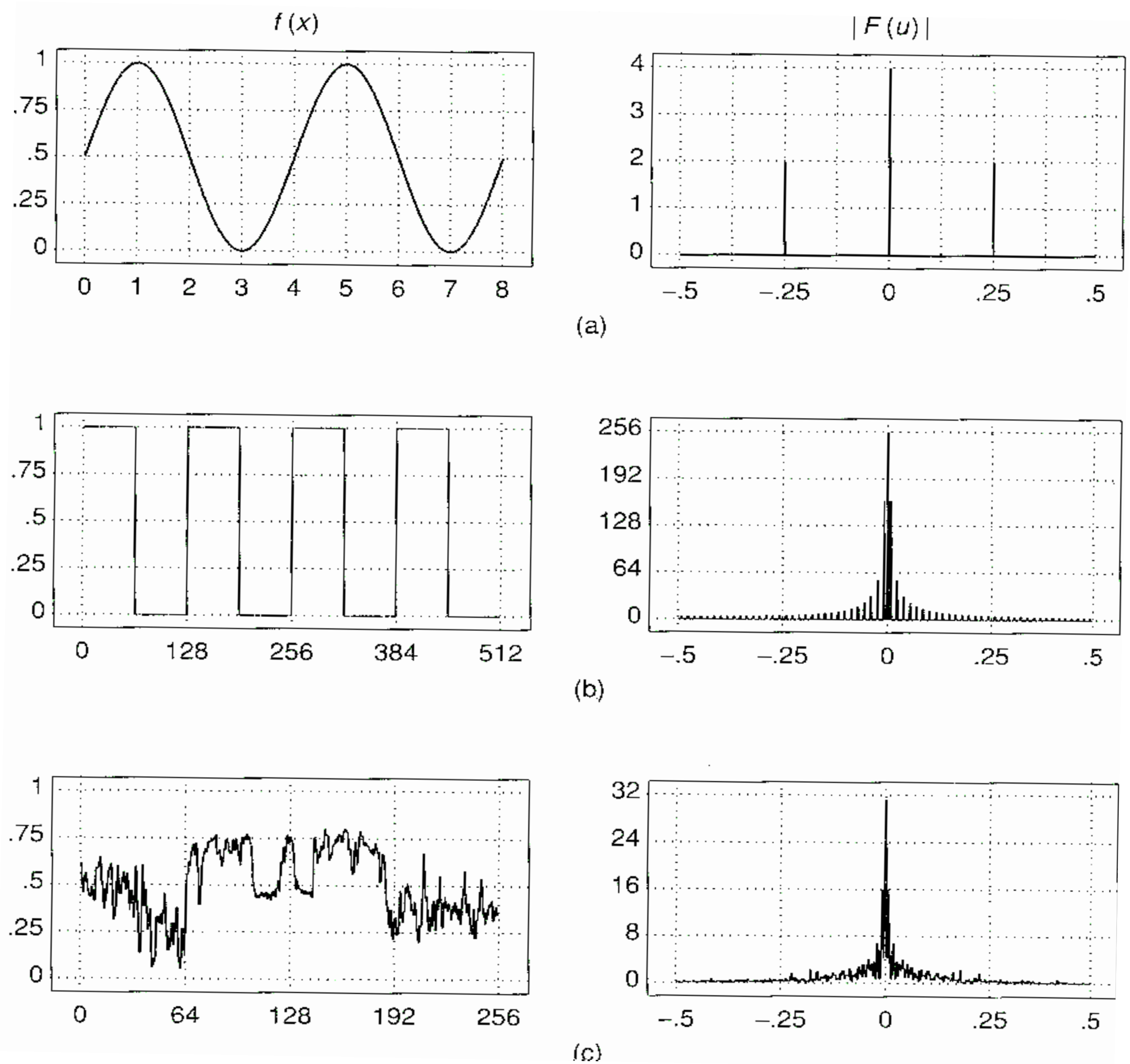
- Sum over all measurement points N
- $k = 0, 1, 2, \ldots$? Highest possible frequency?

⇒Nyquist frequency

- Sampling rate N_i
- 2 samples / period ⇔ 0.5 cycles per pixel
- $\Rightarrow k \leq \frac{N}{2}$

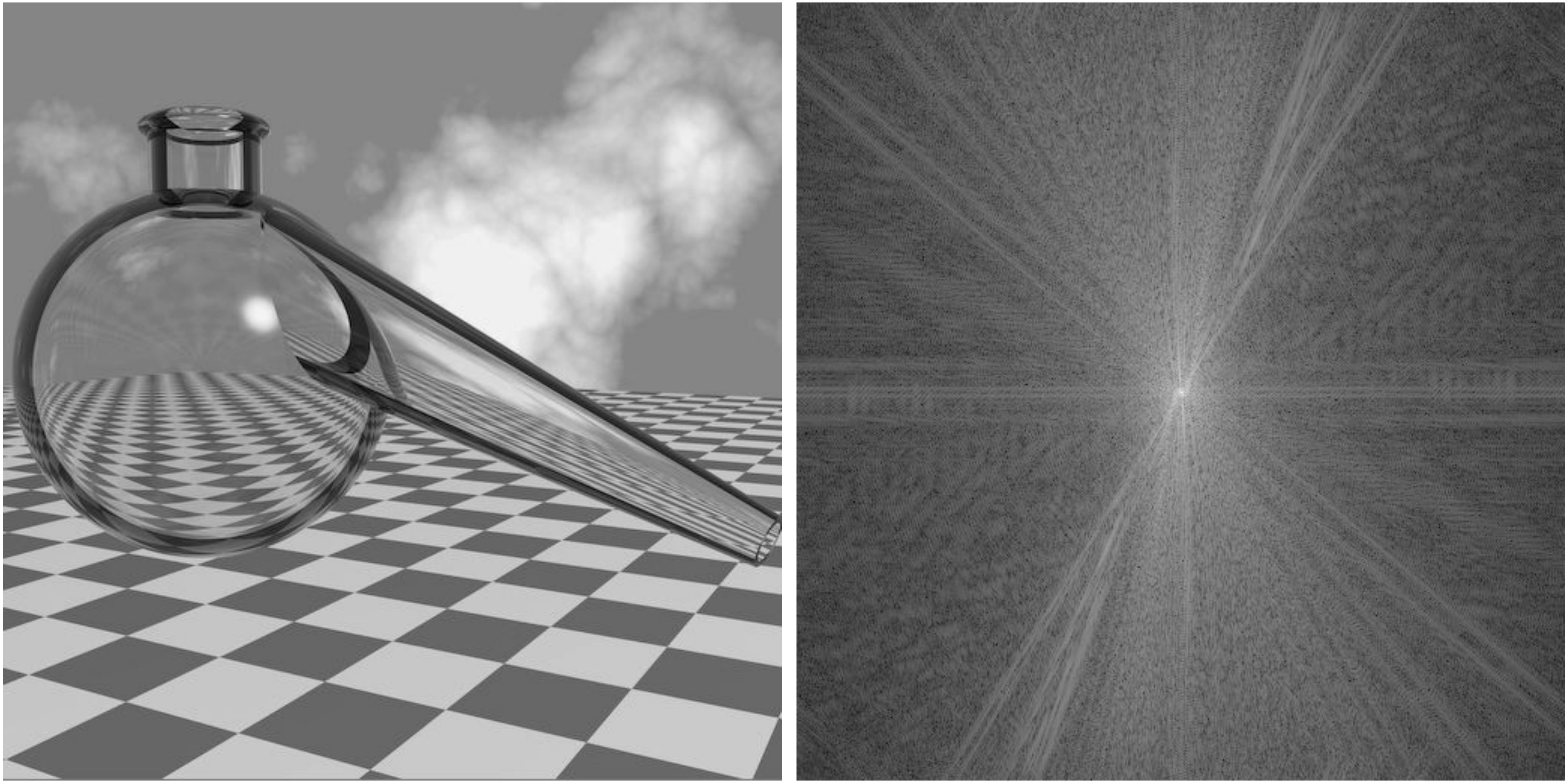
Examples (pixel vs cycles per pixel)

- Sine wave with positive offset
- Square wave
- Scanline of an image

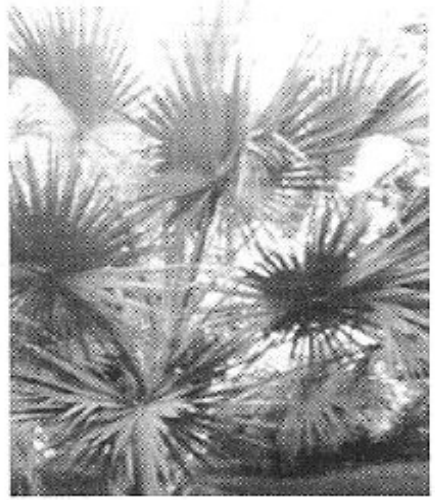


Example for 2D case:

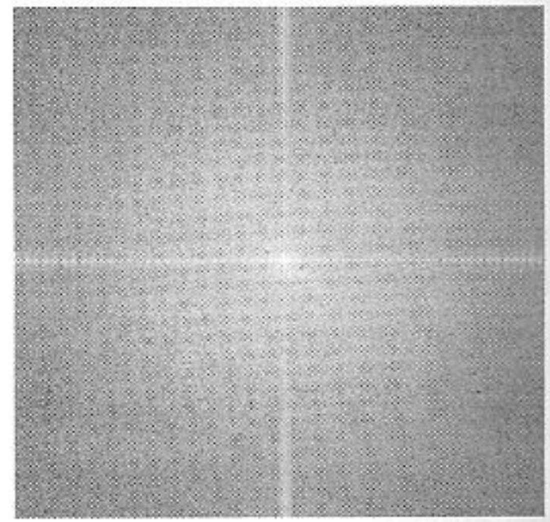
- 2 separate 1D Fourier transformations along x- and y- direction
- Discontinuities: orthogonal direction in Fourier domain!



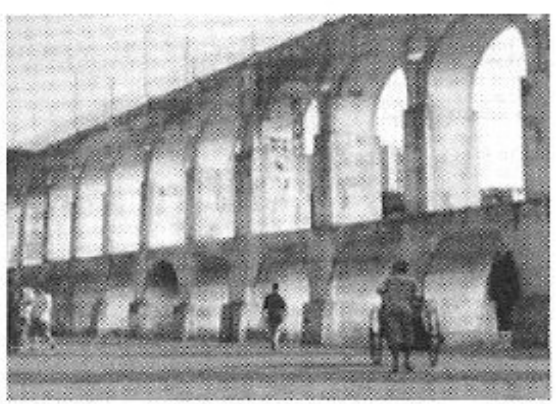
Example for 2D case:



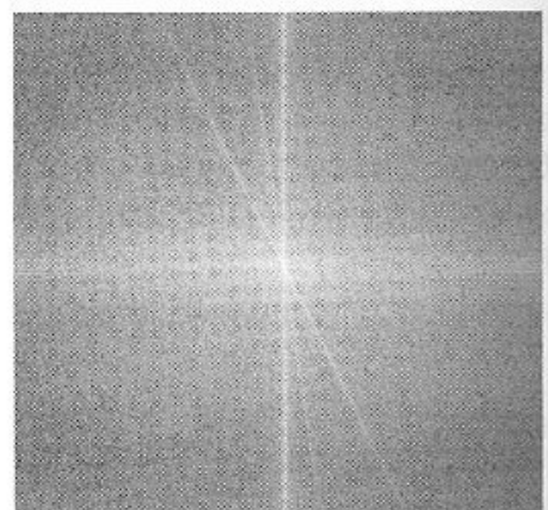
(a) Bush



Fourier transform $|F(u, v)|$



(b) Arcos da Lapa
(Rio de Janeiro)



Fourier transform $|F(u, v)|$

Important basis functions

- Box $\Leftrightarrow$ sinc (*sinus cardinalis*)

$$\mathrm{sinc}(x) = \frac{\sin(x\pi)}{x\pi}$$

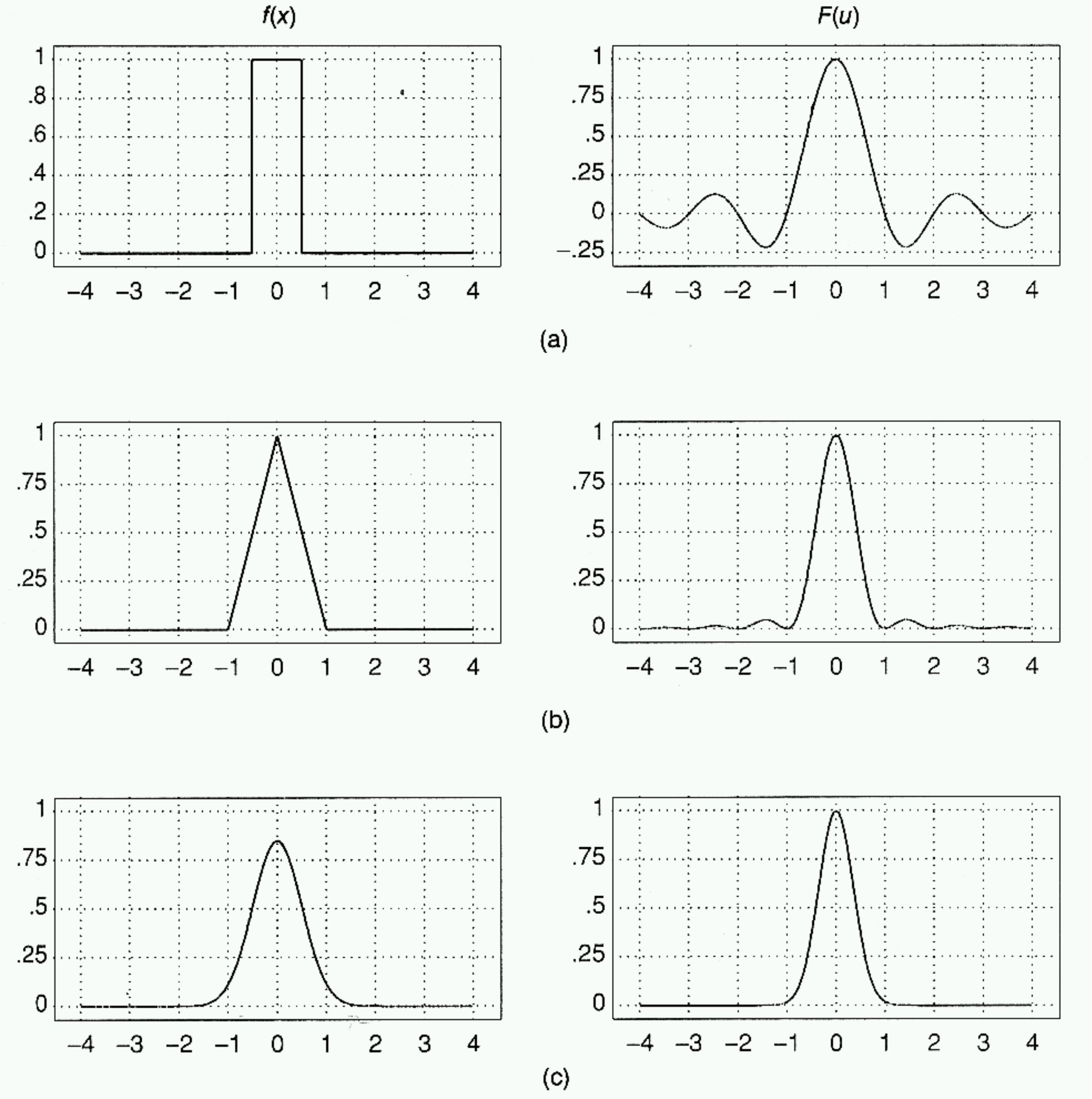
$$\mathrm{sinc}(0) = 1$$

$$\int \mathrm{sinc}(x)dx = 1$$

  - Wide box $\Rightarrow$ small sinc
  - Negative values
  - Infinite support

- Triangle $\Leftrightarrow$ sinc^2

- Gauss $\Leftrightarrow$ Gauss

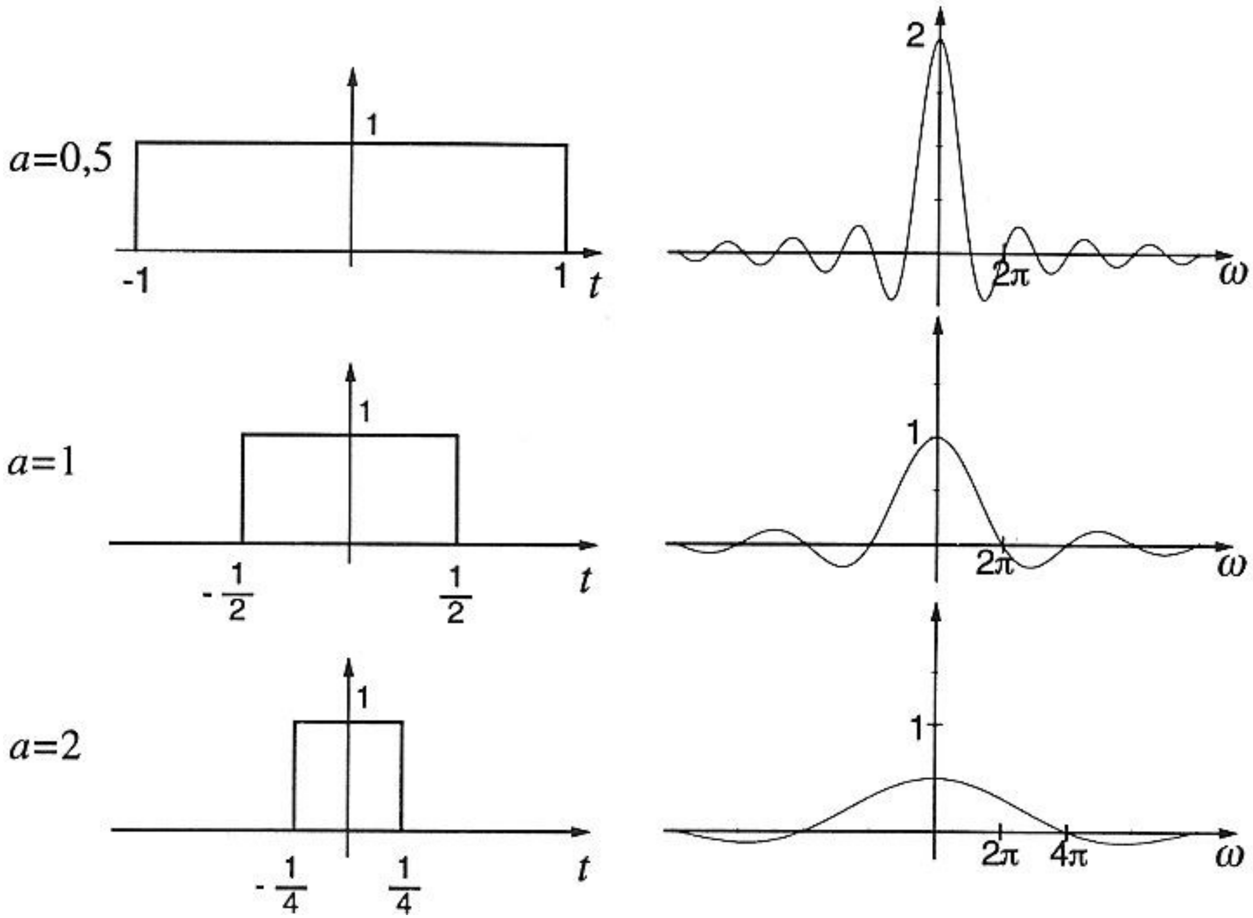


Transform behavior

Example: box function

$$\mathrm{rect}(at) \quad \circ\!\!-\!\!\bullet \quad \frac{1}{|a|}\,\mathrm{si}\left(\frac{\omega}{2a}\right).$$

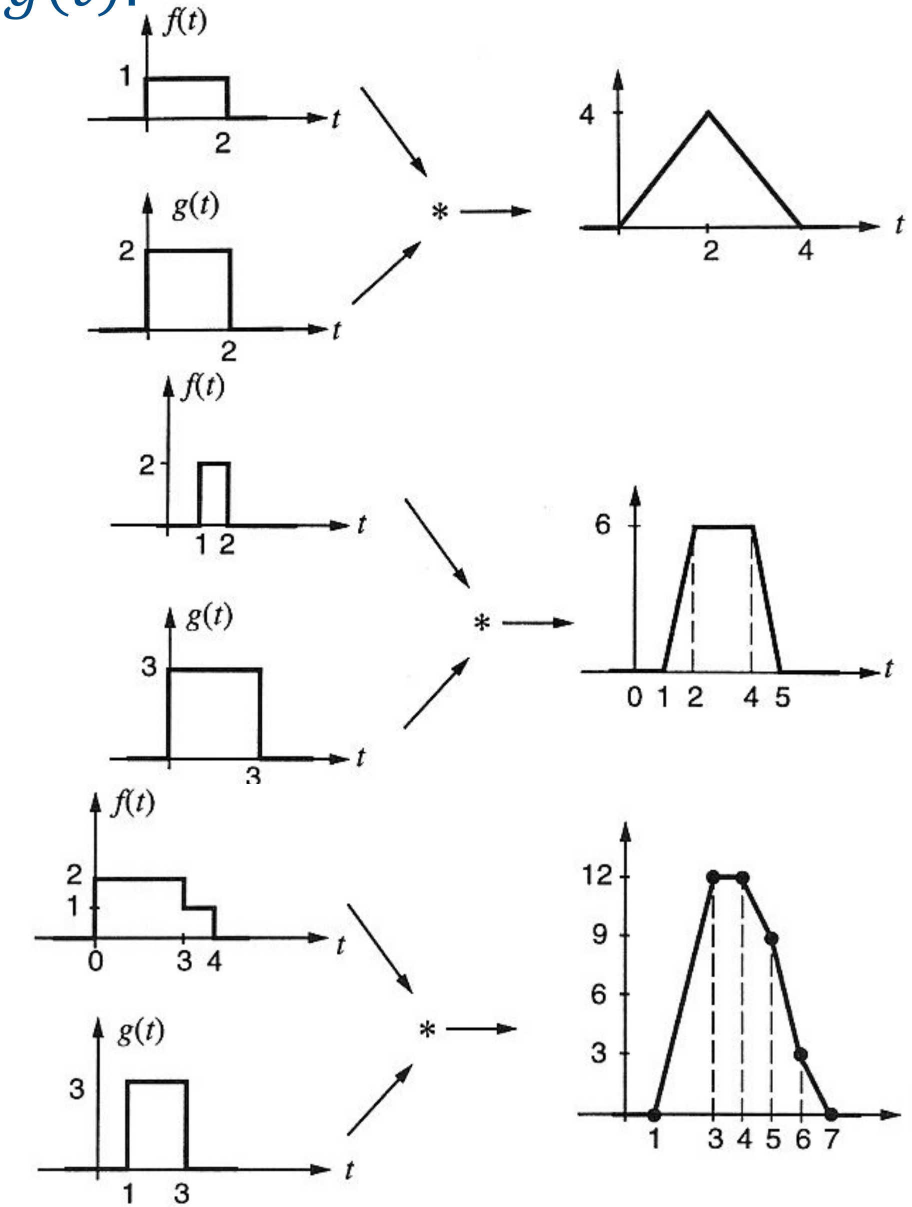
- Fourier transform: sinc
- Wide box: narrow sinc
- Narrow box: wide sinc



Convolution of two functions $f(t)$ and $g(t)$:

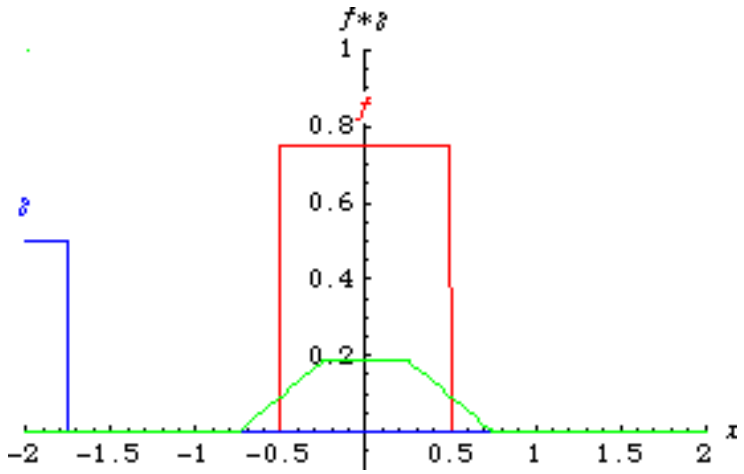
$$(f * g)(t) = \int_{-\infty}^{\infty} f(x)g(t-x)dx$$

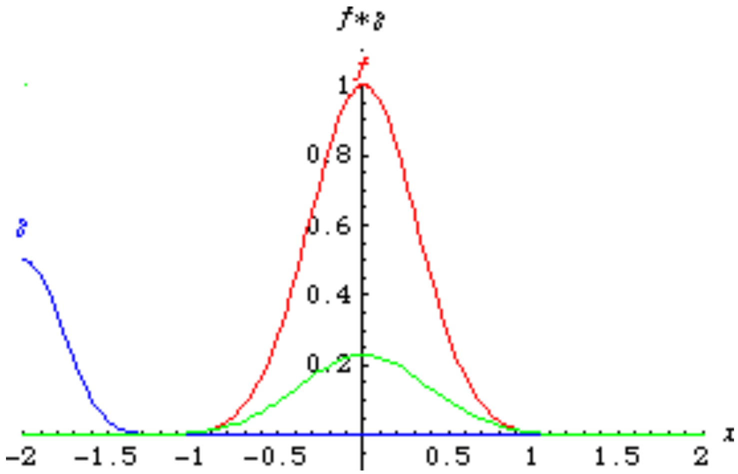
- Shift one function against the other by x
- Multiply function values
- Integrate overlapping region
- Numerical convolution: Expensive operation
  - For each x: integrate over non-zero domain



Examples

- Box functions
- Gauss functions



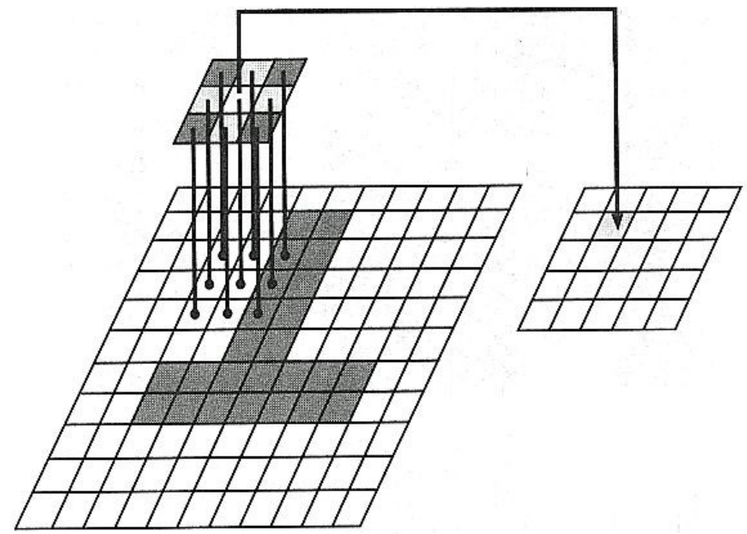


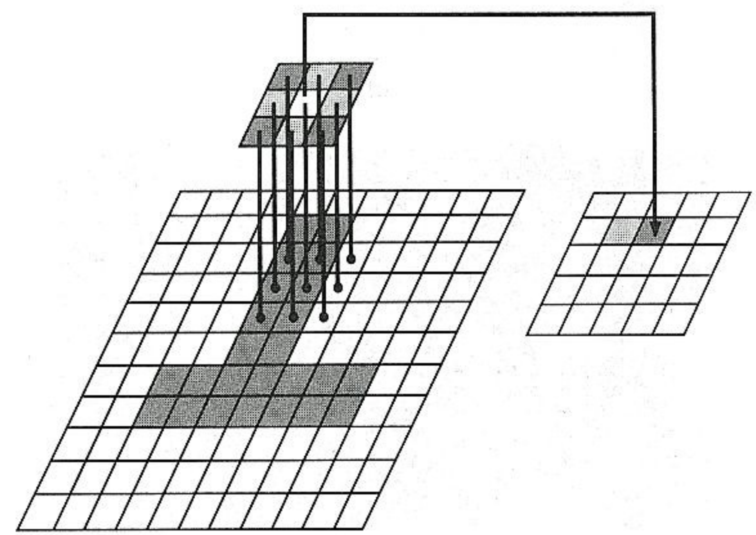
Technical Realization

- In image domain
- Pixel mask with weights
- OpenGL: Convolution extension

Problems (*e.g.* sinc)

- Large filter support
  - Large mask
  - A lot of computation
- Negative weights
  - Negative light?

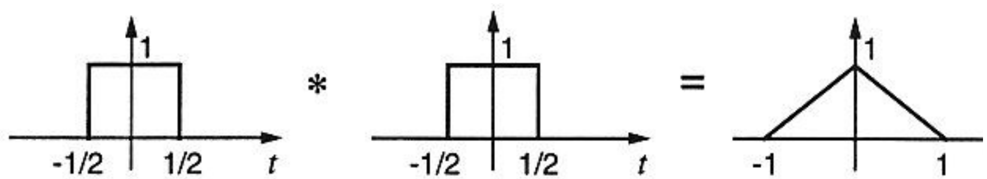




Convolution in image domain: multiplication in Fourier domain

Convolution in Fourier domain: multiplication in image domain

- Multiplication much cheaper than convolution!

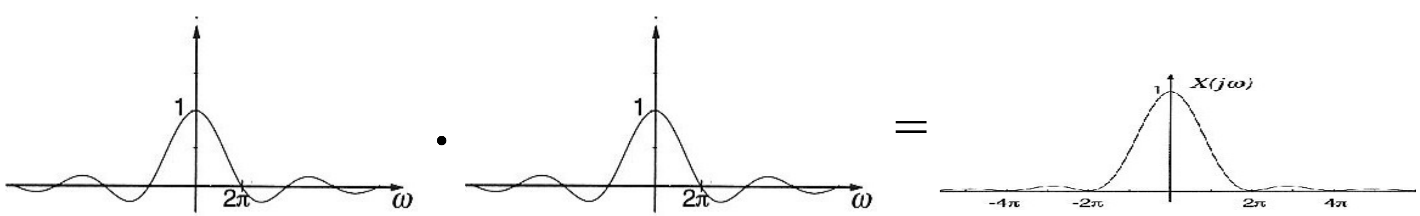


$$x(t) = \begin{cases} t+1 & \text{für } -1 \leq t < 0 \\ -t+1 & \text{für } 0 \leq t < 1 \\ 0 & \text{sonst} \end{cases}$$

$$\operatorname{rect}(t) * \operatorname{rect}(t) = x(t)$$

$$\circ\!\!-\!\!\bullet \qquad \circ\!\!-\!\!\bullet \qquad \circ\!\!-\!\!\bullet$$

$$\operatorname{si}\left(\frac{\omega}{2}\right) \cdot \operatorname{si}\left(\frac{\omega}{2}\right) = X(j\omega)\,.$$



$$X(j\omega) = \operatorname{si}^2\left(\frac{\omega}{2}\right).$$

Low-pass filtering

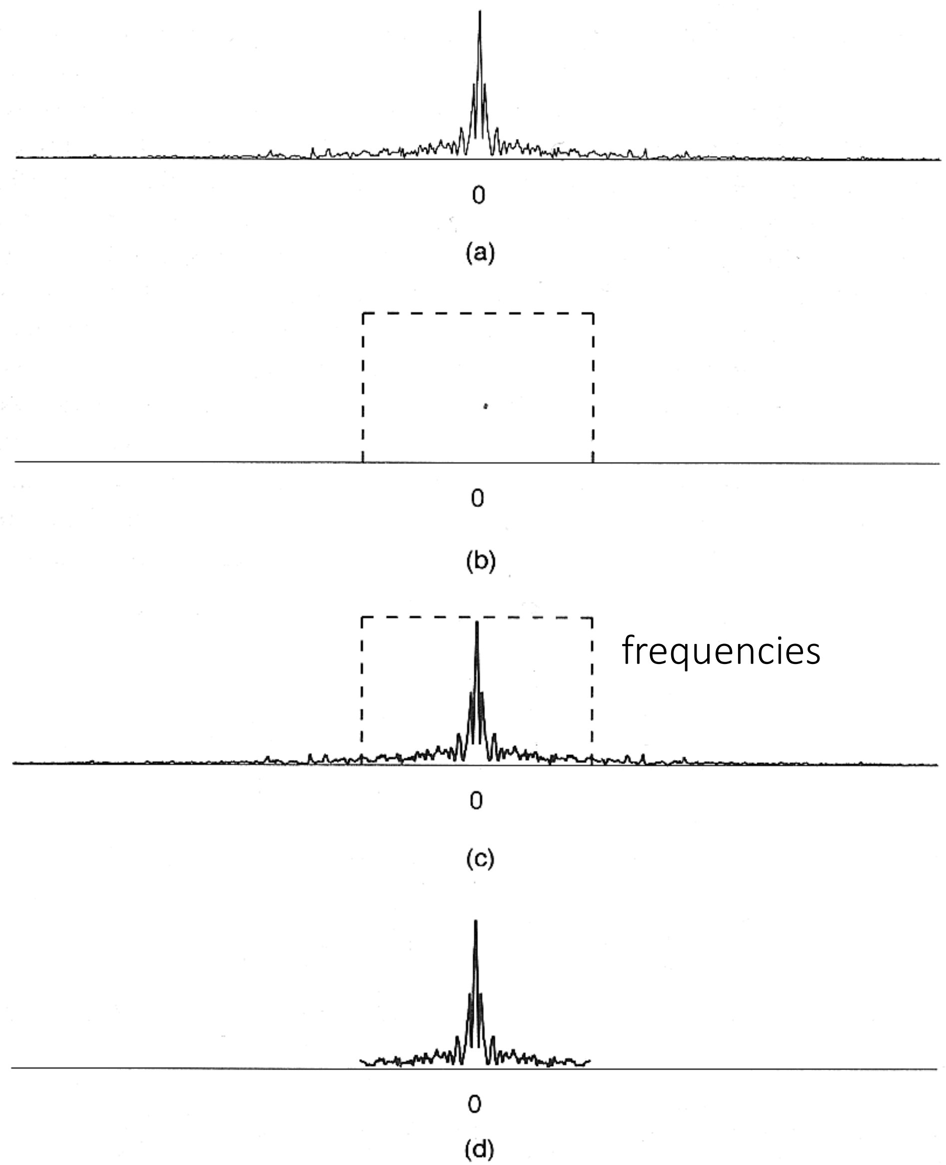
- Convolution with sinc in spatial domain, or
- Multiplication with box in frequency domain

High-pass filtering

- Only high frequencies

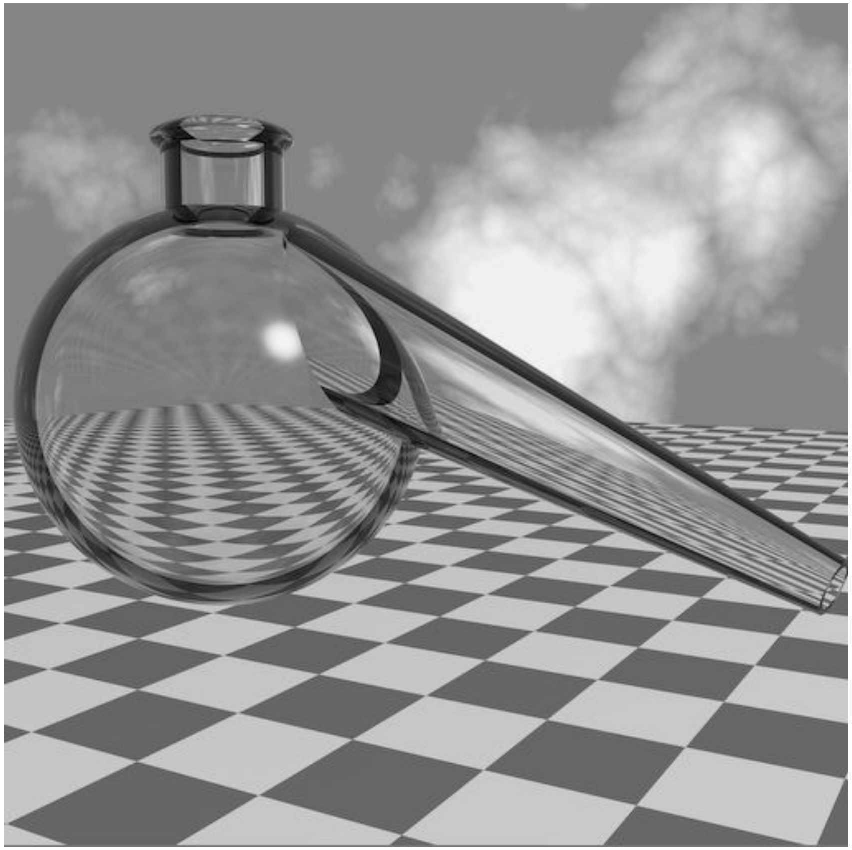
Band-pass filtering

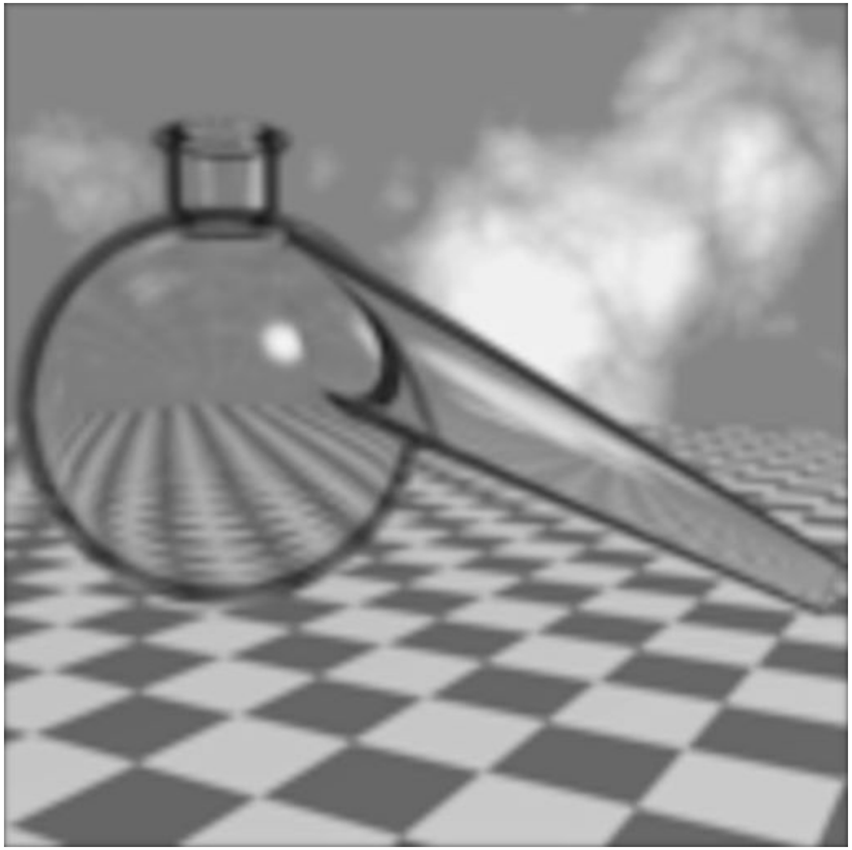
- Only intermediate



Low-pass filtering in frequency domain: multiplication with box

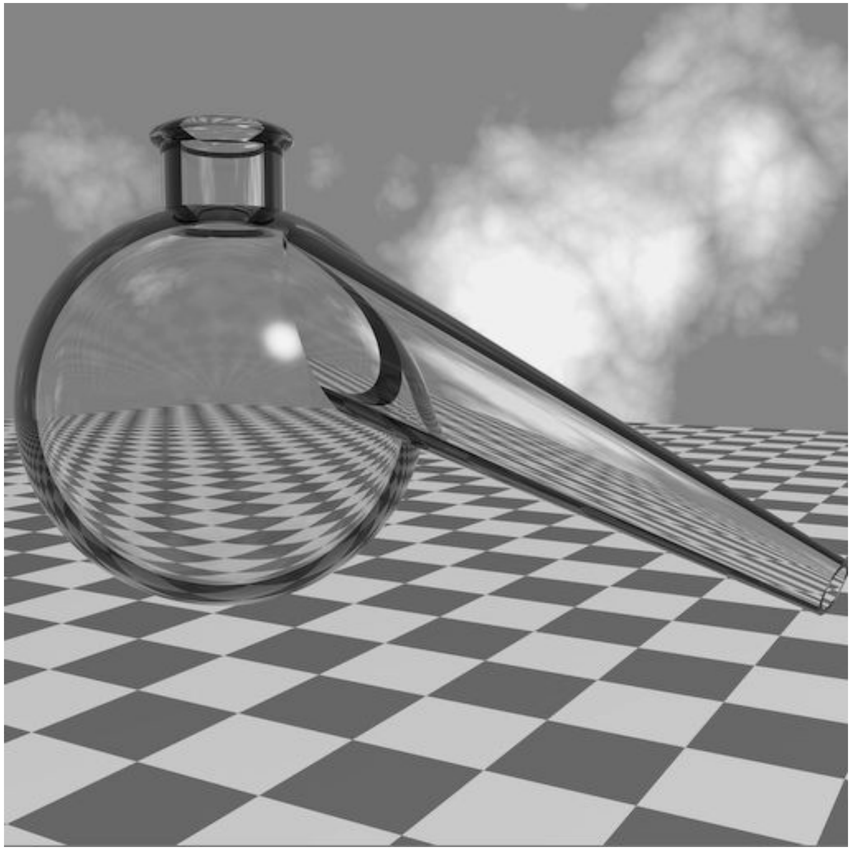
„Blurring“

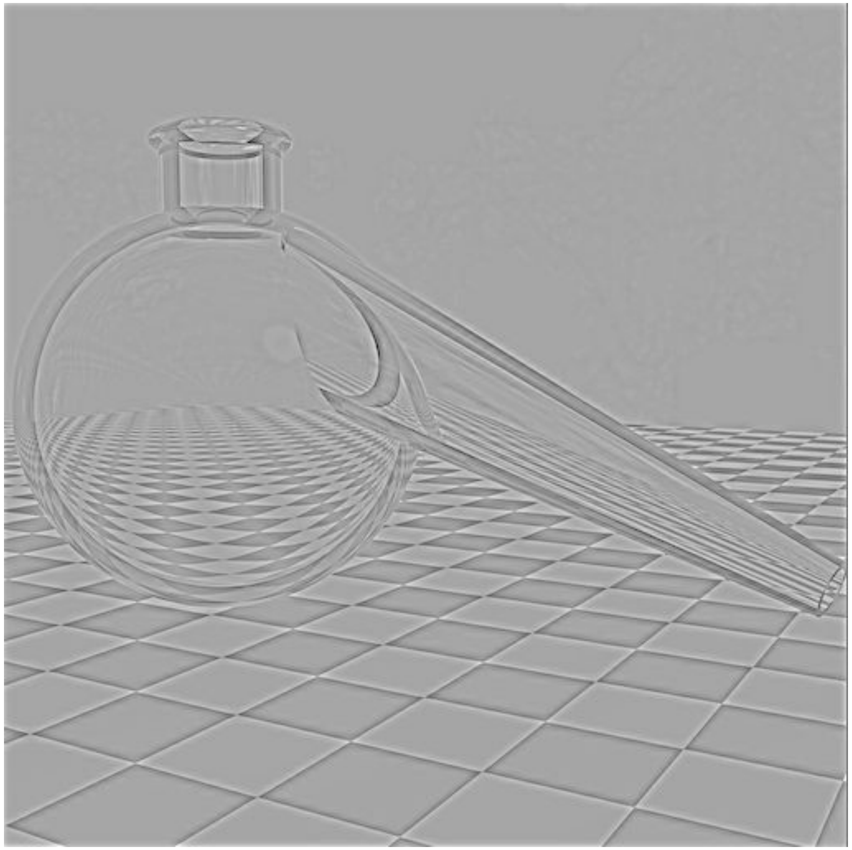




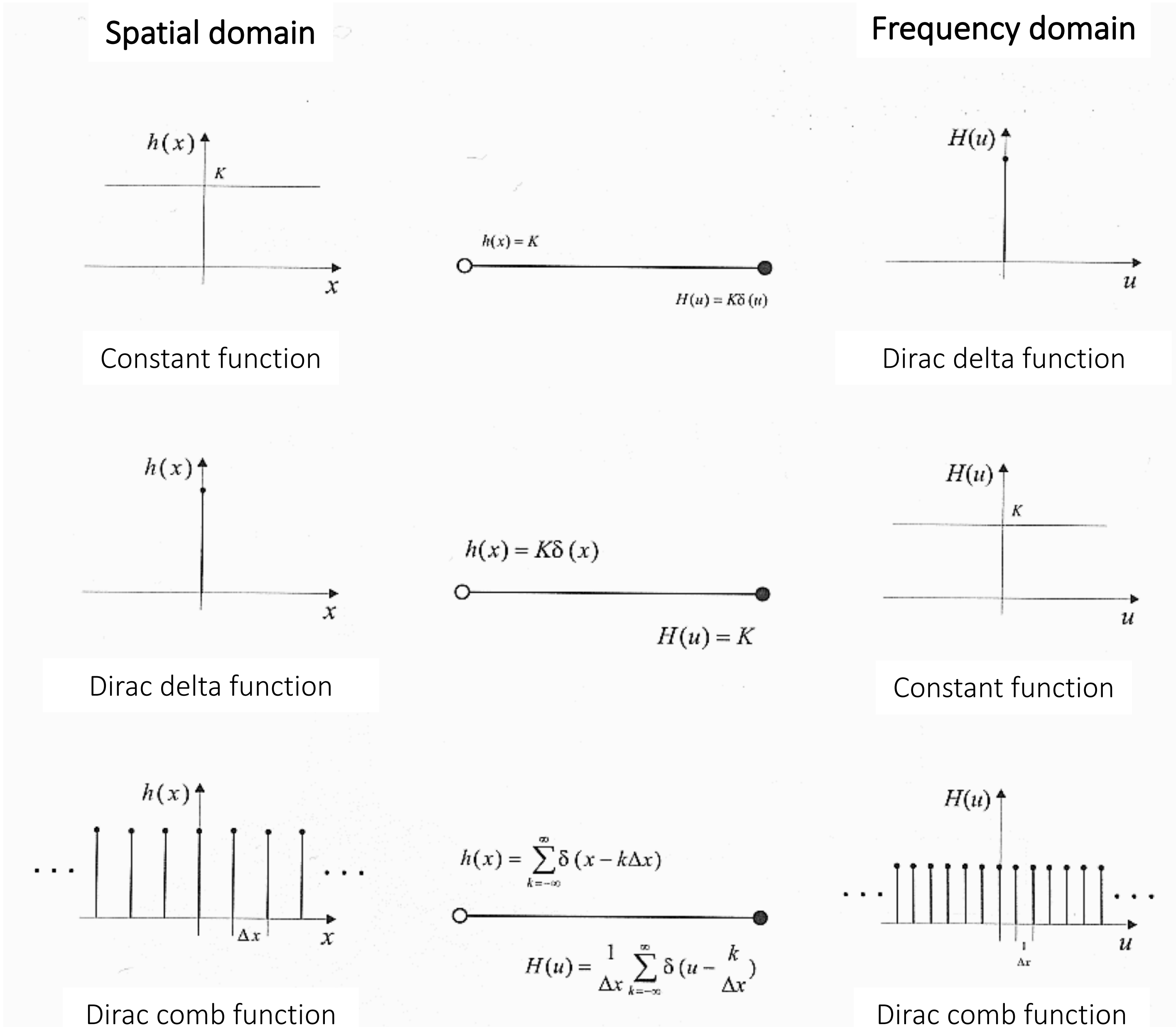
Enhances discontinuities in image

- Useful for edge detection





Sampling


Spatial domain
Frequency domain
h(x)
K
x
h(x) = K
H(u) = Kδ(u)
H(u)
u
Constant function
Dirac delta function
h(x)
x
h(x) = Kδ(x)
H(u) = K
H(u)
K
u
Dirac delta function
Constant function
h(x)
Δx
x
$h(x) = \sum_{k=-\infty}^{\infty} \delta(x - k\Delta x)$
$H(u) = \frac{1}{\Delta x} \sum_{k=-\infty}^{\infty} \delta(u - \frac{k}{\Delta x})$
H(u)
u
Dirac comb function
Dirac comb function

Constant & Dirac-delta function $\delta()$

- Duality

$$f(x) = K$$

$$F(\omega) = K\delta(\omega)$$

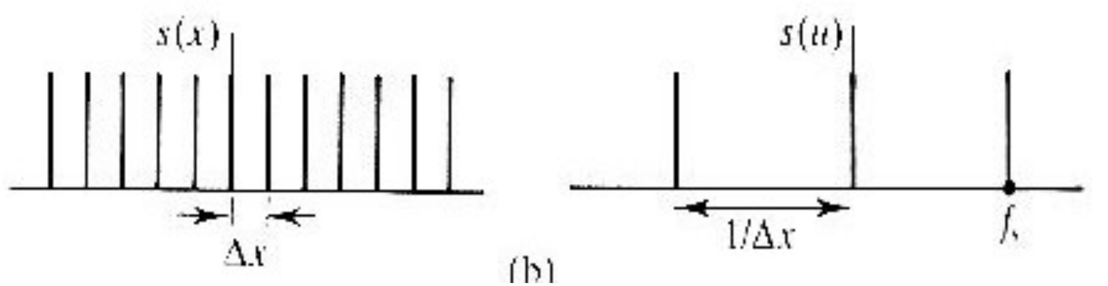
- And vice versa

Dirac-comb function

- Duality: The dual of a comb function is again a comb function
  - Inverse wave length, amplitude scales with inverse wave length

$$f(x) = \sum_{k=-\infty}^{\infty} \delta(x - k\Delta x)$$

$$F(\omega) = \frac{1}{\Delta x} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{1}{\Delta x}\right)$$

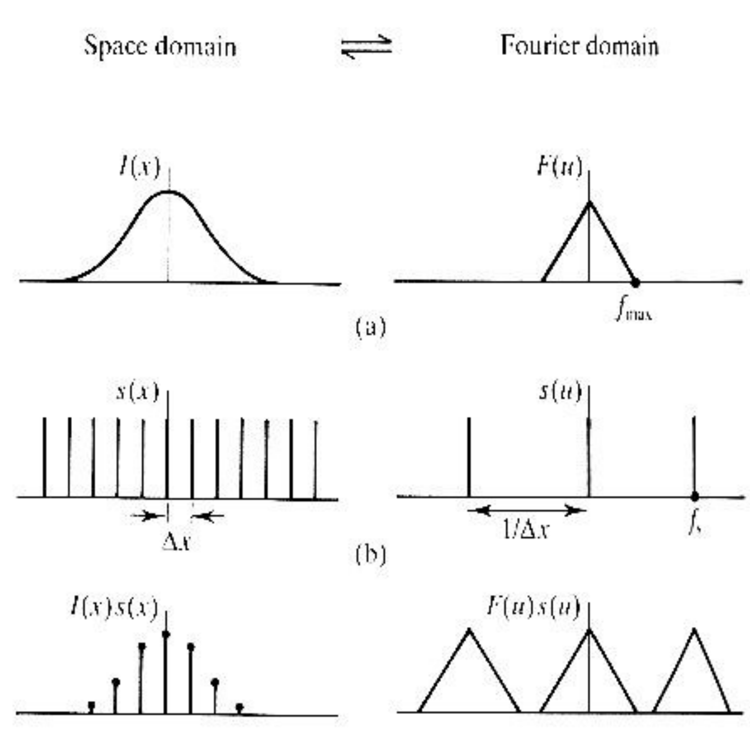


Continuous function

- Band-limited Fourier transform

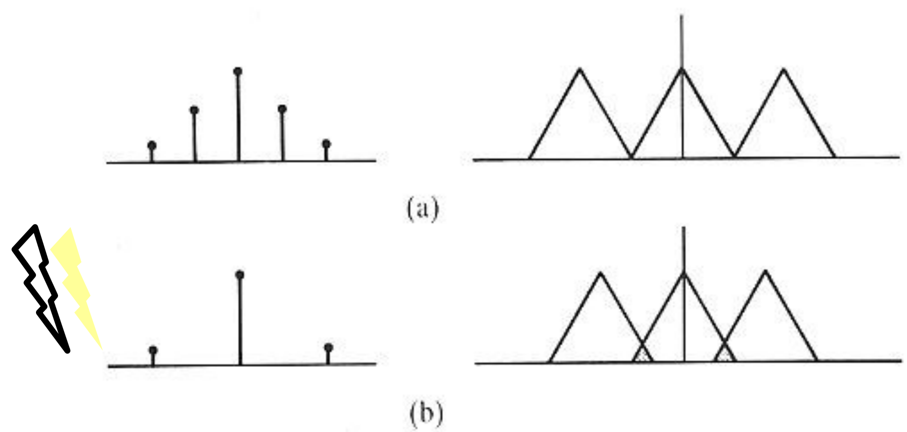
Sampled at discrete points

- Multiplication with Dirac-comb function in spacial domain corresponds to convolution in Fourier domain



Frequency bands overlap?

- No: good
- Yes: bad, **aliasing**



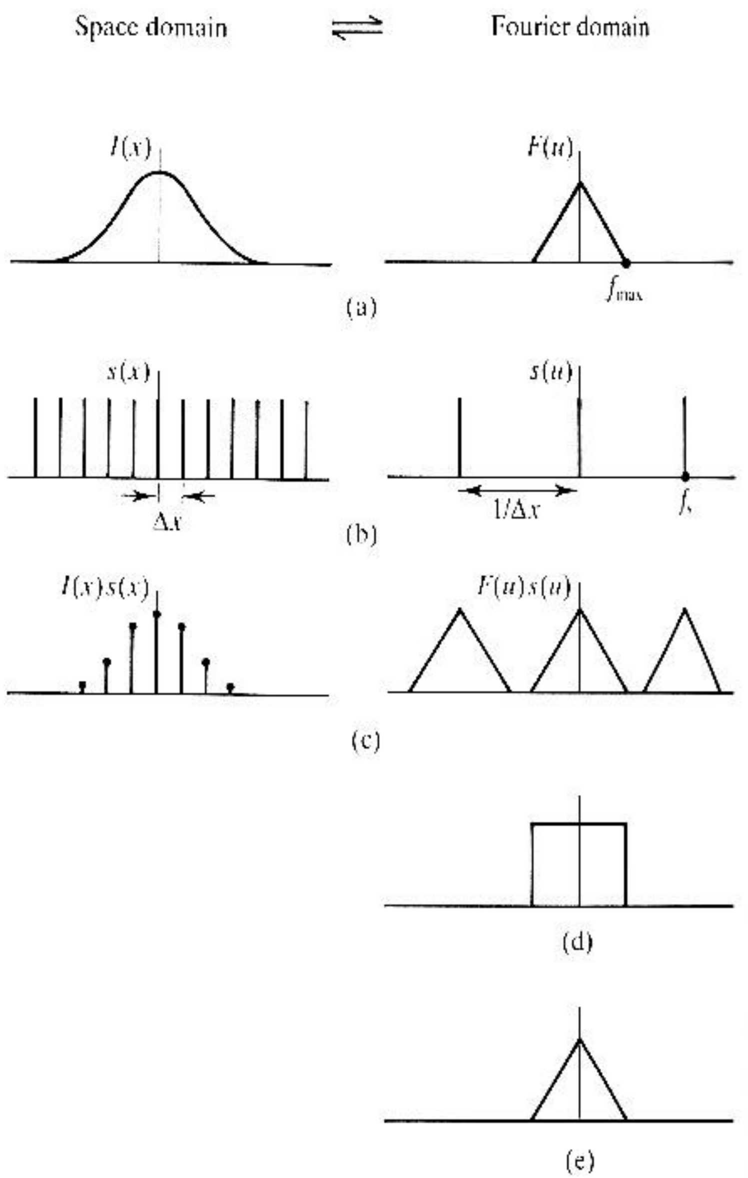
Only original frequency band desired

Filtering

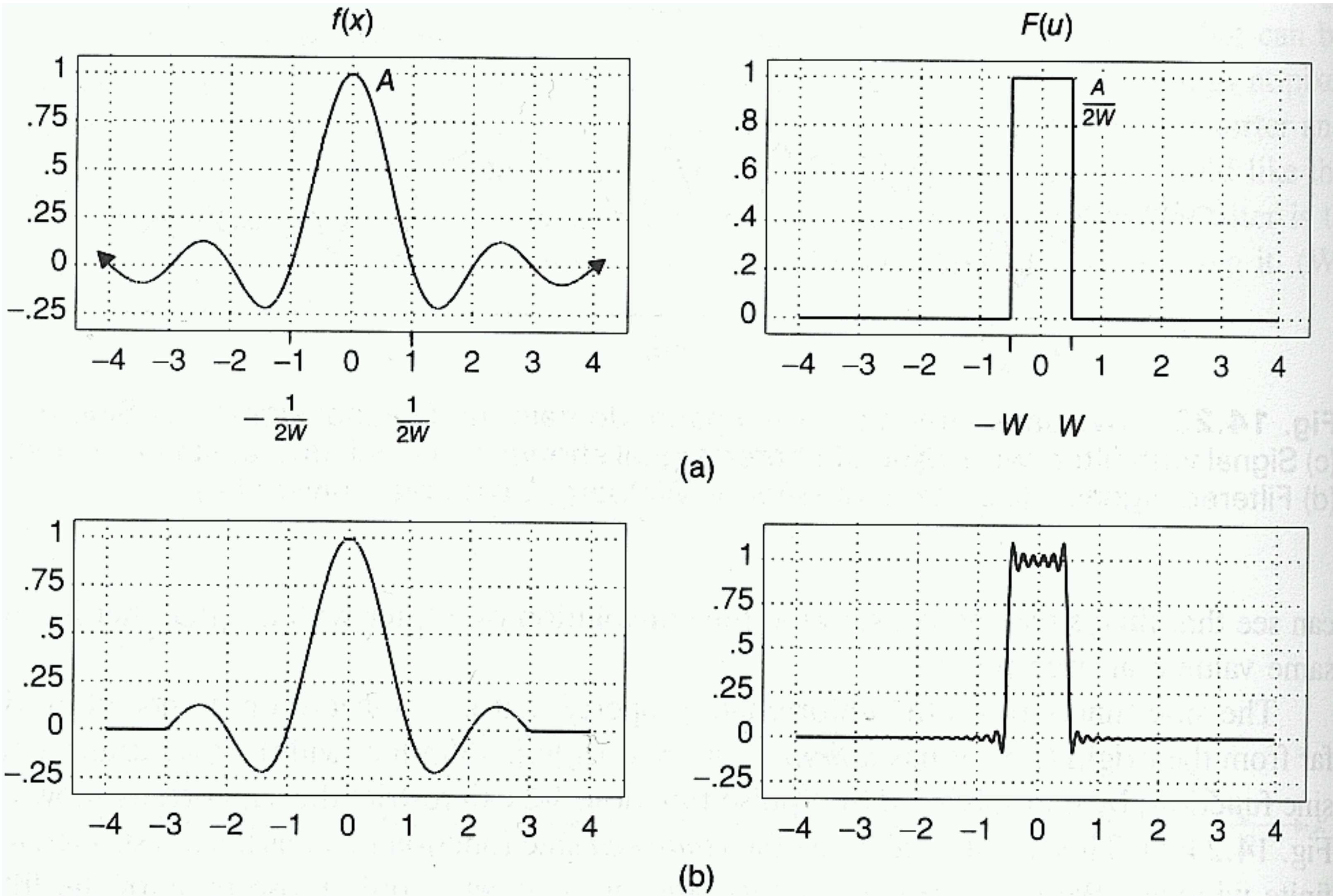
- In Fourier domain: multiplication with windowing function around origin
- In spatial domain: convolution with Fourier transform of windowing function

Optimal filtering function

- Box function in Fourier domain
- Corresponds to $sinc()$ in space domain
  - Unlimited region of support
- Spatial domain only allows approximations (limited support)



Cutting off the support is *not* a good solution



Original function and its band-limited frequency spectrum

Signal sampling:

Multiplication / convolution with comb

Comb dense enough (sampling≥2*bandlimit)

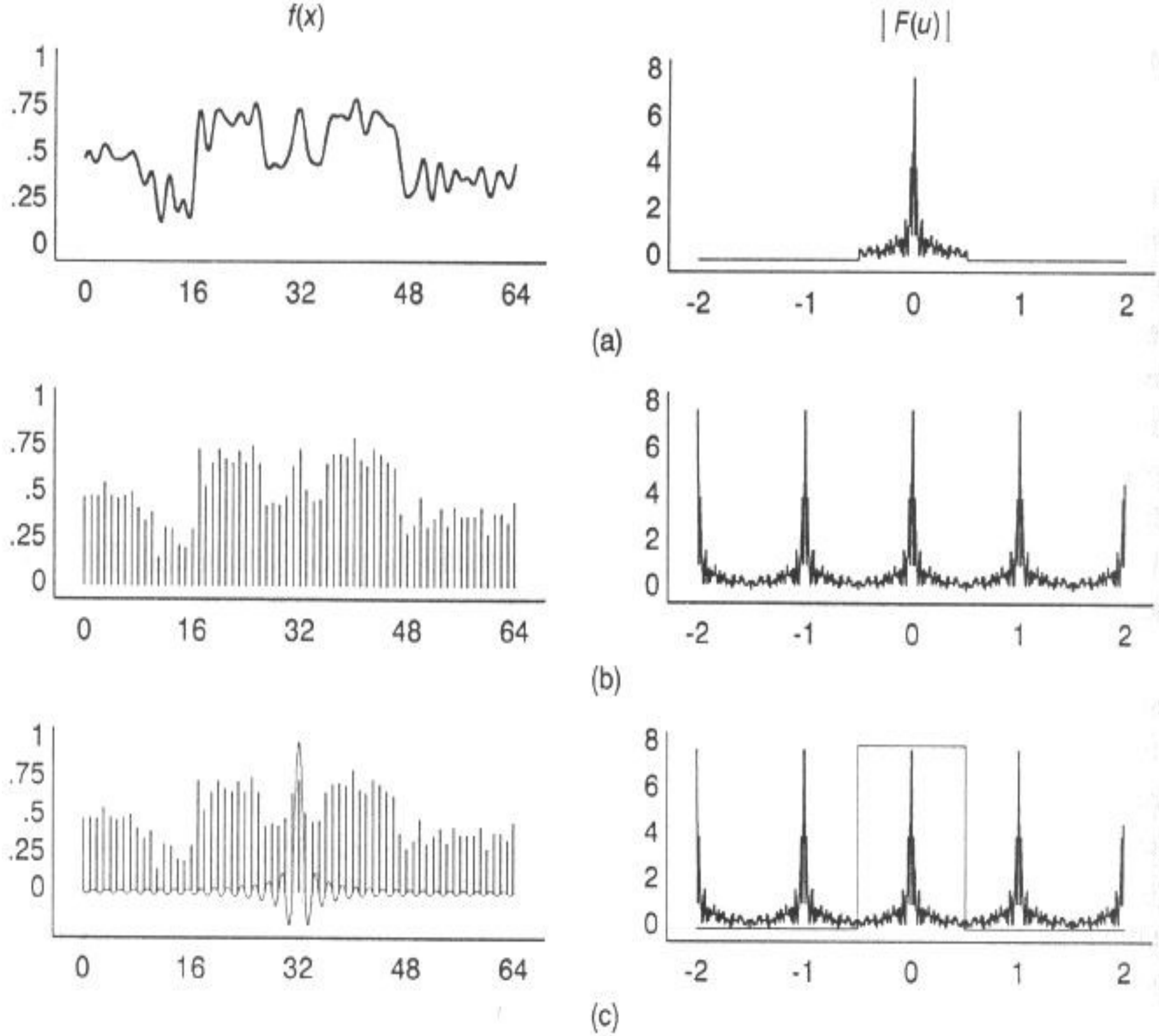
Frequency spectrum is replicated

Bands do not overlap

Correct filtering

Fourier: Box (mult.)
Space: $sinc$ (conv.)

Only one copy



Reconstruction with ideal *sinc*

Identical signal

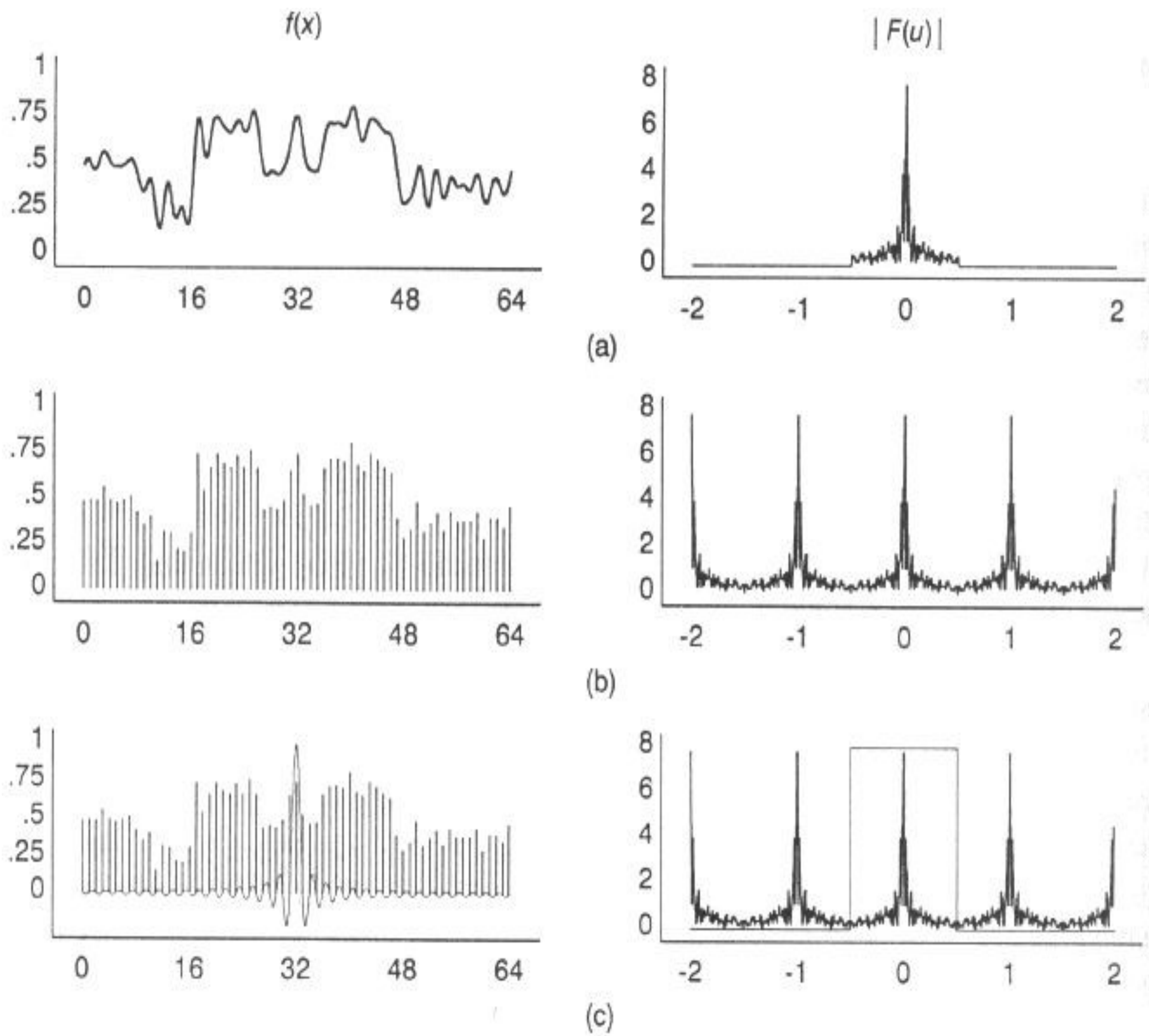
Approximate filtering

Space: triangle (conv.)
Fourier: sinc^2 (mult.)

High frequencies are not ignored

⇒Aliasing

Reconstruction with triangle function (= piecewise linear interpolation)

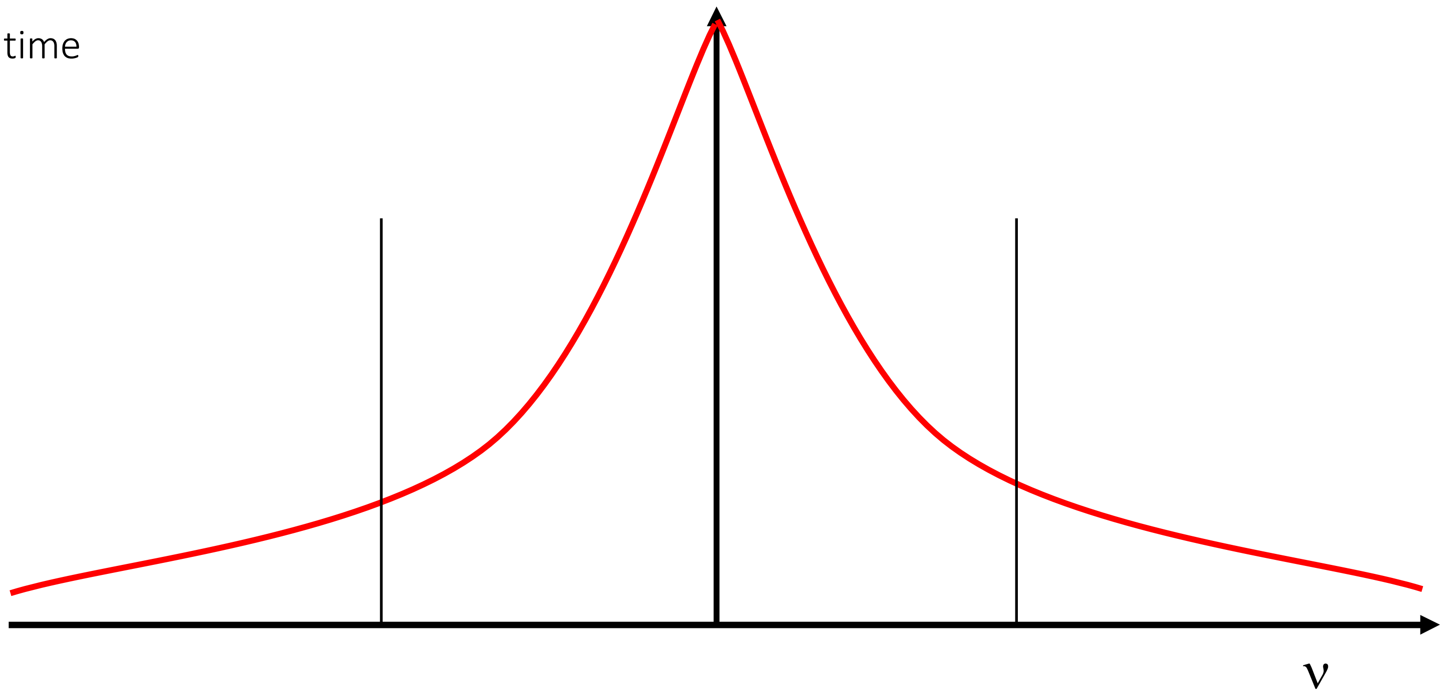


Band-limit unlimited natural signal

- Finite spectral range
- Makes non-aliased reconstruction possible
- Example: thermal noise

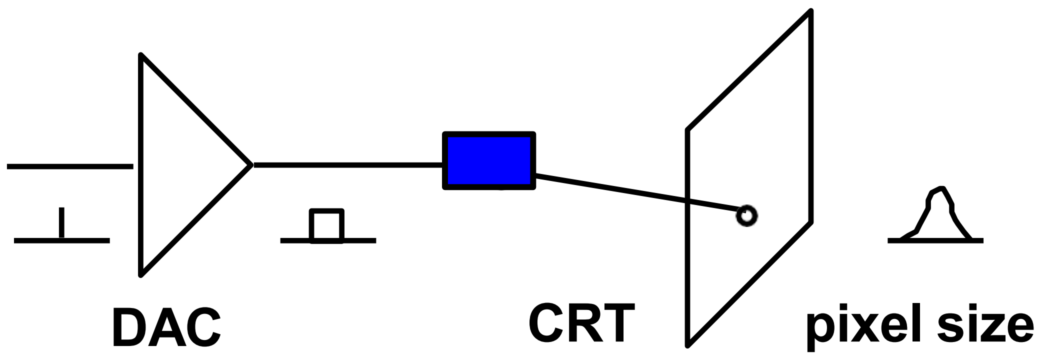
Natural low-pass filtering

- *E.g.*, finite-sized CCD pixels
- Finite exposure time



Reconstruction

- Physical output devices generate a continuous signal, even for discrete input, *e.g.*, on a computer monitor



Example

- DAC (Digital-to-Analog Converter): Sample and hold
  - Capacities and inductivities
- CRT: Phosphor and light spot
  - Afterglow on screen

Sampling with Low Frequency

Original function

Sampling below Nyquist:

Comb spaced to far (sampling<2*bandlimit)

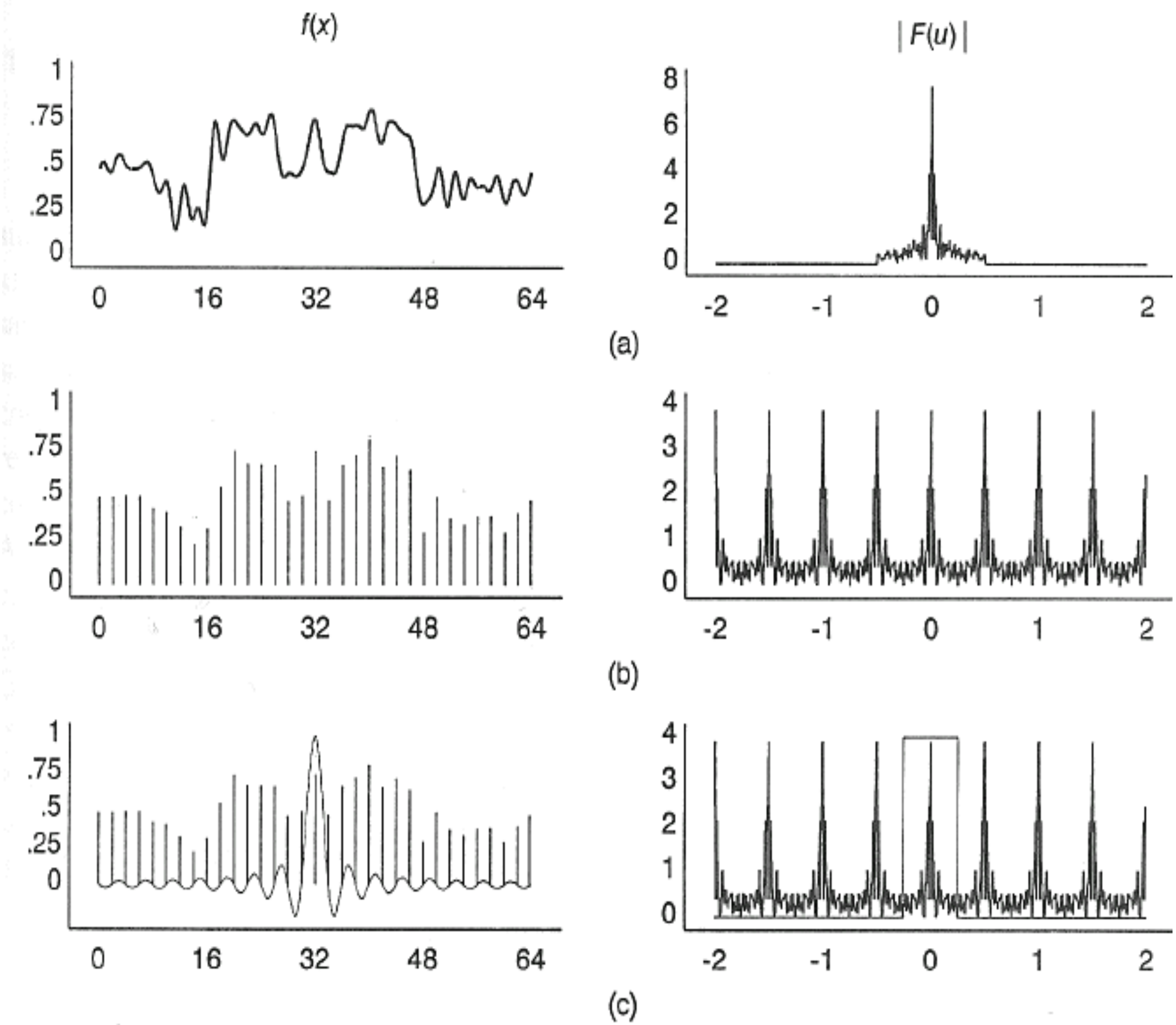
Frequency bands overlap

Correct filtering

Space: $sinc$ (conv.)

Fourier: hat (mult.)

Band overlap in frequency domain cannot be corrected - aliasing



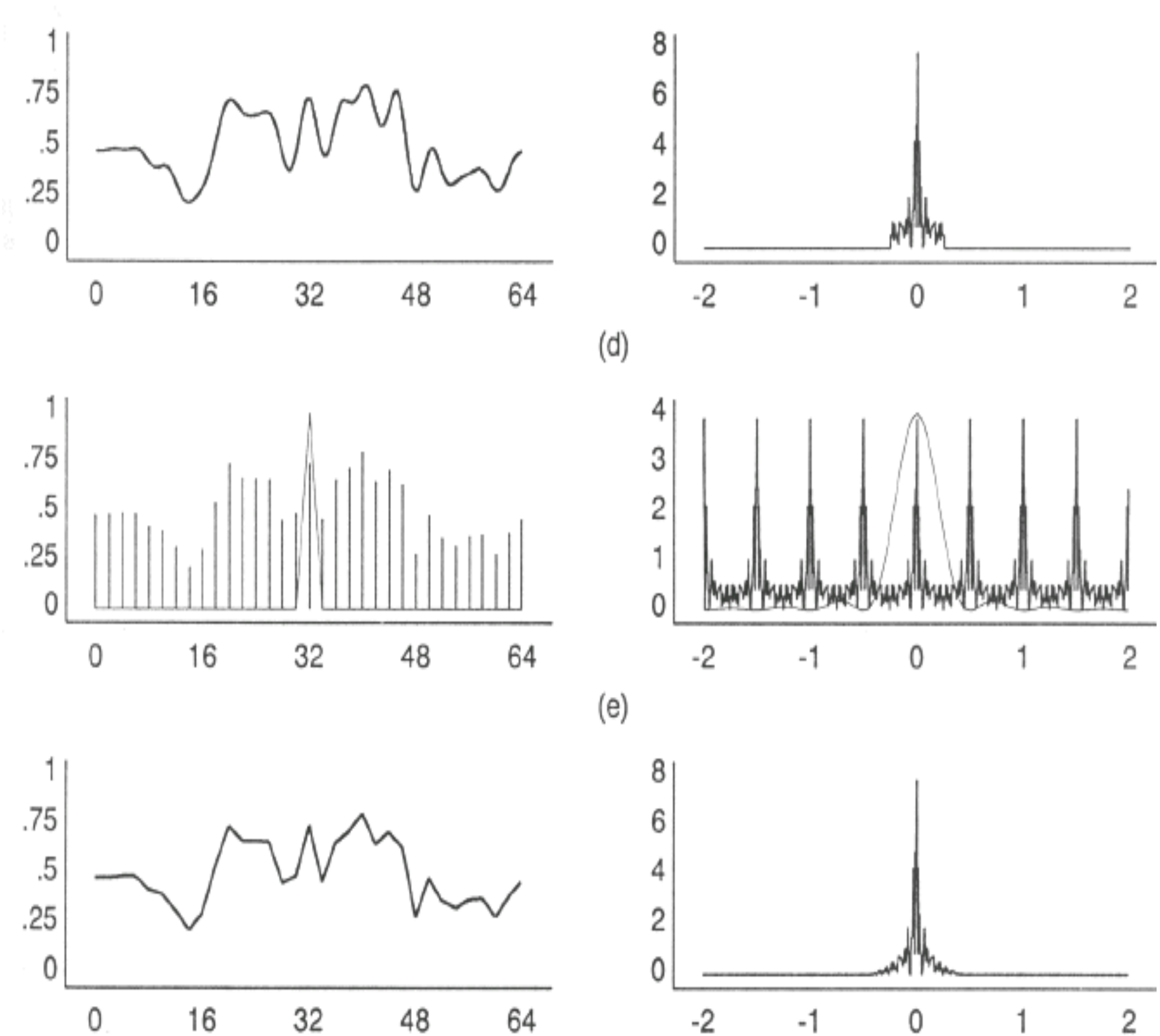
Reconstruction with ideal $sinc$

Reconstruction fails (frequency components wrong due to aliasing!)

Fitlering with $sinc^2$ function

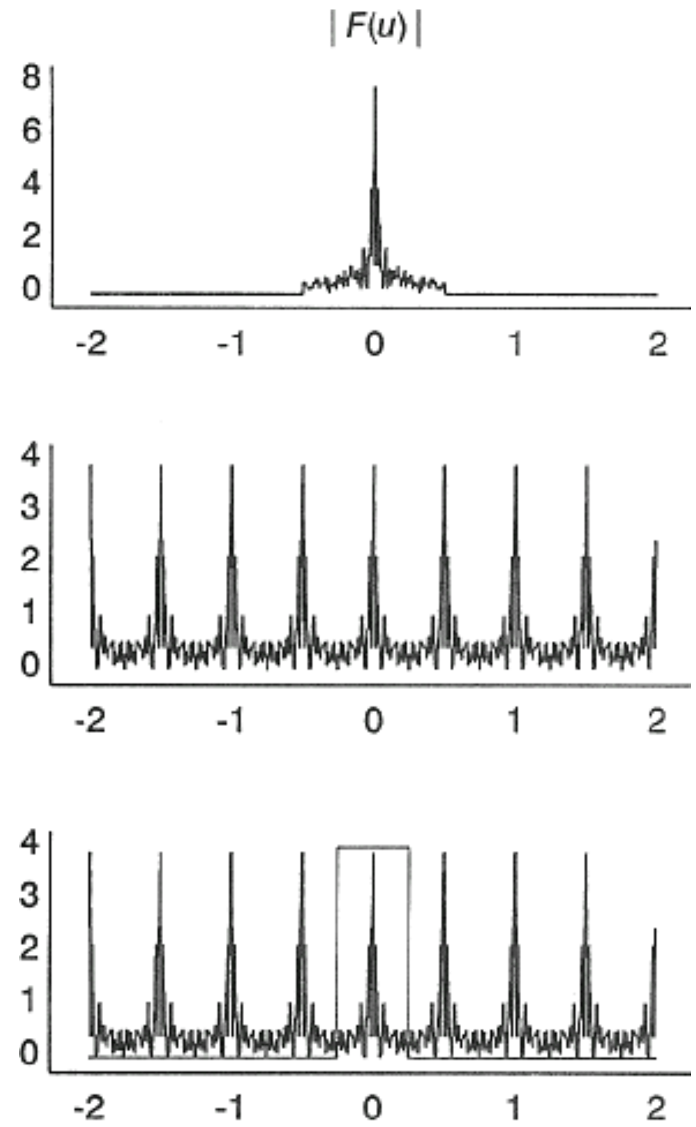
Reconstruction with triangle function (= piecewise linear interpolation)

Even worse reconstruction



Overlap between replicated copies in frequency spectrum

- High frequency components from the replicated copies are treated like low frequencies during the reconstruction process

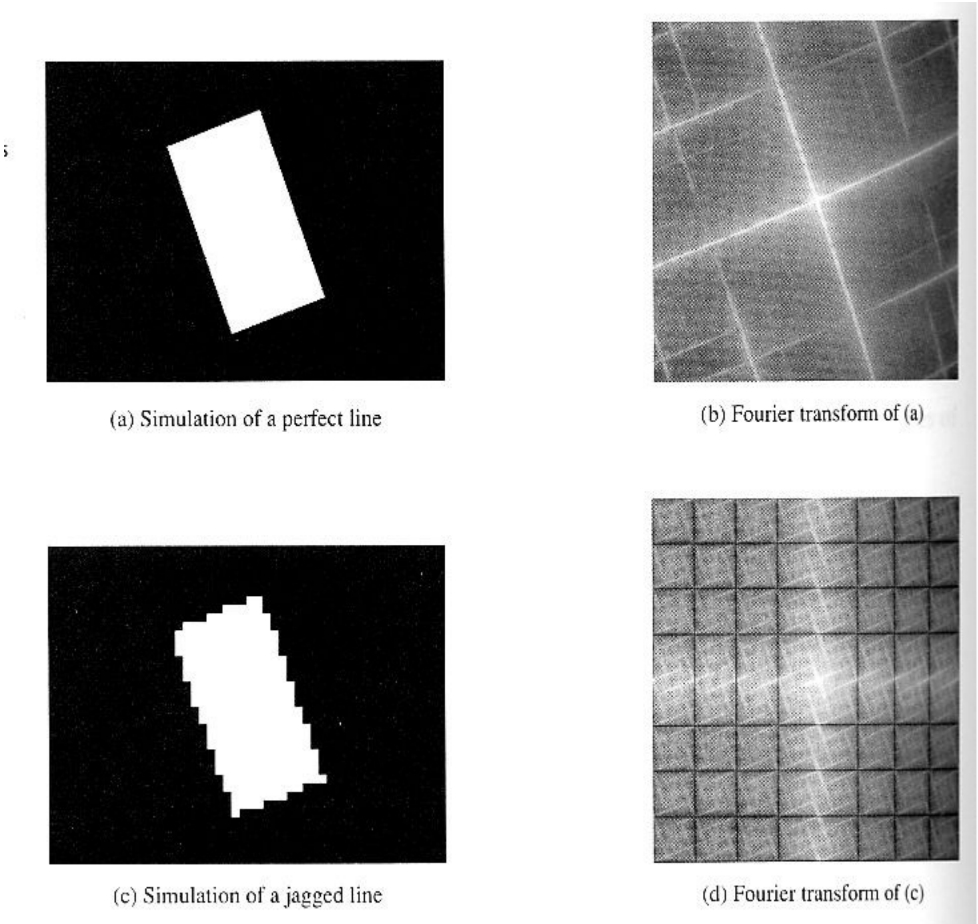


Aliasing Artifacts

Moiré patterns

Aliasing

Jaggies

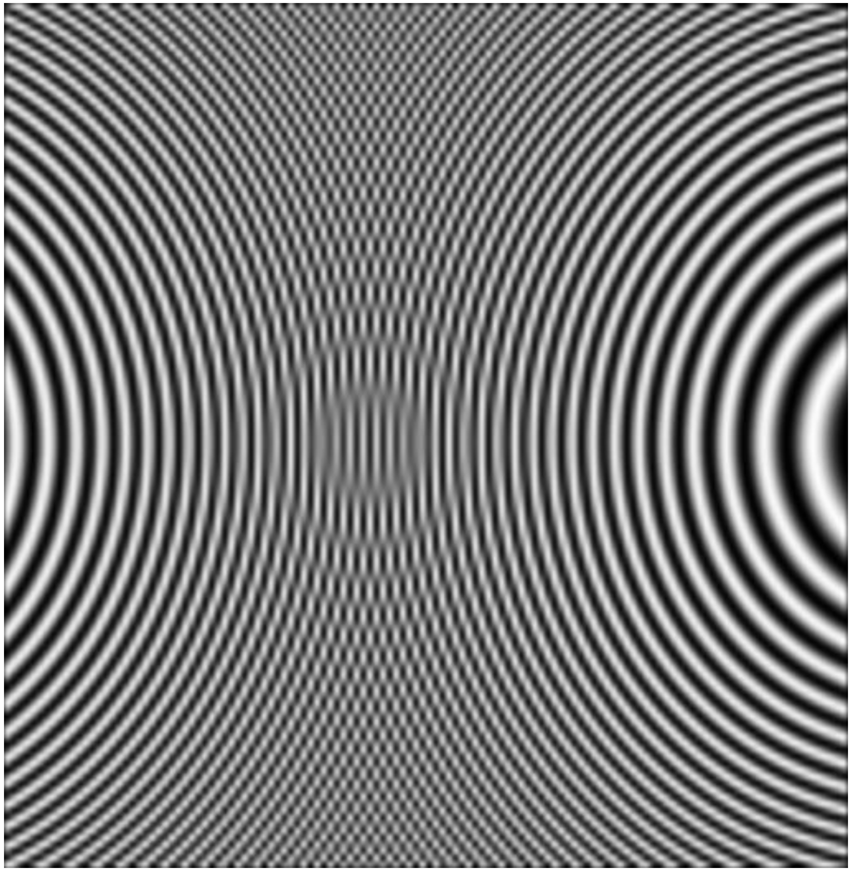


(a) Simulation of a perfect line

(b) Fourier transform of (a)

(c) Simulation of a jagged line

(d) Fourier transform of (c)



Fourier transformation

- Equivalent representation of transformed signal
- Spectral analysis: shows signal's frequency components

Convolution

- Filtering

Sampling

- Multiplication with comb function
- Only at discrete points: no integration over signal
- Frequency spectrum replicated
- Replication distance = sampling rate

Aliasing

- Replicated spectra overlap
- Cannot be separated by filtering anymore
- Erroneous frequency amplitudes – wrong function!