Lecture 18:

Camera & Projective Transformations

Contents

1. Generating 2D image from 3D world
2. Coordinate Spaces
3. Camera Specification
4. Perspective transformation
5. Normalized screen coordinates

Rasterization works on 2D primitives (+ depth)

Need to project 3D world onto 2D screen

Based on

- Positioning of objects in 3D space
- Positioning of the virtual camera

Local (object) coordinate system (3D)

- Object vertex positions
- Can be hierarchically nested in each other (scene graph, transformation stack)

World (global) coordinate system (3D)

- Scene composition and object placement
  - Rigid objects: constant translation, rotation per object, (scaling)
  - Animated objects: time-varying transformation in world-space
- Illumination can be computed in this space

Camera / view / eye coordinate system (3D)

- Coordinates relative to camera pose (position & orientation)
  - Camera itself specified relative to world space
- Illumination can also be done in this space

Normalized device coordinate system (2.5D)

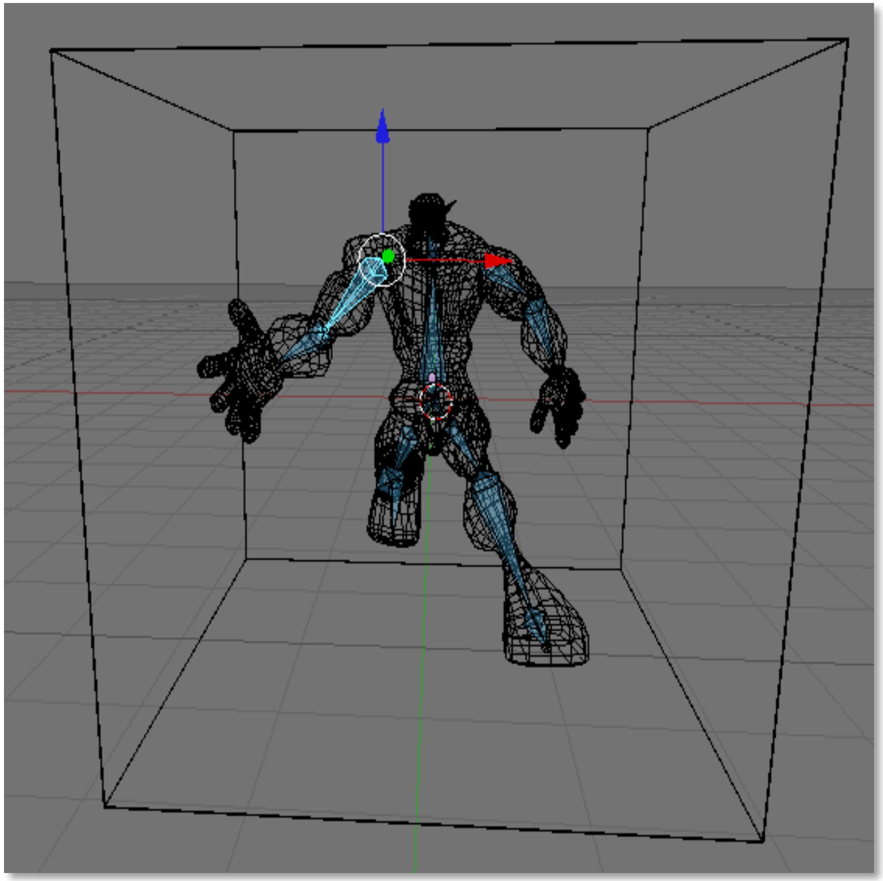
- After perspective transformation, rectilinear, in $[0, 1]^3$
- Normalization to view frustum (for rasterization and depth buffer)
- Shading executed here (interpolation of color across triangle)

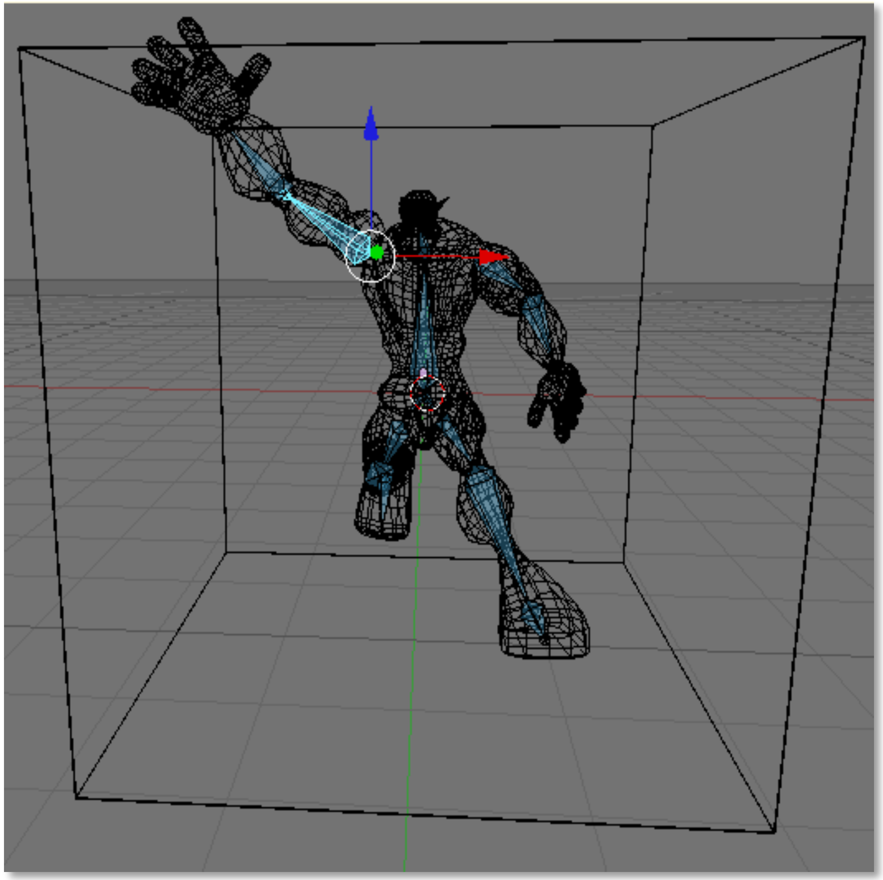
Window/screen (raster) coordinate system (2D)

- 2D transformation to place image in window on the screen

Used in Scene Graphs

- Group objects hierarchically
- Local coordinate system is relative to parent coordinate system
- Apply transformation to the parent to change the whole sub-tree (or sub-graph)





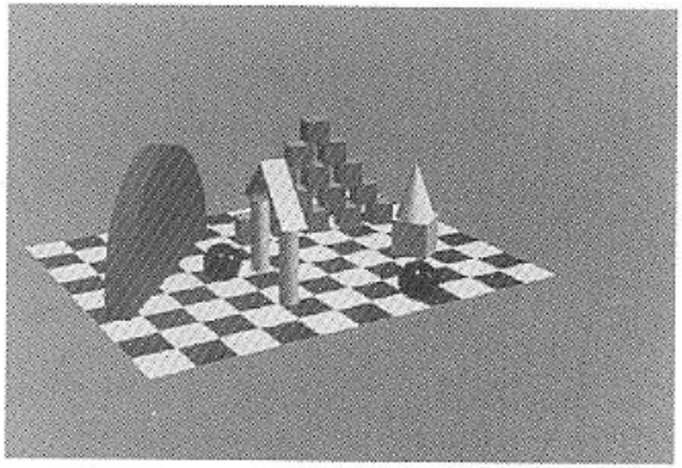
Hierarchy of transformations

T_root	Positions the character in the world	
T_ShoulderR	Moves to the right shoulder	
T_ShoulderRJoint	Rotates in the shoulder	<== User
T_UpperArmR	Moves to the Elbow	
T_ElbowRJoint	Rotates in the Elbow	<== User
T_LowerArmR	Moves to the wrist	
T_WristRJoint	Rotates in the wrist	<== User
.....	Further for the right hand and the fingers	
T_ShoulderL	Moves to the left shoulder	
T_ShoulderLJoint	Rotates in the shoulder	<== User
T_UpperArmL	Moves to the Elbow	
T_ElbowLJoint	Rotates in the Elbow	<== User
T_LowerArmL	Moves to the wrist	
.....	Further for the left hand and the fingers	

- Each transformation is relative to its parent
  - Concatenated my multiplying and pushing onto a stack
  - Going back by popping from the stack
- This transformation stack was so common, it was build into OpenGL

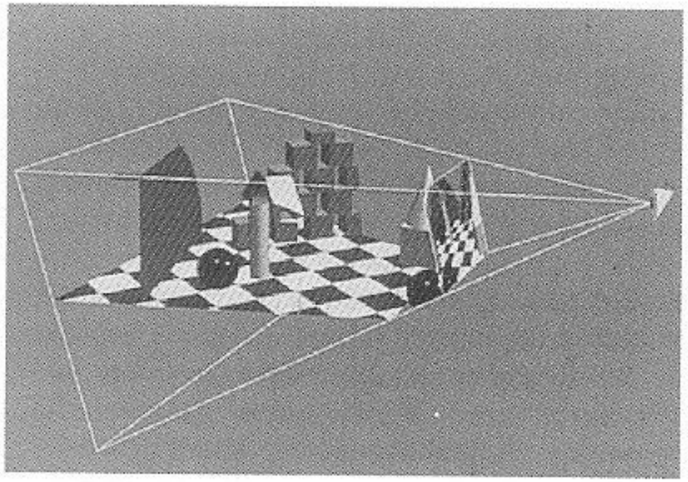
Model transformation

- Object space to world space
- Can be hierarchically nested
- Typically an affine transformation



View transformation

- World space to eye space
- Typically an affine transformation



Combination: Modelview transformation

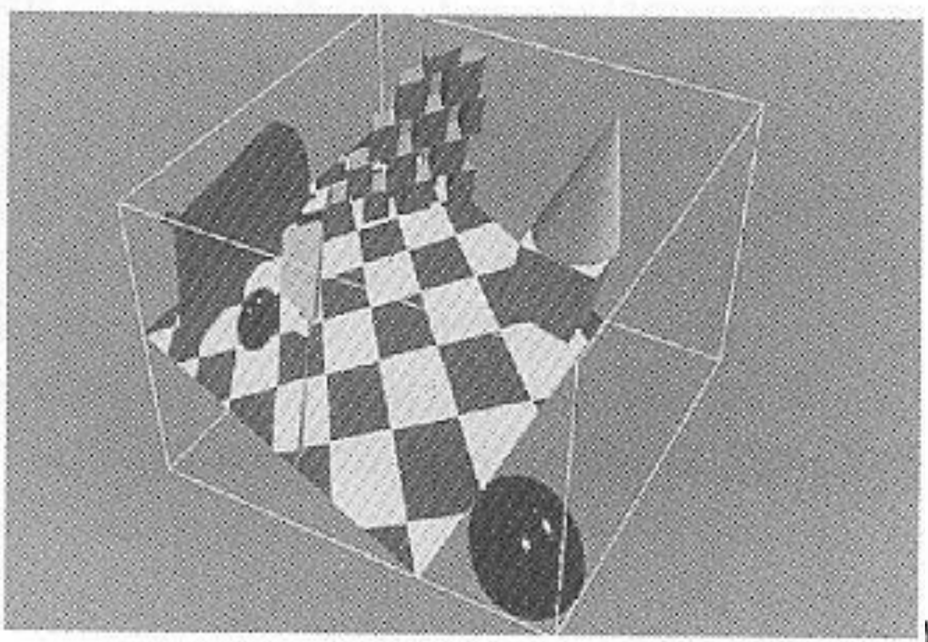
- Used by *traditional* OpenGL (although world space is conceptually intuitive, it isn't explicitly exposed)

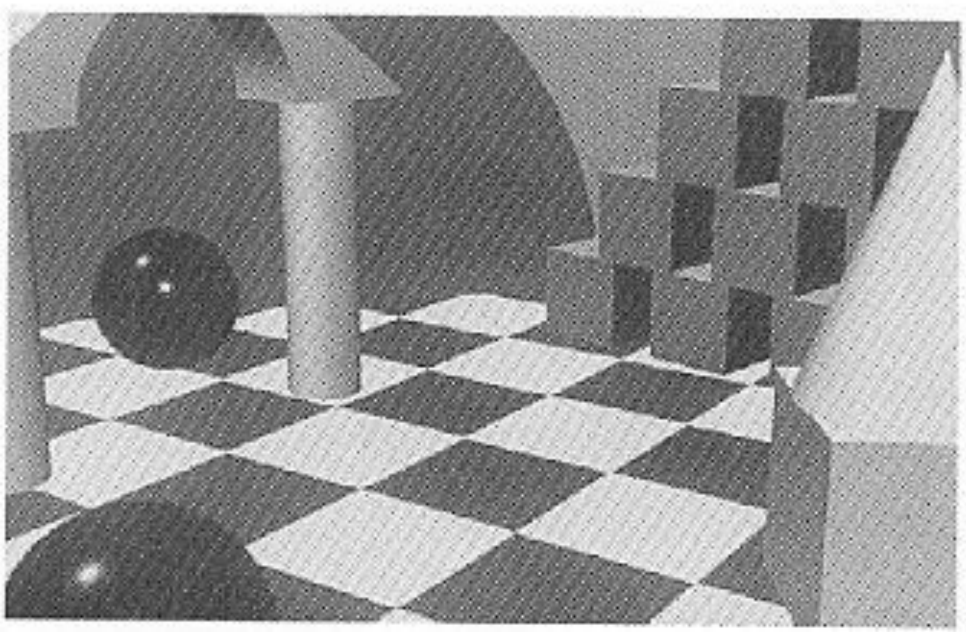
Projective transformation

- Eye space to normalized device space (defined by view frustum)
- Parallel or perspective projection
- 3D to 2D: Preservation of depth in Z coordinate

Viewport transformation

- Normalized device space to window (raster) coordinates



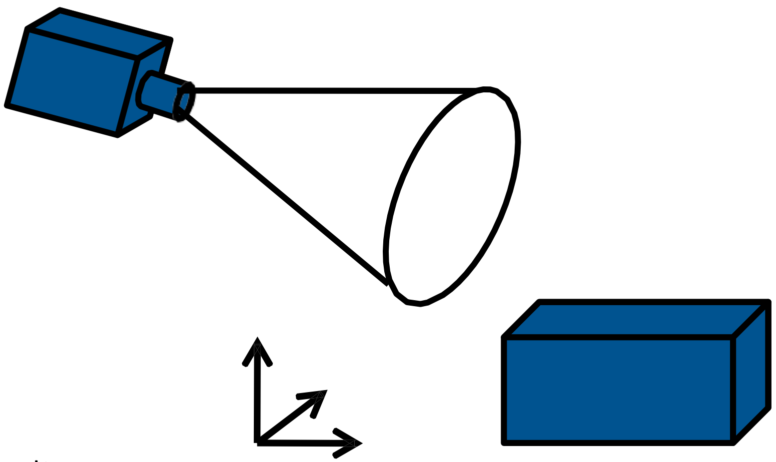


Goal

- Compute the transformation between points in 3D and pixels on the screen
- Required for rasterization algorithms (OpenGL)
  - They project all primitives from 3D to 2D
  - Rasterization happens in 2D (actually 2.5D, XY plus Z attribute)

Given

- Camera pose (position and orientation)
  - *Extrinsic* parameters
- Camera configuration
  - *Intrinsic* parameters
- Pixel raster description
  - Resolution and placement on screen



Following: Stepwise Approach

- Express each transformation step in homogeneous coordinates
- Multiply all 4x4 matrices to combine all transformations

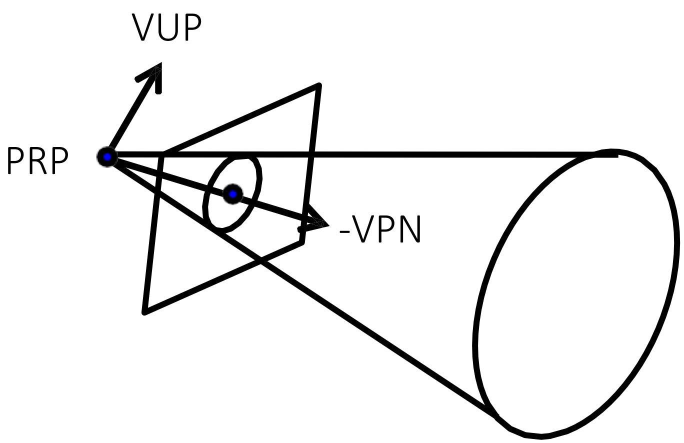
Need camera position and orientation in world space

- External (extrinsic) camera parameters
  - Center of projection: projection reference point (PRP)
  - Optical axis: view-plane normal (VPN)
  - View up vector (VUP)
    - Not necessarily orthogonal to VPN, but not co-linear

Needed Transformations

1) Translation of PRP to the origin (-PRP)
2) Rotation such that viewing direction is along negative Z axis

2a) Rotate such that VUP is pointing up on screen



Define projection (perspective or orthographic)

- Needs internal (intrinsic) camera parameters
- Screen window (Center Window (CW), width, height)
    - Window size/position on image plane (relative to VPN intersection)
    - Window center relative to PRP determines viewing direction (≠ VPN)
- Focal length (f)
    - Distance of projection plane from camera along VPN
    - Smaller focal length means larger field of view
- Field of view (fov) (defines width of view frustum)
    - Often used instead of screen window and focal length
        - Only valid when screen window is centered around VPN (often the case)
    - Vertical (or horizontal) angle plus aspect ratio (width/height)
        - Or two angles (both angles may be half or full angles, beware!)
- Near and far clipping planes
    - Given as distances from the PRP along VPN
    - Near clipping plane avoids singularity at origin (division by zero)
    - Far clipping plane restricts the depth for fixed-point representation

Camera definition (typically used in ray tracers)

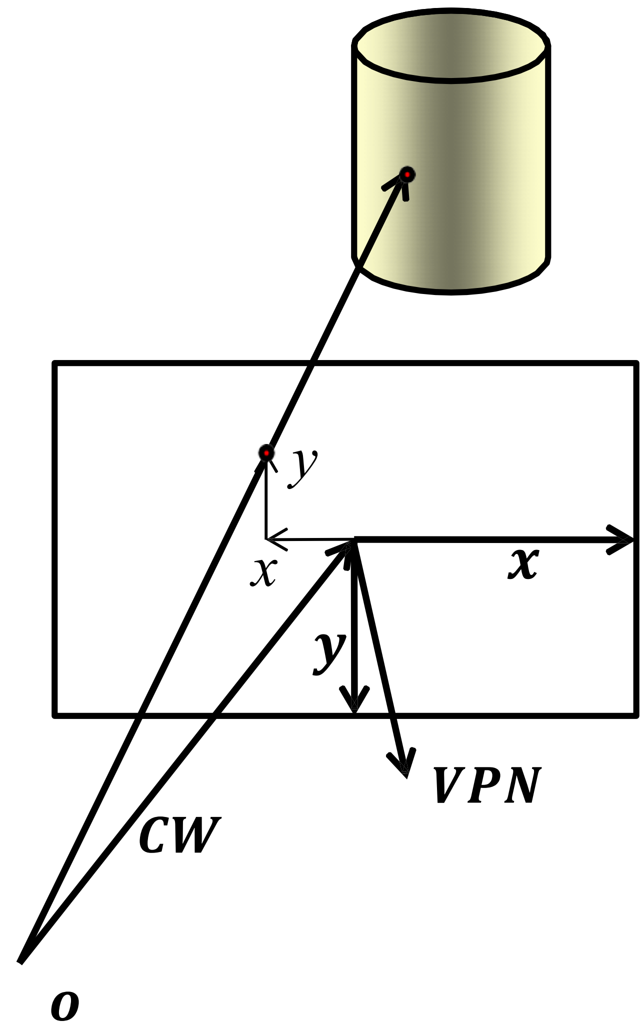
- $o \in \mathbb{R}^3$: center of projection, point of view (PRP)
- $CW \in \mathbb{R}^3$: vector to center of window
  - "Focal length": projection of vector to CW onto VPN
    - $focal = |(CW - o) \cdot VPN|$
- $x, y \in \mathbb{R}^3$: span of half viewing window
  - $VPN = {}^{(\boldsymbol{y} \times \boldsymbol{x})}/_{|(\boldsymbol{y} \times \boldsymbol{x})|}$
  - $VUP = -y$
  - $width = 2|x|$
  - $height = 2|y|$
  - Aspect ratio: $camera_{ratio} = {}^{|x|}/_{|y|}$

PRP: Projection reference point

VPN: View plane normal

VUP: View up vector

CW: Center of window



Normalized Device Coordinates (NDC)

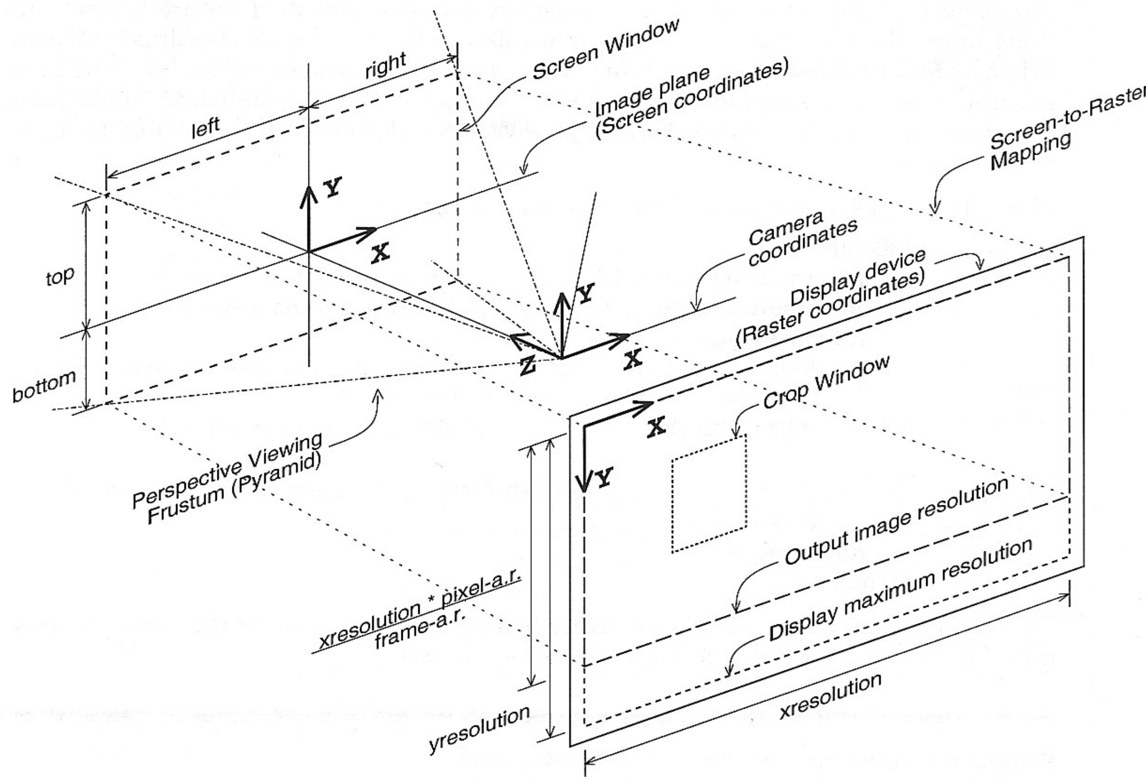
- Intrinsic camera parameters transform to NDC
  - $[0,1]^2$ for x, y across the screen window
  - $[0,1]$ for z (depth)

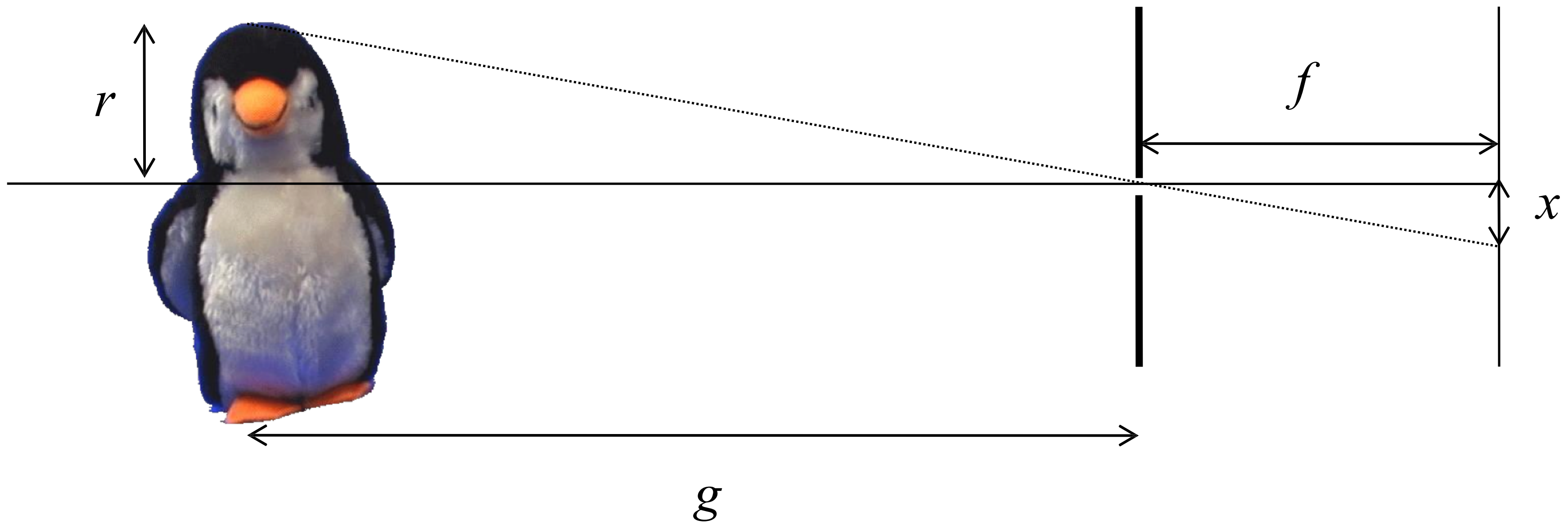
Mapping NDC to raster coordinates on the screen

- $xres, yres$: Size of window in pixels
  - Should have same aspect ratios to avoid distortion
    - $camera_{ratio} = \frac{xres}{yres} \frac{pixelspacing_x}{pixelspacing_y}$
  - Horizontal and vertical pixel spacing (distance between centers)
    - Today, typically the same but can be different *e.g.* for some video formats
- Position of window on the screen
  - Offset of window from origin of screen
    - $posx$ and $posy$ given in pixels
  - Depends on where the origin is on the screen (top left, bottom left)
- “Scissor box” or “crop window” (region of interest)
  - No change in mapping but limits which pixels are rendered

RenderMan camera specification

- Almost identical to above description
  - Distance of Screen Window from origin given by “field of view” (fov)
    - fov: Full angle of segment (-1,0) to (1,0), when seen from origin
  - CW given implicitly
  - No offset on screen



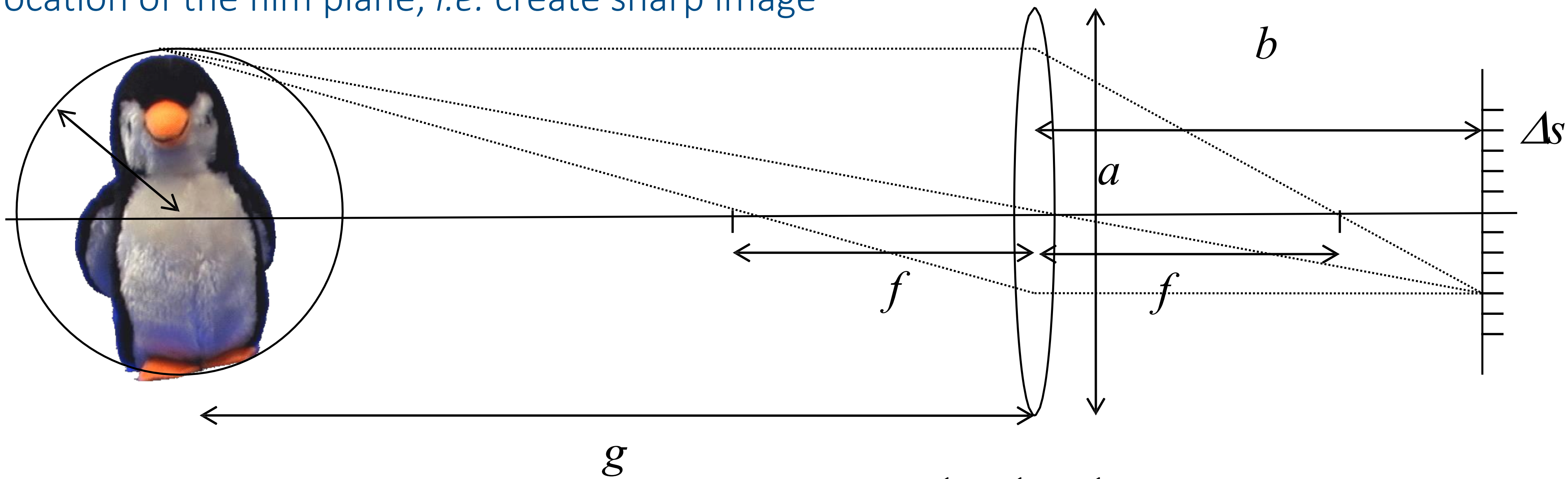


$$\frac{t}{g} = \frac{x}{f} \Rightarrow x = \frac{fr}{g}$$

Infinitesimally small pinhole

- Theoretical (non-physical) model
- Sharp image everywhere
- Infinite depth of field
- Infinitely dark image in reality
- Diffraction effects in reality

Lens focuses light from given position on object through finite-size aperture onto some location of the film plane, *i.e.* create sharp image



- Lens formula defines reciprocal focal length (focus distance from lens of parallel light)

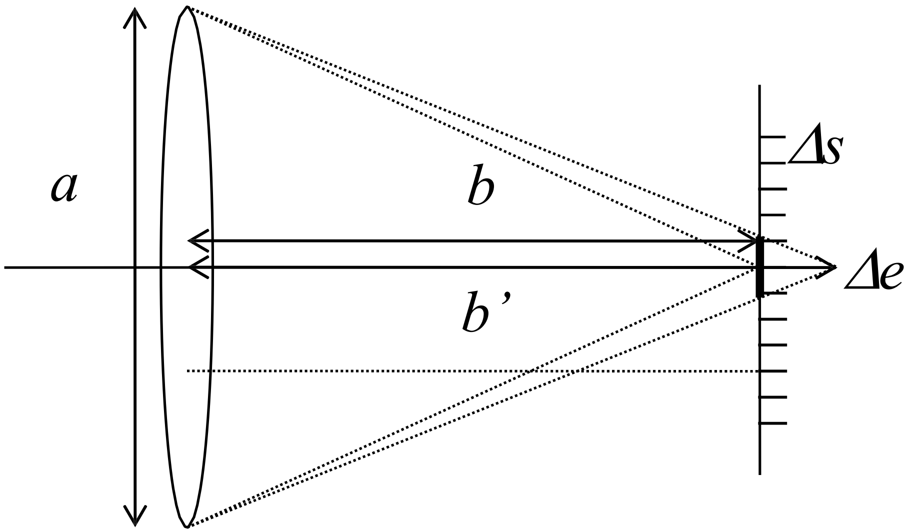
$$\frac{1}{f} = \frac{1}{b} + \frac{1}{g}$$

- Object center at distance g is in focus at

$$b = \frac{f\,g}{g - f}$$

- Object front at distance g-r is in focus at

$$b' = \frac{f(g - r)}{(g - r) - f}$$



Circle of confusion (CoC) $\Delta e = \left| a \left(1 - \frac{b}{b'} \right) \right|$

Sharpness criterion based on pixel size and CoC $\Delta s > \Delta e$

DOF: Defined radius r, such that CoC smaller than Δs

Depth of field (DOF) $$r < \frac{g \Delta s (g - f)}{af + \Delta s (g - f)} \Rightarrow r \propto \frac{1}{a}$$

The smaller the aperture, the larger the depth of field

Let’s put this all together

Goal: Camera: at origin, view along $-Z$, Y upwards

- Assume right handed coordinate system
- Translation of PRP to the origin
- Rotation of VPN to Z-axis
- Rotation of projection of VUP to Y-axis

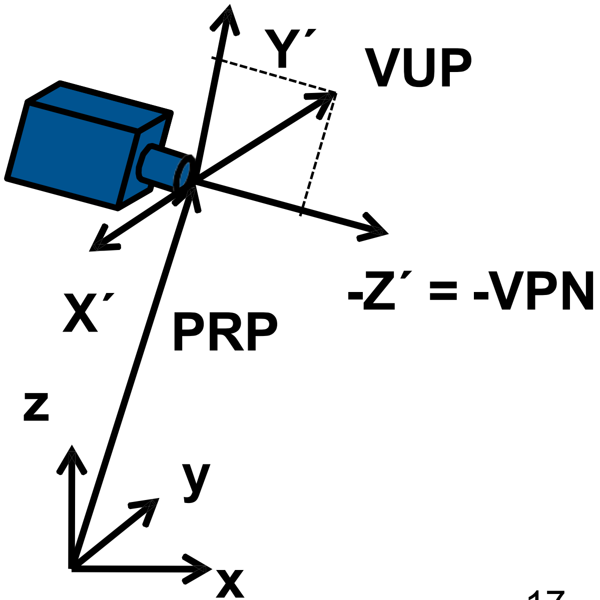
Rotations

- Build orthonormal basis for the camera and form inverse
  - $Z' = VPN, X' = normalize(VUP \times VPN), Y' = Z' \times X'$

Viewing transformation

- Translation followed by rotation

$$V = RT = \begin{pmatrix} X'_x & Y'_x & Z'_x & 0 \\ X'_y & Y'_y & Z'_y & 0 \\ X'_z & Y'_z & Z'_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^T T(-PRP)$$



Step 1: VPN may not go through center of window

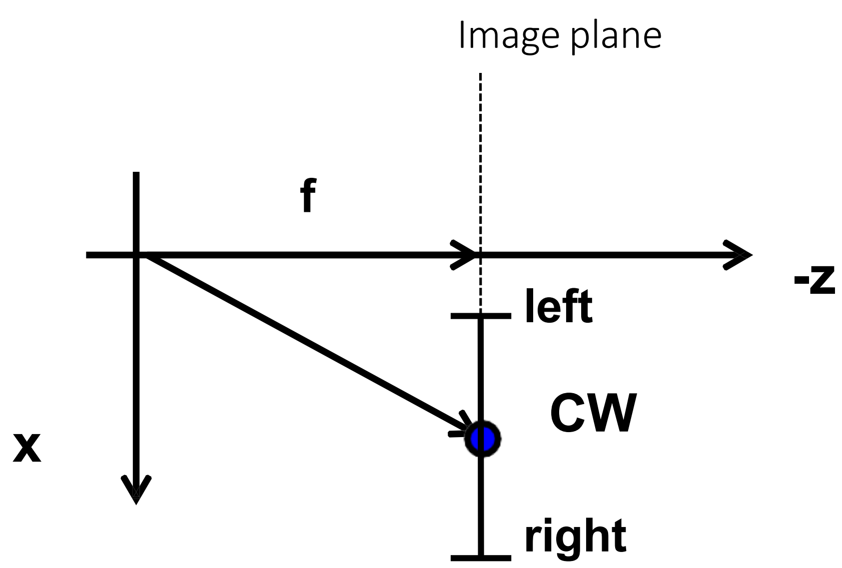
- Oblique viewing configuration

Shear

- Shear space such that window center is along Z-axis
- Window center CW (in 3D view coordinates)
  - $CW = \left(\frac{right+left}{2}, \frac{top+bottom}{2}, -focal\right)^{\mathrm{T}}$

Shear matrix

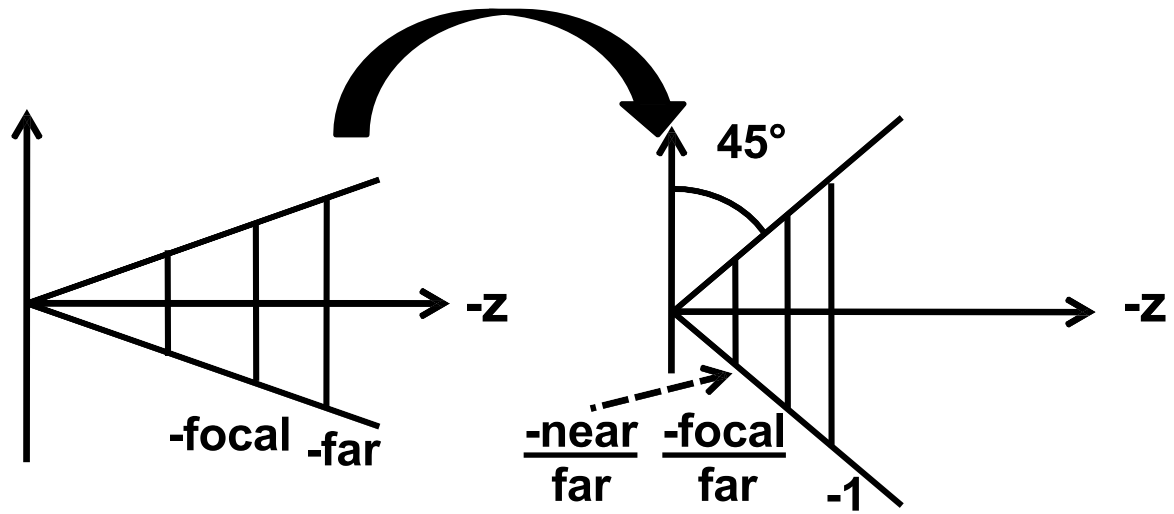
$$H = \begin{pmatrix} 1 & 0 & -\frac{CW_x}{CW_z} & 0 \\ 0 & 1 & -\frac{CW_y}{CW_z} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



View from top

Step 2: Scaling to canonical viewing frustum

- Scale in X and Y such that screen window boundaries open at 45° angles (at focal plane)
- Scale in Z such that far clipping plane is at $Z = -1$



Scaling matrix

$$S = S_{far}S_{xy} = \begin{pmatrix} \frac{1}{far} & 0 & 0 & 0 \\ 0 & \frac{1}{far} & 0 & 0 \\ 0 & 0 & \frac{1}{far} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{2focal}{width} & 0 & 0 & 0 \\ 0 & \frac{2focal}{height} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Step 3: Perspective transformation

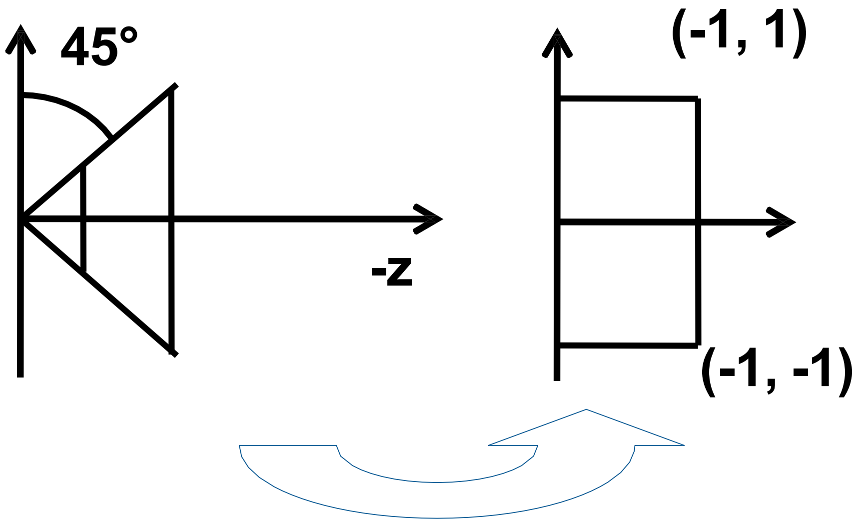
- From canonical perspective viewing frustum (= cone at origin around -Z-axis) to regular box $[-1 .. 1]^2 \times [0 .. 1]$

Mapping of X and Y

- Lines through the origin are mapped to lines parallel to the Z-axis
  - $x' = {}^{x}/_{-z}$ and $y' = {}^{y}/_{-z}$ (coordinate given by slope with respect to z!)
- Do not change X and Y additively (first two rows stay the same)
- Set W to $-z$ so we divide when converting back to 3D
  - Determines last row

Perspective transformation

- $P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ A & B & C & D \\ 0 & 0 & -1 & 0 \end{pmatrix}$ Still unknown



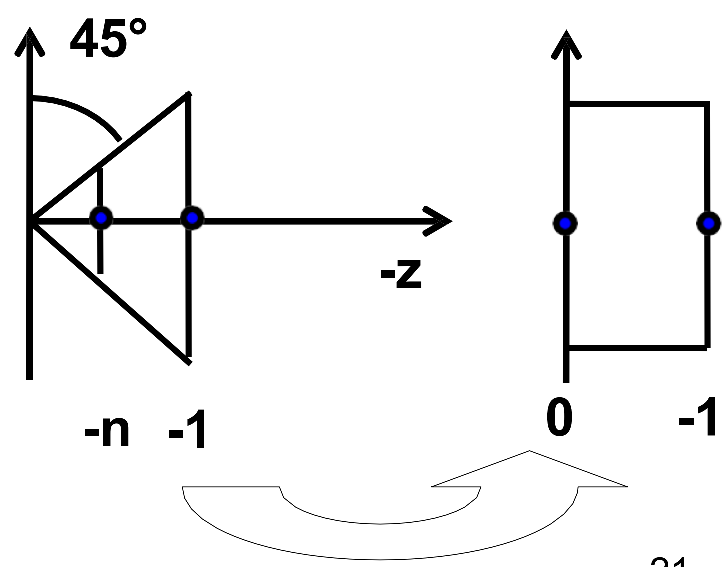
- Note: Perspective projection =perspective transformation + parallel projection

Computation of the coefficients A, B, C, D

- No shear of Z with respect to X and Y
  - $A = B = 0$
- Mapping of two known points
  - Computation of the two remaining parameters C and D
    - $n = {}^{near}/_{far}$ (due to previous scaling by ${}^{1}/_{far}$)
  - Following mapping must hold
    - $(0,0,-1,1)^\top = P(0,0,-1,1)^\top$ and $(0,0,0,1) = P(0,0,-n,1)$

Resulting Projective transformation

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{1-n} & \frac{n}{1-n} \\ 0 & 0 & -1 & 0 \end{pmatrix}$$



- Transform Z non-linearly (in 3D)
  - $z' = \frac{z+n}{z(1-n)}$

Parallel projection to $[-1 .. 1]^2$

- Formally scaling in Z with factor 0
- Typically maintains Z in [0,1] for depth buffering
  - As a vertex attribute (see OpenGL later)

Transformation from $[-1 .. 1]^2$ to NDC ($[0 .. 1]^2$)

- Scaling (by $\frac{1}{2}$ in X and Y) and translation (by $(\frac{1}{2}, \frac{1}{2})$)

Projection matrix for combined transformation

- Delivers normalized device coordinates

$$P_{parallel} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \text{ or } 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Scaling and translation in 2D

- Scaling matrix to map to entire window on screen
  - $S_{raster}(xres, yres)$
  - No distortion if aspects ration have been handled correctly earlier
  - Sometime need to reverse direction of y
    - Some formats have origin at bottom left, some at top left
    - Needs additional translation
- Positioning on the screen
  - Translation $T_{raster}(xpos, ypos)$
  - May be different depending on raster coordinate system
    - Origin at upper left or lower left

Step 2a: Translation (orthographic)

- Bring near clipping plane into the origin

Step 2b: Scaling to regular box $[-1 .. 1]^2$ x [0 .. -1]

Mapping of X and Y

$$P_o = S_{xyz} T_{near} = \begin{pmatrix} \frac{2}{width} & 0 & 0 & 0 \\ 0 & \frac{2}{heigh} & 0 & 0 \\ 0 & 0 & \frac{1}{far-near} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & near \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Complete transformation (combination of matrices)

- Perspective Projection
    - $T_{camera} = T_{raster} S_{raster} P_{parallel} P_{persp} S_{far} S_{xy} H\ R\ T$
- Orthographic Projection
    - $T_{camera} = T_{raster} S_{raster} P_{parallel} S_{xyz} T_{near} H\ R\ T$

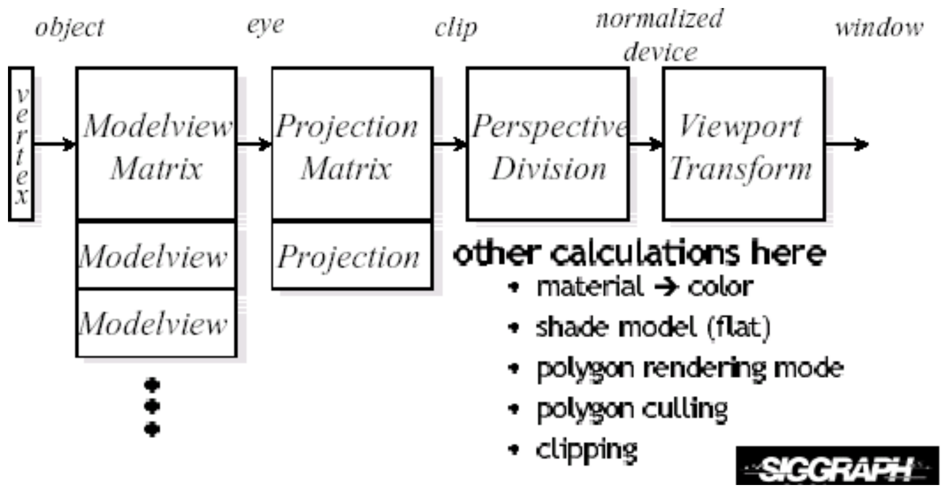
Other representations

- Other literature uses different conventions
    - Different camera parameters as input
    - Different canonical viewing frustum
    - Different normalized coordinates
        - $[-1 .. 1]^3$ versus $[0 ..1]^3$ versus ...
- ...

→ *Results in different transformation matrices – so be careful !!!*

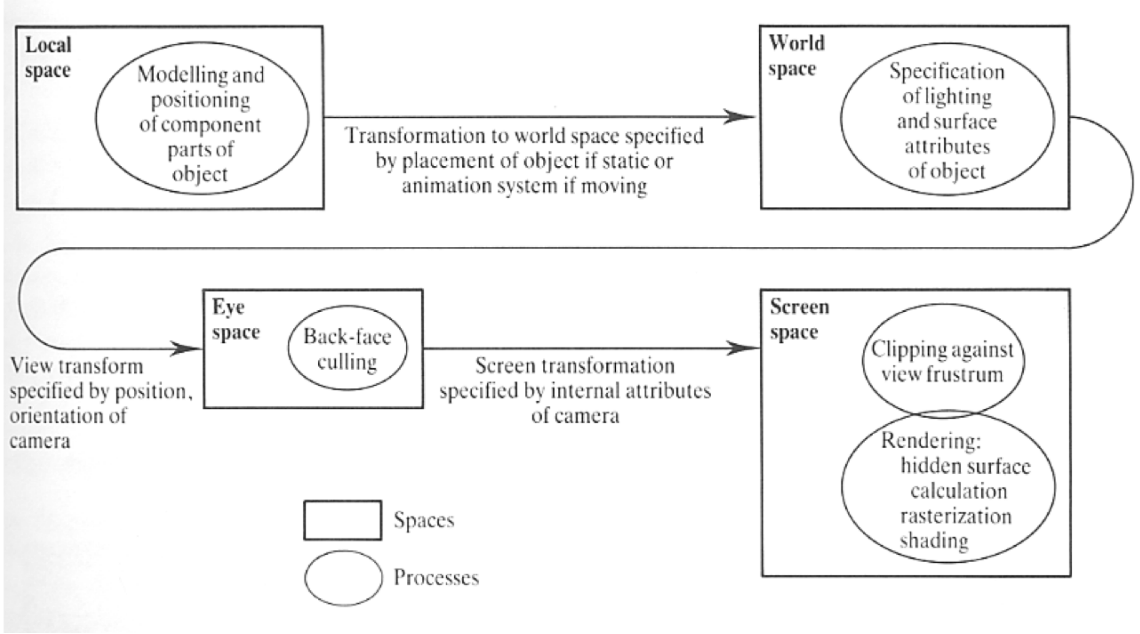
Traditional OpenGL pipeline

- Hierarchical modeling
  - Modelview matrix stack
  - Projection matrix stack
- Each stack can be independently pushed/popped
- Matrices can be applied/multiplied to top stack element



Today

- Arbitrary matrices as attributes to vertex shaders that apply them as they wish (later)
- All matrix stack handling must now be done by application



Traditional ModelView matrix

- Modeling transformations AND viewing transformation
- No explicit world coordinates

Traditional Perspective transformation

- Simple specification
  - glFrustum(left, right, bottom, top, near, far)
  - glOrtho(left, right, bottom, top, near, far)

Modern OpenGL

- Transformation provided by app, applied by vertex shader
- Vertex or Geometry shader must output clip space vertices
  - Clip space: Just before perspective divide (by w)

Viewport transformation

- glViewport(x, y, width, height)
- Now can even have multiple viewports
  - glViewportIndexed(idx, x, y, width, height)
- Controlling the depth range (after Perspective transformation)
  - glDepthRangeIndexed(idx, near, far)

Submission deadline: Friday, 22. November 2019 9:45 (before the lecture)
Written solutions have to be submitted in the lecture room before the lecture. Every assignment sheets counts 100 points (theory and practice)

5.1 Fourier Transformation (30 Points)

Show that the Fourier transformation of the box function $B_d(x)$ is a *sinc* type function. The sinc function is defined as $sinc(x) = \frac{\sin \pi x}{\pi x}$ and a definition of the Fourier transform can be found in Exercise 4.*

$$B_d(x) = \begin{cases} 0 & \text{for} \quad x \leq -d \\ 1 & \text{for} \quad -d < x < d \\ 0 & \text{for} \quad d \leq x \end{cases}$$

5.2 Sampling Theory (10 + 10 Points)

Let $f(x)$ be an infinite signal that fulfills the Nyquist property, thus the highest frequency of the signal is smaller than $\frac{1}{2T}$ if T is the sampling distance. Consider a regular sampling $f_S(x)$ of $f(x)$ with sample distance T.

a) Is an exact signal reconstruction of $f(x)$ possible? If so, why?

b) How has the reconstruction to be performed in image and Fourier space?

5.3 Antialiasing (10 + 10 Points)

a) Describe what aliasing means in Fourier space.

b) Consider an infinite signal $f(x)$ and a regular sampling $f_S(x)$ with sampling distance d that shows no aliasing artefacts. The sampling distance is now increased step by step until the first aliasing artefacts occur.

How can we best get *an aliasing-free* sampled signal from these samples? Describe the filter procedure in Fourier and signal space. You do not have to derive the exact filter kernels (but you can of course).

5.4 Triangle Filter (30 Points)

Show that reconstructing a signal that is sampled at sampling distance 1 with the triangle filter $T(x)$ is equivalent of performing linear interpolation.

$$T(x) = \begin{cases} 0 & \text{for} \quad x \leq -1 \\ \mathrm{x} + 1 & \text{for} \quad -1 < x < 0 \\ -\mathrm{x} + 1 & \text{for} \quad 0 \leq x < 1 \\ 0 & \text{for} \quad 1 \leq x \end{cases}$$