Lecture 15:

Splines 1

Contents

Curve descriptions

- Explicit:
  - $y = f(x)$
    - $y(x) = \pm\sqrt{r^2 - x^2}$ restricted domain
- Implicit:
  - $F(x, y) = 0$
    - $x^2 + y^2 - r^2 = 0$ unknown solution set
- Parametric:
  - $\mathrm{x} = f_x(t),\ y = f_y(t)$
    - $\begin{matrix} x(t) = r\cos 2\pi t \\ y(t) = r\sin 2\pi t \end{matrix}, \quad t \in [0, 1]$ flexibility and ease of use

Polynomials

- $x(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots$
- Avoids complicated functions ($e.g.$ $pow(), \exp(), \sin(), sqrt()$)
- Use simple polynomials of low degree

Monomial basis

- Simple basis: $1, t, t^2, \ldots$ (t usually in $[0, 1]$)

Polynomial representation

$$
\begin{aligned}
x(t) &= a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots \\
y(t) &= b_0 + b_1 t + b_2 t^2 + b_3 t^3 + \cdots \\
z(t) &= c_0 + c_1 t + \boxed{c_2 t^2} + c_3 t^3 + \cdots
\end{aligned}
$$

Degree

Coefficients $p_i \in \mathbb{R}^3$

Monomials

$$
P(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \sum_{i=0}^{n} \begin{pmatrix} a_i \\ b_i \\ c_i \end{pmatrix} t^i
$$

- Coefficients can be determined from a sufficient number of constraints (*e.g.* interpolation of given points)
- Given $(n + 1)$ parameter values t_i and points P_i
- Solution of a linear system in the A_i - possible, but inconvenient

Matrix representation

$$
P(t)^{\mathsf{T}} = \begin{pmatrix} t^n & t^{n-1} & \cdots & t & 1 \end{pmatrix} \begin{pmatrix} a_n & b_n & c_n \\ a_{n-1} & b_{n-1} & c_{n-1} \\ \vdots & \vdots & \vdots \\ a_0 & b_0 & c_0 \end{pmatrix}
$$

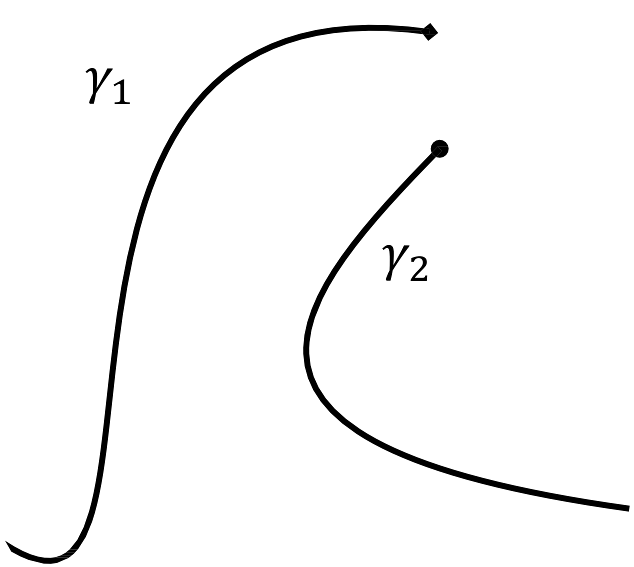
Derivative = tangent vector

- Polynomial of degree $(n-1)$

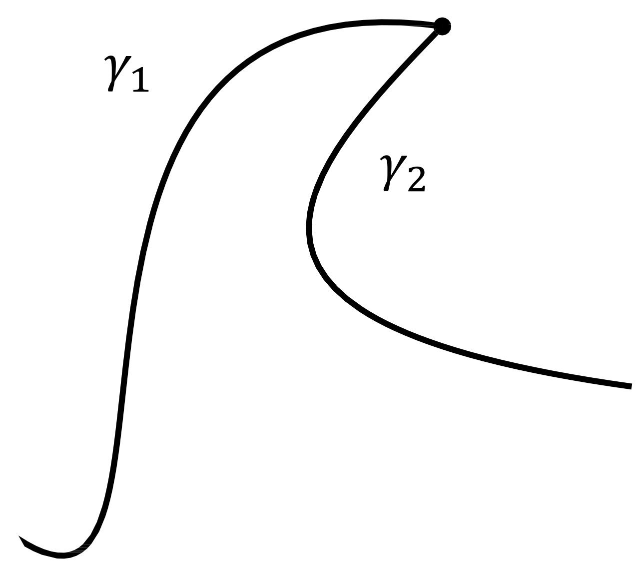
$$\frac{dP(t)}{dt} = P'(t) = \begin{pmatrix} nt^{n-1} & (n-1)t^{n-2} & \cdots & 1 & 0 \end{pmatrix} \begin{pmatrix} a_n & b_n & c_n \\ a_{n-1} & b_{n-1} & c_{n-1} \\ \vdots & \vdots & \vdots \\ a_0 & b_0 & c_0 \end{pmatrix}$$

Continuity and smoothness between parametric curves

- $\gamma_1, \gamma_2 \colon [0,1] \to \mathbb{R}^d$



Not continuous

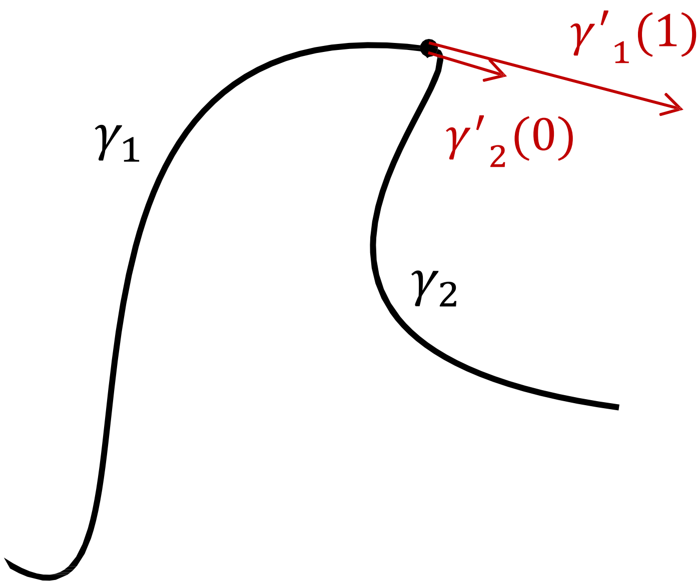


C^0 - G^0 - continuous

$\gamma_1(1) = \gamma_2(0)$

Continuity and smoothness between parametric curves

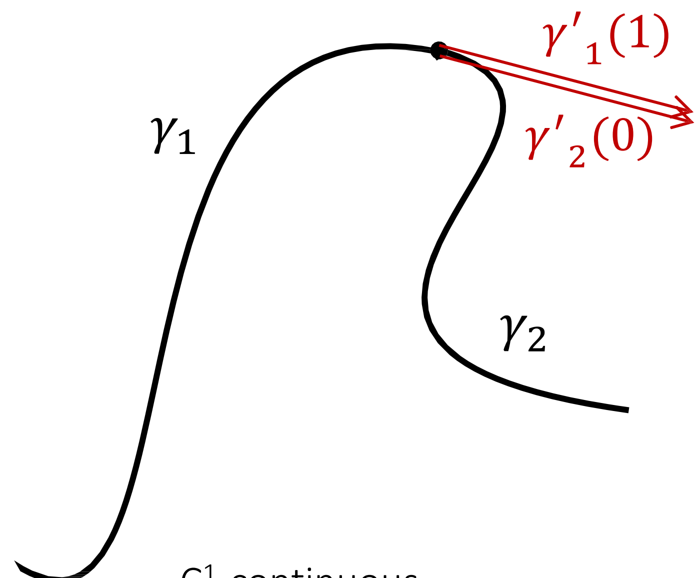
- $C^0 = G^0$ = same point
- Geometric continuity G^1
    - Same direction of tangent vectors
- Parametric continuity C^1
    - Tangent vectors are identical
- Similar for higher derivatives



G^1-continuous

G^0 + tangent vectors parallel

$$\gamma'_1(1) = k\gamma'_2(0)$$



C^1-continuous

C^0 + tangent vectors parallel

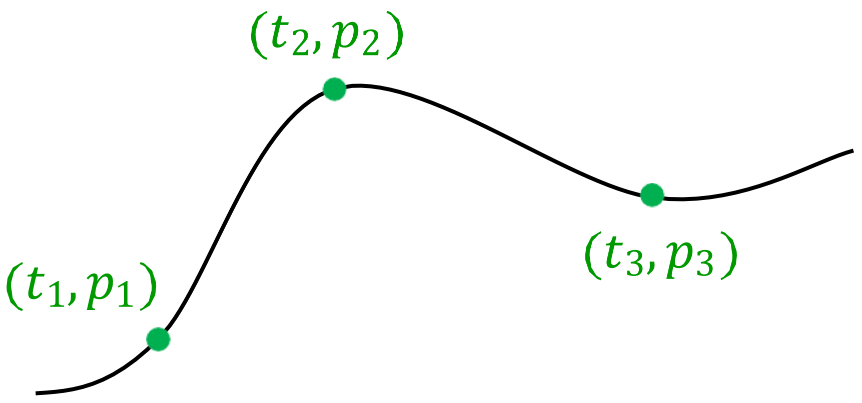
$$\gamma'_1(1) = \gamma'_2(0)$$

Given a set of points:

- $(t_i, p_i),\ t_i \in \mathbb{R},\ \ p_i \in \mathbb{R}^d$

Find a polynomial P such that:

- $\forall i\ \ P(t_i) = p_i$



Given a set of points:

- $(t_i, p_i),\ t_i \in \mathbb{R},\ \ p_i \in \mathbb{R}^d$

Find a polynomial P such that:

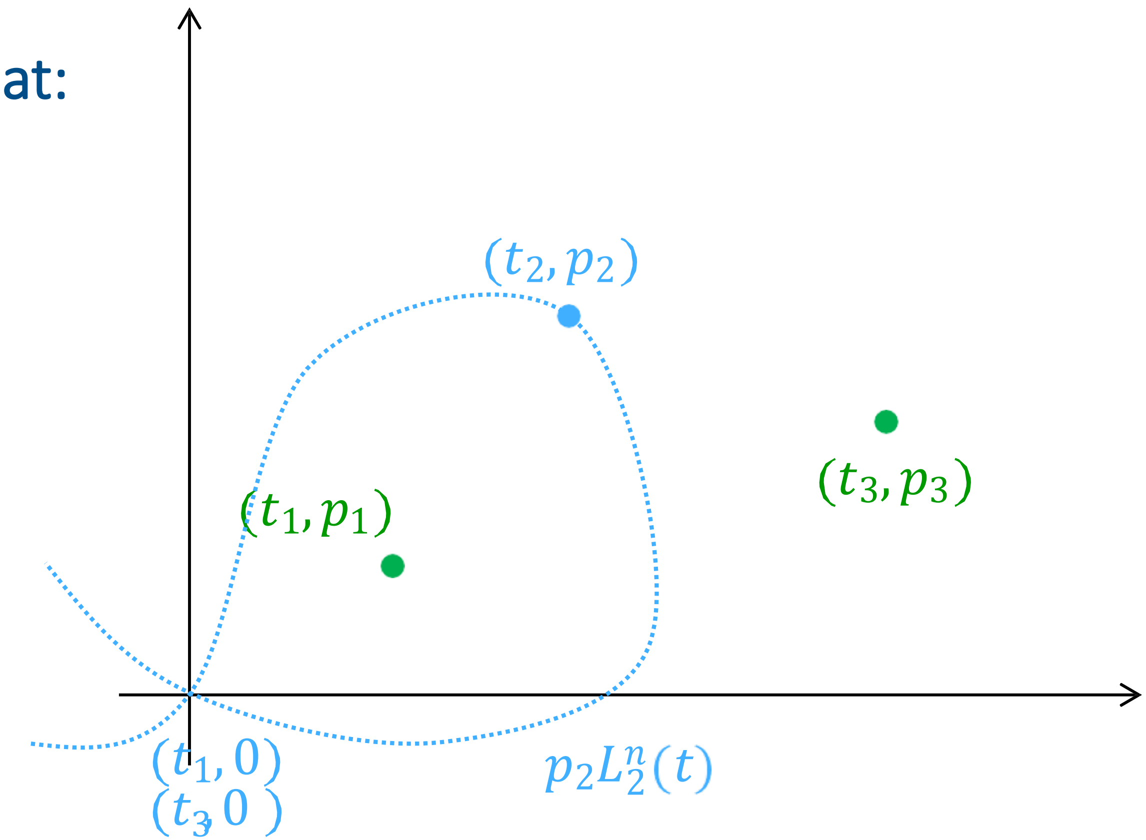
- $\forall i\ \ P(t_i) = p_i$

For each point associate a Lagrange basis polynomial:

$$L_i^n(t) = \prod_{\substack{j=0 \\ i \neq j}}^{n} \frac{t - t_j}{t_i - t_j}$$

where

$$L_i^n(t_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & otherwise \end{cases}$$



Given a set of points:

- $(t_i, p_i),\ t_i \in \mathbb{R},\ \ p_i \in \mathbb{R}^d$

Find a polynomial P such that:

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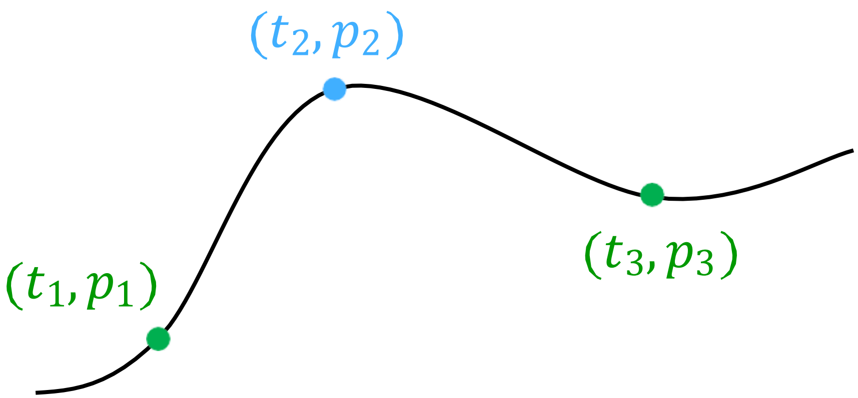
For each point associate a *Lagrange basis polynomial*:



$$L_i^n(t) = \prod_{\substack{j=0 \\ i\neq j}}^{n} \frac{t - t_j}{t_i - t_j}$$

where

$$L_i^n(t_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & otherwise \end{cases}$$

Add the Lagrange basis with points as weights:

$$P(t) = \sum_{i=0}^{n} L_i^n(t) p_i$$

$$P(t)^\top = \begin{pmatrix} L_0^n & L_1^n & \cdots & L_{n-1}^n \end{pmatrix} \begin{pmatrix} p_{0,x} & p_{0,y} & p_{0,z} \\ p_{1,x} & p_{1,y} & p_{1,z} \\ \vdots & \vdots & \vdots \\ p_{n-1,x} & p_{n-1,y} & p_{n-1,z} \end{pmatrix}$$

For each point associate a Lagrange basis polynomial:

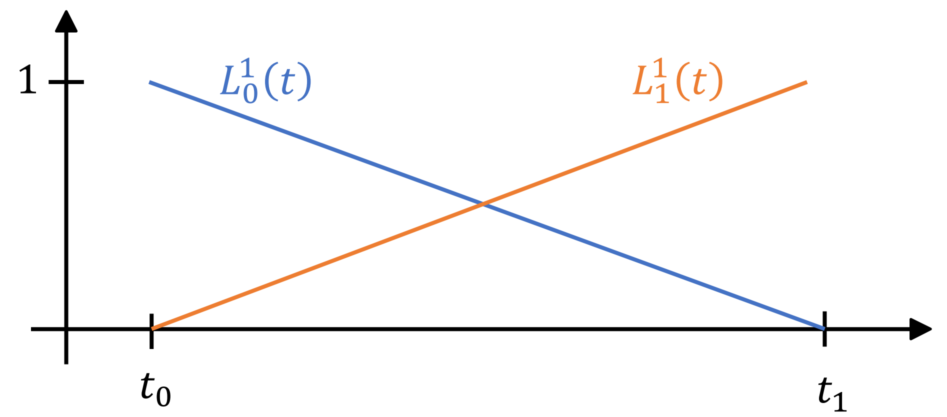
$$L_i^n(t) = \prod_{\substack{j=0 \\ i \neq j}}^{n} \frac{t - t_j}{t_i - t_j}$$

Simple Linear Interpolation

- $T = \{t_0, t_1\}$

$$L_0^1(t) = \frac{t - t_1}{t_0 - t_1}$$

$$L_1^1(t) = \frac{t - t_0}{t_1 - t_0}$$



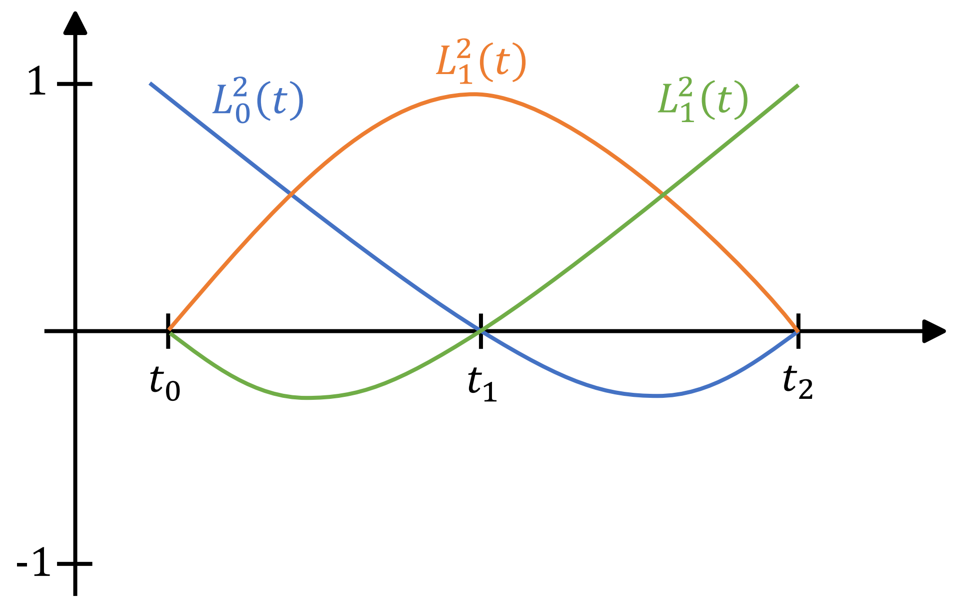
Simple Quadratic Interpolation

- $T = \{t_0, t_1, t_2\}$

$$L_0^2(t) = \frac{t - t_1}{t_0 - t_1} \frac{t - t_2}{t_0 - t_2}$$

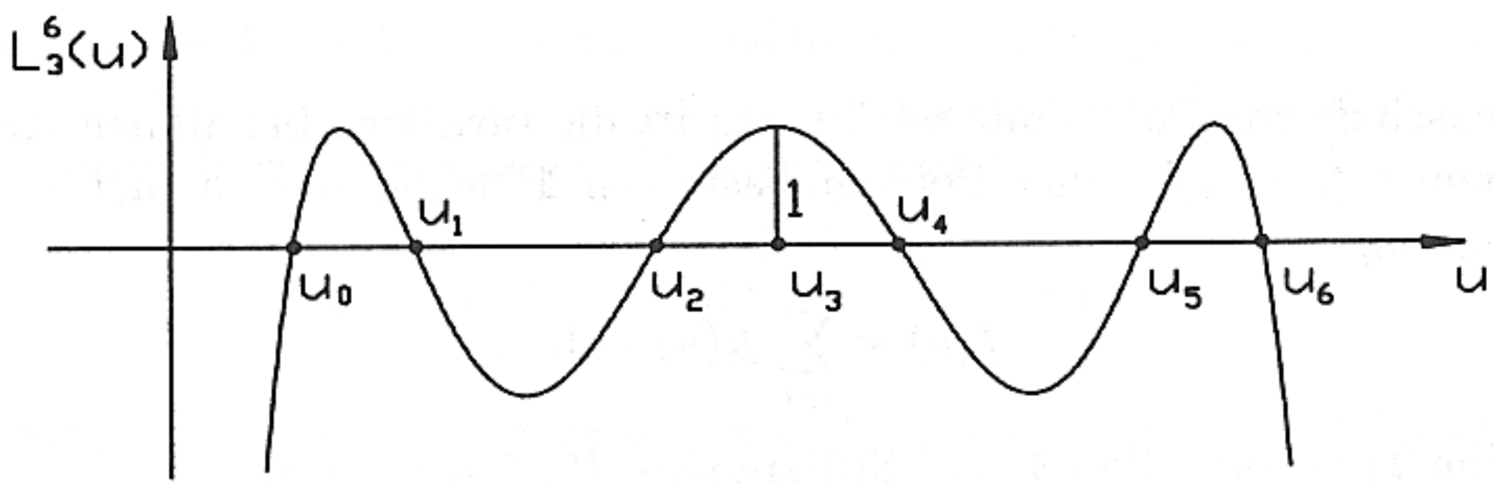
$$L_1^2(t) = \frac{t - t_0}{t_1 - t_0} \frac{t - t_2}{t_1 - t_2}$$

$$L_2^2(t) = \frac{t - t_0}{t_2 - t_0} \frac{t - t_1}{t_2 - t_1}$$



Problems with a single polynomial

- Degree depends on the number of interpolation constraints
- Strong overshooting for high degree ($n > 7$)
- Problems with smooth joints
- Numerically unstable
- No local changes

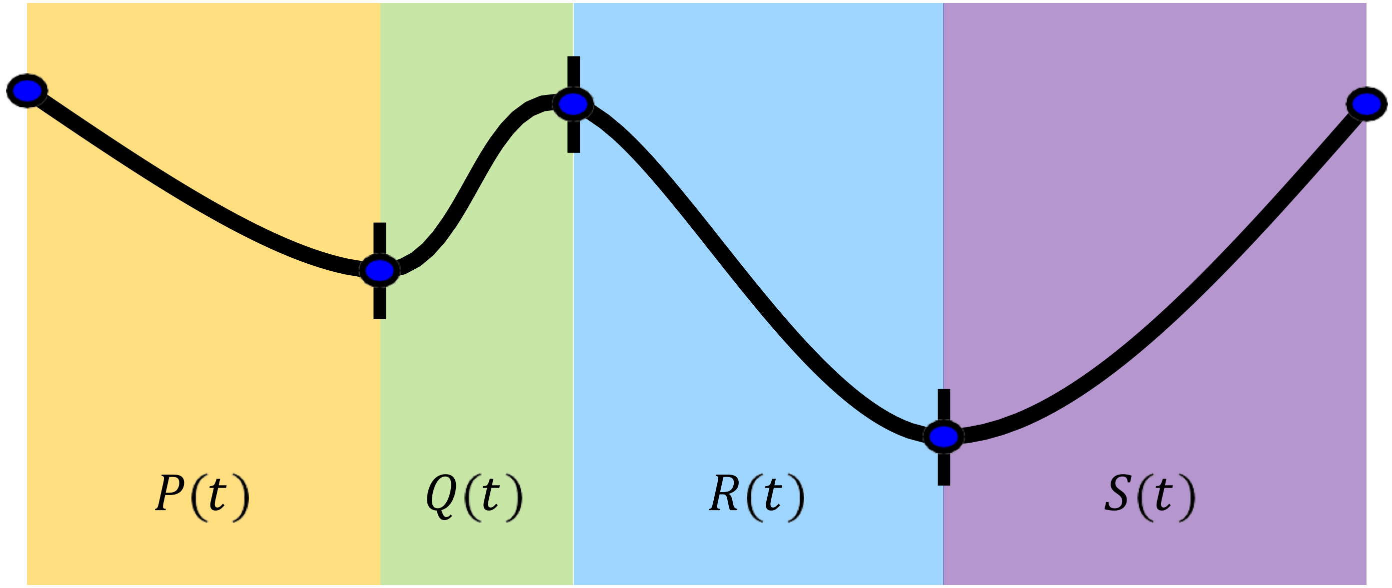


Functions for interpolation & approximation

- Standard curve and surface primitives in geometric modeling
- Key frame and in-betweens in animations
- Filtering and reconstruction of images

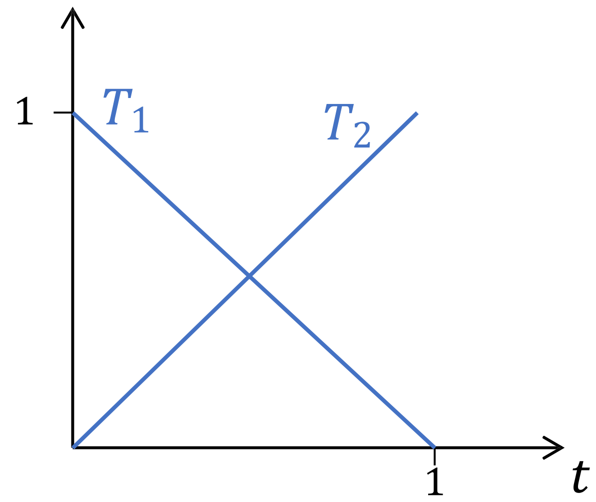
Historically

- Name for a tool in ship building
  - Flexible metal strip that tries to stay straight
- Within computer graphics:
  - Piecewise polynomial function



Linear splines

- Defined by two points: p_1, p_2
- Searching for $P(t)$ such that:
  - $P(0) = p_1$
  - $P(1) = p_2$
  - Degree of P is 1
- Basis:
  - $T_1(t) = 1 - t$
  - $T_2(t) = t$



$$P(t) = p_1 T_1(t) + p_2 T_2(t)$$

Linear basis

$$P(t)^\intercal = \boxed{\begin{pmatrix} 1-t & t \end{pmatrix}} \begin{pmatrix} p_1^\intercal \\ p_2^\intercal \end{pmatrix}$$

Linear Interpolation

monomial basis M

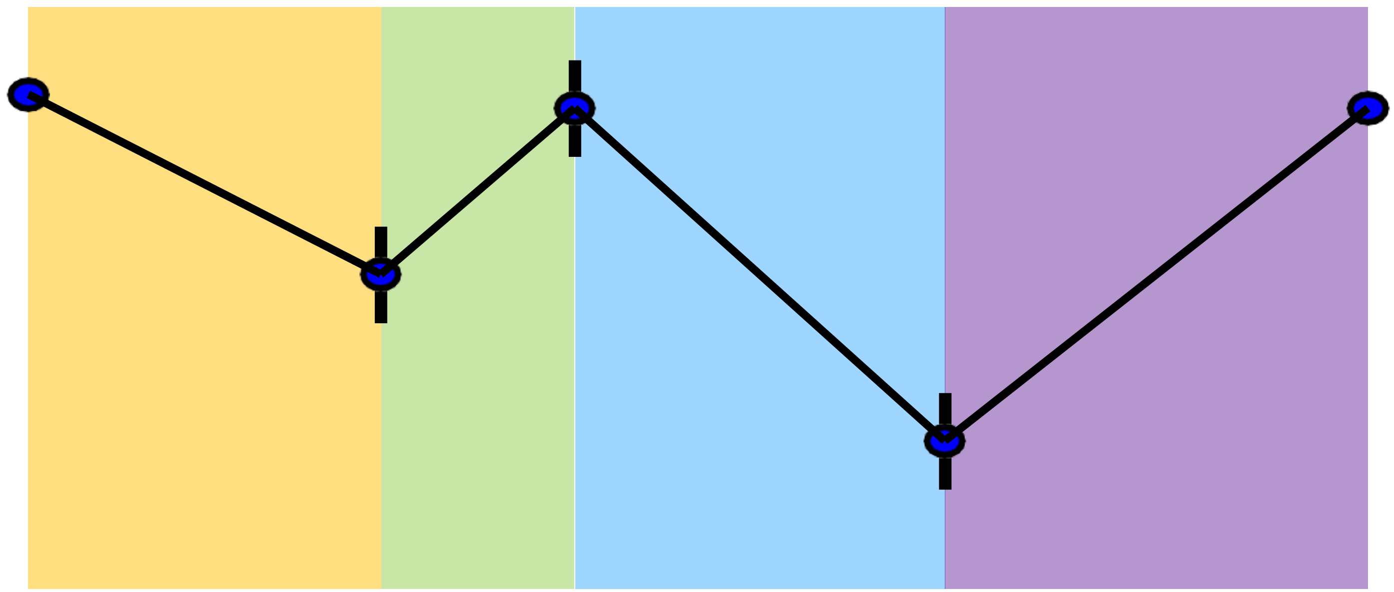
Linear basis

$$\begin{tikzcd}
\begin{pmatrix} t & 1 \end{pmatrix} \arrow[rr, bend left, "\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}"] & & \begin{pmatrix} 1-t & t \end{pmatrix} \arrow[ll, bend left, "\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}"]
\end{tikzcd}$$

$$P(t)^\intercal = M \cdot \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} p_1^\intercal \\ p_2^\intercal \end{pmatrix}$$

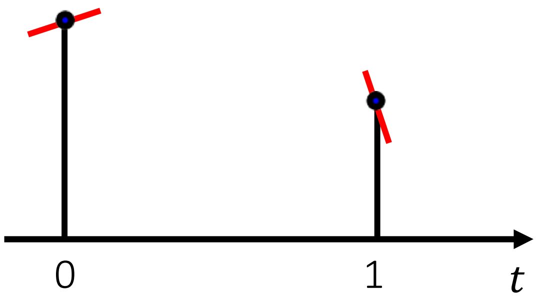
$$P(t)^{\intercal} = M \cdot \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} p_1^{\intercal} \\ p_2^{\intercal} \end{pmatrix}$$

C^0-continuous



Cubic splines

- Defined by two points: p_1, p_2 and two tangents: t_1, t_2
- Searching for $P(t)$ such that:
  - $P(0) = p_1$
  - $P'(0) = t_1$
  - $P'(1) = t_2$
  - $P(1) = p_2$
  - Degree of P is 3
- Basis:
  - $H_0^3(t) =?$
  - $H_1^3(t) =?$
  - $H_2^3(t) =?$
  - $H_3^3(t) =?$



$$P(t) = P_0 H_0^3(t) + P'_0 H_1^3(t) + P'_1 H_2^3(t) + P_1 H_3^3(t)$$

Cubic splines

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- Basis:
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  - $H_1^3(t) = ?$
  - $H_2^3(t) = ?$
  - $H_3^3(t) = ?$

$$
\begin{tikzcd}
(t^3 \quad t^2 \quad t \quad 1) \arrow[r, bend left, "H^{-1}"] & (H_0^3(t) \quad H_1^3(t) \quad H_2^3(t) \quad H_3^3(t)) \arrow[l, bend left, "H"]
\end{tikzcd}
$$

$$
P(t)^\intercal = M \cdot H \cdot \begin{pmatrix} p_1^\intercal \\ t_1^\intercal \\ t_2^\intercal \\ p_2^\intercal \end{pmatrix} = M \cdot H \cdot G
$$

Cubic splines

- Defined by two points: p_1, p_2 and two tangents: t_1, t_2
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  - $H_2^3(t) =?$
  - $H_3^3(t) =?$

- $P(t)^\intercal = (t^3 \quad t^2 \quad t \quad 1) \cdot H \cdot G$
- $P'(t)^\intercal = (3t^2 \quad 2t \quad 1 \quad 0) \cdot H \cdot G$

- $p_1^\intercal = P(0)^\intercal = (0 \quad 0 \quad 0 \quad 1) \cdot H \cdot G$
- $t_1^\intercal = P'(0)^\intercal = (0 \quad 0 \quad 1 \quad 0) \cdot H \cdot G$
- $t_2^\intercal = P'(1)^\intercal = (3 \quad 2 \quad 1 \quad 0) \cdot H \cdot G$
- $p_2^\intercal = P(1)^\intercal = (1 \quad 1 \quad 1 \quad 1) \cdot H \cdot G$

$$\begin{pmatrix} p_1^\intercal \\ t_1^\intercal \\ t_2^\intercal \\ p_2^\intercal \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \cdot H \cdot \begin{pmatrix} p_1^\intercal \\ t_1^\intercal \\ t_2^\intercal \\ p_2^\intercal \end{pmatrix}$$

Cubic splines

- Defined by two points: p_1, p_2 and two tangents: t_1, t_2
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  - $P(1) = p_2$
  - Degree of P is 3
- Basis:
  - $H_0^3(t) = ?$
  - $H_1^3(t) = ?$
  - $H_2^3(t) = ?$
  - $H_3^3(t) = ?$

$$H = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & 1 & 1 & -2 \\ -3 & -2 & -1 & 3 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Cubic splines

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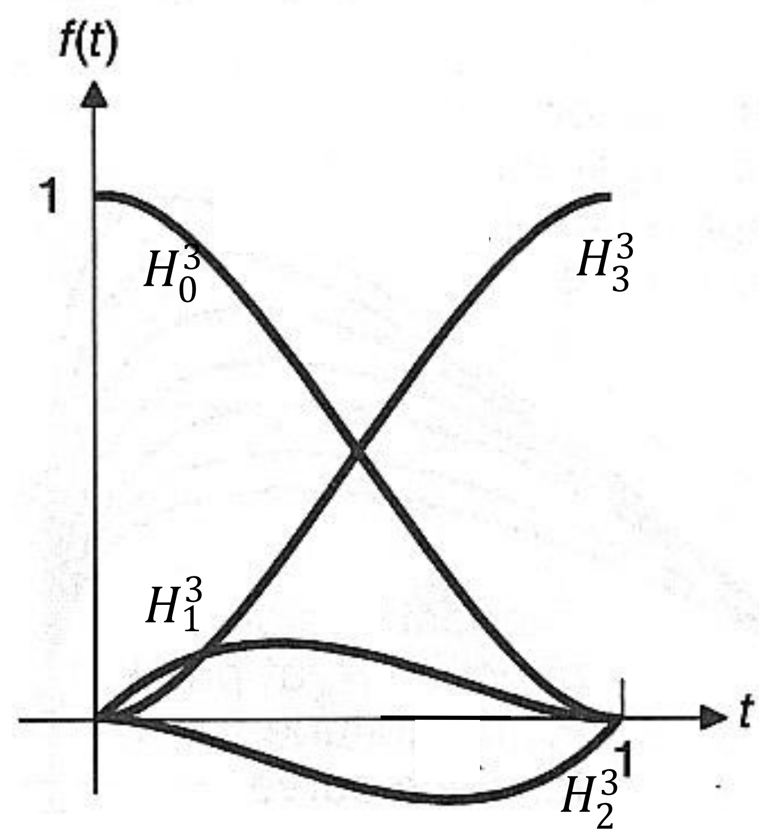
  - $P'(1) = t_2$
  - $P(1) = p_2$
  - Degree of P is 3
- Basis:
  - $H_0^3(t) = (1-t)^2(1+2t)$
  - $H_1^3(t) = t(1-t)^2$
  - $H_2^3(t) = t^2(t-1)$
  - $H_3^3(t) = (3-2t)t^2$

$$H = \begin{pmatrix} \boxed{\begin{matrix} 2 \\ -3 \\ 0 \\ 1 \end{matrix}} & \boxed{\begin{matrix} 1 \\ -2 \\ 1 \\ 0 \end{matrix}} & \boxed{\begin{matrix} 1 \\ -1 \\ 0 \\ 0 \end{matrix}} & \boxed{\begin{matrix} -2 \\ 3 \\ 0 \\ 0 \end{matrix}} \end{pmatrix}$$

$$(H_0^3(t) \quad H_1^3(t) \quad H_2^3(t) \quad H_3^3(t))$$



Cubic splines

- Basis:
  - $H_0^3(t) = (1-t)^2(1+2t)$
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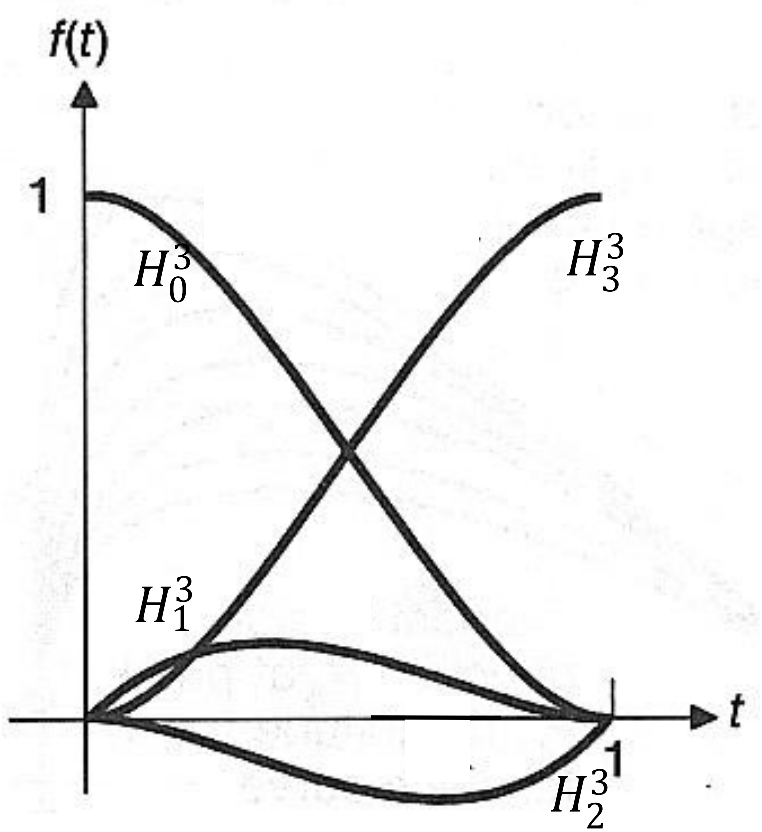
  - $H_3^3(t) = (3-2t)t^2$

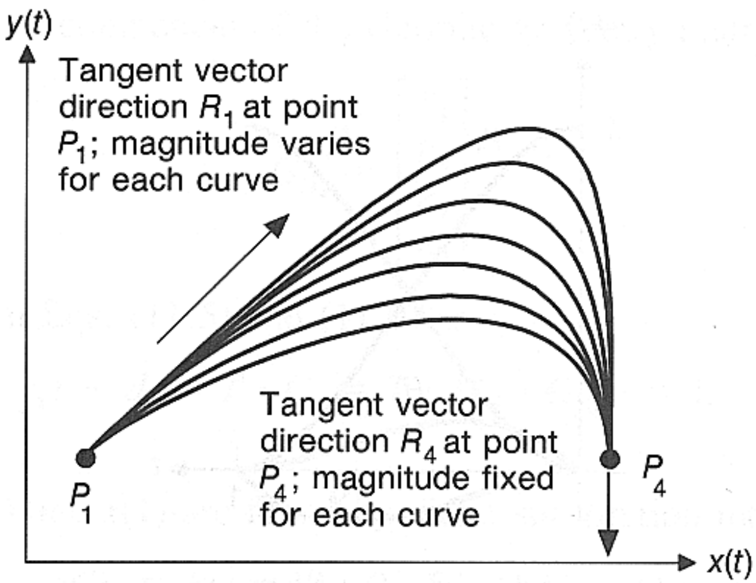
$$H = \begin{pmatrix} 2 & 1 & 1 & -2 \\ -3 & -2 & -1 & 3 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

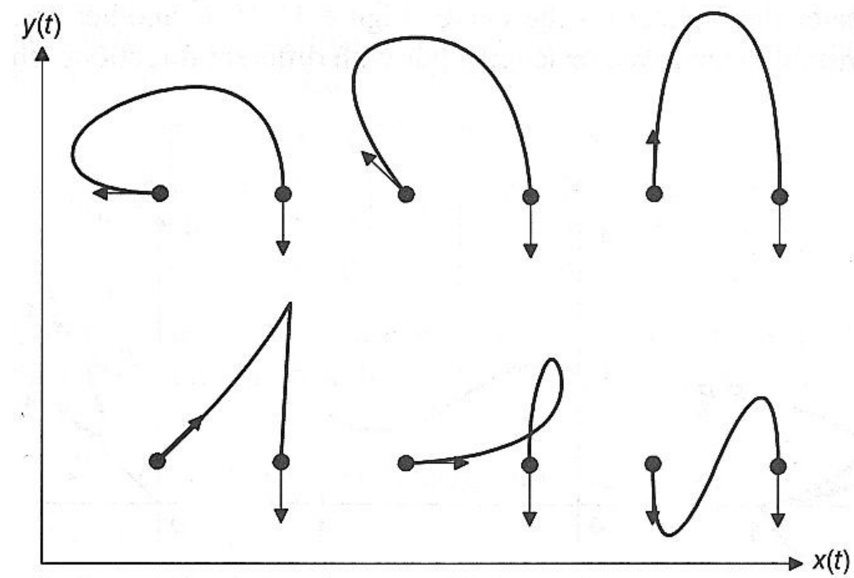
$$\left(H_0^3(t) \quad H_1^3(t) \quad H_2^3(t) \quad H_3^3(t)\right)$$

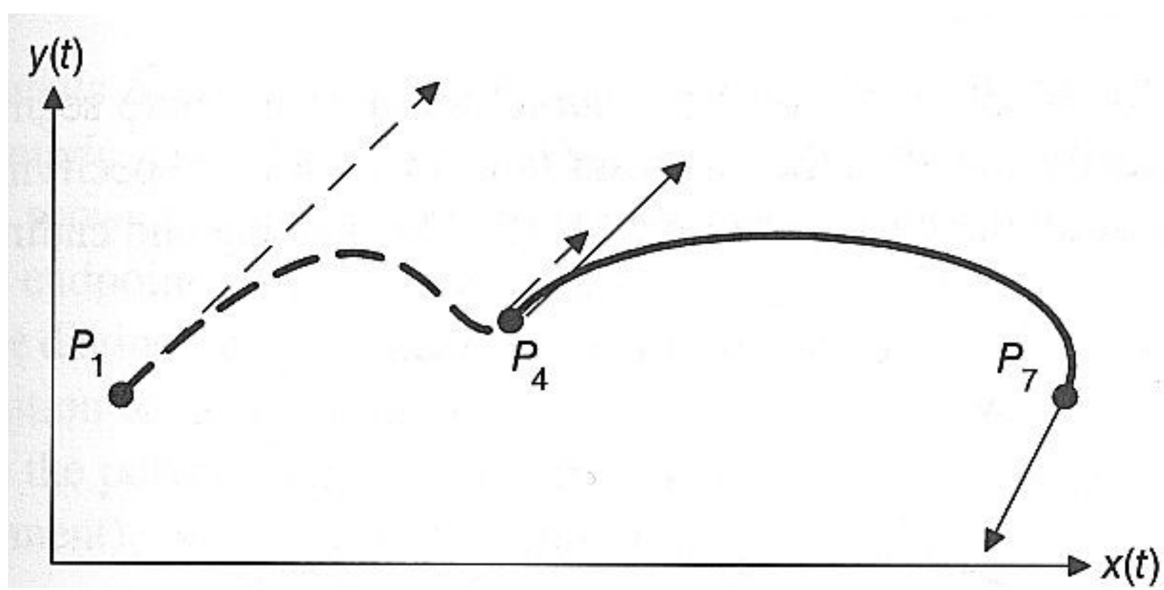
Properties of Hermite Basis Functions

- H_0^3 (H_3^3) interpolates smoothly from 1 to 0
- H_0^3 and H_3^3 have zero derivative at $t = 0$ and $t = 1$
  - No contribution to derivative (H_1^3, H_2^3)
- H_1^3 and H_2^3 are zero at $t = 0$ and $t = 1$
  - No contribution to position (H_0^3, H_3^3)
- H_1^3 (H_2^3) has slope 1 at $t = 0$ ($t = 1$)
  - Unit factor for specified derivative vector



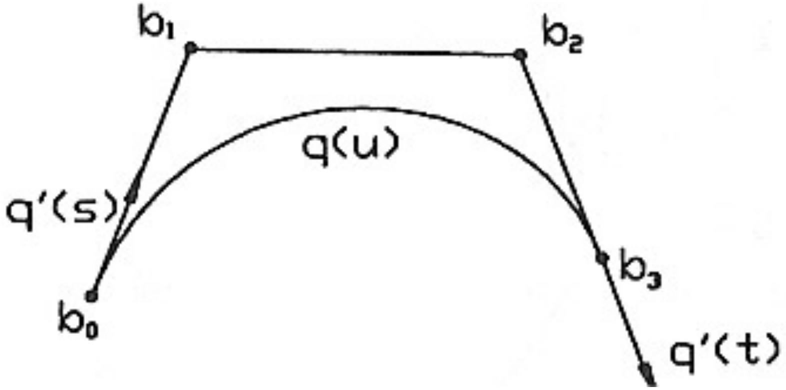

y(t)
Tangent vector direction R_1 at point P_1; magnitude varies for each curve
Tangent vector direction R_4 at point P_4; magnitude fixed for each curve
P_1
P_4
x(t)


y(t)
x(t)


y(t)
P_1
P_4
P_7
x(t)

Bézier splines

- Defined by 4 points:
  - b_0, b_3: start and end points
  - b_1, b_2: control points that are approximated
- Searching for $P(t)$ such that:
  - $P(0) = b_0$
  - $P'(0) = 3(b_1 - b_0)$
  - $P'(1) = 3(b_3 - b_2)$
  - $P(1) = b_3$
  - Degree of P is 3



Bézier splines

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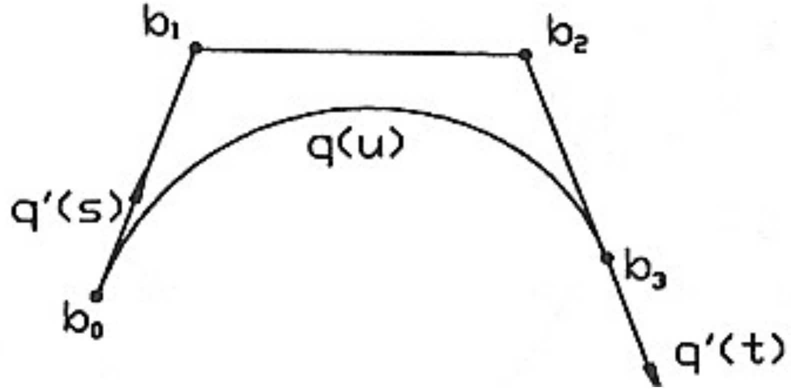

$$\begin{pmatrix} p_1^\top \\ t_1^\top \\ t_2^\top \\ p_2^\top \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & -3 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_0^\top \\ b_1^\top \\ b_2^\top \\ b_3^\top \end{pmatrix}$$

$$P(t)^\top = M \cdot H \cdot T_{BH} \cdot G$$

Bézier splines

- Defined by 4 points:
  - b_0, b_3: start and end points
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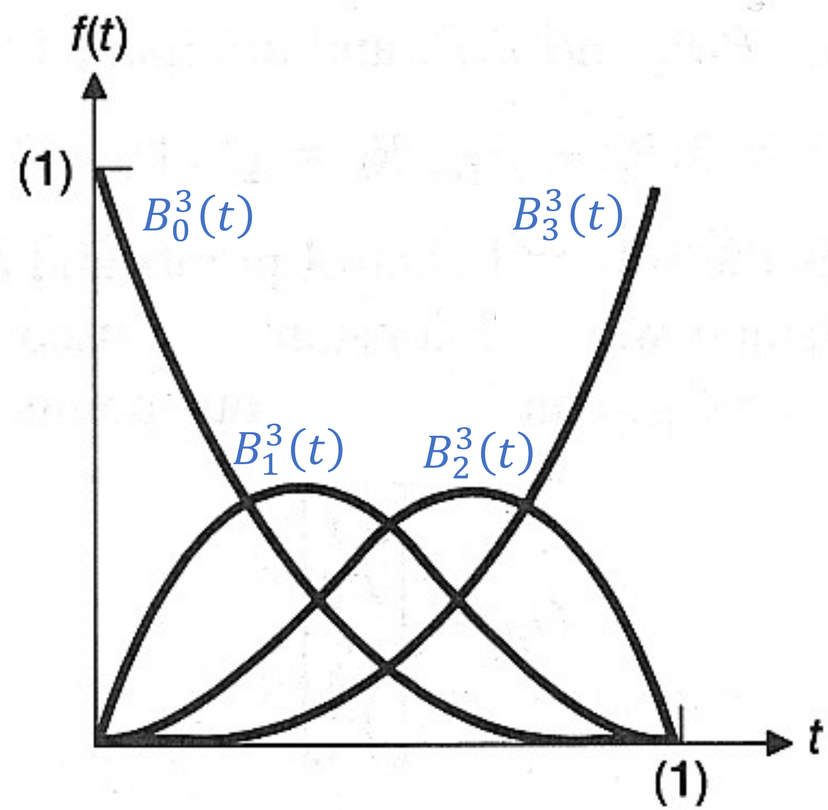
  - $P(1) = b_3$
  - Degree of P is 3
- Basis:
  - $B_0^3(t) = (1-t)^3$
  - $B_1^3(t) = 3t(1-t)^2$
  - $B_2^3(t) = 3t^2(1-t)$
  - $B_3^3(t) = t^3$
- Bernstein polynomial:
  - $B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$

$$B = H \cdot T_{BH} = \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

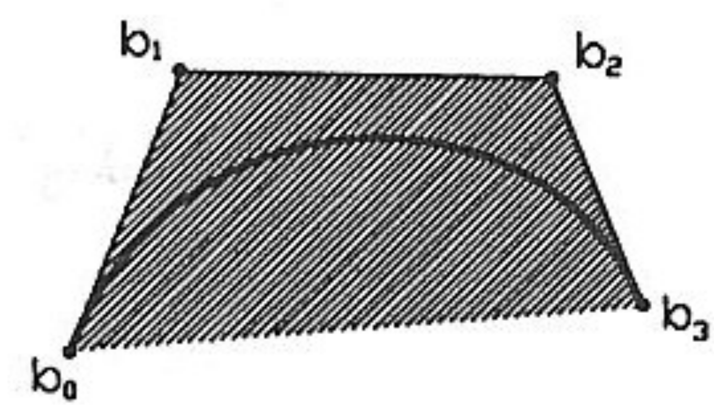


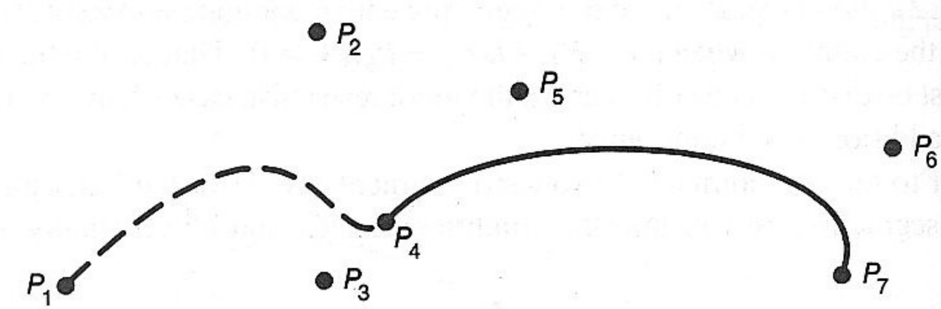
$$P(t) = b_0 B_0^3(t) + b_1 B_1^3(t) + b_2 B_2^3(t) + b_3 B_3^3(t)$$

Bézier Properties

Advantages:

- End point interpolation
- Tangents explicitly specified
- Smooth joints are simple
  - P_3, P_4, P_5 collinear → G^1 continuous
  - $P_5 - P_4 = P_4 - P_3$ → C^1 continuous
- Geometric meaning of control points
- Affine invariance
- Convex hull property
  - For $0 < t < 1$: $B_i(t) \geq 0$
- Symmetry: $B_i(t) = B_{n-i}(1-t)$





Disadvantages

- Smooth joints need to be maintained explicitly
  - Automatic in B-Splines (and NURBS)

Direct evaluation of the basis functions $P(t) = \sum_i b_i B_i^n(t)$

- Simple but expensive

Use recursion

- Recursive definition of the basis functions

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i} = tB_{i-1}^{n-1}(t) + (1-t)B_i^{n-1}(t)$$

- Inserting this once yields:

$$P(t) = \sum_{i=0}^{n} b_i^0 B_i^n(t) = \sum_{i=0}^{n-1} b_i^1 B_i^{n-1}(t)$$

- With the new Bézier points given by the recursion

$$b_i^0(t) = b_i$$

$$b_i^k(t) = tb_{i+1}^{k-1}(t) + (1-t)b_i^{k-1}(t)$$

DeCasteljau Algorithm:

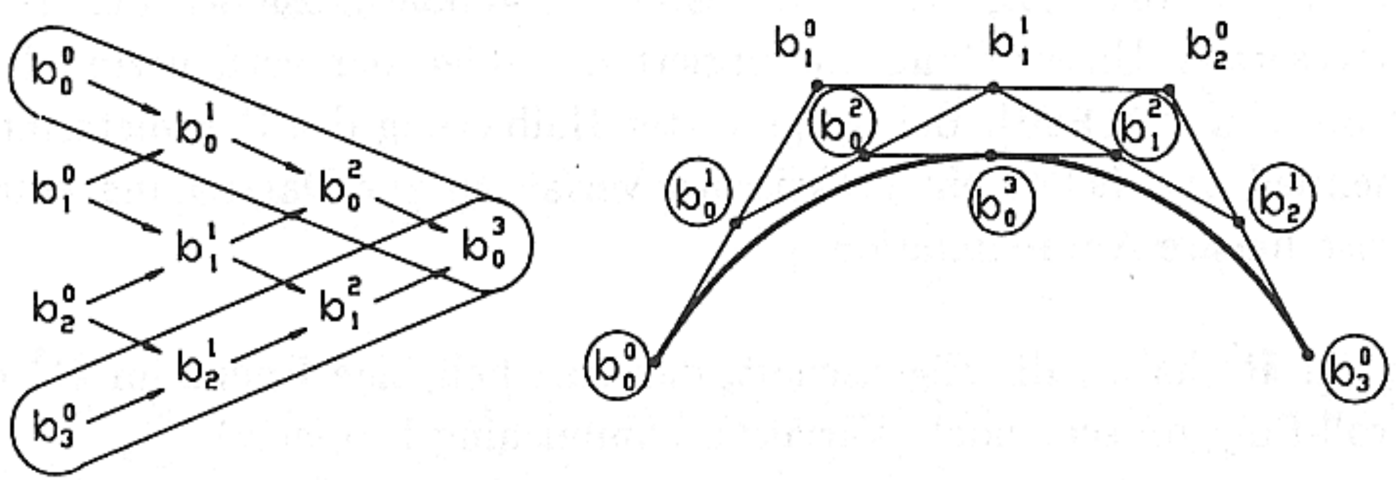
- Recursive degree reduction of the Bezier curve by using the recursion formula for the Bernstein polynomials

$$P(t) = \sum_{i=0}^{n} b_i^0 B_i^n(t) = \sum_{i=0}^{n-1} b_i^1 B_i^{n-1}(t) = \cdots = b_i^n(t) \cdot 1$$

$$b_i^k(t) = t b_{i+1}^{k-1}(t) + (1-t) b_i^{k-1}(t)$$

Example:

- $t = 0.5$

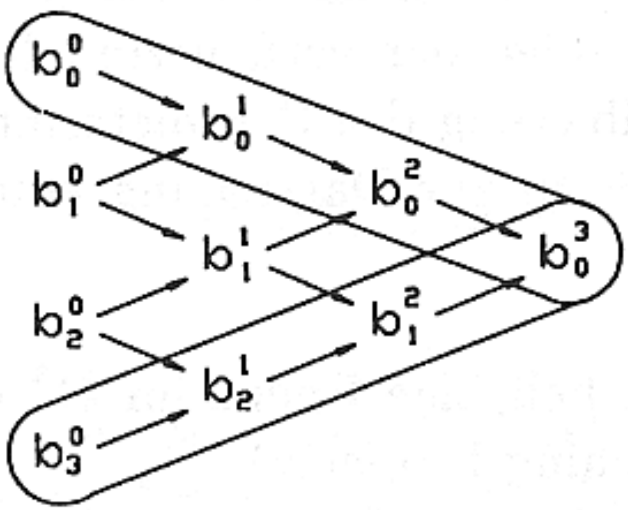


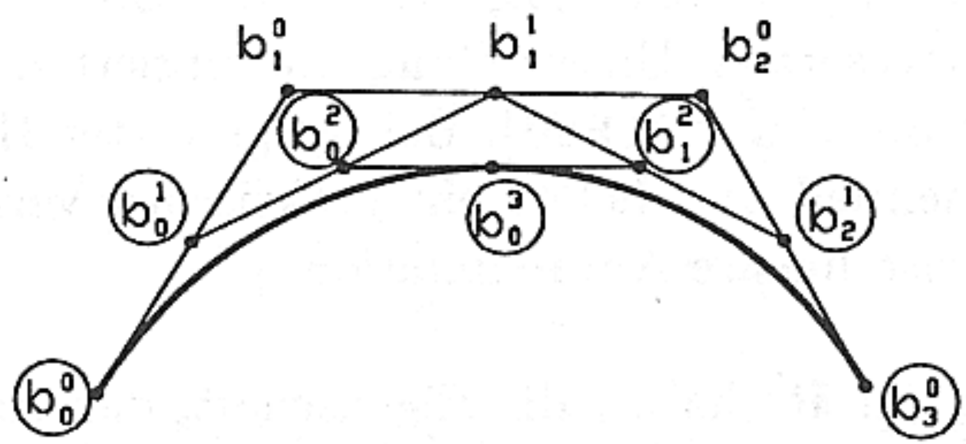
Subdivision using the deCasteljau Algorithm

- Take boundaries of the deCasteljau triangle as new control points for left / right portion of the curve

Extrapolation

- Backwards subdivision
  - Reconstruct triangle from one side



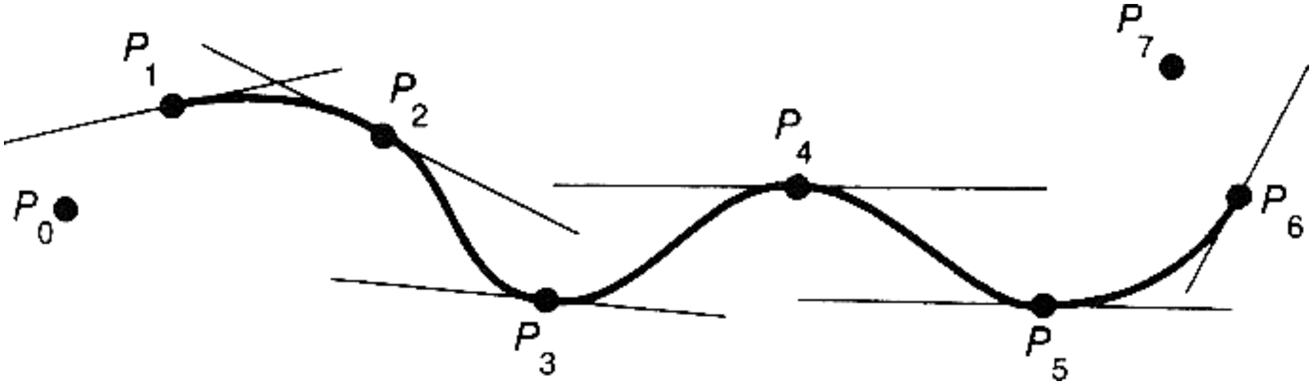


Goal

- Smooth (C^1)-joints between (cubic) spline segments

Algorithm

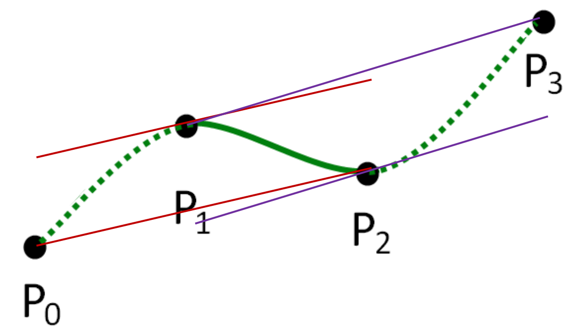
- Tangents given by neighboring points Pi-1 Pi+1
- Construct (cubic) Hermite segments



Advantage

- Arbitrary number of control points
- Interpolation without overshooting
- Local control

Catmull-Rom splines



- Defined by 4 points:
  - c_1, c_2: start and end points
  - c_0, c_3: neighbor segment points
- Searching for $P(t)$ such that:
  - $P(0) = c_1$
  - $P'(0) = \frac{1}{2}(c_2 - c_0)$
  - $P'(1) = \frac{1}{2}(c_3 - c_1)$
  - $P(1) = c_2$
  - Degree of P is 3

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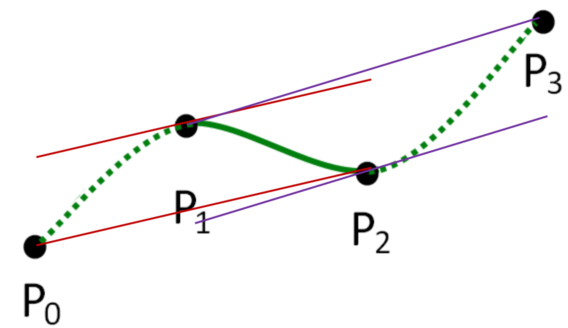
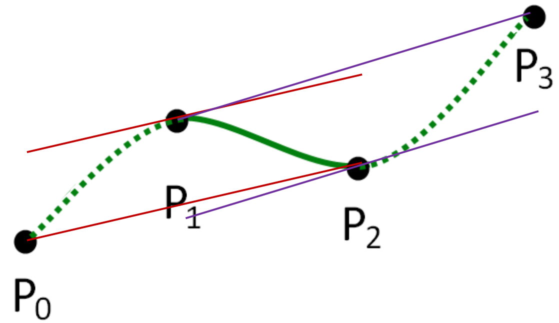


$$\begin{pmatrix} p_1^\top \\ t_1^\top \\ t_2^\top \\ p_2^\top \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -0.5 & 0 & 0.5 & 0 \\ 0 & -0.5 & 0 & 0.5 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} c_0^\top \\ c_1^\top \\ c_2^\top \\ c_3^\top \end{pmatrix}$$

$$P(t)^\top = M \cdot H \cdot T_{CH} \cdot G$$

Catmull-Rom splines



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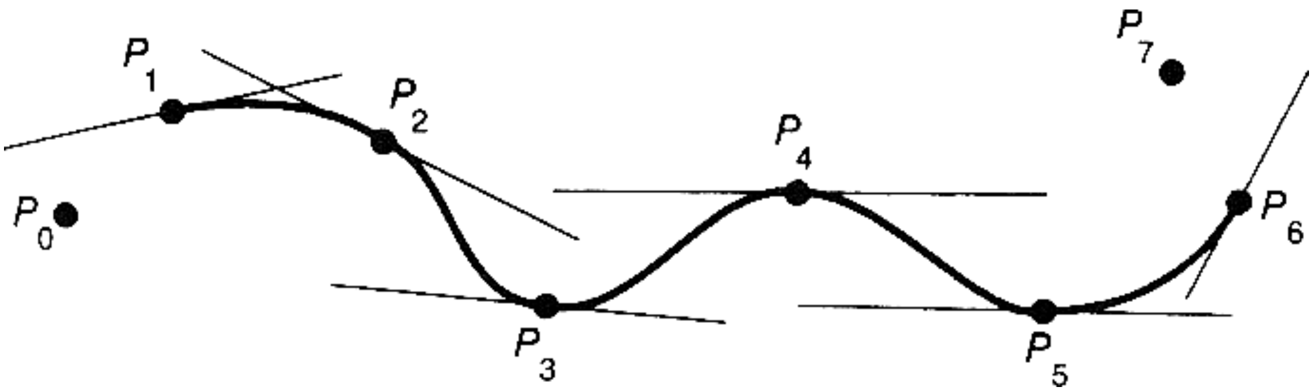
  - $P(1) = c_2$
  - Degree of P is 3
- Basis:
  - $C_0^3(t) = \frac{1}{2}t(1-t)^2$
  - $C_1^3(t) = \frac{1}{2}(t-1)(3t^2-2t-2)$
  - $C_2^3(t) = -\frac{1}{2}t(3t^2-4t-1)$
  - $C_3^3(t) = \frac{1}{2}t^2(t-1)$

$$C = H \cdot T_{CH} = \frac{1}{2}\begin{pmatrix} -1 & 3 & -3 & 1 \\ 2 & -5 & 4 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}$$

Catmull-Rom-Spline

- Piecewise polynomial curve
- Four control points per segment
- For n control points we obtain $(n-3)$ polynomial segments



Application

- Smooth interpolation of a given sequence of points
- Key frame animation, camera movement, *etc.*
- Only G^1-continuity
- Control points should be equidistant in time

Problem

- Often only the control points are given
- How to obtain a suitable parameterization t_i?

Example: *Chord-Length Parameterization*

$$t_0 = 0$$

$$t_i = \sum_{j=1}^{i} dist(P_i - P_{i-1})$$

- Arbitrary up to a constant factor

Warning

- Distances are not affine invariant !
- Shape of curves changes under transformations !!

Chord-Length versus uniform Parameterization

- Analog: Think $P(t)$ as a moving object with mass that may overshoot