Newton's iteration:

$$x_0 = \text{initial guess}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Quadratic convergence

3.3 Secant method:

When the derivative of the function is not easily accessible, we approximate the slope ($f'(x_n)$) by the slope of the secant, i.e.

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

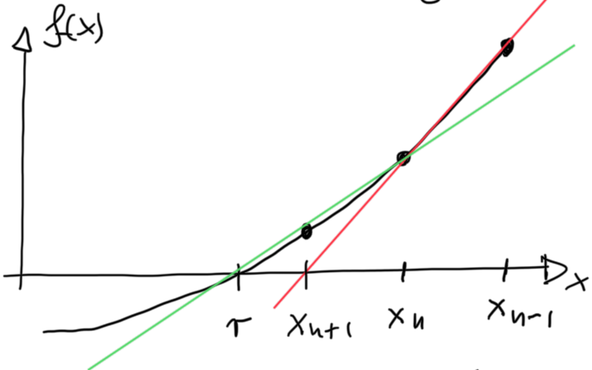
For secant's method, we let $x := x_n$, $h := x_{n-1} - x_n$, so that

$$f'(x_n) \approx \frac{f(x_{n-1}) - f(x_n)}{x_{n-1} - x_n}$$

Substituting this into the Newton sequence we get

$$x_{n+1} = x_n - f(x_n) \frac{x_{n-1} - x_n}{f(x_{n-1}) - f(x_n)}$$

for $n = 1, 2, 3, \dots$



Theorem 28: Let $f \in C^2([a,b])$ and $r \in (a,b)$ s. that $f(r) = 0$ and $f'(r) = 0$. Furthermore let

$$x_{n+1} = x_n - \frac{x_{n-1} - x_n}{f(x_{n-1}) - f(x_n)} f(x_n)$$

$f(x_{n-1}) - f(x_n)$

for $n = 1, 2, 3, \dots$. Then there exists $\delta > 0$ such that when $|r - x_0| < \delta$, $|r - x_1| < \delta$ we have

a) $\lim_{n \to \infty} |r - x_n| = 0$ $\left[\Leftrightarrow \lim_{n \to \infty} x_n = r \right]$

b) $|r - x_{n+1}| \leq M |r - x_n|^{\alpha}$

with $\alpha = \frac{1 + \sqrt{5}}{2} \approx 1.618$

This is actually super-linear convergence, because

$$\lim_{n \to \infty} \frac{|r - x_{n+1}|}{|r - x_n|} \leq \lim_{n \to \infty} M |r - x_n|^{\alpha - 1} = 0$$

Proof: Practice

Summary:

	Regularity	Proximity to r	# initial pts	root between points	# function calls	convergence
Bisection	C^0	No	2	Yes	1	linear
Newton	C^2	Yes	1	No	2	quadratic
Secant	C^2	Yes	2	No	1	super-linear

3.4 Systems of non-linear equations:

Given a vector valued function

$$f: \mathbb{R}^m \longrightarrow \mathbb{R}^m$$

we want to solve the equation $f(\vec{x}) = \vec{0}$, i.e. find a vector $\vec{r} \in \mathbb{R}^m$ such that $f(\vec{r}) = \vec{0}$.

Such function f is represented by

$$f(\vec{x}) = \begin{pmatrix} f_1(\vec{x}) \\ \vdots \end{pmatrix}$$, where $f_i : \mathbb{R}^m \longrightarrow \mathbb{R}$

$$\left. f_m(\vec{x}) \right) \quad \text{for } i = 1, \dots, m$$

and vectors $\vec{x} = (x_{(1)}, \dots, x_{(m)}) \in \mathbb{R}^m$.

Thus, $f(\vec{x}) = \vec{0}$ means that we need to simultaneously solve $\boxed{f_i(\vec{x}) = 0}$ for all $i = 1, \dots, m$

The generalization of derivative for such function is the so called Jacobian matrix:

$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial x_{(1)}} & \frac{\partial f_1}{\partial x_{(2)}} & \dots & \frac{\partial f_1}{\partial x_{(m)}} \\ \vdots & & & \vdots \\ \frac{\partial f_m}{\partial x_{(1)}} & \frac{\partial f_m}{\partial x_{(2)}} & \dots & \frac{\partial f_m}{\partial x_{(m)}} \end{bmatrix} \in \mathbb{R}^{m \times m}$$

$\frac{\partial f_i}{\partial x_{(j)}}$ = partial derivative = slope of the i-th function w.r.t the j-th variable

Example 29:

let $m = 3$, i.e. $f: \mathbb{R}^3 \to \mathbb{R}^3$,

let

$$f(\vec{x}) = \begin{bmatrix} x_{(1)} + x_{(2)} + x_{(3)} - 3 \\ x_{(1)}^2 + x_{(2)} + x_{(3)}^2 - 5 \\ e^{x_{(1)}} - x_{(1)}x_{(2)} - x_{(1)}x_{(3)} - 1 \end{bmatrix}$$

Then the Jacobi matrix (Jacobian) is

$$(Df)(\vec{x}) = \begin{bmatrix} 1 & 1 & 1 \\ 2x_{(1)} & 1 & 2x_{(3)} \\ e^{x_{(1)}} - x_{(2)} - x_{(3)} & -x_{(1)} & -x_{(1)} \end{bmatrix}$$

Now, the Newton's method can be generalized to this setting by the iteration:

$$\vec{x}_{n+1} = \vec{x}_n - (Df)^{-1}(\vec{x}_n)\, f(\vec{x}_n)$$

In practice, the inverse $(Df)^{-1}$ need not be calculated, but a system

$$\left[Df(\vec{x}_n)\right] \vec{y} = f(\vec{x}_n)$$

may be solved. Then $\vec{y} = \left[Df(\vec{x}_n)\right]^{-1} f(\vec{x}_n)$

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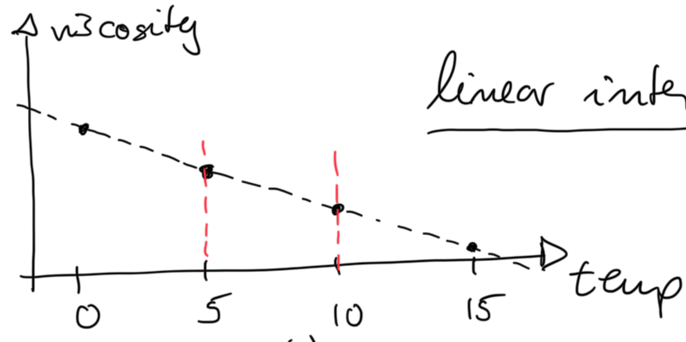
4. Interpolation and approximation:

<u>Motivation:</u> Viscosity of water has been measured for various temperatures:

temp / °C	0	5	10	15
viscosity / cP	1.792	1.519	1.308	1.140

Question: What is the viscosity at temp = 8°C

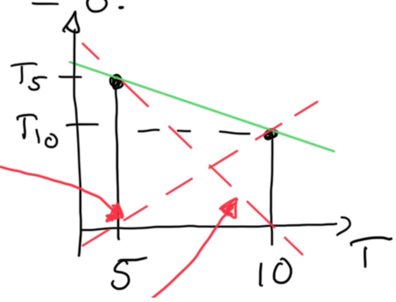
Solution:



<u>linear interpolation</u>

Find a <u>linear</u> function $v(T)$ that attains the measured values at $T = 5$ and $T = 10$. Then evaluate this function at $T = 8$.

$$v(T) = \frac{T-5}{10-5}\, T_{10} + \frac{10-T}{10-5}\, T_5$$



$10-5$

$$= \frac{T-5}{5} T_{10} + \frac{10-T}{5} T_5$$

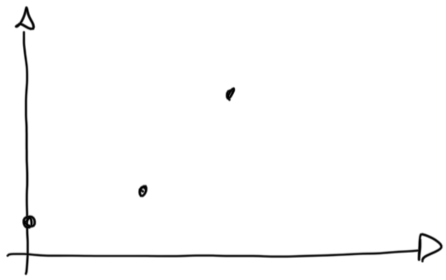
and so

$$v(8) = \frac{3}{5} T_{10} + \frac{2}{5} T_5$$

ratios of how 8 divides $[5, 10]$

that means, in order to interpolate linearly, the given values T_{10} and T_5 need to be weighted with the cut-ratios $\frac{2}{5}, \frac{3}{5}$ and added together.

What, if the measured values looked like this



a) approximate by piecewise linear interpolation

b) polynomial interpolation.