4.1 Polynomial interpolation:

Def 30: Given <u>values</u> $p_0, \dots, p_n \in \mathbb{R}$ and <u>nodes</u> $u_0, \dots, u_n \in \mathbb{R}$ a function $p: \mathbb{R} \to \mathbb{R}$ with $p(u_i) = p_i$ for $i = 0, \dots, n$ is called <u>interpolating</u>.

Note that this definition can be generalized to points $p_i \in \mathbb{R}^d$ and curves $p: \mathbb{R} \to \mathbb{R}^d$.

Goal of polynomial interpolation: Find coefficients $\alpha_0, \dots, \alpha_n \in \mathbb{R}$ such that for a given set of polynomials $\varphi_0, \dots, \varphi_n$ the linear combination

$$\boxed{p(u) := \sum_{i=0}^{n} \alpha_i \varphi_i(u)}$$

is interpolating.

This means $\quad p(u_j) = \sum_{i=0}^{n} \alpha_i \varphi_i(u_j) \overset{!}{=} p_j \, , \; j = 0, \dots$

In other words: $\alpha_1, \dots, \alpha_n$ shall be such that:

$$\boxed{\begin{array}{ccc} \sum \alpha_i \varphi_i(u_0) & = & p_0 \\ \vdots & & \vdots \\ \sum \alpha_i \varphi_i(u_n) & = & p_n \end{array}}$$

This is actually a linear system of equations in the weights α_i:

$$\Phi \vec{\alpha} = \vec{p}$$

where $\vec{\alpha} = (\alpha_0, \dots, \alpha_n)$, $\vec{p} = (p_0, \dots, p_n) \in \mathbb{R}^{n+1}$,

and $\Phi \in \mathbb{R}^{(n+1) \times (n+1)}$

and $\Phi \in \mathbb{K}$.

$$\Phi = \begin{bmatrix} \varphi_0(u_0) & \varphi_1(u_0) & \cdots & \varphi_n(u_0) \\ \varphi_0(u_1) & \varphi_1(u_1) & & \varphi_n(u_1) \\ \vdots & & & \vdots \\ \varphi_0(u_n) & & \cdots & \varphi_n(u_n) \end{bmatrix}$$

is called <u>collocation matrix</u>.

Obviously $\vec{\alpha} = \Phi^{-1} \vec{p}$

Φ is invertible iff the set of functions $\varphi_0, \dots, \varphi_n$ is linearly independent $\Leftrightarrow$ they are a basis of the respective interpolation space, here a basis of the polynomials of degree $\leq n$.

<u>Example 31:</u> Let $\varphi_i(x) = x^i$, $\varphi_0(x) = 1$,
$\varphi_1(x) = x$,
$\varphi_2(x) = x^2$, etc

Further, let $u_i := i+1$, $i = 0, \dots, n$.
Then, the collocation matrix $\Phi \in \mathbb{R}^{(n+1)\times(n+1)}$ is

$$\Phi = \begin{bmatrix} \varphi_0(u_0) & \varphi_1(u_0) & \cdots & \varphi_n(u_0) \\ \varphi_0(u_1) & \varphi_1(u_1) & \cdots & \varphi_n(u_1) \\ \vdots & & & \vdots \\ \varphi_0(u_n) & & \cdots & \varphi_n(u_n) \end{bmatrix}$$

$$= \begin{bmatrix} 1^0 & 1^1 & 1^2 & 1^3 & \cdots & 1^n \\ 2^0 & 2^1 & 2^2 & 2^3 & \cdots & 2^n \\ 3^0 & 3^1 & 3^2 & 3^3 & \cdots & 3^n \\ \vdots & & & & & \vdots \\ (n+1)^0 & & & & \cdots & (n+1)^n \end{bmatrix}$$

$$\begin{array}{cc} \uparrow & \uparrow \\ (u_i)^0 & (u_i)^n \end{array}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 4 & 8 & \cdots & 2^n \\ 1 & 3 & 9 & 27 & \cdots & 3^n \\ & & \cdots & & & \end{bmatrix}$$

Vandermonde matrix

Now, to be more concrete consider $n=2$ and points p_i given as:

x_i	1	2	3
p_i	2	5	10

Then, the system to be solved is:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 10 \end{pmatrix}$$

$\Rightarrow \alpha_0 = 1, \alpha_1 = 0, \alpha_2 = 1$

So that $p(x) = 1 \cdot \varphi_0(x) + 0 \cdot \varphi_1(x) + 1 \cdot \varphi_2(x$

$= 1 \cdot 1 \quad + 0 \cdot x \quad + 1 \cdot x^2$

$= 1 + x^2$ is the interpolating po

Q: Can we find bases for the polynomial spaces that lead to very convenient collocation matrice

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4.2 Lagrange interpolation:

Given $n+1$ points $p_0, \dots, p_n$ and corresponding nodes $x_0, \dots, x_N$ define the interpolating polynomial as

not a power!

$$p(x) = \sum_{i=0}^{n} p_i L_i^n(x)$$

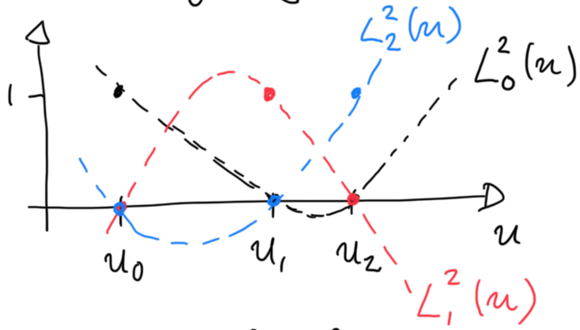
where the so called Lagrange polynomials $L_i^n(x$

of degree n fulfill:

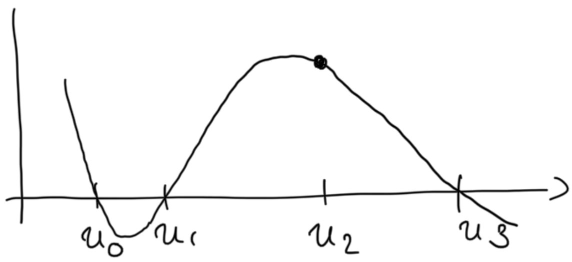
$$L_i^n(u_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{else} \end{cases}$$

Kronecker δ

Example 32: • Lagrange poly of degree 2:



• Lagrange poly of deg 3:



For $n=2$ these polynomials are defined as

$$L_0^2(u) = \frac{(u-u_1)(u-u_2)}{(u_0-u_1)(u_0-u_2)}$$

$$L_1^2(u) = \frac{(u-u_0)(u-u_2)}{(u_1-u_0)(u_1-u_2)}$$

in general, for $n \in \mathbb{N}$ we define

$$\boxed{L_i^n(u) = \prod_{\substack{j=0\\ j\neq i}}^{n} \frac{(u-u_j)}{(u_i-u_j)}}$$

Then, by construction, we have $L_i^n(u_j) = \delta_{ij}$.

Example 33:

u_i	0	1	2
p_i	2	4	3

- step 1: Lagrange polynomials

$$L_0^2(u) = \frac{(u-u_1)(u-u_2)}{(u_0-u_1)(u_0-u_2)} = \frac{(u-1)(u-2)}{(0-1)(0-2)} = \frac{1}{2}u^2 - \frac{3}{2}u \ldots$$

$$L_1^2(u) = \quad \ldots \quad = -u^2 + 2u$$

$$L_2^2(u) = \quad \ldots \quad = \frac{1}{2}u^2 - \frac{1}{2}u$$

- step 2: collocation matrix

$$\Phi = \begin{bmatrix} L_0^2(u_0) & L_1^2(u_0) & L_2^2(u_0) \\ L_0^2(u_1) & L_1^2(u_1) & \ldots \\ & \ldots & \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \ldots$$

- step 3: $\Phi \vec{\alpha} = \vec{p} \iff \vec{\alpha} = \vec{p}$

so $p(u) = 2L_0^2(u) + 4L_1^2(u) + 3L_2^2(u)$
$= \ldots$

> In Lagrange interpolation, the collocation matrix is the identity matrix.

Summary:

- In Lagrange interpolation, basis functions are such that the interpolation scheme becomes trivial.
- If we are interested in a specific value of the interpolating poly, we may use Aitken's algorithm. There is no need to construct th basis and interpolating poly.
- However, if one more node is added, the

computations need to be redone.