GE can be used whenever the pivot elements don't vanish.

But we are already getting problems with very small pivot elements, e.g. let $\varepsilon > 0$ and consider

$$\begin{aligned} \varepsilon x_1 + x_2 &= 1 \\ x_1 + x_2 &= 2 \end{aligned} \quad \Leftrightarrow \begin{bmatrix} \varepsilon & 1 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

for $\varepsilon \ll 1$ the solution will be $x_1 \approx x_2 \approx 1$.

However, GE yields $x_2 = \dfrac{2 - 1/\varepsilon}{1 - 1/\varepsilon} = 1$ in finite floating pt precision, and $x_1 = \dfrac{1 - x_2}{\varepsilon} = 0$.

2.2 Scaled partial pivoting:

- Pivoting means that the pivot element is chosen appropriately and not just row by row.
- partial pivoting means that we will reorder rows (full pivoting would also reorder cols
- Scaled means that we look for the best relative pivot element, i.e. best ratio between pivot element and maximal entry of the row (all in absolute values)

This approach will lead to minimal propagation of finite precision errors.

Algorithm 16:

1) Input $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$
2) Find maximal absolute values of entries in row

2) [illegible]

$s \in \mathbb{R}^n$ s.th. $s_i = \max\limits_{j=1\ldots n} |a_{ij}|$

Forward elimination:

3) For $k = 1, \ldots, n-1$ — for all pivot rows
4) For $i = k, \ldots, n$ — for all rows below the pivot row
5) compute $\left| \dfrac{a_{ik}}{s_i} \right|$
6) end for
7) Find row with largest rel pivot element -> row j
8) Swap rows k and j,
9) swap entries k and j in vector s
10) Do a step of forward elimination
11) End for

Backward substitution: as before

Example 17: (cf. example 13)

$$\left[\begin{array}{cccc|c} 3 & -13 & 9 & 3 & -19 \\ -6 & 4 & 1 & -18 & -34 \\ 6 & -2 & 2 & 4 & 16 \\ 12 & -8 & 6 & 10 & 26 \end{array}\right]$$

initialization: $s = (13, 18, 6, 12)^t$

1st iteration:

- relative pivots: $\dfrac{3}{13}, \dfrac{6}{18}, \dfrac{6}{6}, \dfrac{12}{12}$

- two rows, row 3 and row 4 have relative pivot elements $= 1$.
- We select row 3, and swap with row 1

$\hookrightarrow \left[\begin{array}{cccc|c} 6 & -2 & 2 & 4 & 16 \end{array}\right.$

$$\left[\begin{array}{cccc|c} -6 & 4 & 1 & -18 & -34 \\ 3 & -13 & 9 & 3 & -19 \\ 12 & -8 & 6 & 10 & 26 \end{array}\right]$$

- swap entries 1 and 3 in vector s: $(6, \underline{18}, 13, 1$
- Do a forward elimination step:

$$\left[\begin{array}{cccc|c} 6 & -2 & 2 & 4 & 16 \\ 0 & 2 & 3 & -14 & -18 \\ 0 & -12 & 8 & 1 & -27 \\ 0 & -4 & 2 & 2 & -6 \end{array}\right]$$

<u>2nd iteration</u>:

- <u>relative pivot els</u>: $\frac{2}{18}, \frac{12}{13}, \frac{4}{12}$

  $\uparrow$ best relative pivot el.

- Swap rows 3 and 2
- swap entries in vector s
- do forward elimination step

[...]

Then backward substitution as usual.

<u>Remarks 18</u>:

- In efficient implementations, the step of row swapping can be omitted, just the permutation needs to be stored somehow.
  This will result in "echelon forms" that look like

$$\left[\begin{array}{cccc|c} 0 & * & * & * & * \\ 0 & 0 & 0 & * & * \\ * & * & * & * & * \\ 0 & 0 & * & * & * \end{array}\right]$$

- GE with scaled partial pivoting always works when the matrix is invertible!
- It will fail for a singular matrix, because eventually a division by zero will result.
- Doing GE has a computational complexity of $\mathcal{O}(n^3)$, i.e. asymptotically for $n \to \infty$ the # operations for GE scales like n^3.

2.3 LU decomposition

In many applications the same linear system has to be solved for different right hand sides. For these situations the operations that are done by GE and back substitution can be stored in a convenient way.

Key observation: The row operations of GE can be formulated as linear operators aka matrices.

In forward elimination, i.e. adding a multiple of an upper row to a lower row, thes matrices are lower triangular matices:

E.g.

$$M_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

thus

$$M_1 A = \begin{cases} \cdot \text{ leave } 1^{st} \text{ row unchanged,} \\ \cdot \text{ substract } 2 \cdot R1 \text{ from } R2, \\ \cdot \text{ substract } \frac{1}{2} \cdot R1 \text{ from } R3, \\ \cdot \text{ add } R1 \text{ to } R4. \end{cases}$$

reaching echelon form in 3 steps can then be represented as doing $M_3 M_2 M_1 A$. So the system $Ax=b$ is modified as $M_3M_2M_1Ax = M_3M_2M_1b$

Example 19: (cf. Example 13)

In Example 13, GE w/o pivoting, we actua used the following matrices:

$$M_1 = \begin{bmatrix} 1&0&0&0 \\ -2&1&0&0 \\ -1/2&0&1&0 \\ 1&0&0&1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&-3&1&0 \\ 0&1/2&0&1 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&-2&1 \end{bmatrix}$$

Note that:

- The product of lower triangular matrices is lower triangular
- The inverse of lower triangular matrices is lower triangular again

Consequently: • $M_3M_2M_1$ is lower triangular
• $M_1^{-1}M_2^{-1}M_3^{-1}$ is lower triangular

To summarize

$M_3M_2M_1A$ = echelon form, i.e. upper triangular matrix U

$$M_3M_2M_1A = U \iff A = (M_3M_2M_1)^{-1}U = M_1^{-1}M_2^{-1}M_3^{-1}U = LU$$

where $L := M_1^{-1}M_2^{-1}M_3^{-1}$ is a lower triangular matrix.