Numerical Methods I

Assignment Sheet 5. Due: March 11, 2020

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Exercise 21 [3 + 4 + 4 + 4 Points]: Consider function the function

$$f(x) = \frac{1}{4x^2 + 1}.$$

a) Derive the polynomial $p_1(x)$ in Newton form that interpolates $f(x)$ by taking 5 equally spaced nodes across the interval $[-1, 1]$, i.e. taking nodes $-1, -1/2, 0, 1/2, 1$.

b) Calculate the interpolation error $\max_{x \in [-1,1]} |f(x) - p_1(x)|$ of the solution in a).

c) Re-compute the steps in part a) when using 5 Chebyshev nodes to compute the polynomial $p_2(x)$ in Newton form.

d) Calculate the interpolation error $\max_{x \in [-1,1]} |f(x) - p_2(x)|$.

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Exercise 22 [3 + 4 + 4 + 4 Points]: Consider B-splines over the nodes $u_i \in \{0, 1, 2, 3, 4\}$.

a) Draw the B-splines $N_0^0(u)$, $N_0^1(u)$, and $N_0^2(u)$.

b) Use your construction from a) to estimate $N_0^2(0)$, $N_0^2(1)$, $N_0^2(2)$, and $N_0^2(3)$. Explain your result. Note that there is no need to actually derive and evaluate the recursive formula for any u.

c) Using part b), derive the collocation matrix for spline interpolation at the nodes $2, 3, \dots, n + 1$ with the spline $s(u) = \sum_{i=0}^{n-1} s_i N_i^2(u)$ defined over the node set $\{0, 1, 2, \dots, n + 2\}$.

d) Solve the interpolation problem when assuming nodes $2, 3, 4$ and the values $3, 2, 5$. Provide the interpolating spline $s(u)$ and sketch it.

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Exercise 23 [not graded, w/o Points]: Given a spline $b(u) = \sum_{i=0}^{1} b_i N_i^2(u)$ in B-spline representation over the nodes $\{0, 1, 3, 4, 7\}$.

a) Derive the B-splines $N_i^2(u)$ for $i = 0, 1$ in piecewise monomial form for the given nodes.

b) Consider $b_i = i + 1$ for $i = 0, 1$ and derive the spline $b(u)$ in piecewise monomial form.

c) Evaluate $b(u)$ at all the nodes and sketch the spline.

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Exercise 24 [not graded, w/o Points]: Show that the B-splines fulfill the partition of unity property, i.e. $\sum_i N_i^n(u) = 1$.

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Exercise 25 [not graded, w/o Points]: *De Boor's algorithm* provides a method for evaluating a spline in B-spline representation without actually constructing the spline. It basically follows the recursion formula that has been given in class (Cox-de Boor recursion formula):

$$N_i^0(u) = \begin{cases} 1 & \text{if } u_i \le u < u_{i+1}, \\ 0 & \text{else.} \end{cases}$$

$$N_i^n(u) = \frac{u - u_i}{u_{i+n} - u_i} N_i^{n-1}(u) + \frac{u_{i+n+1} - u}{u_{i+n+1} - u_{i+1}} N_{i+1}^{n-1}(u)$$

but it avoids to compute terms that are multiplied with zero in the recursion. Study the de Boor's algorithm from the textbook and understand the derivation of the recursion formula.