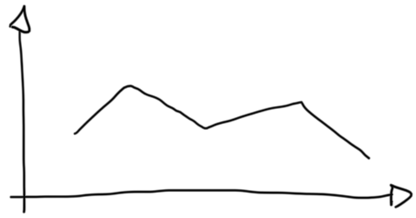
Spline interpolation:

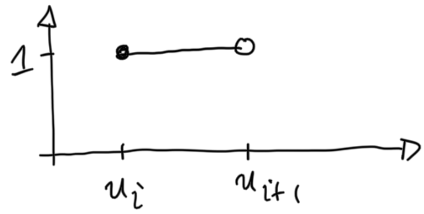
Splines are locally polynomials,
Spline of degree k is of class C^{k-1} and locally a poly of degree k.

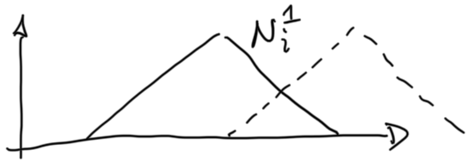


Spline of degree 1, i.e.
C^0 continuous
and locally poly of deg 1
i.e. linear

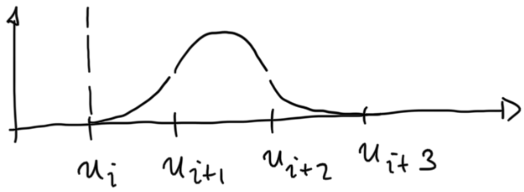
Basis splines (B-splines) N_i^k

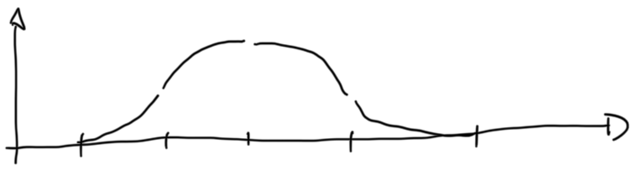
N_i^0





$\mathrm{supp}\, N_i^k = [u_i, u_{i+k+1}$





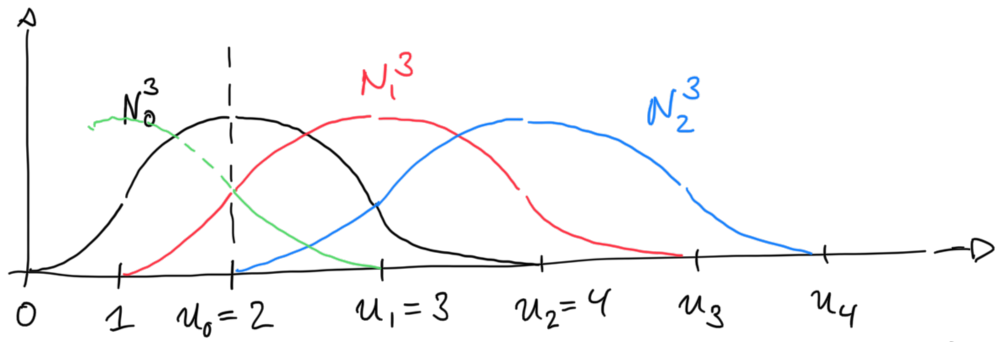
Example 41: Cubic spline interpolation

Given values p_i, $i = 0, \ldots, m$

Nodes $u_i = i + 2$

[illegible]

Assume degree of splines is 3



! Note that the numbering / indexing of the B-splines is not matching the numbering of the nodes.

Interpolation means: Find $\sum_{i=0}^{m} c_i N_i^3(u) =: s(u)$

such that $\boxed{s(u_i) = p_i}$.

This leads to the collocation matrix ϕ

$$\phi = \begin{bmatrix} N_0^3(u_0) & N_1^3(u_0) & N_2^3(u_0) & \cdots & N_m^3(u_0) \\ N_0^3(u_1) & N_1^3(u_1) & \cdots & & \vdots \\ \vdots & & & & \\ & & \cdots & & N_m^3(u_m) \end{bmatrix}$$

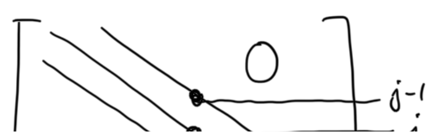
Obviously,

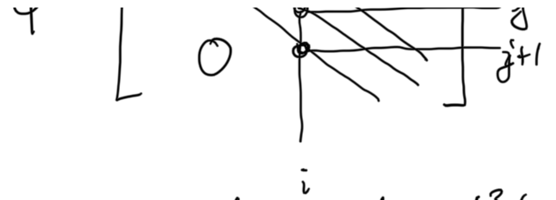
- $N_i^3(u_j) \neq 0$ when $i = j$ or $j = i+1$ or $j = i-1$

  $N_i^3(u_j) = 0$ else

- $N_i^3(u_{i-1}) = N_i^3(u_{i+1})$

This means, the collocation matrix is symmetric and has <u>3 bands</u>, i.e. <u>tridiagona</u>

$\phi =$ 



We find out that $N_i^3(u_i) = \frac{4}{6} = \frac{2}{3}$

and $N_i^3(u_{i\pm 1}) = \frac{1}{6}$

Thus

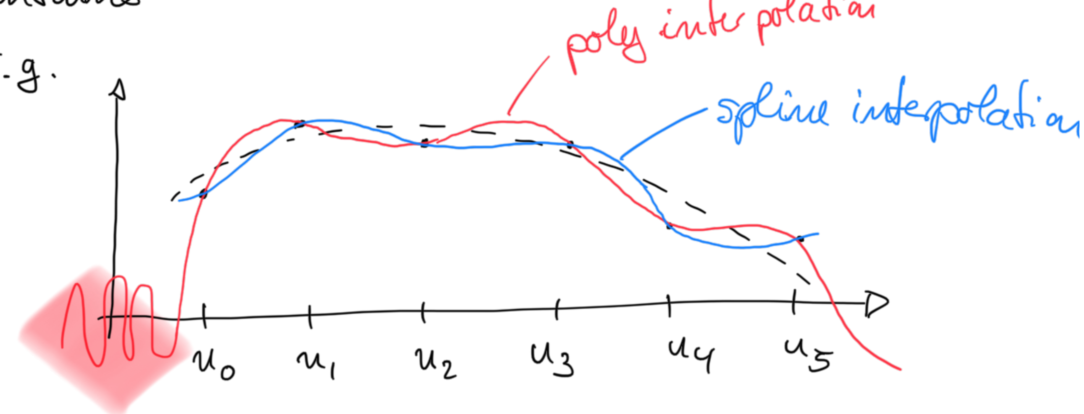
$$\Phi = \frac{1}{6}\begin{bmatrix} 4 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & 4 \end{bmatrix}$$

This matrix is SPD, consequently a unique solution to $\Phi \vec{c} = \vec{p}$ exists.

4.7 Least squares approximation

Polynomial interpolation, piecewise interpolation, Hermite even spline interpolation may yield unsatisfactory results when applied to measurement data that contains errors:

E.g.



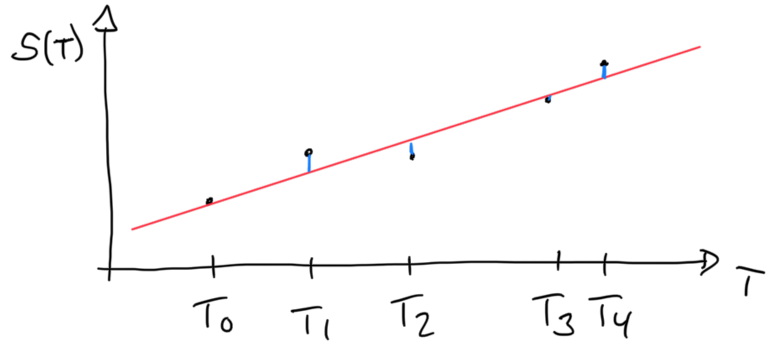
Instead of doing an interpolation we should look for an approximation, i.e. a curve that stays close to the measurements and is of a certain type.

Example 42: Surface tension S of a liquid depends on the liquid's temperature. It is known that the dependence is linear, i.e.

$$S = S(T) = aT + b$$

for some $a, b \in \mathbb{R}$.

In order to find a and b for a specific liquid we do measurements:



Obviously, no straight line interpolates all points. Of all lines that could be considered a fit, which is the best?

Consider the approximation error and minimize

- Suppose we have $(n+1)$ measurements: $S_0, \dots, S_n$ for nodes $T_0, \dots, T_n$
- At any node T_i the error is

$$\left| S_i - (aT_i + b) \right|$$

- The sum of all local errors is

$$E_1(a,b) := \sum_{i=0}^{n} \left| S_i - (aT_i + b) \right|$$

This is the <u>approximation error</u> measured in the ℓ_1-norm.

- In order to find the minimum we need to identify critical pts of $E_1(a,b)$

BUT: $E_1(a,b)$ is not continuously differentiable for all a, b.

- Remedy: go over to squares of the local errors.

- Approximation error in the l_2-norm:

$$E_2(a,b) := \sum_{i=0}^{n} \left(S_i - (aT_i + b) \right)^2$$

this will not change the optimal a, b.

So, now, to find the critical pts we consider

$$\frac{\partial E_2}{\partial a} = \frac{\partial}{\partial a}\left[\sum_{i=0}^{n} (S_i - (aT_i + b))^2\right]$$

$$= \sum \frac{\partial}{\partial a} \left(S_i - (aT_i + b)\right)^2$$

$$= -\sum 2\left(S_i - (aT_i + b)\right) T_i$$

$$= -\sum 2 S_i T_i + 2a \sum T_i^2 + 2b \sum T_i$$

critical pt:

$$0 = \frac{\partial E_2}{\partial a} \iff \boxed{a \sum T_i^2 + b \sum T_i = \sum S_i T_i} \quad (\mathrm{I})$$

and

$$\frac{\partial E_2}{\partial b} = \frac{\partial}{\partial b}\left[\sum \left(S_i - (aT_i + b)\right)^2\right]$$

$$= -\sum 2 (S_i - (aT_i + b))$$

critical point:

$$0 = \frac{\partial E_2}{\partial b} \iff a \sum T_i + b \sum 1 = \sum S_i$$

$$\iff \boxed{a \sum T_i + b(n+1) = \sum S_i} \quad (\mathrm{II})$$

(I) and (II) determine a linear system of equations in a and b.

equations in a and b:

$$\begin{bmatrix} \sum_i T_i^2 & \sum_i T_i \\ \sum_i T_i & (n+1) \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{bmatrix} \sum_i S_i T_i \\ \sum_i S_i \end{bmatrix}$$

HW: Existence of solutions?

<u>In more generality</u>: The <u>least squares approach</u> assumes that we have measurements y_i, $i=0,\dots,n$ at nodes x_i, $i=0,\dots,n$. We are searching for a function:

$$y(x) = \sum_{j=0}^{m} c_j\, g_j(x)$$

for linearly independent functions $g_j(x)$, $j=0,\dots,m$

In the least squares approach we are minimizing the ℓ_2-error

$$\varphi(c_0,\dots,c_m) = \sum_{i=0}^{n} \Big(y_i - \sum_j c_j g_j(x_i) \Big)^2$$

A necessary condition is vanishing partial derivatives

$$\frac{\partial \varphi}{\partial c_k} = 0 \qquad \text{for } k=0,\dots,m$$

In fact:

$$\frac{\partial \varphi}{\partial c_k} = -\sum_i 2\Big(y_i - \sum_j c_j g_j(x_i) \Big) g_k(x_i) \overset{!}{=} 0$$

$$\Longleftrightarrow \boxed{\sum_{j=0}^{m} \Big(\sum_{i=0}^{n} g_j(x_i) g_k(x_i) \Big) c_j = \sum_{i=0}^{n} y_i\, g_k(x_i)}$$

for $k=0,\dots,m$

These equations are <u>called normal equations.</u>

It yields an $(m+1)\times(m+1)$ system.

We need $n \geq m$ in order to have unique solution