Problem Set # 1

Math 215

Please box your answers for each of the exercises below. Also, be mindful of your presentation, I will deduct 10 points for disorganized or unintelligible answers.

Hi all – Please do ALL exercises in 1.5 (page 3), 2.2 (page 5), and prove Theorem 2.4 part (b). Enjoy!

Also, please read carefully as much of the whole chapter to complement our earlier lectures or to prepare for next week.

Preface

The fundamentals of general topology are basic to modern analysis. Furthermore, abstract algebra and general topology are increasingly recognized as essential for the mathematics major. Some difference of opinion exists as to when topology should be introduced to the student.

After observing my undergraduate students in topology for several years, I am convinced that the fullest understanding and the greatest economy in effort result when the student follows an early course in abstract algebra by a course in topology before he studies advanced calculus or real analysis. It seems that a background in topology greatly simplifies his work in advanced calculus, while a background in the latter does not noticeably simplify his work in topology.

Topology for the undergraduate is not something for the honor student only, but benefits the general mathematics major and the mathematics education student. The idea that mathematical maturity is a prerequisite for a study of topology is widely replaced by the belief that such a study provides an excellent opportunity to attain rapid maturity.

For the reasons given above I have written this book with the undergraduate in mind. Many items (from set theory and topology) which might be difficult to the inexperienced student if introduced at the earliest logical entry place, are postponed. Often these items are very simple to the student after he is more experienced.

In Chapter 2, I begin the study of topology with the general topological space (rather than a metric space or the space of reals) for three reasons.

First, I feel that the early theorems are, in fact, easier for the undergraduate to prove for the general topological space than for the space of reals or for metric spaces. Second, there is a real economy in proving the general theorems and then applying them to the special spaces. Third, the student who has chosen mathematics as a major and who has studied abstract algebra can appreciate the abstract space from the outset and will realize the motivation soon after the general theory is applied to the special cases. Applications to the space of reals are stressed in this book.

The rigor of this text is adequate for graduate students who have not had a course in topology, and this material has been used as a text for these students as well as for undergraduates. Chapters 1 through 3, the first five sections of Chapter 4, and the first section of Chapter 5 can be covered in a one-semester undergraduate course.

Throughout this book it is emphasized that the student should give his own proofs of the exercises and many of the theorems before reading the proofs given in the text. This is an attempt to build into the book some of the features of the popular "do-it-yourself" types of courses in topology (where the class proves all the theorems). This text is intended to speed progress and to offer polished proofs with which the student may compare his own proofs. The "do-it-yourself" method may be used to cover some topics omitted from the text.

I am especially grateful to Professor W. L. Strother, who read parts of the manuscript and made valuable comments. I am also grateful to Professor R. G. Blake for many discussions concerning the manuscript.

I am grateful to many of my students who have given me great pleasure by their enthusiasm, interest, and success.

I am indebted to Professors L. M. Blumenthal, P. B. Burcham, W. R. Utz, V. W. Adkisson, and D. P. Richardson for invaluable assistance when I was a student.

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Contents

I

Elementary Set Theory

1. Sets and Subsets

A precise, sophisticated development of set theory will not be given here. Rather, we give some of the language and ideas from set theory which are essential to our present work and to abstract mathematics in general.

The word *set* will be used as an undefined term. However, we may associate with the phrase "set of things" an intuitive idea of a collection of things, and we shall use the words "set" and "collection" synonymously—though certainly without defining either! "Class" and "family" are sometimes used as synonyms for "set".

Let A denote a particular collection of objects. Each member of this collection is called an *element of A*. (Thus if A denotes the set of all books on a given shelf, the elements of A are the individual books on that shelf.) Now if x denotes a specific object, exactly one of the following statements is correct:

(1) x is an element of A (indicated by $x \in A$),

or

(2) x is not an element of A (written $x \notin A$).

The notation $\{a, b, g, k\}$ denotes the set consisting of the objects represented by the symbols listed and no other objects. Thus {Arkansas, Florida, Missouri} denotes the set consisting of the things which the writer represents by the symbols listed. (Naturally the writer is using the last symbols listed to represent three states in the United States.)

The notation $\{x\colon\ P(x)\}$, where $P(x)$ is a statement concerning x, denotes the set of all x's for which $P(x)$ is true and no other objects. This notation is read "the set of all x's such that $P(x)$". Of course a letter other than "x" may be used at times. Thus $\{y\colon\ y$ is a real number greater than $4\}$ denotes the set of all real numbers greater than 4, and $\{p\colon\ p$ is a person now in Florida$\}$ denotes the set of all people now in Florida.

If A and B are sets, B is said to be a *subset of* A if and only if each element of B is also an element of A. We indicate that B is a subset of A by writing $B \subset A$ (read "B is contained in A"), or by writing $A \supset B$ (read "A contains B"). We formally state this definition using language to be adopted henceforth. We use "iff" as an abbreviation for "if and only if". (The reader should be sure to note that "if and only if" means more than just "if".)

Definition 1.1. Let A and B be sets. Then $B \subset A$ iff $x \in B$ implies $x \in A$.

[We shall use $A, C \subset B$ to mean $A \subset B$ and $C \subset B$; similarly, $x, y \in A$ means $x \in A$ and $y \in A$.]

Definition 1.2. Two sets A and B are said to be *equal* (written $A = B$) iff $A \subset B$ and $B \subset A$.

In view of 1.1, $\{a, d\} \subset \{a, b, d, h\}$ and $\{2, 5, 7\} \subset \{2, 5, 7\}$ even though $\{2, 5, 7\} = \{2, 5, 7\}$. In general, 1.1 gives: if A is a set, then $A \subset A$. Finally we note that

$$\{x\colon\ x \text{ is a positive integer less than } 2\} = \{1\}.$$

Throughout this book we shall reserve certain letters to denote specific sets as indicated:

Notation 1.3.

$R = \{x\colon\ x \text{ is a real number}\}$, $\quad N = \{x\colon\ x \text{ is a positive integer}\}$,

$N_k = \{x\colon\ x \in N \text{ and } x \leq k\}$, where $k \in N$,

$Q = \{x\colon\ x \in R \text{ and } 0 \leq x \leq 1\}$, $\quad I = \{x\colon\ x \in R \text{ and } 0 < x < 1\}$.

In the natural course of events we frequently write $\{x\colon\ P(x)\}$, where it may turn out that there is no x for which $P(x)$ is true; for example, consider $\{x\colon\ x \in N \text{ and } x < k\}$, where k is some positive integer. If $k = 1$, it turns out that there is no x such that x is a positive integer and $x < k$. So it is convenient to introduce a set which has no elements; we call this set the *null set*, *void set*, or *empty set* and denote it by $\varnothing$.

Note that 1.1 gives: if A is a set, then $\varnothing \subset A$.

In a particular mathematical discussion there is usually a set which consists of all primary elements under consideration. For lack of a better term, we shall call this set the *underlying set* and denote it by S throughout sections 1 and 2 of this chapter. We shall be concerned with subsets of S and with collections of subsets of S.

(In a study of the system of real numbers, the underlying set S would be R, and we would have occasions to mention various subsets of R such as Q, I, or the set of all rational numbers. If one is studying Euclidean plane geometry, then S would be the set of all points in the Euclidean plane.)

Definition 1.4. Let $A, B \subset S$. Then $\{x\colon\ x \in A \text{ and } x \notin B\}$ is called the *complement of B relative to A* and is denoted by $A \sim B$. The set $S \sim B$ is simply called the *complement of B*.

Exercises 1.5

1. List the eight subsets of $\{a, b, c\}$.
2. If the underlying set S is the set of all people now in the United States, what is the complement of

   (a) the set of all people now in Florida,

   (b) $\{x\colon\ x$ is a person now in Florida or x is a person now in Georgia$\}$?
3. Let $A = \{1, 2, 3, 4\}$, $\quad B = \{1, 2\}$, $\quad C = \{4, 5\}$. What is the set

   (a) $A \sim B$, $\quad$ (c) $B \sim A$,

   (b) $A \sim C$, $\quad$ (d) $B \sim C$?
4. Let A be any subset of the underlying set S. What is the complement of

   (a) S, $\quad$ (b) $\varnothing$, $\quad$ (c) $(S \sim A)$?

Answers and proofs for many exercises in this book are given following the sets of exercises. It should be emphasized that the student should do these exercises before reading the answers in the book. Also he should receive the greatest pleasure in trying his hand at proving some of the theorems in the text before reading the proofs given. This is his most valuable mathematical experience, and soon he should be able to give his own proofs for some "very good" theorems. The following theorem gives the answer to 4(c) in 1.5.

Theorem 1.6. If $A \subset S$, then $S \sim (S \sim A) = A$.

Proof. Suppose $p \in A$. Then

$$p \notin S \sim A, \qquad \text{(Def 1.4)}$$

and so

$$p \in S \sim (S \sim A) \qquad \text{(Def 1.4)}.$$

Therefore

(3) $$A \subset S \sim (S \sim A) \qquad \text{(Def 1.1)}.$$

Now suppose $q \in S \sim (S \sim A)$. Then $q \notin S \sim A$ and so $q \in A$. Hence

(4) $$S \sim (S \sim A) \subset A \qquad \text{(Def 1.1)}.$$

Therefore

$$S \sim (S \sim A) = A \qquad [(3), (4), \text{(Def 1.2)}].$$

Theorem 1.7. Let $A, B \subset S$. Then $B \subset A$ iff $S \sim A \subset S \sim B$.

Proof. Suppose $B \subset A$ and $p \in S \sim A$. Then $p \notin A$ and, by 1.1, $p \notin B$ since $B \subset A$. Hence $p \in S \sim B$. Therefore $S \sim A \subset S \sim B$, and we have proved the "only if" part of our theorem (the part obtained by writing "only if" in the place of "iff"). Changing notation we have now proved that if $C \subset D$ then $S \sim D \subset S \sim C$, and we may use this result to prove the other "half" of 1.7.

To prove the "if" part of 1.7, suppose $S \sim A \subset S \sim B$. Then from the above result (taking C to be $S \sim A$ and D to be $S \sim B$) we have

$$S \sim (S \sim B) \subset S \sim (S \sim A).$$

Thus

$$\begin{aligned} B &= S \sim (S \sim B) && \text{(Th 1.6)} \\ &\subset S \sim (S \sim A) && \text{(above result)} \\ &= A && \text{(Th 1.6)}. \end{aligned}$$

Therefore $B \subset A$ and 1.7 is proved.

Theorem 1.8. Let $A, B \subset S$. Then $A = B$ iff $S \sim A = S \sim B$.

Proof. Suppose $A = B$. Then (by 1.2) $B \subset A$ and, by 1.7,

(5) $$S \sim A \subset S \sim B.$$

Similarly

(6) $$S \sim B \subset S \sim A \quad (\text{since } A \subset B).$$

Therefore from (5), (6) and 1.2,

$$S \sim A = S \sim B.$$

To prove the converse suppose $S \sim A = S \sim B$. Then

$$\begin{aligned} A &= S \sim (S \sim A) && \text{(Th 1.6)} \\ &= S \sim (S \sim B) && \text{(first part 1.8)} \\ &= B && \text{(Th 1.6)}. \end{aligned}$$

2. Unions and Intersections

Definition 2.1. Let $\mathcal{C}$ be a collection of subsets of the underlying set S.

(a) The set $\{x\colon\ x \in A$ for *at least one* A in $\mathcal{C}\}$ is called the *union* of the collection $\mathcal{C}$ and is denoted by $\bigcup \mathcal{C}$.

(b) The set $\{x\colon\ x \in S$ and $x \in A$ for *each* A in $\mathcal{C}\}$ is called the *intersection* of the collection $\mathcal{C}$ and is denoted by $\bigcap \mathcal{C}$.

Notice that an element of S, say p, belongs to $\bigcup \mathcal{C}$ iff *there exists at least one* A in $\mathcal{C}$ such that $p \in A$, while $p \in \bigcap \mathcal{C}$ iff $A \in \mathcal{C}$ implies $p \in A$.

Since "$\{A\colon\ A \in \mathcal{C}\}$" is just another symbol for the collection $\mathcal{C}$, we may write "$\bigcup\{A\colon\ A \in \mathcal{C}\}$" to denote $\bigcup \mathcal{C}$. Indeed, any symbol which denotes a collection of sets may be written after "$\bigcup$" to denote the union of the collection; similarly for intersections.

Exercises 2.2

1. Let $S = R$, $A = \{x\colon\ x \in S \text{ and } 0 < x < 4\}$, $B = \{x\colon\ x \in S \text{ and } 1 < x \leq 3\}$ and $C = \{x\colon\ x \in S \text{ and } 2 < x \leq 5\}$. Specify

   (a) $\bigcup \{A, B, C\}$,

   (b) $\bigcap \{A, B, C\}$.

Usually we shall use the notation $A \cup B \cup C$ for the set in (a), $A \cap B \cap C$ for the set in (b) and similar (self-explaining) notations for unions and intersections.

2. Let $A = \{-2, 0, 3, 4, 5\}$, $B = \{0, 1, 2, 3, 5\}$, $C = \{0, 6, 7\}$ and $D = \{8, 9\}$. List the elements (if there are any) of

   (a) $A \cap B$, (c) $A \cap C$,

   (b) $A \cup B \cup C$, (d) $A \cap D$.

3. Before reading 2.3, consider 2.1 and decide what $\bigcup \mathcal{C}$ is when $\mathcal{C}$ is the null collection of subsets of S. What is $\bigcap \mathcal{C}$ in case $\mathcal{C}$ is null? (See the first paragraph after 2.1.)

4. $S \sim \bigcup \mathcal{C}$ is equal to which of the sets: $\bigcup \{S \sim A\colon\ A \in \mathcal{C}\}$ or $\bigcap \{S \sim A\colon\ A \in \mathcal{C}\}$, where $\{S \sim A\colon\ A \in \mathcal{C}\}$ denotes $\{B\colon\ B = S \sim A$ for some A in $\mathcal{C}\}$. The notation introduced here will be used freely in the future.

5. What can you say about $S \sim \bigcap \mathcal{C}$?

Remark 2.3. If $\mathcal{C}$ is the null collection of subsets of S, then

$$\text{(a)}\quad \bigcup \mathcal{C} = \varnothing \qquad \text{and} \qquad \text{(b)}\quad \bigcap \mathcal{C} = S.$$

Proof. (a) Suppose $p \in S$. Now in order that $p \in \bigcup \mathcal{C}$, p would have to belong to *at least one* A in $\mathcal{C}$; but this is impossible since there is no A in $\mathcal{C}$. Hence $p \notin \bigcup \mathcal{C}$. Since p was an arbitrary element of S, it follows that $\bigcup \mathcal{C} = \varnothing$.

(b) Suppose $p \in S$. Then since $\mathcal{C}$ is null, it is correct that p belongs to each member of $\mathcal{C}$. Hence, by 2.1(b), $p \in \bigcap \mathcal{C}$ and so $S \subset \bigcap \mathcal{C}$. But by 2.1(b), $\bigcap \mathcal{C} \subset S$; and therefore $\bigcap \mathcal{C} = S$.

(To convince the skeptical reader of (b), we might say that in view of 2.1(b) the only way an element p of S can fail to belong to $\bigcap \mathcal{C}$ is that there be an A in $\mathcal{C}$ such that $p \notin A$; but this cannot happen since there is no A in $\mathcal{C}$.) (Or we may say to the reader experienced in formal logic that the second sentence in the proof of (b) can be written as: "Then since $\mathcal{C}$ is null, the following proposition is true—since the hypothesis in this proposition is false: If $A \in \mathcal{C}$, then $p \in A$.")

Theorem 2.4 (DeMorgan). If $\mathcal{C}$ is a collection of subsets of the underlying set S, then

$$\text{(a)}\qquad S \sim \bigcup \mathcal{C} = \bigcap \{S \sim A\colon\ A \in \mathcal{C}\}$$

and

$$\text{(b)}\qquad S \sim \bigcap \mathcal{C} = \bigcup \{S \sim A\colon\ A \in \mathcal{C}\}.$$

Proof. (a) Suppose $p \in S \sim \bigcup \mathcal{C}$. Then $p \notin \bigcup \mathcal{C}$, and so

$$\text{for each } A \text{ in } \mathcal{C},\quad p \notin A \qquad \text{(Def 2.1(a))}.$$

Hence

$$\text{for each } A \text{ in } \mathcal{C},\quad p \in S \sim A \qquad \text{(Def 1.4)}.$$

Therefore

$$p \in \bigcap \{S \sim A\colon\ A \in \mathcal{C}\} \qquad \text{(Def 2.1(b))},$$

and

$$\text{(1)}\qquad S \sim \bigcup \mathcal{C} \subset \bigcap \{S \sim A\colon\ A \in \mathcal{C}\} \qquad \text{(Def 1.1)}.$$

Now suppose $q \in \bigcap \{S \sim A\colon\ A \in \mathcal{C}\}$. Then $q \in S \sim A$ for each A in $\mathcal{C}$; so for each A in $\mathcal{C}$, $q \notin A$. Therefore $q \notin \bigcup \mathcal{C}$ so that $q \in S \sim \bigcup \mathcal{C}$. Thus

$$\text{(2)}\qquad \bigcap \{S \sim A\colon\ A \in \mathcal{C}\} \subset S \sim \bigcup \mathcal{C}.$$

Consequently

$$S \sim \bigcup \mathcal{C} = \bigcap \{S \sim A\colon\ A \in \mathcal{C}\} \qquad [(1), (2), 1.2].$$

(b) To prove (b) we first apply result (a) to the collection $\{S \sim A\colon\ A \in \mathcal{C}\}$. We have

$$\begin{aligned} S \sim \bigcup \{S \sim A\colon\ A \in \mathcal{C}\} &= \bigcap \{S \sim (S \sim A)\colon\ A \in \mathcal{C}\} && \text{[Part (a)]} \\ &= \bigcap \{A\colon\ A \in \mathcal{C}\} && \text{(Th 1.6)} \\ &= \bigcap \mathcal{C}. \end{aligned}$$

Taking complements of the first and last members of this equality, we have

$$\bigcup \{S \sim A\colon\ A \in \mathcal{C}\} = S \sim \bigcap \mathcal{C}$$

and (b) is proved.

(The reader should note that each sentence in the proof of 2.4 is valid even when $\mathcal{C}$ is null.)

Exercises 2.5

Let B, C and D be subsets of the underlying set S and let $\mathcal{C}$ be a collection of subsets of S. Prove each of the following.

1. If $H \in \mathcal{C}$, then $H \subset \bigcup \mathcal{C}$. (Note the corollary: $B \subset C \cup B$.)
2. $\bigcap \mathcal{C} \subset H$ for each H in $\mathcal{C}$.
3. $B \cup C = B$ iff $C \subset B$.
4. $B \cap C = B$ iff $B \subset C$.
5. If $B \cup C = B$ and $B \cap C = B$, then $B = C$. (Use half of 3 and half of 4.)

6. The distributive property of $\cap$ over $\bigcup$:

$$B \cap (\bigcup \mathcal{C}) = \bigcup \{B \cap A: \quad A \in \mathcal{C}\}$$

where $\{B \cap A: \quad A \in \mathcal{C}\}$ denotes

$$\{X: \quad X = B \cap A \text{ for some } A \text{ in } \mathcal{C}\}.$$

(We shall use similar obvious notations in the future without explanation.) Note the corollary:

$$B \cap (C \cup D) = (B \cap C) \cup (B \cap D).$$

7. Distributive property of $\cup$ over $\bigcap$:

$$B \cup (\bigcap \mathcal{C}) = \bigcap \{B \cup A: \quad A \in \mathcal{C}\}.$$

8. $B \cup (\bigcup \mathcal{C}) = \bigcup \{B \cup A: \quad A \in \mathcal{C}\}$ *if* $\mathcal{C}$ *is not null.* Show that the condition that $\mathcal{C}$ be non-null is needed here but was not needed in 1, 2, 6 and 7.

9. $B \cap (\bigcap \mathcal{C}) = \bigcap \{B \cap A: \quad A \in \mathcal{C}\}$ *if* $\mathcal{C}$ *is not null.* Is the hypothesis that $\mathcal{C}$ be non-void needed here?

3. Binary Relations: Cartesian Products and Mappings

If $x \in X$ and $y \in Y$ where X and Y are sets, we shall use (x, y) to denote the "ordered pair" whose "first member" is x and whose "second member" is y. We say $(x, y) = (u, v)$ iff $x = u$ and $y = v$. Thus $(x, y) \neq (y, x)$ unless $x = y$. Hence (x, y) is not the same thing as $\{x, y\}$, for always $\{x, y\} = \{y, x\}$.

We use $\{(x, y): \quad x \in X, y \in Y\}$ to denote

$$\{w: \quad w = (x, y) \text{ for some } x \text{ in } X \text{ and some } y \text{ in } Y\},$$

i.e., the set of all ordered pairs (x, y) for x in X and y in Y.

Definition 3.1. Let X and Y be sets. The set

$$\{(x, y): \quad x \in X, y \in Y\}$$

is called the *Cartesian product of X and Y* and is denoted by $X \times Y$.

Of course X and Y may be the same set in many examples. For instance, from one point of view, the set of all points in the Euclidean plane is precisely $R \times R$, where (see 1.3) R is the set of all real numbers.

Definition 3.2. Let X and Y be sets. Any subset E of $X \times Y$ is called a *binary relation from X to Y.* We call E a *binary relation on X* iff $Y = X$.

Suppose $E \subset X \times Y$. We write xEy iff $(x, y) \in E$, and in this case we say that *x is E-related to y* or that *y is an E-relative of x.*

Examples 3.3. (a) Let X be the set of all male students in a given university at a given time. Let E be the binary relation on X (i.e., $E \subset X \times X$) defined as follows: for $x, y \in X$,

$$(x, y) \in E \text{ iff } x \text{ is a brother of } y.$$

(Here we do not call a man his own brother.)

(b) Let X be the set of all men in the U.S. and Y the set of all women in the U.S. at a given time. Let $E \subset X \times Y$ be defined so that

$$xEy \text{ iff } x \text{ is the husband of } y \text{ at the given time.}$$

(c) Let X be the set of all straight lines in the Euclidean plane, and let $E \subset X \times X$ be defined so that

$$xEy \text{ iff } x \text{ is parallel to } y.$$

Here we do not call a line parallel to itself.

(d) Let X be the same as in (c), and define E so that for $x, y \in X$,

$$(x, y) \in E \text{ iff } x \text{ is perpendicular to } y.$$

(e) Let X be a non-empty set and let c be some particular element in X. Define a binary relation E on X so that for x, y in X,

$$xEy \text{ iff } x = c \text{ or } y = c.$$

Let $E \subset X \times Y$. The set $\{x: \quad (x, y) \in E \text{ for some } y \text{ in } Y\}$ is called the *domain of E.* (Clearly the domain of E is a subset of X.) For each subset A of X, we use $E[A]$ to denote

$$\{y: \quad (x, y) \in E \text{ for some } x \text{ in } A\}.$$

(If no element of A is in the domain of E, then $E[A]$ turns out to be the null set.) The set $E[X]$ is called the *range of E.*

Since $X \times Y \subset X \times Y$, it follows from 3.2 that the Cartesian product $X \times Y$ is a special binary relation. Another important type of binary relation is a mapping or function. We usually use small letters to denote functions.

Definition 3.4. Let X and Y be sets. A subset f of $X \times Y$ is called a *mapping of X into Y* (or a *function on X to Y*) iff for each x in X there is exactly one y in Y such that $(x, y) \in f$. We often denote this mapping by $f\colon \; X \longrightarrow Y$.

Thus:

Remark 3.5. A binary relation f from X to Y is a mapping of X into Y iff

(a) $x \in X$ implies there is a y in Y such that $(x, y) \in f$,

and

(b) $(x, y) \in f$ and $(x, z) \in f$ implies $y = z$.

If we say "f associates the element y with x" iff $(x, y) \in f$, we see that a mapping f of X into Y associates with each element in X exactly one element in Y. We call y the *image of* x (*under* f) iff $(x, y) \in f$. For each x in X we denote the image of x under f by $f(x)$. Since the mapping f is a binary relation, the terminology and notation given in the first paragraph following 3.3 are applicable to f. If $A \subset X$, then $f[A]$ (as given in that paragraph) may be written in our special notation for functions as

$$\{y: \ y = f(x) \text{ for some } x \text{ in } A\} \quad \text{or as} \quad \{f(x): \ x \in A\}$$

and is called the *image of* A (*under* f). Part (a) of 3.5 tells us that the domain of f is the whole set X (if f is a mapping of X into Y). Of course the range of f, i.e., $f[X]$ (which is a subset of Y) may not be equal to Y. (See 3.6(b).) If $B \subset Y$, then $\{x: \ f(x) \in B\}$ is called the *inverse image of* B (*under* f) and is denoted by $f^{-1}[B]$.

A familiar example of a mapping of the set R of all real numbers into R is determined by the following criterion: For each x in R, $f(x)$ is $3x^2$. Here the mapping f is the set

$$\{(x, y): \ x \in R \quad \text{and} \quad y = 3x^2\}.$$

Some writers call this set the "graph" of the mapping, but we call this set the mapping itself (Def 3.4). Those writers may have defined a mapping of X into Y as a "correspondence" or "rule" which associates with each element x in X a unique element $f(x)$ in Y. We do not prefer to define a mapping as a "rule" or as a "correspondence" since these terms have not been precisely defined in a mathematical sense. The reader might find himself hard pressed to give a good definition of "correspondence" if asked to do so. Ultimately he might use the idea of a subset of a Cartesian product.

Apart from aesthetic reasons, we have adopted Definition 3.4 so as to define a mapping in terms of our original undefined concept "set" (a mapping of X into Y being a certain kind of set contained in $X \times Y$). We know that in a formal mathematical system something must remain undefined. However, we like to use as few undefined terms as possible.

Although we have defined the notion of a mapping of X into Y as a certain type of subset of $X \times Y$, we usually indicate a specific example of a mapping by giving the criterion which determines for each x in X the image $f(x)$ in Y which is paired with x.

Definition 3.6. Let f be a mapping of X into Y.

(a) f is said to be a *one-to-one* mapping (of X into Y) iff $x, z \in X$ and $x \neq z$ implies $f(x) \neq f(z)$.

(b) f is said to be a mapping of X *onto* Y iff $f[X] = Y$.

Notice that a mapping of X into Y may be one-to-one without being onto Y. Of course it may be onto Y without being one-to-one.

Examples 3.7. (a) Let f be the mapping of R into R defined as follows: For each x in R, $f(x) = x^3$. This mapping is one-to-one since $a, b \in R$ and $a \neq b$ implies $a^3 \neq b^3$. It is onto R since for each y in R there is some x in R such that $x^3 = y$.

(b) Let X be the set of all complex numbers and let f: $X \longrightarrow X$ be defined so that for each x in X, $f(x) = x^3$. This mapping is onto X but not one-to-one (since there are three distinct elements in X each of which maps into 1 under f; they are the three cube roots of 1).

(c) Let f: $N \longrightarrow N$ be defined as follows (where N is as given in 1.3): For each x in N, let $f(x) = 2x$. This mapping is one-to-one but not onto N since there is no x in N such that $f(x) = 5$.

Suppose f is a one-to-one mapping of X into Y. In this case we see that for each y in $f[X]$, there is exactly one x in X such that $f(x) = y$. Thus if we associate with each y in $f[X]$ the unique x in X such that $f(x) = y$, we determine a one-to-one mapping of $f[X]$ onto X. This mapping is called the inverse mapping of f.

Definition 3.8. Let f: $X \longrightarrow Y$ be a one-to-one function on X to Y. The function which associates with each y in $f[X]$ the unique x in X such that $f(x) = y$ is called the *inverse mapping* (or *inverse function*) of f and is denoted by f^{-1} or sometimes by f^{-1}: $f[X] \longrightarrow X$.

Let the reader determine the inverse mapping of the function given in 3.7(a).

We give one further definition in this section, which is of general interest.

Definition 3.9. Let X be a set. A subset E of $X \times X$ (i.e., a binary relation on X) is called an *equivalence relation on* X iff E is

(a) reflexive on X [for each x in X, $(x, x) \in E$],

(b) symmetric [$(x, y) \in E$ implies $(y, x) \in E$], and

(c) transitive [$(x, y) \in E$ and $(y, z) \in E$ implies $(x, z) \in E$].

Let X be the set of all integers ($X = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$), and let E be the subset of $X \times X$ defined as follows: The ordered pair (x, y) belongs to E iff there is some k in X such that $x - y = 5k$. The reader may verify that E is an equivalence relation on X. Also, if X is any set, then $X \times X$ is an equivalence relation on X.

Exercises 3.10

In 1–5 let X and Y be sets and let f be a mapping of X into Y. Let $\mathcal{C}$ be a collection of subsets of X and let $\mathcal{F}$ be a family (i.e., collection) of subsets of Y.

1. Prove:

   (a) $f[\bigcup \mathcal{C}] = \bigcup \{f[A]: A \in \mathcal{C}\}$,

   (b) $f[\bigcap \mathcal{C}] \subset \bigcap \{f[A]: A \in \mathcal{C}\}$.

   (Note the corollary $f[B \cap C] \subset f[B] \cap f[C]$ and indicate an example in which $f[B \cap C] \neq f[B] \cap f[C]$.)

   (c) $f[\bigcap \mathcal{C}] = \bigcap \{f[A]: A \in \mathcal{C}\}$ if f is one-to-one.

2. Prove:

   (a) $f^{-1}[\bigcup \mathcal{F}] = \bigcup \{f^{-1}[A]: A \in \mathcal{F}\}$ and

   (b) $f^{-1}[\bigcap \mathcal{F}] = \bigcap \{f^{-1}[A]: A \in \mathcal{F}\}$.

3. Let $B \subset Y$. Prove $f[f^{-1}[B]] = B \cap f[X]$.

4. Let $A \subset X$. Prove $A \subset f^{-1}[f[A]]$, and give an example to show that equality need not hold.

5. Prove $f^{-1}[f[A]] = A$ for each $A \subset X$ iff f is one-to-one.

6. Let $X = \{a, b\}$ and $Y = \{a, c, d\}$. List the six elements of $X \times Y$.

7. When making examples of binary relations to satisfy specified conditions, one may make perfectly "artificial" relations just to fit his needs (as in 3.3(e)) rather than look for "natural" relations such as "is less than", "is a brother of", "is parallel to", etc. Give an example of a binary relation on a set X which is

   (a) reflexive on X and symmetric but not transitive,

   (b) reflexive on X and transitive but not symmetric,

   (c) symmetric and transitive but not reflexive on X.

   (*Caution:* If a binary relation E is to be transitive, xEy and yEz must imply xEz even when $x = z$. In 3.9(c) nothing is said about x, y and z being distinct.)

8. (a) Show that the relations in 3.3(c) and (d) are not transitive.

   (b) Under what circumstance is the relation in 3.3(a) transitive?

   (c) When is the relation in 3.3(e) transitive?

9. (a) Show that the relation in 3.3(a) is transitive iff there is no pair of brothers in the given university (i.e., E is null).

   (b) Show that the relation in 3.3(e) is transitive iff $X = \{c\}$ (i.e., X consists of the single element c).

10. Let E be a symmetric, transitive binary relation on a set X. Prove that E is reflexive on X if for each x in X there is some y in X such that xEy. (This makes it clear as to the type of example one must seek for 7(c). Do 7(c) before reading the next exercise.)

11. (We give here an answer to 7(c). In view of 10, we must make sure there is at least one element which has no relative at all. We "build" an "artificial" example just to fit our purpose.) Let X be a non-void set and let b be some particular element of X. Let E be the binary relation on X defined as follows: for $x, y \in X$, xEy iff $x \neq b$ and $y \neq b$. Show that E is symmetric and transitive but not reflexive on X.

4. Infinite Sets

Definition 4.1. A set X is said to be *finite* iff it is null or there is a mapping of N_k onto X for some k in N. (See 1.3 for the meaning of N and N_k.) X is said to be *infinite* iff it is not finite.

Definition 4.2. A set X is said to be *countable* iff it is null or there is a mapping of N onto X. A set is said to be *uncountable* iff it is not countable.

Before stating our next theorem, we comment on the system of logic which we use. Let P and W denote propositions and let P' and W' denote the negations of P and W respectively. Then each of the following statements is a proposition:

(1) If P, then W.

(2) P implies W.

(3) If W', then P'.

(4) W' implies P'.

Statements (1) and (2) are just two ways of saying the same thing; (3) and (4) are two ways of saying the same thing. In our work we shall consider (1) and (3) to be logically equivalent. Statement (3) is called the *contrapositive* of (1).

Theorem 4.3. Let A and B be sets with $A \subset B$.

(a) If B is finite, then A is finite.

(b) If A is infinite, then B is infinite.

Proof. (a) In case A is void, then A is finite by 4.1. So suppose A is not null, and suppose B is finite. Then B is not null, and so there is a mapping f of N_k onto B for some k in N. Let p be some particular element in A, and define a mapping g of N_k into A as follows. For each n in N_k,

$$g(n) = \begin{cases} f(n) & \text{if } f(n) \in A \\ p & \text{if } f(n) \notin A. \end{cases}$$

Clearly g is a mapping of N_k *onto* A (since f is onto B) and (a) is proved.

(b) The proposition in (b) is simply the contrapositive of the one in (a) since "A is infinite" is the negation of "A is finite" and similarly for B.

Let the reader prove the following theorem.

Theorem 4.4. Let A and B be sets with $A \subset B$.

(a) If B is countable, then A is countable.

(b) If A is uncountable, then B is uncountable.

Remark 4.5. The set N is countable.

Proof. The "identity mapping" f of N into N defined by $f(x) = x$ for each x in N is clearly a mapping of N *onto* N. Hence N is countable by 4.2.

Corollary 4.6. Each subset of N is countable.

Theorem 4.7. Let f be a mapping of a set A onto a set B.

(a) If A is finite, then B is finite.

(b) If A is countable, then B is countable.

Proof. (a) If $B \neq \varnothing$ and A is finite, then there is a mapping g of N_k onto A for some k in N. For each x in N_k define

$$h(x) = f(g(x)).$$

Thus we have a mapping h of N_k *onto* B. (The reader should verify that h is onto B.) Hence B is finite.

(b) The proof of (b) is similar to that of (a).

The mapping h used in the above proof is called the composition mapping of g and f. Since composition mappings are used frequently, we give the following definition.

Definition 4.8. If $g\colon X \longrightarrow Y$ and $f\colon Y \longrightarrow Z$ are mappings, then the mapping $h\colon X \longrightarrow Z$ defined by

$$h(x) = f(g(x)) \quad \text{for each} \quad x \text{ in } X$$

is called the *composition mapping of g and f* (or *g followed by f*) and is denoted by $f \circ g$.

Exercises 4.9

1. Prove that each finite set is countable.
2. Let f be a one-to-one mapping of a set A into a set B. Prove: If A is infinite, then B is infinite.
3. Let f be a one-to-one mapping of a set A into a set B. Prove: If B is countable, then A is countable.

Preparatory for the next theorem we observe that a real number may or may not have two different decimal representations (by our usual method of representing numbers in decimal form). For instance 7.320000 . . . and 7.31999 . . . represent the same number, while 4.963333 . . . is the only decimal representation (in the usual form) for the number which it does represent. The important observation for us is that if a number has two different decimal representations, then one of these representations repeats

0's from some place onward and the other repeats 9's from some place onward.

Theorem 4.10. The set R of all real numbers is uncountable.

Proof. Let f be *any* mapping of N into R. (We shall show that $f[N] \neq R$, i.e., f is not onto R; and since f is an *arbitrary* mapping of N into R, it will follow that there is *no* mapping of N onto R and that R is uncountable.)

Let each element in R have a definite decimal representation. Let r be the real number whose representation is $3.d_1d_2d_3 \ldots d_n \ldots$, where for each n in N, $d_n = 4$ if the nth decimal place in our representation of the number $f(n)$ is 7 and $d_n = 7$ otherwise. Thus, for each n in N, the number r differs from $f(n)$ in the nth decimal place. But r has only one decimal representation since the given one has no 9's and no 0's. Therefore, for each n in $N, f(n) \neq r$, so that $r \notin f[N]$. Hence f is not onto R, and R is uncountable.

Theorem 4.11. The set $N \times N$ is countable.

Proof. Let $A = \{x\colon\ x = 2^n3^m \text{ for } n, m \in N\}$. Then $A \subset N$, and so A is countable by 4.6. We now define a mapping f of A into $N \times N$. If $x \in A$, then there is exactly one pair n, m in N such that $x = 2^n3^m$. For each x in A we define $f(x)$ to be that pair (n, m) in $N \times N$ such that $x = 2^n3^m$. The mapping thus defined is clearly a mapping of the countable set A *onto* $N \times N$. Therefore $N \times N$ is countable by 4.7(b).

Theorem 4.12. The union of a countable collection of countable sets is countable.

Proof. Let $\mathcal{C}$ be a countable collection of sets each of which is countable. If $\bigcup \mathcal{C} = \varnothing$, then this union is countable by Def 4.2. Suppose then that $\bigcup \mathcal{C} \neq \varnothing$, and let $\mathcal{B}$ denote the collection of all non-void members of $\mathcal{C}$.

Since $\mathcal{B} \subset \mathcal{C}$, $\mathcal{B}$ is countable; and (since $\mathcal{B}$ is non-null) there is a mapping, say f, of N *onto* $\mathcal{B}$. So

$$\mathcal{B} = \{f(1), f(2), \ldots, f(n), \ldots\} = \{A_1, A_2, \ldots, A_n, \ldots\}$$

if we denote the set $f(n)$ by A_n for each n in N. For each n in N, A_n is a non-empty countable set. Thus, for each n in N, there is a mapping, say g_n, of N *onto* A_n. If we denote the image of m under g_n by a_{nm}, then

$$\begin{aligned} A_1 &= \{a_{11}, a_{12}, a_{13}, \ldots, a_{1m}, \ldots\} \\ A_2 &= \{a_{21}, a_{22}, a_{23}, \ldots, a_{2m}, \ldots\} \\ &\ldots \\ A_n &= \{a_{n1}, a_{n2}, a_{n3}, \ldots, a_{nm}, \ldots\} \\ &\ldots . \end{aligned}$$

We now define a mapping h of $N \times N$ into $\bigcup \mathcal{B}$. For each (n, m) in $N \times N$, let $h((n, m))$ be a_{nm} (which is an element of A_n).

The reader should verify that h is onto $\bigcup \mathcal{B}$. Now since $N \times N$ is countable, it follows from 4.7(b) that $\bigcup \mathcal{B}$ is countable. Finally, since $\bigcup \mathcal{C} = \bigcup \mathcal{B}$, we have $\bigcup \mathcal{C}$ is countable.

Exercises 4.13

1. Prove the set of all positive rational numbers is countable.
2. Prove the set of all rational numbers is countable.
3. A circle in the Euclidean plane is said to have a *rational center* iff both coordinates of its center are rational numbers. Prove that the set of all circles in the Euclidean plane with rational radii and rational centers is countable.

5. Set Theory and the Foundations of Mathematics

In the preceding sections we have tried to give the bare essentials from set theory necessary for the very beginning of a sane study of topology. We have tried to do so with a minimum of motivating commentary, which would tend to clutter up our presentation and obscure our main precise ideas—thus distracting from the clarity and beauty of these concepts and the theory as a whole.

Now that the student is thoroughly familiar with the concepts presented, we feel he may enjoy, and benefit from, a discussion of some factors which motivated us in our choice of topics and particular definitions. A glance at the definition of a topology (Ch 2, 1.1) as a certain kind of collection of sets makes it clear as to why some knowledge of set theory is a must for a study of topology. But what is more profound, set theory is basic for the whole of modern mathematics.

Set theory and logic constitute the foundations of mathematics. The development of set theory was initiated by the research of George Cantor around 1870, and Cantor is generally regarded as the founder of set theory. A most remarkable claim for set theory is that all present-day mathematics can be derived from the concept of a set.

We know that in a formal mathematical system (i.e., an axiomatic system) one begins with some undefined (or primitive) terms and some axioms concerning these primitive terms and then proceeds by laying down definitions and using logic to derive conclusions from the axioms. The remarkable claim of the last paragraph is that the collection of primitive terms for all mathematics of today may be reduced to the primitive terms in set theory (and it has been claimed that the primitive terms in set theory may be reduced to a single term).

Let us indicate briefly how one particular part of mathematics may be built upon the concept of set as a foundation. The Euclidean plane E_2 may be considered as the Cartesian product $R \times R$ together with a certain distance function on $E_2 \times E_2$ which assigns to each element (p, q) in $E_2 \times E_2$ a non-negative real number called the distance from p to q. We see then that this approach to the Euclidean plane involves real numbers, Cartesian products and a function (or mapping). Even our brief treatment of set theory has indicated that Cartesian products and mappings are just certain kinds of sets. So if the system of real numbers can be built in such a way that it rests upon the set concept, then the Euclidean plane can be made to rest upon this concept. Indeed the system of real numbers can be so developed, and this development is itself considered a part of set theory. Such development here would too long delay our study of topology.

Of course for a program which makes all mathematics rest on the concept of set, set theory itself must be developed on an axiomatic basis. By "a precise, sophisticated development" in our opening sentence of this chapter we mean an axiomatic development. (Such development is not given here.)

We hope that by now the student has a feeling for the reason we defined a binary relation as we did. First we have an intuitive idea of a relation between ordered pairs of things—a relation such as "is married to" or "is less than". We observe that any one example of an "intuitive relation" (among ordered pairs) has associated with it the set of all those ordered pairs (x, y) such that x is related to y and that this set seems to be as explicitly and completely identifiable with the "intuitive relation" as anything we can think of. We then realize that the simplest thing to do is to call this set of ordered pairs the *relation*. The pleasant thought then comes to mind that perhaps we have mathematically defined a relation in terms of the concept of a set.

We hope, however, that the critical reader noticed that we did not give adequate attention to the concept of an ordered pair just before Definition 3.1. We did not rid that concept of an intuitive idea of one element being listed first or the intuitive idea of going from left to right. We thought it best to bypass this complication at that time. In the next paragraph we indicate how the concept of an ordered pair is made mathematically precise and is given in terms of sets alone.

Suppose $x \in X$ and $y \in Y$, where X and Y are sets. Then the set $\{\{x\}, \{x, y\}\}$ is called the *ordered pair with first coordinate x and second coordinate y* and is denoted by (x, y). (Note that the order in which things are listed is immaterial; $\{\{y, x\}, \{x\}\}$ is the same set as the last one written. Also note that the words "first coordinate" and "second coordinate" are just words used in naming the thing being defined. Finally (x, y) is just a notation, and the ordered pair denoted by (y, x) is the set $\{\{y\}, \{x, y\}\}$.) If $x = y$, then

$$(x, y) = \{\{x\}, \{x, x\}\} = \{\{x\}, \{x\}\} = \{\{x\}\} = (y, x).$$

Now that we have indicated how to define an ordered pair in terms of the concept "set", we would like to emphasize that:

> Definition 3.2 is often adopted in order to give the concept "binary relation" in terms of the concept "set".

A similar statement in the third paragraph after 3.5 indicates the motivation for our particular definition of a function.

In this chapter we have not even given all of the items from set theory to be used later in this book. We have given what we consider to be a reasonable minimum of these items for the beginning of our study of topology. Other items will be mentioned when they are used in our main work.

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2

Topological Spaces

1. Basic Concepts

Definition 1.1. Let X be a set. A collection $\mathscr{T}$ of subsets of X is called a *topology for* X iff

(a) the union of *each* subcollection of $\mathscr{T}$ is a member of $\mathscr{T}$ and

(b) the intersection of *each finite* subcollection of $\mathscr{T}$ is a member of $\mathscr{T}$ (where X is the underlying set).

Definition 1.2. Let $\mathscr{T}$ be a topology for a set X. Then the pair $(X, \mathscr{T})$ is called a *topological space*.

It should be verified that in each of the following examples the given family $\mathscr{T}$ is a topology.

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Examples 1.3. (a) Let X be any set, and let $\mathscr{T}$ be the family of all subsets of X. Then $(X, \mathscr{T})$ is a topological space—a rather trivial one! This topology $\mathscr{T}$ is called the *discrete topology* for X.