Problem Set # 2

# Math 215

Please box your answers for each of the exercises below. Also, be mindful of your presentation, I will deduct 10 points for disorganized or unintelligible answers.

**Exercises**[1]

1) If $A$ and $B$ are sets, show that $A \subseteq B$ iff $A \cap B = A$.

2) $\forall\, n \in \mathbb{N}$, let $A_n := \{(n+1)k : \ k \in \mathbb{N}\}$.
   - What is $A_1 \cap A_2$?
   - Determine the sets $\bigcup\{A_n : \ n \in \mathbb{N}\}$ and $\bigcap\{A_n : \ n \in \mathbb{N}\}$

3) Draw diagrams in the plane of the Cartesian products $A \times B$ for the given sets $A$ and $B$.
   - $A = \{x \in \mathbb{R} : \ 1 \leq x \leq 2 \ \vee \ 3 \leq x \leq 4\}$, $B = \{x \in \mathbb{R} : \ x = 1 \ \vee \ x = 2\}$.
   - $A = \{1, 2, 3\}$, $B = \{x \in \mathbb{R} : \ 1 \leq x \leq 3\}$.

4) Let $A := \{x \in \mathbb{R} : \ -1 \leq x \leq 1\}$, $B := A$ and consider the subset $C := \{(x, y) : \ x^2 + y^2 = 1\}$ of $A \times B$. Is this set a function? Explain.

5) Prove that $\mathbb{Q} := \{\frac{p}{q} : \ p, q \in \ \mathbb{Z} \ \wedge \ q \neq 0\}$ is countable.

6) Prove that $\forall n \in \mathbb{N}$, $1^3 + 2^3 + \cdots + n^3 = \left[\frac{1}{2}n(n+1)\right]^2$.

7) Prove that $\forall n \in \mathbb{N}$, $3 + 11 + \cdots + (8n - 5) = 4n^2 - n$.

8) Prove that $\forall n \in \mathbb{N}$, $\sum\limits_{i=1}^{n} (2i - 1)^2 = \frac{4n^3 - n}{3}$.

9) Find the fallacy with the following 'proof' by mathematical induction?
   **Claim:**
   If $n \in \mathbb{N}$ and if the maximum of the natural numbers $p$ and $q$ is $n$ (namely $n = \max\{p, q\}$), then $p = q$.

   **"Proof:"**
   Let $S \subseteq \mathbb{N}$ for which the claim above is true. Evidently, $1 \in S$. Now assume that $k \in S$ and that $k + 1 = \max\{p, q\}$. Then $k = \max\{p - 1, q - 1\}$. Since $k \in S$, then $p - 1 = q - 1$ and therefore $p = q$. Thus, $k + 1 \in S$, and we conclude that the assertion is true $\forall n \in \mathbb{N}$.

10) **Bernoulli's Inequality:** Prove that if $x > -1$, then $(1 + x)^n \geq 1 + nx$.

[1] All exercises have been borrowed directly from *Introduction to Real Analysis 3$^{rd}$ Ed.*, by Robert G. Bartle and Donald R. Sherbert.

**More difficult problems** 1 point.[2]

11. Produce an infinite collection of sets $A_1, A_2, \ldots$ with the property that every $A_i$ has an infinite number of elements, $\forall i \neq j$ $A_i \cap A_j = \emptyset$, and $\cup_{i=1}^{\infty} A_i = \mathbb{N}$.

[2]Taken from *Understanding Analysis $2^{nd}$ Ed.* By Stephen Abbott.