

Problem 1:

A binary search of a sorted array with 3 unique elements where the search is always successful can be viewed as an array with each element having a probability of being selected = $1/3$ and the number of comparisons a binary search requires to find the element is shown inside the array.

2	1	2
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Then the average number of comparisons would be $A(3) = (1/3)[2 + 1 + 2] = 5/3$
Find the $A(7)$ for an array with 7 elements for the same binary search.

(a) Draw the 7-element array showing the number of comparisons needed to find each element.

3	2	3	1	3	2	3
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(b) Determine $A(7)$

$$A(7) = (1/7)[3 + 2 + 3 + 1 + 3 + 2 + 3] = (1/7)(17) = \frac{17}{7} = 2\frac{3}{7}$$

(c) Using what you learned from parts (a) and (b), find the summation expression for $A(n)$. To simplify your work, assume that $n = 2^k - 1$. Use k in your summation. Leave your answer in closed form in terms of n .

$$A(n) = \frac{\sum_{i=1}^k i 2^{i-1}}{2^k - 1} = \frac{(k-1)2^k + 1}{2^k - 1} = \frac{(\lg(n+1)-1)(n+1)+1}{n}$$

(d) Simplify your $a(n)$ for large values of n .

$$\frac{(\lg(n+1)-1)(n+1)+1}{n} \approx \frac{(\lg(n))(n)}{n} \approx \lg(n) \text{ for large values of } n$$

Problem 2:

Consider the algorithm below
Precondition: n is a non-negative integer

```
function f(n)
{
    temp = 0
    if (n != 0)
    {
        for (i = 1; i != 3; i++)
            temp = temp + n * f(n-1)
        return temp
    }
    else
        return 1;
}
```

Solve for the closed form by repeated substitution

$$\begin{aligned} T(0) &= 1 \\ T(n) &= 3nT(n-1) \\ T(n-1) &= 3(n-1)T(n-2) \\ T(n-2) &= 3(n-2)T(n-3) \\ T(n) &= 3nT(n-1) \\ &= 3n[3(n-1)T(n-2)] \end{aligned}$$

$$\begin{aligned}
&= 3n[3(n-1)[3(n-2)T(n-3)]] \\
&= 3^3 n(n-1)(n-2)T(n-3)
\end{aligned}$$

$$T(n) = 3^n (n-1)(n-2) \dots (n-(k-2))(n-(k-1))T(n-k)$$

Problem 3:

The recurrence relation is given as

$$a_n = 2a_{n-1} + 3a_{n-2}$$

Use the method linear homogeneous characteristic roots to solve for the closed form, with given initial conditions

$$a_0 = 2 \text{ and } a_1 = 4$$

(a) Find the general solution

$$\begin{aligned}
a_n &= r^n \\
r^n &= 2r^{n-1} + 3r^{n-2} \\
r^2 &= 2r + 3 \\
r^2 - 2r - 3 &= 0 \\
(r - 3)(r + 1) &= 0 \\
r &= 3, -1 \\
a_n &= \alpha(3^n) + \beta(-1^n)
\end{aligned}$$

(b) Find the Specific solution

$$\begin{aligned}
a_0 = 2 &= \alpha(3^0) + \beta(-1^0) \\
&= \alpha + \beta \\
a_1 = 4 &= \alpha(3^1) + \beta(-1^1) \\
&= \alpha(3) + \beta(-1)
\end{aligned}$$

$$\begin{aligned}
2 &= \alpha + \beta \\
4 &= 3\alpha - \beta \\
6 &= 4\alpha, \quad \frac{3}{2} = \alpha
\end{aligned}$$

$$\begin{aligned}
2 &= \left(\frac{3}{2}\right) + \beta \\
2 - \frac{3}{2} &= \beta, \quad \frac{1}{2} = \beta
\end{aligned}$$

$$a_n = \frac{3}{2}(3)^n + \frac{1}{2}(-1)^n$$

(c) Use your closed form result to find a_5

$$\begin{aligned}
a_5 &= \frac{3}{2}(3)^5 + \frac{1}{2}(-1)^5 \\
a_5 &= \frac{3}{2}(243) - \frac{1}{2} \\
a_5 &= 364.5 - 0.5 = 364
\end{aligned}$$

Problem 4:

Two average complexities are given below.

Variable p is probability value in the interval of [0.0, 1.0]. Variable n is the problem size.

$$\begin{aligned}
A_1 &= p(n^2 + 1) + (1 - p)3n \\
A_2 &= (1 - p)(3n^2 + n) + p(2n)
\end{aligned}$$

(a) Determine the condition for which A_1 is faster than A_2 .

$$\begin{aligned}
 A_1 &< A_2 \\
 p(n^2 + 1) + (1 - p)3n &< (1 - p)(3n^2 + n) + p(2n) \\
 p(n^2) + p + 3n - 3pn &< 3n^2 + n - 3p(n^2) - pn + 2pn \\
 p(n^2) + p - 3pn + 3p(n^2) + pn - 2pn &< 3n^2 + n - 3n \\
 4p(n^2) - 4pn + p &< 3n^2 - 2n \\
 p(4(n^2) - 4n + 1) &< 3n^2 - 2n \\
 p &< \frac{3n^2 - 2n}{4n^2 - 4n + 1}
 \end{aligned}$$

(b) Approximate the range of p values for which A_1 is faster than A_2 for large values of n.

$$\lim_{p \rightarrow \infty} \left[\frac{3n^2 - 2n}{4n^2 - 4n + 1} \right] = \frac{3}{4}$$

(c) Given n=5, find the range of values for p where A_2 is faster than A_1 .

$$\begin{aligned}
 A_1 &> A_2 \\
 p &> \frac{3n^2 - 2n}{4n^2 - 4n + 1}
 \end{aligned}$$

$$n = 5$$

$$\begin{aligned}
 p &> \frac{3(5^2) - 2(5)}{4(5^2) - 4(5) + 1} \\
 p &> \frac{75 - 10}{100 - 20 + 1} \\
 p &> \frac{65}{81}
 \end{aligned}$$

(d) Can you determine the problem size where A_1 is always faster? Why or why not?

No because A_1 and A_2 both depend on p which varies independently from the problem size n
 So given a fixed problem size n, A_1 can still be fast or slower than A_2 depending on the value of p