12 Binary Space Partitions

The Painter's Algorithm

These days pilots no longer have their first flying experience in the air, but on the ground in a flight simulator. This is cheaper for the air company, safer for the pilot, and better for the environment. Only after spending many hours in the simulator are pilots allowed to operate the control stick of a real airplane. Flight simulators must perform many different tasks to make the pilot forget that she is sitting in a simulator. An important task is visualization: pilots must be able to see the landscape above which they are flying, or the runway on which they are landing. This involves both modeling landscapes and rendering the models. To render a scene we must determine for each pixel on the screen the object that is visible at that pixel; this is called *hidden surface removal*. We must also perform shading calculations, that is, we must compute the intensity of the light that the visible object emits in the direction of the view point. The latter task is very time-consuming if highly realistic images are desired: we must compute how much light reaches the object—either directly from light sources or indirectly via reflections on other objects—and consider the interaction of the light with the surface of the object to see how much of it is reflected in the direction of the view point. In flight simulators rendering must be performed in real-time, so there is no time for accurate shading calculations. Therefore a fast and simple shading technique is employed and hidden surface removal becomes an important factor in the rendering time.

The *z-buffer algorithm* is a very simple method for hidden surface removal. This method works as follows. First, the scene is transformed such that the viewing direction is the positive z-direction. Then the objects in the scene are scan-converted in arbitrary order. Scan-converting an object amounts to determining which pixels it covers in the projection; these are the pixels where the object is potentially visible. The algorithm maintains information about the already processed objects in two buffers: a frame buffer and a z-buffer. The frame buffer stores for each pixel the intensity of the currently visible object, that is, the object that is visible among those already processed. The z-buffer stores for each pixel the z-coordinate of the currently visible object. (More precisely, it stores the z-coordinate of the point on the object that is visible at the pixel.) Now suppose that we select a pixel when scan-converting an object.

If the z-coordinate of the object at that pixel is smaller than the z-coordinate stored in the z-buffer, then the new object lies in front of the currently visible object. So we write the intensity of the new object to the frame buffer, and its z-coordinate to the z-buffer. If the z-coordinate of the object at that pixel is larger than the z-coordinate stored in the z-buffer, then the new object is not visible, and the frame buffer and z-buffer remain unchanged. The z-buffer algorithm is easily implemented in hardware and quite fast in practice. Hence, this is the most popular hidden surface removal method. Nevertheless, the algorithm has some disadvantages: a large amount of extra storage is needed for the z-buffer, and an extra test on the z-coordinate is required for every pixel covered by an object. The *painter's algorithm* avoids these extra costs by first sorting the objects according to their distance to the view point. Then the objects are scan-converted in this so-called *depth order*, starting with the object farthest from the view point. When an object is scan-converted we do not need to perform any test on its z-coordinate, we always write its intensity to the frame buffer. Entries in the frame buffer that have been filled before are simply overwritten. Figure 12.1 illustrates the algorithm on a scene consisting of three triangles. On the left, the triangles are shown with numbers corresponding to the order in which they are scan-converted. The images after the first, second, and third triangle have been scan-converted are shown as well. This approach is correct because we scan-convert the objects in back-to-front order: for each pixel the last object written to the corresponding entry in the frame buffer will be the one closest to the viewpoint, resulting in a correct view of the scene. The process resembles the way painters work when they put layers of paint on top of each other, hence the name of the algorithm.

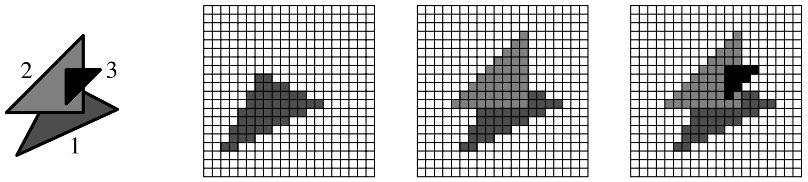
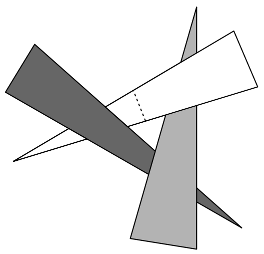


Figure 12.1
The painter's algorithm in action

To apply this method successfully we must be able to sort the objects quickly. Unfortunately this is not so easy. Even worse, a depth order may not always exist: the in-front-of relation among the objects can contain cycles. When such a *cyclic overlap* occurs, no ordering will produce a correct view of this scene. In this case we must break the cycles by splitting one or more of the objects, and hope that a depth order exists for the pieces that result from the splitting. When there is a cycle of three triangles, for instance, we can always split one of them into a triangular piece and a quadrilateral piece, such that a correct displaying order exists for the resulting set of four objects. Computing which objects to split, where to split them, and then sorting the object fragments is an expensive process. Because the order depends on the position of the view point, we must recompute the order every time the view point moves. If we want to use the painter's algorithm in a real-time environment such as flight simulation, we should preprocess the scene such that a correct displaying order can be found



quickly for any view point. An elegant data structure that makes this possible is the binary space partition tree, or BSP tree for short.

12.1 The Definition of BSP Trees

To get a feeling for what a BSP tree is, take a look at Figure 12.2. This figure shows a binary space partition (BSP) for a set of objects in the plane, together with the tree that corresponds to the BSP. As you can see, the binary space partition is obtained by recursively splitting the plane with a line: first we split the entire plane with ℓ_1, then we split the half-plane above ℓ_1 with ℓ_2 and the half-plane below ℓ_1 with ℓ_3, and so on. The splitting lines not only partition the plane, they may also cut objects into fragments. The splitting continues until there is only one fragment left in the interior of each region. This process is

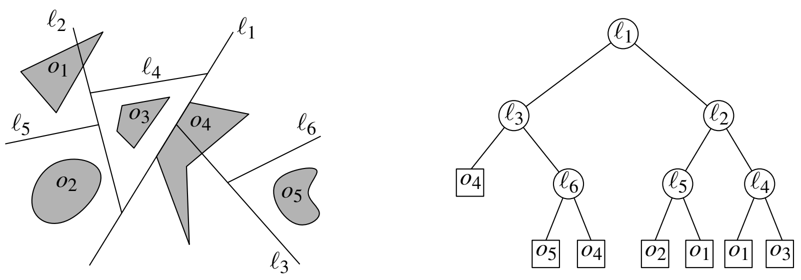


Figure 12.2
A binary space partition and the corresponding tree

naturally modeled as a binary tree. Each leaf of this tree corresponds to a face of the final subdivision; the object fragment that lies in the face is stored at the leaf. Each internal node corresponds to a splitting line; this line is stored at the node. When there are 1-dimensional objects (line segments) in the scene then objects could be contained in a splitting line; in that case the corresponding internal node stores these objects in a list.

For a hyperplane $h : a_1x_1 + a_2x_2 + \cdots + a_dx_d + a_{d+1} = 0$, we let h^+ be the open positive half-space bounded by h and we let h^- be the open negative half-space:

$$h^+ := \{(x_1, x_2, \ldots, x_d) : a_1x_1 + a_2x_2 + \cdots + a_dx_d + a_{d+1} > 0\}$$

and

$$h^- := \{(x_1, x_2, \ldots, x_d) : a_1x_1 + a_2x_2 + \cdots + a_dx_d + a_{d+1} < 0\}.$$

A binary space partition tree, or BSP tree, for a set S of objects in d-dimensional space is now defined as a binary tree $\mathcal{T}$ with the following properties:

- If $\text{card}(S) \leqslant 1$ then $\mathcal{T}$ is a leaf; the object fragment in S (if it exists) is stored explicitly at this leaf. If the leaf is denoted by ν, then the (possibly empty) set stored at the leaf is denoted by $S(\nu)$.

- If $\text{card}(S) > 1$ then the root ν of $\mathcal{T}$ stores a hyperplane h_ν, together with the set $S(\nu)$ of objects that are fully contained in h_ν. The left child of ν is the root of a BSP tree $\mathcal{T}^-$ for the set $S^- := \{h_\nu^- \cap s : s \in S\}$, and the right child of ν is the root of a BSP tree $\mathcal{T}^+$ for the set $S^+ := \{h_\nu^+ \cap s : s \in S\}$.

The *size* of a BSP tree is the total size of the sets $S(\nu)$ over all nodes ν of the BSP tree. In other words, the size of a BSP tree is the total number of object fragments that are generated. If the BSP does not contain useless splitting lines—lines that split off an empty subspace—then the number of nodes of the tree is at most linear in the size of the BSP tree. Strictly speaking, the size of the BSP tree does not say anything about the amount of storage needed to store it, because it says nothing about the amount of storage needed for a single fragment. Nevertheless, the size of a BSP tree as we defined it is a good measure to compare the quality of different BSP trees for a given set of objects.

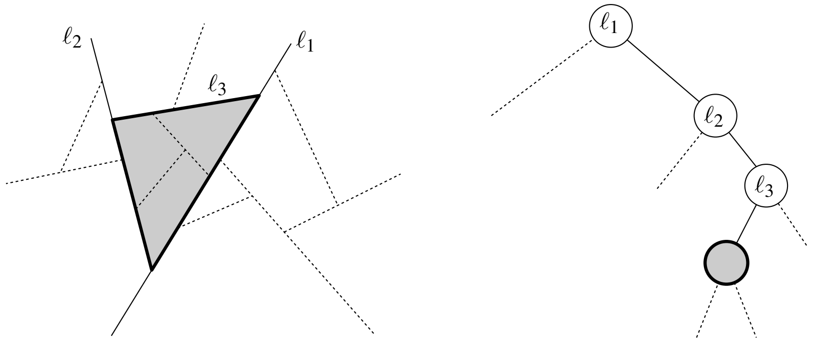
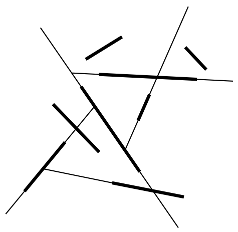


Figure 12.3
The correspondence between nodes and regions

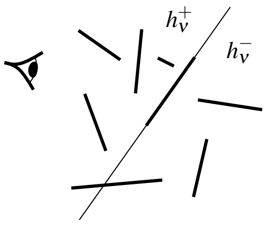
The leaves in a BSP tree represent the faces in the subdivision that the BSP induces. More generally, we can identify a convex region with each node ν in a BSP tree $\mathcal{T}$: this region is the intersection of the half-spaces $h_\mu^\diamond$, where μ is an ancestor of ν and $\diamond = -$ when ν is in the left subtree of μ, and $\diamond = +$ when it is in the right subtree. The region corresponding to the root of $\mathcal{T}$ is the whole space. Figure 12.3 illustrates this: the grey node corresponds to the grey region $\ell_1^+ \cap \ell_2^+ \cap \ell_3^-$.



The splitting hyperplanes used in a BSP can be arbitrary. For computational purposes, however, it can be convenient to restrict the set of allowable splitting hyperplanes. A usual restriction is the following. Suppose we want to construct a BSP for a set of line segments in the plane. An obvious set of candidates for the splitting lines is the set of extensions of the input segments. A BSP that only uses such splitting lines is called an *auto-partition*. For a set of planar polygons in 3-space, an auto-partition is a BSP that only uses planes through the input polygons as splitting planes. It seems that the restriction to auto-partitions is a severe one. But, although auto-partitions cannot always produce minimum-size BSP trees, we shall see that they can produce reasonably small ones.

12.2 BSP Trees and the Painter's Algorithm

Suppose we have built a BSP tree $\mathcal{T}$ on a set S of objects in 3-dimensional space. How can we use $\mathcal{T}$ to get the depth order we need to display the set S with the painter's algorithm? Let p_{view} be the view point and suppose that p_{view} lies above the splitting plane stored at the root of $\mathcal{T}$. Then clearly none of the objects below the splitting plane can obscure any of the objects above it. Hence, we can safely display all the objects (more precisely, object fragments) in the subtree $\mathcal{T}^-$ before displaying those in $\mathcal{T}^+$. The order for the object fragments in the two subtrees $\mathcal{T}^+$ and $\mathcal{T}^-$ is obtained recursively in the same way. This is summarized in the following algorithm.



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Algorithm PAINTERSALGORITHM($\mathcal{T}$, $p_{\text{view}}$)
1.  Let $\nu$ be the root of $\mathcal{T}$.
2.  if $\nu$ is a leaf
3.     then Scan-convert the object fragments in $S(\nu)$.
4.     else if $p_{\text{view}} \in h_\nu^+$
5.            then PAINTERSALGORITHM($\mathcal{T}^-$, $p_{\text{view}}$)
6.                 Scan-convert the object fragments in $S(\nu)$.
7.                 PAINTERSALGORITHM($\mathcal{T}^+$, $p_{\text{view}}$)
8.            else if $p_{\text{view}} \in h_\nu^-$
9.                   then PAINTERSALGORITHM($\mathcal{T}^+$, $p_{\text{view}}$)
10.                       Scan-convert the object fragments in $S(\nu)$.
11.                       PAINTERSALGORITHM($\mathcal{T}^-$, $p_{\text{view}}$)
12.                  else (* $p_{\text{view}} \in h_\nu$ *)
13.                       PAINTERSALGORITHM($\mathcal{T}^+$, $p_{\text{view}}$)
14.                       PAINTERSALGORITHM($\mathcal{T}^-$, $p_{\text{view}}$)
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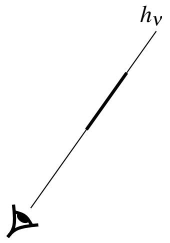
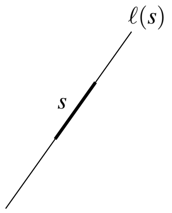


Note that we do not draw the polygons in $S(\nu)$ when p_{view} lies on the splitting plane h_ν, because polygons are flat 2-dimensional objects and therefore not visible from points that lie in the plane containing them.

The efficiency of this algorithm—indeed, of any algorithm that uses BSP trees—depends largely on the size of the BSP tree. So we must choose the splitting planes in such a way that fragmentation of the objects is kept to a minimum. Before we can develop splitting strategies that produce small BSP trees, we must decide on which types of objects we allow. We became interested in BSP trees because we needed a fast way of doing hidden surface removal for flight simulators. Because speed is our main concern, we should keep the type of objects in the scene simple: we should not use curved surfaces, but represent everything in a polyhedral model. We assume that the facets of the polyhedra have been triangulated. So we want to construct a BSP tree of small size for a given set of triangles in 3-dimensional space.

12.3 Constructing a BSP Tree

When you want to solve a 3-dimensional problem, it is usually not a bad idea to gain some insight by first studying the planar version of the problem. This is also what we do in this section.



Let S be a set of n non-intersecting line segments in the plane. We will restrict our attention to auto-partitions, that is, we only consider lines containing one of the segments in S as candidate splitting lines. The following recursive algorithm for constructing a BSP immediately suggests itself. Let $\ell(s)$ denote the line that contains a segment s.

```
Algorithm 2DBSP($S$)
Input. A set $S = \{s_1, s_2, \dots, s_n\}$ of segments.
Output. A BSP tree for $S$.
1.  if card($S$) $\leqslant 1$
2.    then Create a tree $\mathcal{T}$ consisting of a single leaf node, where the set $S$ is
           stored explicitly.
3.         return $\mathcal{T}$
4.    else (∗ Use $\ell(s_1)$ as the splitting line. ∗)
5.         $S^+ \leftarrow \{s \cap \ell(s_1)^+ : s \in S\}$;    $\mathcal{T}^+ \leftarrow$2DBSP($S^+$)
6.         $S^- \leftarrow \{s \cap \ell(s_1)^- : s \in S\}$;    $\mathcal{T}^- \leftarrow$2DBSP($S^-$)
7.         Create a BSP tree $\mathcal{T}$ with root node $\nu$, left subtree $\mathcal{T}^-$, right sub-
           tree $\mathcal{T}^+$, and with $S(\nu) = \{s \in S : s \subset \ell(s_1)\}$.
8.         return $\mathcal{T}$
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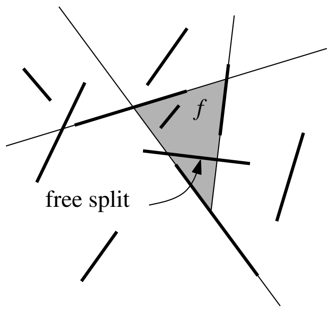
The algorithm clearly constructs a BSP tree for the set S. But is it a small one? Perhaps we should spend a little more effort in choosing the right segment to do the splitting, instead of blindly taking the first segment, s_1. One approach that comes to mind is to take the segment $s \in S$ such that $\ell(s)$ cuts as few segments as possible. But this is too greedy: there are configurations of segments where this approach doesn't work well. Furthermore, finding this segment would be time consuming. What else can we do? Perhaps you already guessed: as in previous chapters where we had to make a difficult choice, we simply make a random choice. That is to say, we use a random segment to do the splitting. As we shall see later, the resulting BSP is expected to be fairly small.

To implement this, we put the segments in random order before we start the construction:

```
Algorithm 2DRANDOMBSP($S$)
1.  Generate a random permutation $S' = s_1, \dots, s_n$ of the set $S$.
2.  $\mathcal{T} \leftarrow$2DBSP($S'$)
3.  return $\mathcal{T}$
```

Before we analyze this randomized algorithm, we note that one simple optimization is possible. Suppose that we have chosen the first few partition lines. These lines induce a subdivision of the plane whose faces correspond to nodes in the BSP tree that we are constructing. Consider one such face f. There can be segments that cross f completely. Selecting one of these crossing segments to split f will not cause any fragmentation of other segments inside f, while the segment itself can be excluded from further consideration. It would be foolish not to take advantage of such *free splits*. So our improved strategy is to make free splits whenever possible, and to use random splits otherwise. To implement this optimization, we must be able to tell whether a segment is a free split. To this end we maintain two boolean variables with each segment, which indicate whether the left and right endpoint lie on one of the already added splitting lines. When both variables become true, then the segment is a free split.



We now analyze the performance of algorithm 2DRANDOMBSP. To keep it simple, we will analyze the version without free splits. (In fact, free splits do not make a difference asympotically.)

We start by analyzing the size of the BSP tree or, in other words, the number of fragments that are generated. Of course, this number depends heavily on the particular permutation generated in line 1: some permutations may give small BSP trees, while others give very large ones. As an example, consider the

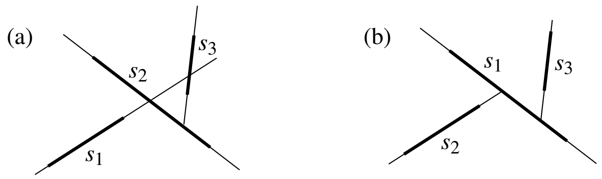


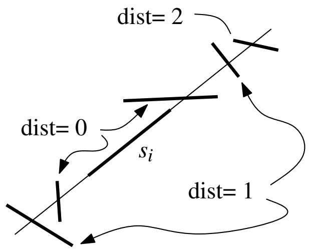
Figure 12.4
Different orders give different BSPs

collection of three segments depicted in Figure 12.4. If the segments are treated as illustrated in part (a) of the figure, then five fragments result. A different order, however, gives only three fragments, as shown in part (b). Because the size of the BSP varies with the permutation that is used, we will analyze the *expected* size of the BSP tree, that is, the average size over all $n!$ permutations.

Lemma 12.1 *The expected number of fragments generated by the algorithm* 2DRANDOMBSP *is* $O(n \log n)$.

Proof. Let s_i be a fixed segment in S. We shall analyze the expected number of other segments that are cut when $\ell(s_i)$ is added by the algorithm as the next splitting line.

In Figure 12.4 we can see that whether or not a segment s_j is cut when $\ell(s_i)$ is added—assuming it can be cut at all by $\ell(s_i)$—depends on segments that are also cut by $\ell(s_i)$ and are 'in between' s_i and s_j. In particular, when the line through such a segment is used before $\ell(s_i)$, then it shields s_j from s_i. This is what happened in Figure 12.4(b): the segment s_1 shielded s_3 from s_2. These considerations lead us to define the distance of a segment with respect to the fixed segment s_i:



$$\mathrm{dist}_{s_i}(s_j) = \begin{cases} \text{the number of segments intersecting } \ell(s_i) \text{ in between } s_i \text{ and } s_j & \text{if } \ell(s_i) \text{ intersects } s_j \\ +\infty & \text{otherwise} \end{cases}$$

For any finite distance, there are at most two segments at that distance, one on either side of s_i.

Let $k := \text{dist}_{s_i}(s_j)$, and let $s_{j_1}, s_{j_2}, \ldots, s_{j_k}$ be the segments in between s_i and s_j. What is the probability that $\ell(s_i)$ cuts s_j when added as a splitting line? For this to happen, s_i must come before s_j in the random ordering and, moreover, it must come before any of the segments in between s_i and s_j, which shield s_j from s_i. In other words, of the set $\{i, j, j_1, \ldots, j_k\}$ of indices, i must be the smallest one. Because the order of the segments is random, this implies

$$\Pr[\ell(s_i) \text{ cuts } s_j] \leqslant \frac{1}{\text{dist}_{s_i}(s_j)+2}.$$

Notice that there can be segments that are not cut by $\ell(s_i)$ but whose *extension* shields s_j. This explains why the expression above is not an equality.

We can now bound the expected total number of cuts generated by s_i:

$$\begin{aligned} \text{E}[\text{number of cuts generated by } s_i] &\leqslant \sum_{j \neq i} \frac{1}{\text{dist}_{s_i}(s_j)+2} \\ &\leqslant 2\sum_{k=0}^{n-2} \frac{1}{k+2} \\ &\leqslant 2\ln n. \end{aligned}$$

By linearity of expectation, we can conclude that the expected total number of cuts generated by all segments is at most $2n\ln n$. Since we start with n segments, the expected total number of fragments is bounded by $n+2n\ln n$. $\square$

We have shown that the expected size of the BSP that is generated by 2DRANDOMBSP is $n+2n\ln n$. As a consequence, we have proven that a BSP of size $n+2n\ln n$ *exists* for any set of n segments. Furthermore, at least half of all permutations lead to a BSP of size $n+4n\ln n$. We can use this to find a BSP of that size: After running 2DRANDOMBSP we test the size of the tree, and if it exceeds that bound, we simply start the algorithm again with a fresh random permutation. The expected number of trials is two.

We have analyzed the size of the BSP that is produced by 2DRANDOMBSP. What about the running time? Again, this depends on the random permutation that is used, so we look at the expected running time. Computing the random permutation takes linear time. If we ignore the time for the recursive calls, then the time taken by algorithm 2DBSP is linear in the number of fragments in S. This number is never larger than n—in fact, it gets smaller with each recursive call. Finally, the number of recursive calls is obviously bounded by the total number of generated fragments, which is $O(n\log n)$. Hence, the total construction time is $O(n^2\log n)$, and we get the following result.

Theorem 12.2 *A BSP of size $O(n\log n)$ can be computed in expected time $O(n^2\log n)$.*