Nonlinear ODEs and Linear PDEs Equivalence in Fluid Dynamics

John D. Ramshaw
Department of Physics, Portland State University, Portland, Oregon 97207

By Romerico David, Micah Lawal, James Minter, Mauro Sgromo

Problem Statement

Exploring the relations between nonlinear ordinary differential equations (ODEs) and linear partial differential equations (PDEs) within the context of fluid dynamics since this is not widely taught and is undervalued.

Solution Method

Construct solutions to certain PDEs occurring in fluid dynamics in terms of the solutions to corresponding ODEs by using the equivalence between nonlinear ODEs and linear PDEs.

Assumptions that need to be made...

01

The fluid velocity field, $\boldsymbol{u}(\boldsymbol{x}, t)$, is known or can be determined.

02

The fluid velocity field are relatively simple.

03

The scalar quantities being advected are non-diffusive.

04

The fluid motion can be described using Lagrangian surfaces

Unfamiliar Terms:

Advection:

The movement of a mass/particle because of the fluid.

Non-Diffusive Scalar:

A scalar quantity that remains constant for individual fluid particles as they move through the fluid.

Lagrangian Surfaces:

A surface that moves with the velocity field and the points on the surface stay in the same orientation.

Lagrangian Trajectory:

The path a fluid particle follows as it moves through space and time

Lagrangian Trajectories

Definition: Lagrangian Trajectories describe the paths of fluid particles as they move through the fluid via differential equations.

Key Concept: The focus is following individual particles than looking at specific locations in space (Eulerian approach)

Key Equations:

- Position vector of a Lagrangian fluid particle x is governed by the nonlinear ODE $\dot{x} = u(x, t)$ where $u(x, t)$ is the velocity field
- Solution $x = F(t, X)$ with X as the initial position.
- Reverse transformation: $X = G(t, x)$

Lagrangian Trajectories

Key Equations:

Position vector of a Lagrangian fluid particle x, is governed by the nonlinear ODE

$$\dot{x} = u(x, t)$$

where $u(x, t)$ is the velocity field.

The formal solution of the ODE can be expressed using the initial position X at time $t = 0$

$$x = F(t, X)$$

where $F(0, X) = X$.

We can solve X to get the reverse transformation from Eulerian to Lagrangian coordinates

$$X = G(t, x)$$

where $G(0, x) = x$.

Lagrangian Trajectories cont...

Transformations:

- From Eulerian to Lagrangian coordinates: Defined by $X = G(t, x)$
- Reverse process: Defined by $x = F(t, X)$

Let's differentiate $\mathrm{X} = G(t, x)$ with respect to time,

$$\frac{d\boldsymbol{X}}{dt} = \frac{\partial G(t,\boldsymbol{x})}{\partial t}$$

$$\Rightarrow \frac{d\boldsymbol{X}}{dt} = 0 \; (\boldsymbol{X} \textit{ is constant since } t = 0) = \frac{\partial G(t,\boldsymbol{x})}{\partial t} = \frac{\partial G(t,\boldsymbol{x})}{\partial t} + \frac{\partial G(t,\boldsymbol{x})}{\partial \boldsymbol{x}} \cdot \frac{d\boldsymbol{x}}{dt}$$

$$\Rightarrow 0 = \frac{\partial G(t,\boldsymbol{x})}{\partial t} + \frac{\partial G(t,\boldsymbol{x})}{\partial \boldsymbol{x}} \cdot \frac{d\boldsymbol{x}}{dt}$$

Lagrangian Trajectories cont…

$$\Rightarrow 0 = \frac{\partial G(t,\boldsymbol{x})}{\partial t} + \frac{\partial G(t,\boldsymbol{x})}{\partial \boldsymbol{x}} \cdot \boldsymbol{u}(\boldsymbol{x},t)$$

$$\Rightarrow 0 = \frac{\partial G(t,\boldsymbol{x})}{\partial t} + \boldsymbol{u}(\boldsymbol{x},t) \cdot \nabla G(t,\boldsymbol{x}(t))$$

$$\Rightarrow \frac{\partial G}{\partial t} + u \cdot \nabla G = 0$$

G satisfies the first-order linear PDE

$$\frac{\partial G}{\partial t} + u \cdot \nabla G = 0$$

which is important because it shows how transformations related to fluid particle trajectories adhere to linear dynamics even though the original ODE is nonlinear.

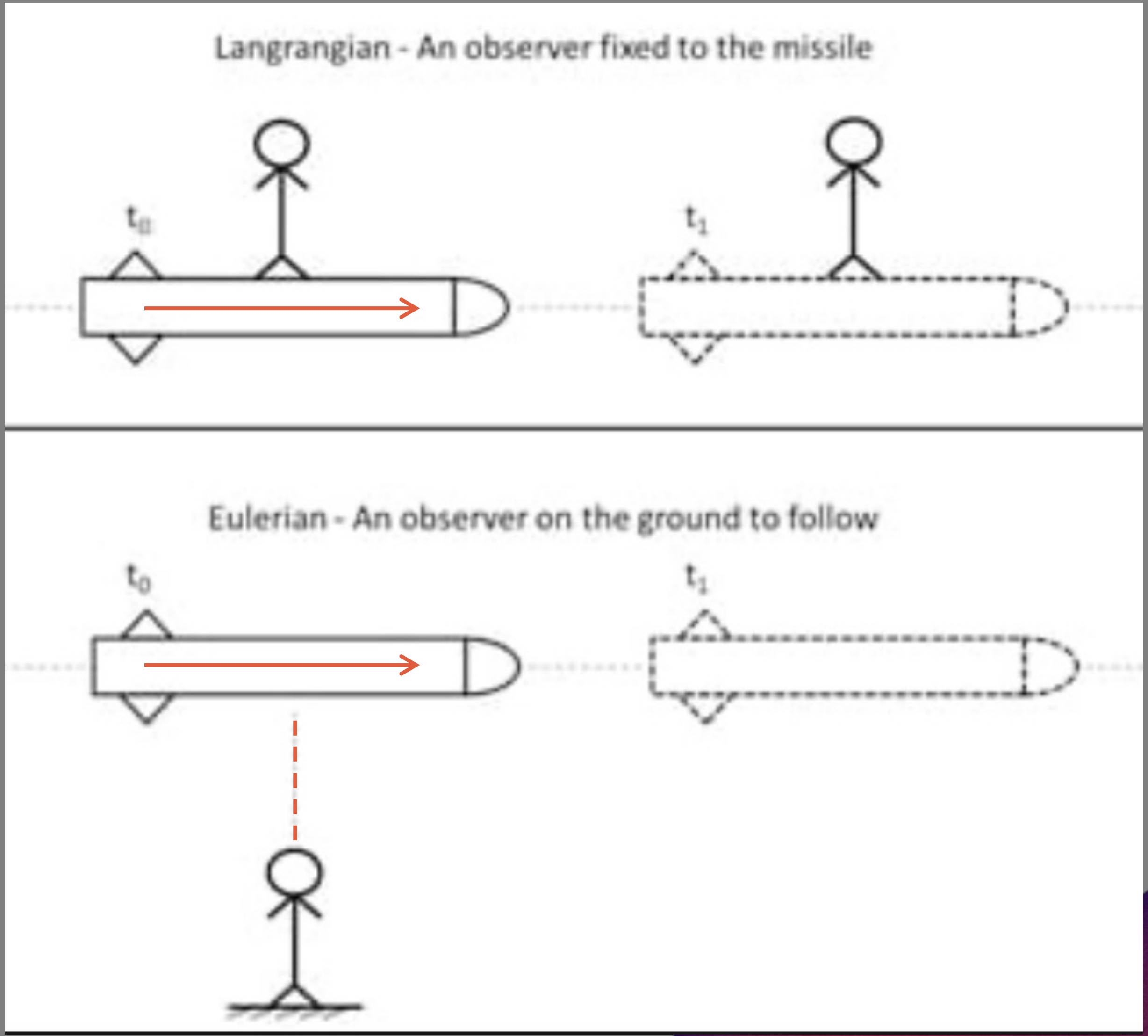
Background: Difference Between Lagrangian & Eulerian

Lagrangian:

- Focuses on tracking individual fluid particles over time

Eulerian:

- Focuses on how fluid properties change at specific locations in space over time



Credit: SIMSCALE

Understanding the Non-Diffusive Scalar

<u>Definition:</u> A non-diffusive scalar $f(x,t)$ in fluid dynamics is a scalar quantity that remains constant along the trajectory of any fluid particle.

Following the equation for x

$$f(x,t) = f_0(X) \text{ where } X \text{ is the initial position of the particle}$$

which transforms to exact general solution

$$f(x,t) = f_0(G(t,x))$$

from the initial condition $X = G(t,x)$

Understanding the Non-Diffusive Scalar cont…

In terms of Eulerian coordinates, the scalar field satisfies the advection equation

$$\frac{\partial f}{\partial t} + u \cdot \nabla f = 0$$

which means that the rate of f is only determined by the velocity field u and not other forces.

<u>**General Properties**</u>:

- All non-diffusive scalars satisfy the same fundamental PDE but differ in initial conditions.
- Any function of a non-diffusive scalar is itself non-diffusive.
- This PDE can be solved using the method of characteristics.

What is the Method of Characteristics?

The method of characteristics is a technique to solve first-order linear, semi-linear, and quasi-linear PDEs by reducing a PDE to a system of simpler ODEs.

- Linear: $a(x,y)u_x + b(x,y)u_y = f(x,y)$
- Semi-Linear: $a(x,y)u_x + b(x,y)u_y = f(x,y,u)$
- Quasi-Linear: $a(x,y,u)u_x + b(x,y,u)u_y = f(x,y,u)$

We will only focus on first-order linear PDEs.

Simple Example using Method of Characteristics

Consider the first-order linear PDE:

$$a(x,t)u_x + b(x,t)u_t = c(x,t)$$

$$= au_x + bu_t = c$$
$$\Rightarrow au_x + bu_t - c = 0$$

a, b, c dictate the behavior of the partial derivatives u_x and u_y for some surface $S = u(x,y)$

By the definition of a directional derivative $D\vec{u}f(x,y) = f_x(x,y) \cdot a + f_y(x,y) \cdot b$ and the dot product $\vec{v} \cdot \vec{u} = v_1u_1 + v_2u_2 + v_3u_3$

$$au_x + bu_t - c = < a, b, c > \cdot < u_x, u_t, -1 > = 0$$

If $< a, b, c >$ is perpendicular (orthogonal) to $< u_x, u_t, -1 >$, $< a, b, c >$ lies in the tangent plane to S and we can construct a curve C parametrized by s such that each point on C, the vector $< a(x,t), b(x,t), c(x,t) >$ is tangent to the curve.

Simple Example using Method of Characteristics cont…

Parametrization along s called characteristic curves: $C = \{\, x = x(s), t = t(s), z = z(s) \,\}$

Following chain rule $\frac{dz}{ds} = \frac{\partial z}{\partial x}\frac{dx}{ds} + \frac{\partial z}{\partial t}\frac{dt}{ds} = u_x a + u_t b = c$

Matching coefficients:

$$\frac{dx}{ds} = a(x,t)$$

$$\frac{dt}{ds} = b(x,t)$$

$$\frac{dz}{ds} = c(x,t)$$

Simple Example using Method of Characteristics cont...

Thus we have the set of characteristic equations given as:

$$\frac{dx}{ds} = a(x(s), t(s))$$

$$\frac{dt}{ds} = b(x(s), t(s))$$

$$\frac{dz}{ds} = c(x(s), t(s))$$

Our linear PDE was reduced into a system of ODES where we can use our knowledge of ODEs to solve the characteristic equations.

Understanding the Non-Diffusive Scalar cont…

So, when we solve the PDE

$$\frac{\partial f}{\partial t} + u \cdot \nabla f = 0$$

using the method of characteristics will result in a system of characteristic equations that will give us the same exact general solution under the same initial conditions

$$f(x,t) = f_0(G(t,x))$$

Lagrangian Surfaces in Fluid Dynamics

Definition: A Lagrangian surface $S(t)$ is a collection of fluid particles that move with the fluid velocity field $u(x, t)$ to form a coherent surface.

The initial surface $S(0)$ can be parameterized by a function $Q_0(\alpha, \beta)$ defined by

$$X = Q_0(\alpha, \beta)$$

Example: A sphere parameterized by spherical coordinates evolving over time.

The Lagrangian surface at a later time t can be defined by

$$x = Q(t, \alpha, \beta) \text{ where } Q(t, \alpha, \beta) = F(t, Q_0(\alpha, \beta))$$

which represents the new position of the fluid particles as they move with the fluid.

Example: Incompressible Spherically Symmetric Radial Velocity Field in Spherical Coordinates

Given the velocity field:

$$\vec{u} = u(r,t)\overrightarrow{e_r}$$

where $\overrightarrow{e_r}$ is the radial unit vector.

The advection equation in spherical coordinates for an incompressible fluid is given by:

$$\nabla \cdot \vec{u} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u) = 0$$

Example: Incompressible Spherically Symmetric Radial Velocity Field in Spherical Coordinates

Integrating the advection equation:

$$\frac{\partial}{\partial r}(r^2 u) = 0$$

$$\Rightarrow r^2 u = a(t)$$

$$\Rightarrow u(r,t) = \frac{a(t)}{r^2}$$

The position vector x in spherical coordinates is given as:

$$x = r\overrightarrow{e_r}$$

So

$$\dot{x} = \frac{d}{dt}(r\overrightarrow{e_r}) = \dot{r}e_r + r\dot{e}_r = \frac{a(t)}{r^2}\overrightarrow{e_r}$$

Example: Incompressible Spherically Symmetric Radial Velocity Field in Spherical Coordinates

The position vector x in spherical coordinates is given as:

$$x = r\overrightarrow{e_r}$$

So

$$\dot{x} = \frac{d}{dt}(r\overrightarrow{e_r}) = \frac{dr}{dt}\overrightarrow{e_r} + r\frac{d\overrightarrow{e_r}}{dt} = \frac{a(t)}{r^2}\overrightarrow{e_r}$$

Using the identity $e_r \cdot e_r = 1 \Rightarrow e_r \cdot \dot{e_r} = 0$. This mean that $\dot{e_r}$ is orthogonal to e_r meaning it does not contribute to the radial component.

$$\left[\frac{dr}{dt}\overrightarrow{e_r} + r\frac{d\overrightarrow{e_r}}{dt}\right] \cdot \overrightarrow{e_r} = \left[\frac{a(t)}{r^2}\overrightarrow{e_r}\right] \cdot \overrightarrow{e_r}$$

So, the equation reduces to $\dot{r} = \frac{a(t)}{r^2}$

Example: Incompressible Spherically Symmetric Radial Velocity Field in Spherical Coordinates

$a(t)$ can be eliminated in terms of the time dependence by using a particular reference Lagrangian radius $R(t)$ satisfying our radial velocity,

This means we can define $\mathrm{r} = R(t)$ as the radius of the unperturbed interface which is the boundary between the particle and surface but remains unaffected by other variables since the surface is a nearly spherical interface thus its geometry cancels opposing inertial forces.

Thus, $\dot{R} = \frac{a(t)}{R^2} \Rightarrow a(t) = R^2\dot{R}$.

So, $\dot{r} = \frac{R^2\dot{R}}{r^2}$

Example: Incompressible Spherically Symmetric Radial Velocity Field in Spherical Coordinates

Solving $\dot{r} = \frac{R^2 \dot{R}}{r^2}$

$$\Rightarrow r^2 dr = R^2 dR$$

Assuming $r(0) = r_0$ and $R(0) = R_0$

$$\Rightarrow \int_{r_0}^{r} r^2 dr = \int_{R_0}^{R} R^2 dR$$

$$\Rightarrow r = [R^3(t) - R_0^3 + r_0^3]^{\frac{1}{3}}$$

OR

$$\Rightarrow r_0 = [R^3(t) - R_0^3 + r^3]^{\frac{1}{3}}$$

Where r_0 is the radial coordinate of X

Example: Incompressible Spherically Symmetric Radial Velocity Field in Spherical Coordinates

According to our non-diffusive scalar function with the initial condition $f_0(r_0, \theta, \phi)$ at $t = 0$, the exact general solution is

$$f(r, \theta, \phi, t) = f_0\left([R^3(t) - R_0^3 + r^3]^{\frac{1}{3}}, \theta, \phi\right)$$

Additionally, using the method of characteristics on the advection equation $\frac{\partial f}{\partial t} + u \cdot \nabla f = 0$ where $u(r, t) = \frac{a(t)}{r^2}$ will give us the same exact general solution

$$f(r, \theta, \phi, t) = f_0\left([R^3(t) - R_0^3 + r^3]^{\frac{1}{3}}, \theta, \phi\right)$$

with the initial condition $f_0(r_0, \theta, \phi)$ at $t = 0$.

Example: Incompressible Spherically Symmetric Radial Velocity Field in Spherical Coordinates

The surface is initially defined by $r_0 = Q_0(\theta, \phi)$ at $t = 0$.

The radius of the surface at a later time t is defined by $r = [R^3(t) - R_0^3 + Q_0^3(\theta, \phi)]^{\frac{1}{3}}$.

This shows how the radius r evolves from its initial configuration through some Lagrangian deformation $R(t)$. The aim is to ensure that deformation from r_0 to r preserves the volumed enclosed by the surface meaning every point on the initial surface retains its volume at later times.

Since they both states are Lagrangian, if the initial condition is satisfied, then all later conditions will also be satisfied. So, we can define R_0 by

$$\frac{4\pi}{3}R_0^3 = \int\int sin\theta d\theta\, d\phi \int_0^{Q_0} r^2 dr = \frac{1}{3}\int\int Q_0^3(\theta,\phi) sin\theta d\theta d\phi\,, 0 \le \theta \le 2\pi, -\pi \le \phi \le \pi$$

Example: Incompressible Spherically Symmetric Radial Velocity Field in Spherical Coordinates

Volume Conservation: Since both states are Lagrangian, if the initial condition is satisfied, then all later conditions will also be satisfied. So, we can define R_0 by

$$\frac{4\pi}{3}R_0^3 = \int\int sin\theta d\theta\, d\phi \int_0^{Q_0} r^2 dr$$

$$= \frac{1}{3}\int\int Q_0^3(\theta,\phi) sin\theta d\theta d\phi\,, 0 \leq \theta \leq 2\pi, -\pi \leq \phi \leq \pi$$

This step describes the mathematical formulation for how a deformable and spherically influenced fluid surface evolves over time while conserving its volume to set the stage for further analysis of such a system.

Simulation

Using the nonlinear ODE:

$$\dot{r} = \frac{a(t)}{r^2}$$

We set $a(t) = t$, a simple function of time

which results to

$$\dot{r} = \frac{t}{r^2}$$

Simulation: Euler's Method

General Formula:

$$r_{n+1} = r_n + h \cdot \dot{r}(t_n, r_n)$$

where h is the time step and $\dot{r}(t_n, r_n) = \frac{t_n}{r_n^2}$

Initial Conditions:

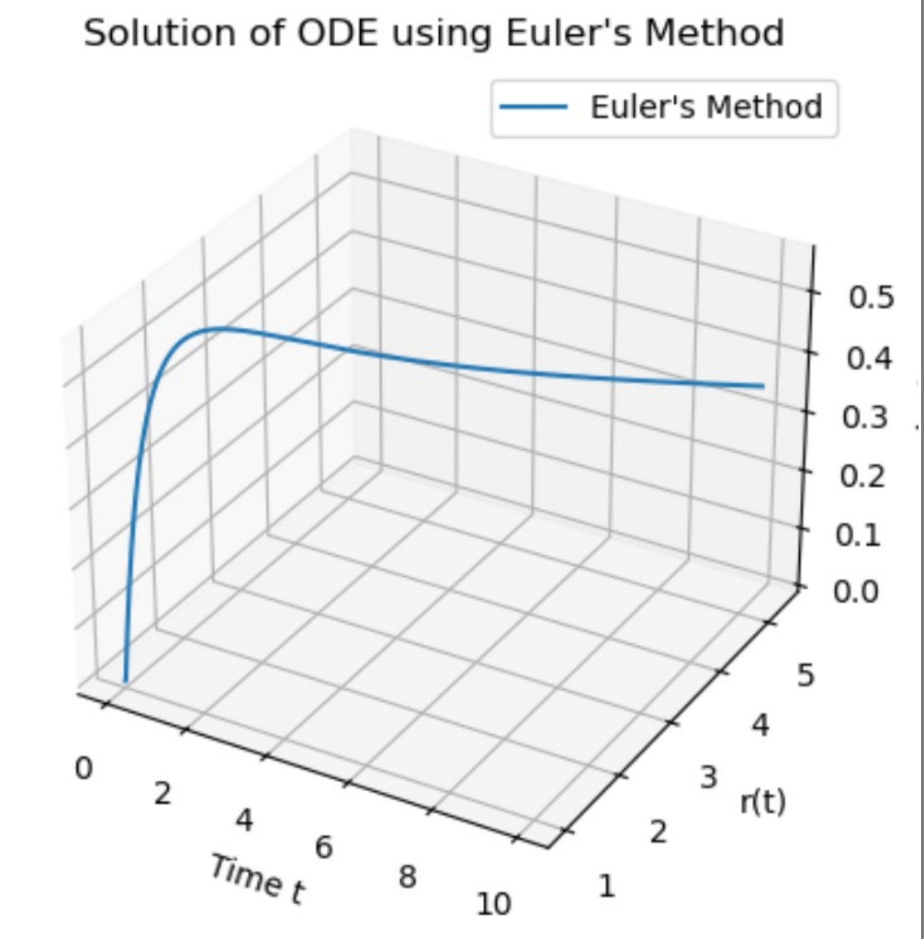
$r_0 = 1$

$t_0 = 0$

$t_f = 10$

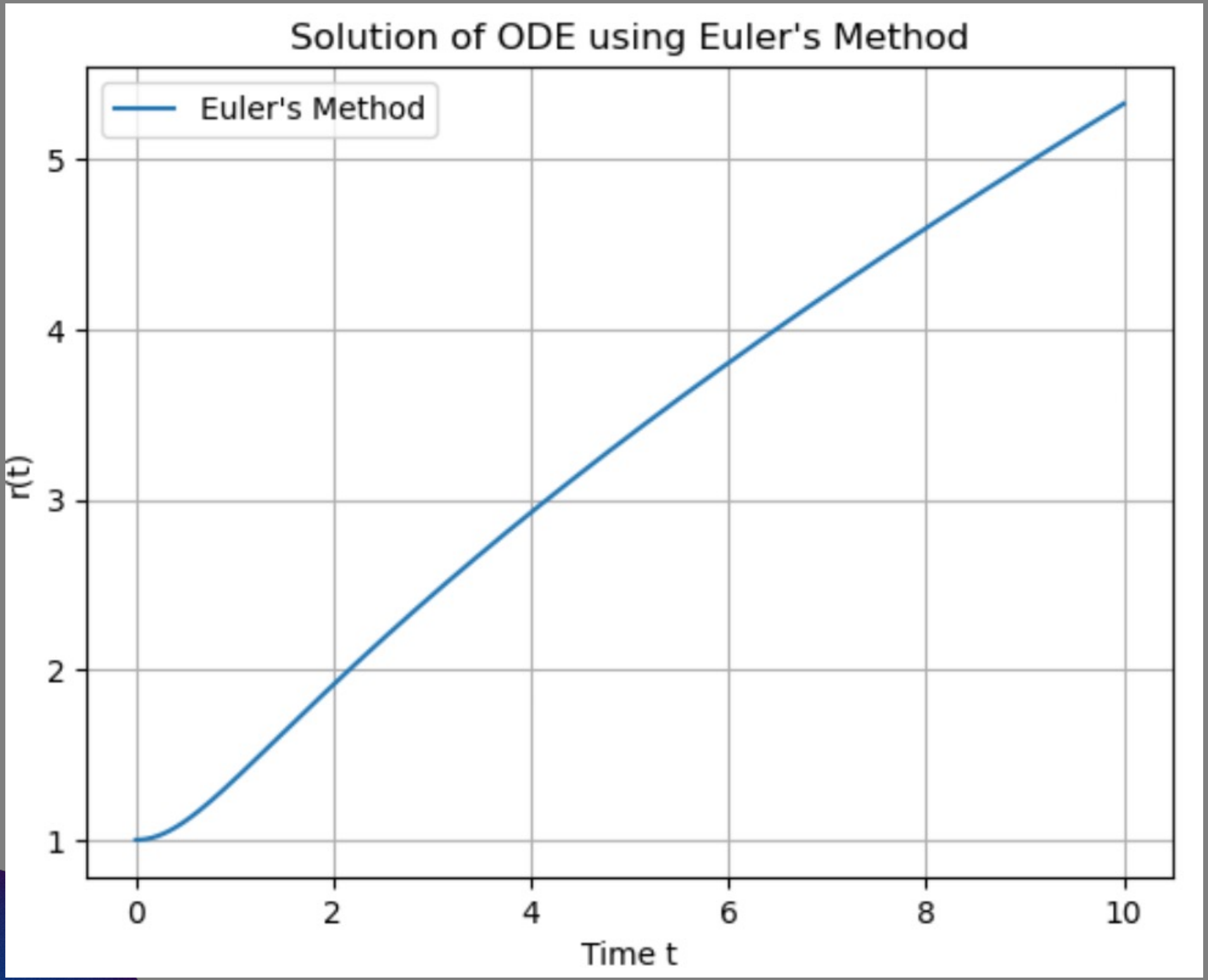
$h = 0.01$

Simulation: Euler's Method Graphical Plot in 3D and Table Results



Time (t)	r(t)	r_dot(t)
0.00	1.00	0.00
0.01	1.00	0.01
0.02	1.00	0.02
0.03	1.00	0.03
0.04	1.00	0.04
0.05	1.00	0.05
0.06	1.00	0.06
0.07	1.00	0.07
0.08	1.00	0.08
0.09	1.00	0.09

Simulation: Euler’s Method Graphical Plot in 2D



Simulation: Runge-Kutta Method

Runge-Kutta Method uses four intermediate step to compute the next value of r

Intermediate Slopes:

$$k_1 = h \cdot f(t_n, r_n)$$

$$k_2 = h \cdot f\left(t_n + \frac{h}{2}, r_n + \frac{k_1}{2}\right)$$

$$k_3 = h \cdot f(t_n + \frac{h}{2}, r_n + \frac{k_2}{2})$$

$$k_4 = h \cdot f(t_n + h, r_n + k_3)$$

Simulation: Runge-Kutta Method

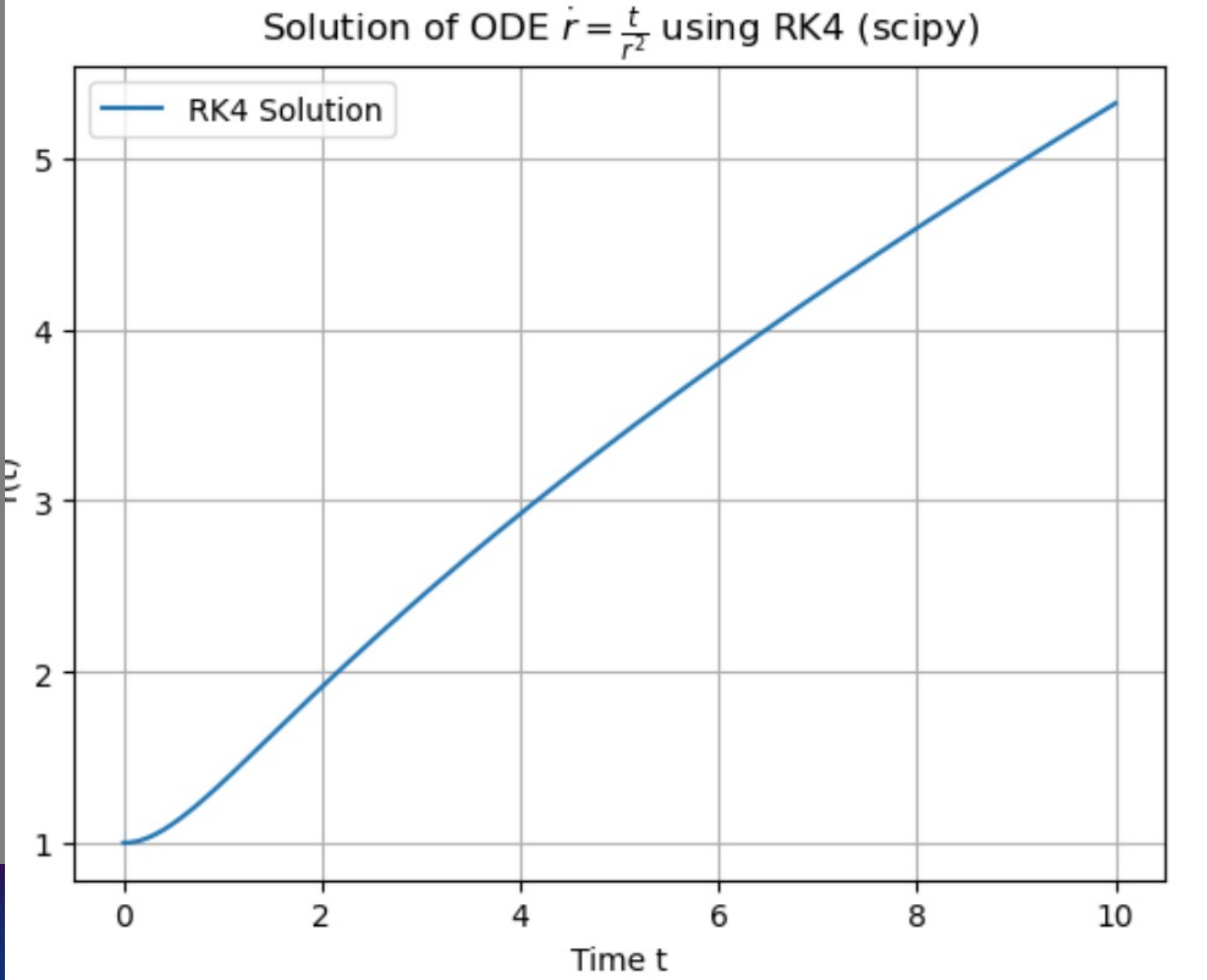
We can now define our formula

$$r_{n+1} = r_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

and updating time using the formula

$$t_{n+1} = t_n + h$$

Runge-Kutta Method Graphical Plot in 2D



Simulation: Results

Velocity Field:

- In spherical coordinates, our spherically symmetric radial velocity field $\vec{u} = u(r,t)\overrightarrow{e_r}$ means that the velocity of fluid particles is directed radially outward and depends only on the radial distance r and time t.

Incompressibility:

- Incompressibility means the volume of any fluid element remains constant over time. This means volume is conserved if the deformation from r_0 to r remains constant

Simulation: Results

Our simulations show that the radial displacement r steadily increases radially outward with time t under the influence of the velocity field. The deformation (slope) from initial radius r_0 to any radius r remains constant, showing volume is conserved in our system.

Given that the ODE $\dot{r} = \frac{a(t)}{r^2}$ is equivalent to the PDE $\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u) = 0$, we expect similar results if we knew methods of modeling PDEs.

New Variable

We include the effects of pressure and viscosity in the radial component of the Navier-Stokes equations for an incompressible fluid. The general form of the Navier-Stokes equation for radial velocity $u(r)$ in spherical coordinates, ignoring external forces like gravity for simplicity, is:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}\right)=-\frac{\partial p}{\partial r}+\mu\left(\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}-\frac{2u}{r^2}\right)$$

Let's assume that the velocity field $u(r)=\frac{C}{r^2}$ is steady, so $\frac{\partial u}{\partial t}=0$, then compute the derivatives:

$$\frac{\partial u}{\partial r}=-\frac{2C}{r^3}$$

$$\frac{\partial^2 u}{\partial r^2}=\frac{6C}{r^4}$$

New Variable cont…

Substituting these into the viscous term

$$\mu\left(6\frac{C}{r^4}+\frac{2}{r}(-\frac{2C}{r^3})-\frac{2\frac{C}{r^2}}{r^2}\right)=\mu\left(6\frac{C}{r^4}-\frac{4C}{r^4}-\frac{2C}{r^4}\right)=0$$

We can get a simplified equation

$$-\frac{\partial p}{\partial r}=\rho\left(-\frac{2C}{r^5}\right)\Rightarrow\frac{\partial \rho}{\partial r}=2\rho\frac{C}{r^5}$$

We then integrate the pressure gradient and are left with

$$p(r)=-2\rho\frac{C^2}{3r^4}+p_0$$

New Variable cont...

$$p(r) = -2\rho \frac{C^2}{3r^4} + p_0$$

With C^2 determined by boundary conditions, such as specifying the velocity at a certain radius. The pressure varies inversely with the fourth power of the radius, illustrating how pressure gradients necessary to maintain such a flow increase sharply as r decreases.

This derivation encapsulates the interaction between the radial velocity profile and the pressure in a spherically symmetric, incompressible flow, showcasing the direct dependence of the pressure gradient on the square of the radial velocity field.

If we had more time, we would use this pressure equation added to our radial velocity to confirm whether the equivalence still holds true.

Q&A

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