Fundamental limits in RSSI-based direction-of-arrival estimation

Conference Paper · October 2016

DOI: 10.1109/WPNC.2016.7822837

4 authors, including:

Thorsten Nowak
Friedrich-Alexander-University of Erlangen-Nürnberg

Markus Hartmann
Friedrich-Alexander-University of Erlangen-Nürnberg

Lucila Patino Studencki
University of Applied Sciences Coburg

Fundamental Limits in RSSI-based Direction-of-Arrival Estimation

Thorsten Nowak, Markus Hartmann, Lucila Patino-Studencki, Jörn Thielecke
Institute of Information Technology (Communication Electronics)
University of Erlangen-Nürnberg, Germany
Email: thorsten.nowak@fau.de

***Abstract*—The use of wireless sensor networks is rapidly increasing. Also the demand of ubiquitous location sensors is swiftly expanding. Hence, energy and location-awareness come into focus of research today. A prospective approach for low-power locating sensor networks is received signal strength indicator (RSSI)-based direction finding. The presented approach is based on RSSI difference measurements retrieved by a array of directed antennas. In this paper, fundamental limits of RSSI-based direction finding are evaluated, beyond the Cramér-Rao Lower Bound (CRLB). That is not applicable for the design of a localization system topology due to the nature of the gain difference function that leads to an unbounded variance of the unbiased estimator. Thus, a maximum likelihood (ML) approach to the RSSI-based direction finding is presented. The ML estimator yields a limited variance for all signal directions. However, that benefit comes at the expense of being biased. Beyond treating direction estimates, mean square position errors are compared for both, the unbiased and the ML estimator.**

I. Introduction

In the past years, locating wireless sensor networks (WSNs) have become popular in many fields of life. Location-awareness is an essential feature of todays sensor networks [1]. One of those emerging applications of WSNs is wildlife monitoring. Recent advances of WSNs enable biologists to apply sensor networks to research questions on habitat use [2] and foraging strategies [3] including observation of social interactions in groups of animals and the study of behavioral structures of individual animals [4].

The *BATS*[1] system [5] is an energy-efficient sensor network tracking bats in the wild. In the WSN position information is obtained from received signal strength indicator (RSSI) measurements [6] gathered within the sensor network. RSSI-based localization is a good choice for a low-power locating systems since the complexity of signal processing within the WSN is comparably low [7]. Furthermore, RSSI-based techniques do not require exhaustive synchronization of the sensor nodes.

In general localization is a problem of parameter estimation. Hence, fundamental limits from estimation theory, such as the Cramér-Rao Lower Bound (CRLB), may be applied to the problem of direction-of-arrival (DOA) estimation. In [8] a CRLB for RSSI-based DOA for switched antenna beams has been presented. This concept has been extended to arrays of directed antennas in [9]. Furthermore, in [9] the CRLB for RSSI-based DOA is evaluated in the position domain and the impact of network topology is considered. Since the CRLB for the direction estimation is not bounded for all received signal directions, it is reasonable that an unbiased estimator, as considered by the CRLB, may not exist for RSSI-based direction finding. Moreover, a biased estimator, most prominent being the maximum likelihood (ML) estimator, allows to trade-off bias for variance [10]. Therefore, in this paper the ML estimator for RSSI-based DOA estimation is derived and is assessed applying the CRLB for biased estimators [11].

The paper is organized as follows. Section II gives an introduction to the system model, CRLB and ML estimation in general. A brief review of RSSI-based DOA is given in Section III. In Section IV ML estimation and biased CRLB are illustrated with a simple representative sine function. A comparison of position estimation errors for both DOA estimators, the unbiased one and the ML estimator, is given in Section V evaluating the mean squared error (MSE) in the position domain. Section VI concludes this paper.

[1]Dynamic Adaptable Applications for Bats Tracking by Embedded Communicating Systems, http://www.for-bats.org/

II. Preliminaries

In this introductory section the system model is presented and fundamentals of parameter estimation are covered which includes CRLB for single parameter estimation in white Gaussian noise (WGN) and ML estimation.

A. System Model

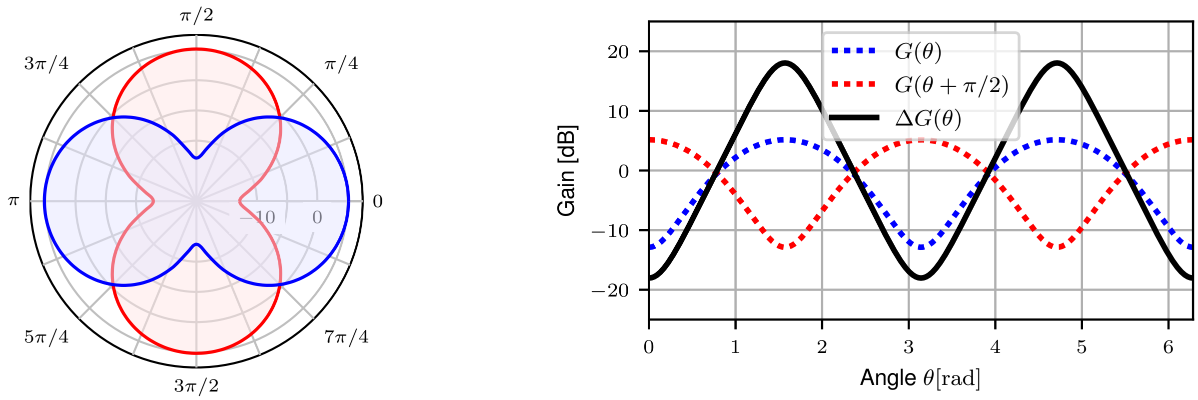
A hidden parameter is observed by measurements. Observations are considered to be given by

$$r = f(\theta) + w, \tag{1}$$

where f is an arbitrary function, θ is the desired parameter and w is a WGN process with $\mathcal{N}(0, \sigma_r^2)$.

B. Cramér-Rao Lower Bound

Due to the stochastic nature of measurements every parameter estimation exhibits some sort of variance. The variance represents the uncertainty of a measurement. A process estimating a parameter from a measurement, i.e. an estimator, should have two basic properties: unbiasedness and minimum variance. Such estimators are called minimum variance unbiased estimators (MVUE). For these unbiased estimators the



(a) Radiation pattern

(b) Gain and gain difference functions

Fig. 1: The radiation patterns and gain functions for both antenna configurations, rotated by 0° and 90° are depicted in blue and red, respectively. The black line denotes the gain difference function.

Cramer-Rao Lower Bound (CRLB) provides a lower bound on the error variance of the estimated state [12].

Recalling the alternative form of the CRLB [13]

$$\operatorname{var}\left(\hat{\theta}\right) \geq E\left[\left(\frac{\partial \ln p(r|\theta))}{\partial \theta}\right)^2\right]^{-1} \tag{2}$$

and assuming that an unknown parameter θ of a deterministic signal is observed in white Gaussian noise by a series of measurements, described by

$$r[n] = f[n;\theta] + w[n] \qquad n = 0, 1, \dots, N-1 \tag{3}$$

Following [13] using the assumptions stated above the general CRLB for estimating an unknown parameter of a signal in white Gaussian noise can be expressed as

$$\operatorname{var}\left(\hat{\theta}\right) \geq \frac{\sigma_r^2}{\sum_{n=0}^{N-1}\left(\frac{\partial f[n;\theta]}{\partial \theta}\right)^2} \tag{4}$$

Equation (4) simplifies to

$$\operatorname{var}\left(\hat{\theta}\right) \geq \frac{\sigma_r^2}{\left(\frac{\partial f(\theta)}{\partial \theta}\right)^2} \tag{5}$$

for a single observation in presence of a WGN.

C. Maximum Likelihood Estimation

In this section one of the most prominent estimators, the ML estimator, is covered. The ML estimator $\hat{\theta}_{ML}$ maximizes the likelihood function [11]

$$\hat{\theta}_{ML} = \underset{\theta}{\operatorname{argmax}}\ p(r|\theta). \tag{6}$$

In this case it is assumed that $\hat{\theta}$ is estimated for a particular realization of a random variable θ. Taking the logarithm yields to

$$0 = \frac{\partial}{\partial \theta} \ln\left(p(r|\theta)\right)\bigg|_{\theta=\hat{\theta}_{ML}}. \tag{7}$$

Assuming WGN in equation (1) the expression further simplifies to

$$0 = \left(r - f(\theta)\right) \frac{\partial}{\partial \theta} f(\theta)\bigg|_{\theta=\hat{\theta}_{ML}}. \tag{8}$$

Obviously, the inverse function f^{-1} maximizes the likelihood function. Hence, the ML estimator is given by:

$$\hat{\theta}_{ML}(r) = f^{-1}(r) \tag{9}$$

III. RSSI-based Direction Finding

The direction-of-arrival (DOA) is inferred from the RSSI difference of a signal received at multiple directed antennas. The field strength of the received signal is expressed by

$$S_a(\theta) = G_a(\theta) + P_{\mathrm{RX},a} + w_a \tag{10}$$

where P_{RX} is the signal power at the node without antenna gain, $G_k(\theta)$ the gain of the respective antenna a, and w_a an additive white noise process. The difference in signal strength becomes

$$\Delta S(\theta) = S_1(\theta) - S_2(\theta) + w_1 - w_2. \tag{11}$$

and further assuming a single signal source and the noise processes w_i to be uncorrelated the expression simplifies to

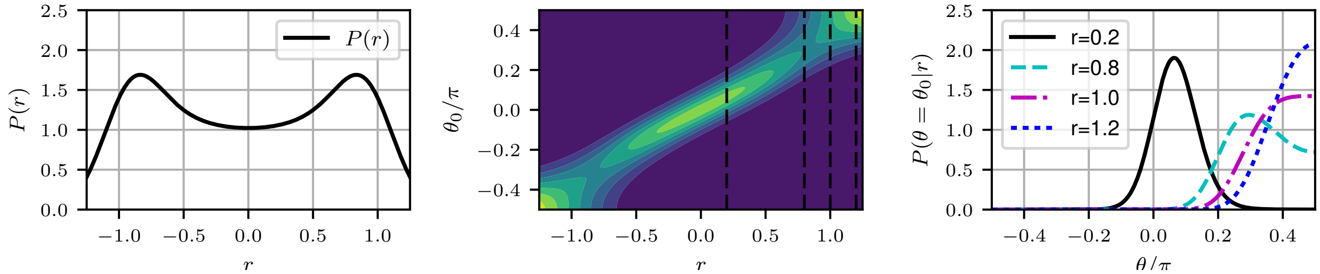
$$\Delta S(\theta) = \Delta G(\theta) + w, \tag{12}$$

with $\Delta G(\theta) = G_1(\theta) - G_2(\theta)$ being the gain difference function for the two considered antennas, and w a Gaussian noise process. Following [9] and [14] the gain function for the considered arrangement of two half-wave dipoles at distance of $\sim \lambda/2$ is given by

$$G(\theta) \propto \left[\cos(2\pi d \cos(\theta))\right]^2, \tag{13}$$

and apparently depends on the rotation angle θ only. Restricting the setup to two antennas with the same gain patterns and rotated by 90° towards each other, the gain difference function becomes

$$\Delta G(\theta) = G(\theta) - G(\theta + {}^{\pi}/_{2}). \tag{14}$$



(a) Marginal likelihood (b) A-posteriori PDF (c) A-posteriori PDF for given r]

Fig. 2: Densities: Marginal likelihood (a) and a-posteriori density (b, c).

Considered antenna patterns are depicted in Figure 1a. For the considered antennas at rotation angles of 0° and 90°, respectively, the gain functions and gain difference function are shown in in Figure 1b. The DOA of a received signal is inferred from the gain difference function as described in the next section.

IV. DOA Estimation

For the sake of simplicity in this section a simple representative measurement model

$$r = \sin(\theta) + w, \tag{15}$$

where θ is the desired parameter and w is a WGN process with $\mathcal{N}(0, \sigma_r^2)$, is considered. The considered sine function features the same essential properties as the gain difference function of the antenna described above in equation (14). For the given problem the likelihood function states as

$$P(r|\theta_0) = \frac{1}{\sqrt{2\pi\sigma_r^2}} \exp\left[-\frac{1}{2}\left(\frac{r - \sin(\theta_0)}{\sigma_r}\right)^2\right]. \tag{16}$$

The CRLB for the sine problem is fairly simple. Applying equation (5) leads to

$$\operatorname{var}\left(\hat{\theta}\right) \geq \frac{\sigma_r^2}{(\cos(\theta))^2} \tag{17}$$

and figures out to have an unbounded variance for θ_0 approaching $-\pi/2$ or $\pi/2$ as shown as dashed line in Figure 3. In terms of the CRLB it makes sense that the variance is unbounded as there is no *curvature* [13] at that positions and pointing out that CRLB is a local measure.

However, practically one would expect the error of an estimator for θ to be bounded in any case. And indeed, the CRLB just states that an unbiased estimator has an infinite variance. Actually, a biased estimator might exist with limited variance for the given problem. Moreover, the CRLB approaching infinity allows to infer that an unbiased estimator for the problem does not exist for θ_0 approaching $-\pi/2$ or $\pi/2$.

It is reasonable to derive a more appropriate bound for the estimation error. A nearby approach is calculating the full a-posteriori probability density $P(\theta_0|r)$. Applying the Bayes' theorem the full a-posteriori density is obtained from

$$P(\theta_0|r) = \frac{P(r|\theta_0)P(\theta_0)}{P(r)} \tag{18}$$

Without loss of generality $P(\theta_0)$ may be limited to $\theta_0 \in [-\pi/2, -\pi/2]$. Assuming now a-priori knowledge on the distribution, the a-priori density is given by

$$P(\theta_0) = \mathcal{U}(-\pi/2, -\pi/2). \tag{19}$$

With the given non-informative a-priori density, the marginal likelihood $P(r)$ is obtained from

$$P(r) = \int_\theta P(r|\theta)P(\theta)\, \partial\theta \tag{20}$$

and depicted in Figure 2a. The full a-posteriori density as a function of r and θ is shown in Figure 2b and some probability density functions (PDFs) for selected measurements $r \in [0.2, 0.8, 1.0, 1.2]$, denoted by dashed lines in Figure 2b, are sketched in Figure 2c.

Considering the ML estimator introduced in (9), the inverse function of the stated problem is

$$\hat{\theta} = \arcsin(r), \tag{21}$$

which also evidently is the ML estimator for the given observation. However, some points have to be noted about the function $f = \sin(\theta)$. Due to the nature of the sine function r is limited to $r \in [-1, 1]$, which is also true for the inverse function $\sin^{-1}(r)$. For the ability to transform arbitrary measurements $r \in [-\infty, \infty]$ into the parameter space, the definition of the arcsin function is expanded to

$$\arcsin(r) = \begin{cases} \pi/2 & \text{for } r > 1 \\ -\pi/2 & \text{for } r < -1 \\ \sin^{-1}(r) & \text{else} \end{cases}. \tag{22}$$

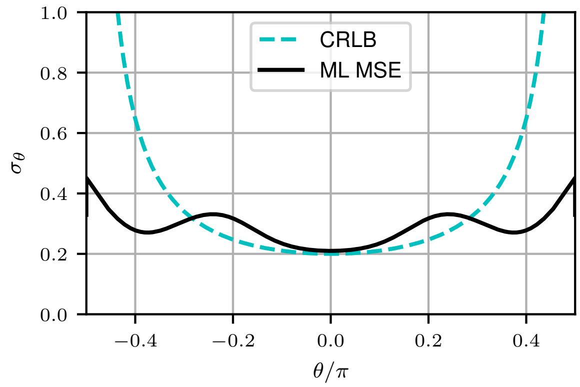
Given the a-posteriori density the error of the ML estimator is defined by

$$\operatorname{mse}(\hat{\theta}) = \int_{-\pi/2}^{\pi/2} P(\theta_0|r)\left(\theta_0 - \arcsin(r)\right)^2 \partial\theta_0. \tag{23}$$

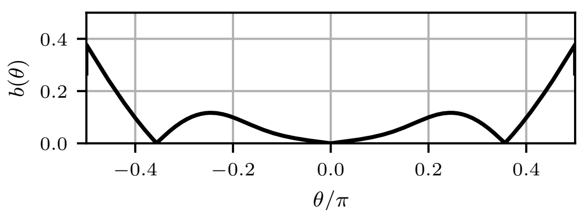
The results are shown in Figure 3. As expected, the variance of the ML estimator is limited for all θ. Nevertheless, it has to be noted, that this estimator is biased. Hence, the standard CRLB may not be applied.

The obtained estimator can be evaluated applying the CRLB for a biased estimator with known bias [11]. Consider a biased estimator $\hat{\theta}$ with its bias $b(\theta)$ known for all θ

$$b(\theta) = E\{\hat{\theta}\} - \theta \tag{24}$$



(a) CRLB and ML MSE



(b) ML Bias

Fig. 3: CRLB, ML MSE and ML Bias

Defining

$$\psi(\theta) = b(\theta) + \theta \tag{25}$$

any unbiased estimator with expectation $\psi(\theta)$ fulfills:

$$\text{var}\left(\hat{\theta}\right) \geq I(\theta)^{-1} \left(\frac{\partial \psi(\theta)}{\partial \theta}\right)^2 . \tag{26}$$

Applying (25) the lower limits for the variance of the biased estimator is expressed by

$$\text{var}\left(\hat{\theta}\right) \geq I(\theta)^{-1} \left(1 + \frac{\partial b(\theta)}{\partial \theta}\right)^2 . \tag{27}$$

For the sine problem, denoted by (16), the CRLB is computed by

$$\text{var}\left(\hat{\theta}\right) \geq \frac{\sigma_r^2}{(\cos(\theta))^2} \left(1 + \frac{\partial b(\theta)}{\partial \theta}\right)^2 . \tag{28}$$

Obviously, the CRLB in case of the biased estimator is limited for all θ. Apparently looking at equation (27), the derivative of the bias term $b(\theta)$ allows to compensate the zero for θ_0 approaching $-\pi/2$ or $\pi/2$ in the denominator of the CRLB. Thus, bias can be traded off for variance and vice versa. The ML estimator approaches the CRLB for a biased estimator with exactly the bias of the ML estimator.

V. Position Estimation

In the last section MSEs of the minimum variance unbiased estimator (MVUE) and the ML estimator have been derived. Now, position errors resulting from these DOA estimators are evaluated. The position error of multiple receivers in a sensor network is considered for noisy DOA estimates. Especially, the impact of the network topology is studied for both, the MVUE and the ML estimator.

A. Mean Square Positioning Error

Considering the position estimation errors, the MSE in the position domain denotes a bound on the minimum MSE of the position estimator [13]. The MSE for a position estimate is given by

$$\text{mse}(\hat{\boldsymbol{x}}) = E\left[(\hat{\boldsymbol{x}} - \boldsymbol{x})(\hat{\boldsymbol{x}} - \boldsymbol{x})^T\right]. \tag{29}$$

Incorporating arbitrary non-zero mean measurement noise, the measured angles $\boldsymbol{\theta}$ can be expressed as

$$\boldsymbol{\theta} = \boldsymbol{g}(\boldsymbol{x}) + \boldsymbol{n} \tag{30}$$

in dependence of the user position $\boldsymbol{x}$ with

$$g_k(\boldsymbol{x}) = \tan^{-1} \frac{\Delta y_k}{\Delta x_k}, \tag{31}$$

and

$$\Delta x_k = x - x_k \quad and \quad \Delta y_k = y - y_k. \tag{32}$$

Given by the error propagation law, the covariance matrix is expressed by [15]

$$\left[H(\boldsymbol{x})^{-1}\right]_{i,j} = \left[\frac{\partial \boldsymbol{g}(\boldsymbol{x})}{\partial x_i}\right]^T \frac{1}{\text{mse}_\theta(\boldsymbol{x})} \boldsymbol{I} \left[\frac{\partial \boldsymbol{g}(\boldsymbol{x})}{\partial x_j}\right], \tag{33}$$

which for the presented case leads to

$$H(\boldsymbol{x})^{-1} = \sum_k \begin{bmatrix} \frac{\Delta y_k^2}{\text{mse}_\theta(\boldsymbol{x})\|\boldsymbol{x}-\boldsymbol{x}_k\|_2^4} & -\frac{\Delta x_k \cdot \Delta y_k}{\text{mse}_\theta(\boldsymbol{x})\|\boldsymbol{x}-\boldsymbol{x}_k\|_2^4} \\ -\frac{\Delta x_k \cdot \Delta y_k}{\text{mse}_\theta(\boldsymbol{x})\|\boldsymbol{x}-\boldsymbol{x}_k\|_2^4} & \frac{\Delta x_k^2}{\text{mse}_\theta(\boldsymbol{x})\|\boldsymbol{x}-\boldsymbol{x}_k\|_2^4} \end{bmatrix}$$

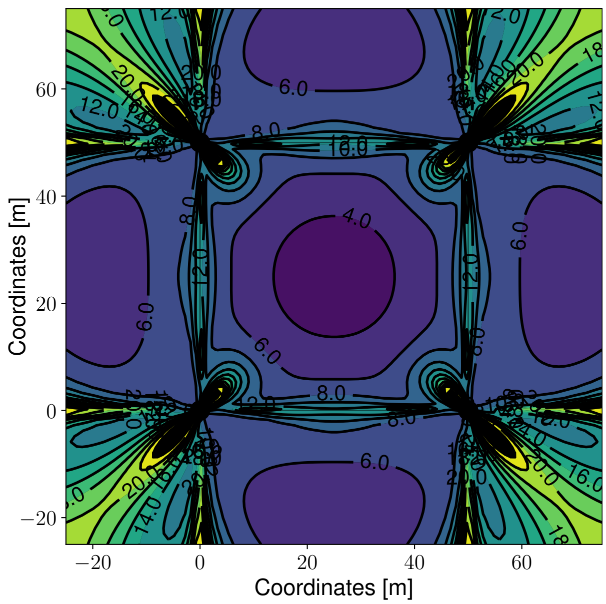
for position estimation from RSSI-based DOA measurements, with $||\boldsymbol{x}||_2$ denoting the euclidean norm. Finally, the position estimation error is retrieved with

$$\text{mse}_{\boldsymbol{x}}(\boldsymbol{x}) = \sum \text{tr}\left(\boldsymbol{H}(\boldsymbol{x})\right) \tag{34}$$

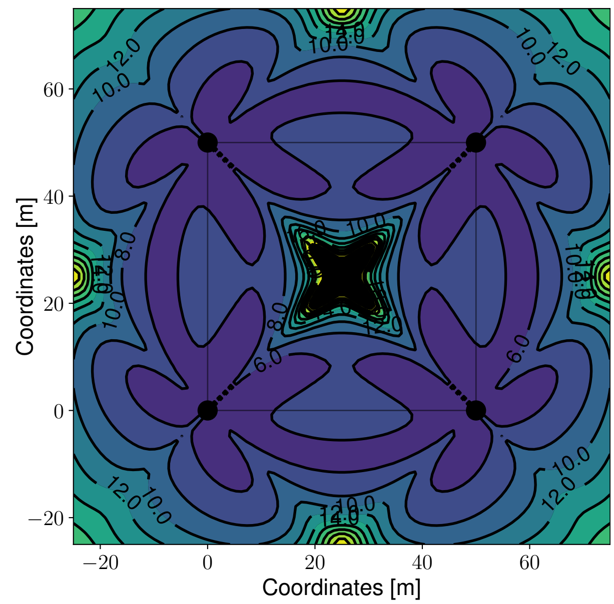
for DOA-based localization in a WSNs. At this point it has to noted again, that the DOA estimation error mse_θ inherently depends on the received signal direction θ, thus it depends on the user position $\boldsymbol{x}$.

B. Comparison of Position Estimation Errors

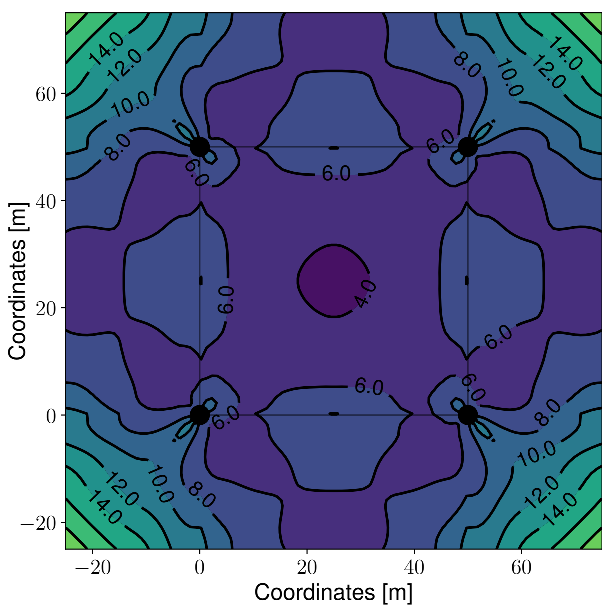
The derivation of mean square position errors is now utilized to compare the performance of the MVUE with the performance of the ML-based estimator, which is known to be a biased estimator. The DOA estimation error mse_θ is computed according to (17) for the MVUE and for the ML approach according to equation (23). Mean square position errors are evaluated for two different WSN topologies. For the first network the measurement function is given by $f_1 = \cos(2\theta)$, which is just a scaled and shifted version of the measurement function studied in the section above. The measurement function of the second network is defined by $f_2 = \sin(2\theta)$, which apparently is the same measurement function as above, except of being shifted by $\pi/4$. This can be interpreted as nodes with the antenna patterns but the receivers being rotated by 0° and 45° for the examined network topologies 1 and 2, respectively. For both the sensor nodes have been arranged in rectangular shape with a node distance of 50 m.



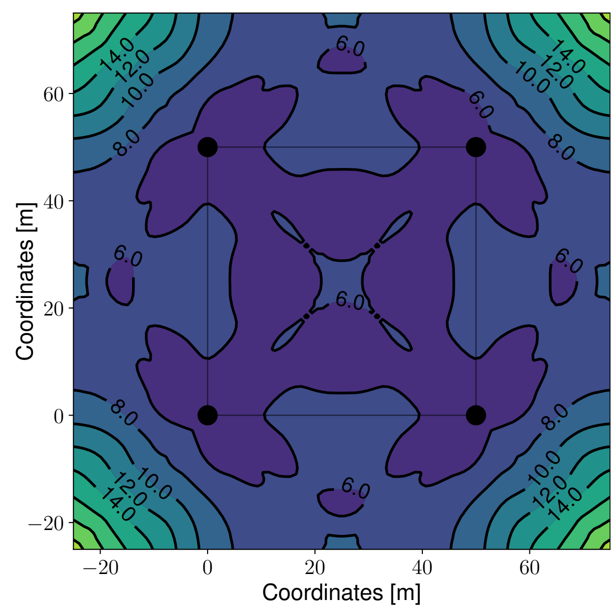
(a) MVUE position error for measurement function $\cos(2\theta)$



(b) MVUE position error for measurement function $\sin(2\theta)$



(c) ML position error for measurement function $\cos(2\theta)$

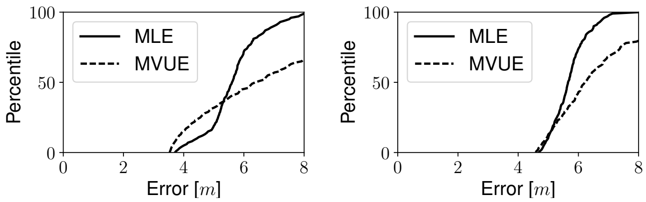


(d) ML position error for measurement function $\sin(2\theta)$

Fig. 4: Position errors for both estimators, MVUE and ML estimator for two different measurement functions.

In [9] it has been stated, that the network topology, i.e, orientation of sensor nodes, is a crucial parameter for the localization performance. Indeed, this is correct for the MVUE. But for the ML estimator results considerably differ from those of the unbiased estimator. Comparing Figure 4c and Figure 4d the effect of different network topologies is negligible, whereas changing topology for the MVUE substantially affects the mean square position error as seen in Figure 4a and Figure 4b. Therefore, network topology is not that crucial for the ML estimator as the mean square position error is quite homogeneous and hardly depends on node orientation. This fact is not really surprising as the MSE for DOA estimates, comparing Figure 3 is fairly constant over all signal directions.

From Figure 4 it can be easily seen that the CRLB is a rather pessimistic measure evaluating the MSE of position estimates. These results show the importance of selecting the right criteria to assess the performance of a localization system and to draw the right conclusion for design of a localization system. Furthermore, unbiasedness comes at the cost of an increasing variance leading to larger MSE values for the position estimates in the end. Figure 5 depicts the position error percentiles. For network 1 the MVUE at least shows a better performance at smaller MSEs. However, this swiftly change at increasing MSEs. In case of network 2 there is no



(a) $\cos(2\theta)$ (b) $\sin(2\theta)$

Fig. 5: Position errors (percentiles)

benefit for unbiasedness at all.

These results hold true for a snapshot position estimate, i.e. based on a set single DOA observations. When processing a series of DOA measurements, e.g. recursive Bayesian filtering, unbiasedness might be the more important property of an estimator, though. Bayesian filters are able to average out a larger variance of a DOA estimator when the motion of tracked target can be modeled appropriately. Therefore, in Bayesian filtering, variance is not that harmful. However, a unknown bias of a sensor is critical, even when considering perfect motion models.

VI. Conclusion

In this paper the fundamentals of RSSI-based DOA have been presented. For both, the MVUE and the ML estimator, DOA estimation errors have been computed. The derived CRLB for unbiased DOA estimation highly depends on direction of the received signals and is unbounded for some signal directions. In contrast to that, the ML estimation error is limited for all signal directions and features a rather constant MSE. But on the other, the ML estimator is biased. Mean square position errors have been derived for a sensor network with erroneous DOA estimators. It has been shown that network topology matters in the MVUE case, but is negligible when applying the ML approach to DOA estimation. Furthermore, the ML estimator shows smaller MSE compared to the MVUE considering snapshot localization. However, in recursive Bayesian filtering, unbiasedness might be the major objective. In that case, the MVUE would be first choice then.

Acknowledgment

This work is funded by the German Science Foundation DFG grant FOR 1508, Research Unit BATS.

References

[1] M. Vossiek, L. Wiebking, P. Gulden, J. Wieghardt, C. Hoffmann, and P. Heide. Wireless local positioning. *Microwave Magazine, IEEE*, 4(4):77 – 86, dec. 2003.

[2] Jakob Fahr, Michael Abedi-Lartey, Thomas Esch, Miriam Machwitz, Richard Suu-Ire, Martin Wikelski, and Dina KN Dechmann. Pronounced seasonal changes in the movement ecology of a highly gregarious central-place forager, the african straw-coloured fruit bat (eidolon helvum). *PloS one*, 10(10):e0138985, 2015.

[3] Noam Cvikel, Katya Egert Berg, Eran Levin, Edward Hurme, Ivailo Borissov, Arjan Boonman, Eran Amichai, and Yossi Yovel. Bats aggregate to improve prey search but might be impaired when their density becomes too high. *Current Biology*, 25(2):206–211, 2015.

[4] Christian Rutz, Zackory T Burns, Richard James, Stefanie MH Ismar, John Burt, Brian Otis, Jayson Bowen, and James JH St Clair. Automated mapping of social networks in wild birds. *Current Biology*, 22(17):R669–R671, 2012.

[5] Falko Dressler, Simon Ripperger, Martin Hierold, Thorsten Nowak, Christopher Eibel, Björn Cassens, Frieder Mayer, Klaus Meyer-Wegener, and Alexander Koelpin. From radio telemetry to ultra-low power sensor networks - tracking bats in the wild. *IEEE Communications Magazine*, 54(1):129–135, 2016.

[6] Markus Hartmann, Thorsten Nowak, Lucila Patino-Studencki, Jörg Robert, Albert Heuberger, and Jörn Thielecke. A low-cost rssi-based localization system for wildlife tracking. *IOP Conference Series: Materials Science and Engineering*, 120(1):012004, 2016.

[7] A. Boukerche, H. A. B. F. Oliveira, E. F. Nakamura, and A. A. F. Loureiro. Localization systems for wireless sensor networks. *IEEE Wireless Communications*, 14(6):6–12, December 2007.

[8] G. Giorgetti, S. Maddio, A. Cidronali, S. K S Gupta, and G. Manes. Switched beam antenna design principles for angle of arrival estimation. In *Wireless Technology Conference, 2009. EuWIT 2009. European*, pages 5–8, Sept 2009.

[9] Thorsten Nowak, Markus Hartmann, Thomas Lindner, and Jörn Thielecke. Optimal network topology for a locating system using rssi-based direction finding. In *IPIN 2015 Sixth International Conference on Indoor Positioning and Indoor Navigation (IPIN 2015)*, Banff, Canada, October 2015.

[10] Stuart Geman, Elie Bienenstock, and René Doursat. Neural Networks and the Bias/Variance Dilemma. *Neural Computation*, 4(1):1–58, January 1992.

[11] Harry L Van Trees. *Detection, estimation, and modulation theory*. John Wiley & Sons, Hoboken, N.J, 2004.

[12] Dimitris G Manolakis, Vinay K Ingle, and Stephen M Kogon. *Statistical and adaptive signal processing: spectral estimation, signal modeling, adaptive filtering, and array processing*, volume 46. Artech House Norwood, 2005.

[13] Steven Kay. *Fundamentals of statistical signal processing*. Prentice-Hall PTR, Englewood Cliffs, N.J, 1993.

[14] Markus Hartmann, Oliver Pfadenhauer, Lucila Patino-Studencka, Hans-Martin Tröger, Albert Heuberger, and Jörn Thielecke. Antenna pattern optimization for a rssi-based direction-of-arrival localization system. In *ION's Pacific PNT Conference 2015*, pages 429–433, Honolulu, Hawaii, April 2015.

[15] Pratap Misra and Per Enge. *Global Positioning System: Signals, Measurements, and Performance*. Ganga-Jamuna Press, 2006.