Quantum Fourier Transform

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Contents

Fourier Analysis

- A mathematical tool
- time-domain signal to a frequency-domain signal
- and vice versa
- 4 classes

Time-Domain	Periodic	Aperiodic
Continuous	Fourier Series (FS)	Fourier Transform (FT)
Discrete	Discrete Fourier Transform (DFT)	Discrete Time Fourier Transform (DTFT)

Freq.-Domain	Periodic	Aperiodic
Continuous	DTFT	FT
Discrete	DFT	FS

Discrete Fourier Transform (DFT)

- discrete, periodic time-domain signal to frequency-domain signal.
- $x_0, x_1, \ldots, x_{\{N-1\}} \longmapsto y_0, y_1, \ldots, y_{\{N-1\}}$
- DFT

$$y_k = \sum_{n=0}^{N-1} x_n \cdot e^{-2\pi\iota\frac{k}{N}n}$$

- Inverse DFT (IDFT)

$$x_n = \frac{1}{N} \sum_{k=0}^{N-1} y_k \cdot e^{2\pi\iota\frac{k}{N}n}$$

Quantum Fourier Transform (QFT)

- QFT = Inverse DFT

$$|j\rangle \rightarrow \frac{1}{2^{n/2}} \sum_{k=0}^{2^n-1} e^{2\pi\iota\frac{k}{2^n}j} \cdot |k\rangle$$

$$= \frac{1}{2^{n/2}} \sum_{k_1=0}^{1} \sum_{k_2=0}^{1} \cdots \sum_{k_n=0}^{1} e^{2\pi\iota j\left(\sum_{l=1}^{n} k_l 2^{-l}\right)} \cdot |k_1 k_2 \cdots k_n\rangle$$

$$= \frac{1}{2^{n/2}} \sum_{k_1=0}^{1} \sum_{k_2=0}^{1} \cdots \sum_{k_n=0}^{1} e^{2\pi\iota j\left(\sum_{l=1}^{n} k_l 2^{-l}\right)} \cdot |k_1 k_2 \cdots k_n\rangle$$

$$= \frac{1}{2^{n/2}} \sum_{k_1=0}^{1} \sum_{k_2=0}^{1} \cdots \sum_{k_n=0}^{1} \prod_{l=1}^{n} e^{2\pi\iota j \quad {}_l 2^{-l}} \cdot |k_l\rangle$$

$$= \frac{1}{2^{n/2}} \prod_{l=1}^{n} \left[\sum_{k_l=0}^{1} e^{2\pi\iota j k_l 2^{-l}} \cdot |k_l\rangle\right]$$

$$= \frac{1}{2^{n/2}} \prod_{l=1}^{n} \left[|0\rangle + e^{2\pi\iota j 2^{-l}} \cdot |1\rangle\right]$$

$$= \frac{(|0\rangle + e^{2\pi\iota j/2} \cdot |1\rangle)(|0\rangle + e^{2\pi\iota j/2^2} \cdot |1\rangle) \cdots (|0\rangle + e^{2\pi\iota j/2^n} \cdot |1\rangle)}{2^{n/2}}$$

$$= \frac{(|0\rangle + e^{2\pi\iota 0.j_n} \cdot |1\rangle)(|0\rangle + e^{2\pi\iota 0.j_{n-1}j_n} \cdot |1\rangle) \cdots (|0\rangle + e^{2\pi\iota 0.j_1 j_2 \cdots j_n} \cdot |1\rangle)}{2^{n/2}}$$

Quantum Fourier Transform (QFT)

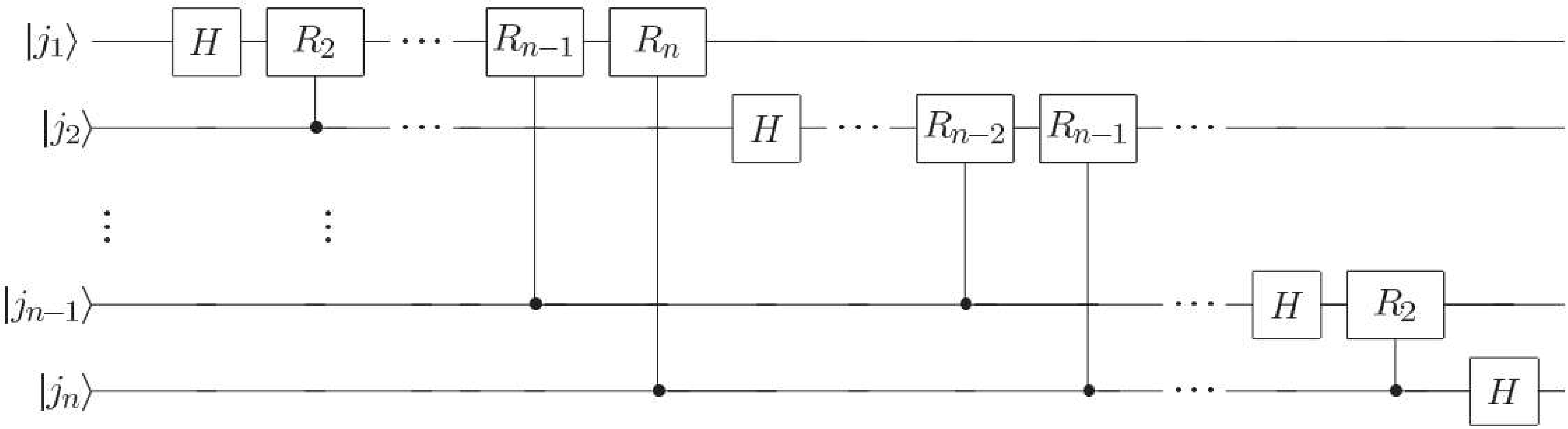
$$\frac{(|0\rangle + e^{2\pi\iota j/2} \cdot |1\rangle)(|0\rangle + e^{2\pi\iota j/2^2} \cdot |1\rangle) \cdots (|0\rangle + e^{2\pi\iota j/2^n} \cdot |1\rangle)}{2^{n/2}}$$

Phase	State
1	$\|00\cdots 0\rangle$
$e^{\frac{2\pi\iota j}{2^n}}$	$\|00\cdots 1\rangle$
$e^{\frac{2\pi\iota j}{2^{n-1}}}$	$\|0\cdots 10\rangle$
$e^{\frac{2\pi\iota j}{2}}$	$\|10\cdots 0\rangle$
$e^{\frac{2\pi\iota j}{2}+\frac{2\pi\iota j}{2^2}+\frac{2\pi\iota j}{2^3}+\cdots+\frac{2\pi\iota j}{2^n}}$	$\|11\cdots 1\rangle$

$$\mathrm{H}|j_k\rangle = \begin{cases} \frac{|0\rangle+|1\rangle}{\sqrt{2}}, j_k = 0 \\ \frac{|0\rangle-|1\rangle}{\sqrt{2}}, j_k = 1 \end{cases} = \frac{|0\rangle + e^{\frac{2\pi\iota j_k}{2}}|1\rangle}{\sqrt{2}} \qquad \mathrm{R}_i|j_k\rangle = e^{\frac{2\pi\iota j_k}{2^i}}|j_k\rangle$$

Quantum Fourier Transform (QFT)

$$\frac{(|0\rangle + e^{2\pi\iota j/2} \cdot |1\rangle)(|0\rangle + e^{2\pi\iota j/2^2} \cdot |1\rangle) \cdots (|0\rangle + e^{2\pi\iota j/2^n} \cdot |1\rangle)}{2^{n/2}}$$



Example

- QFT on $j = |01\rangle$

$$|01\rangle \rightarrow \frac{1}{2^{n/2}} \sum_{k=0}^{2^n-1} e^{2\pi\iota \frac{k}{2^n} j} \cdot |k\rangle$$

$$= \frac{(|0\rangle + e^{2\pi\iota j/2} \cdot |1\rangle)(|0\rangle + e^{2\pi\iota j/2^2} \cdot |1\rangle)}{2^{n/2}}$$

$$= \frac{(|0\rangle + e^{2\pi\iota 1/2} \cdot |1\rangle)(|0\rangle + e^{2\pi\iota 1/2^2} \cdot |1\rangle)}{2^{2/2}}$$

$$= \frac{|00\rangle + \iota|01\rangle - |10\rangle - \iota|11\rangle}{2}$$

Applications and Open Problems

Applications

- Quantum phase estimation
- Quantum period finding
- Quantum algorithms
- Optimization problems

Open Problems

- Efficiency and Resource Optimization
- Noise
- Limited applications
- Parallelization