Deutsch's Algorithm and the Deutsch-Jozsa Algorithm

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Quantum Computing & Algorithms
Department of Information Technology, NIT Raipur

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Outline

- Constant and Balanced Functions
- Deutsh's Problem
- Query Model of Computation
- Deutsch's Algorithm
- Deutsh-Jozsa Problem
- Deutsh-Jozsa Algorithm
- Summary
- References

Function : Definition

A function f from a set X to a set Y, denoted $f : X \rightarrow Y$, is a rule that assigns to every element $x \in X$ exactly one element $y \in Y$, where $y = f(x)$.

Example:

Determine the number of distinct functions from $\{0, 1\}$ to $\{0, 1\}$.

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0	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
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Constant Function

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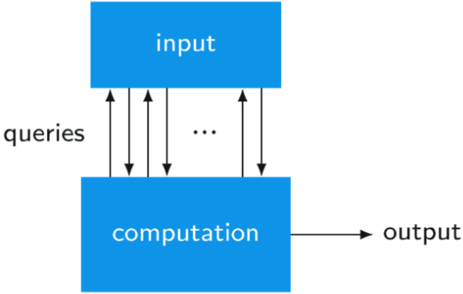
The Deutsch algorithm, a quantum algorithm, solves the Deutsch problem using only one query.

What is query model of computation?

- The query model of computation, also known as the black-box model, is a computational framework where an algorithm gains information about a problem instance by querying an oracle.



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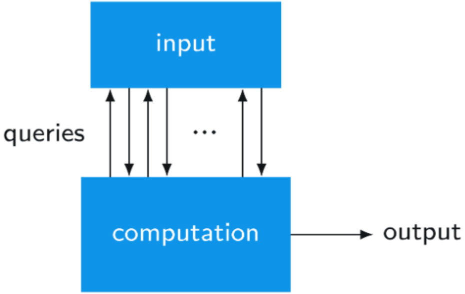

- In the query model of computation, the entire input is not provided to the computation. Rather, the input is made available in the form of a function, which the computation accesses by making queries.

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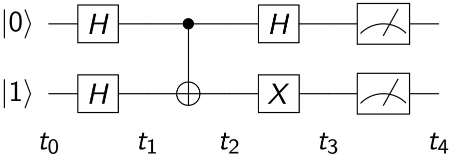
Example:



Quantum Algorithms Timeline

1. Deutsch Algorithm (1985)
2. Deutsch-Jozsa Algorithm (1992)
3. Bernstein-Vazirani Algorithm (1992)
4. Simon's Algorithm (1993)
5. Shor's Algorithm (1994)
6. Grover's Algorithm (1996)
7. Variational Quantum Eigensolver (2014)
8. Quantum Approximate Optimization Algorithm (2014)
9. Quantum Computing in Cloud by IBM (2016)
10. Quantum Machine Learning (2018)

In the study of quantum query complexity, one is given a black box U_f implementing some function f, and asked what the minimum number of required queries to U_f is in order to determine some desired property of f.

Deutsch's Algorithm

- Deutsch Algorithm was proposed by David Deutsch in 1985.

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- Deutsch algorithm was one of the first examples demonstrating the potential of quantum computing to solve certain problems more efficiently than classical computing.

Prof. David Deutsch

Quantum Circuit for the Deutsch's Algorithm

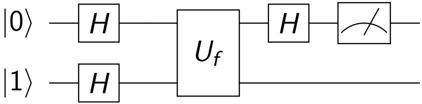
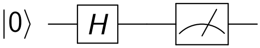


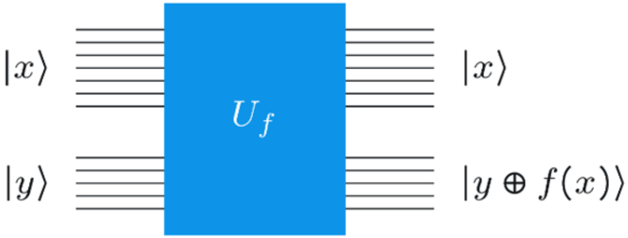
Figure 1: Quantum circuit for the Deutsch algorithm

Hadamard Gate

- The Hadamard gate is a single qubit gate.
- The Hadamard gate has the matrix : $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$
- Circuit with H gate: $|q\rangle \;—\boxed{H}—$
- 

- Matrix for $|0\rangle$ is : $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- Output of the above circuit is : $\mathrm{H}(|0\rangle) =$
$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle.$
- Similarly, $\mathrm{H}(|1\rangle) =$
$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle.$

Quantum Query Gates



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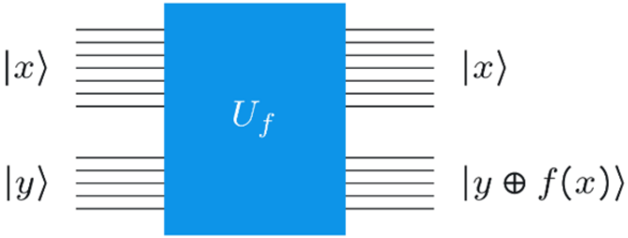

- In circuit model of computation, queries are made by special gates called query gates.
- An operation which takes qubits $|x\rangle$ and $|y\rangle$ as input and produces $|x\rangle$ and $|y \oplus f(x)\rangle$ as output is an unitary operation.

Analysis of Deutsch Algorithm

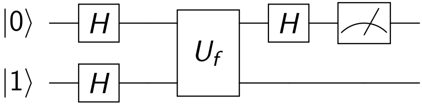

Figure 2: Quantum circuit for the Deutsch algorithm

Constant and Balanced Functions - A Generalization

- A Boolean function $f : \{0,1\}^n \to \{0,1\}$, is called constant if the function produces the same output (either all 0s or all 1s) for every input x, i.e.,

$$f(x) = f(y) \text{ for all } x, y \in \{0,1\}^n.$$

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  i.e.,

$$f(x) = 0 \text{ for half of the } 2^n \text{ inputs, and } f(x) = 1 \text{ for the other half.}$$

Example

Determine the number of distinct constant functions and the balanced functions from $\{0,1\}^2 = \{00, 01, 10, 11\}$ to $\{0,1\}$.

x_1	x_2	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}	f_{15}	f_{16}
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0	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
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Minimum number of queries, i.e., evaluations of f_1, needed to determine if f_1 is a constant function is :

Example

p	q	r	$f(p,q,r)$	$g(p,q,r)$	$h(p,q,r)$
0	0	0	0	1	0
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- The function f is :

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- The function f is : constant
- The function g is :

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- The function f is : constant
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- The function f is : constant
- The function g is : balanced
- The function h is : neither constant nor balanced
- Minimum number of queries, i.e., evaluations of g, needed to determine if g is a balanced function is :

Deutsch-Jozsa Problem

Given a Boolean function $f(x)$, where $f : \{0,1\}^n \to \{0,1\}$ for an n-bit input string x, and where the function is either constant or balanced, determine whether the function $f(x)$ is constant or balanced.

Classical Algorithm for the Deutsh-Jozsa Problem

- Given a Boolean function $f\colon \{0,1\}^n \to \{0,1\}$, there are 2^n different input strings corresponding to the decimal values $0, 1, 2, \ldots, 2^n - 1$.

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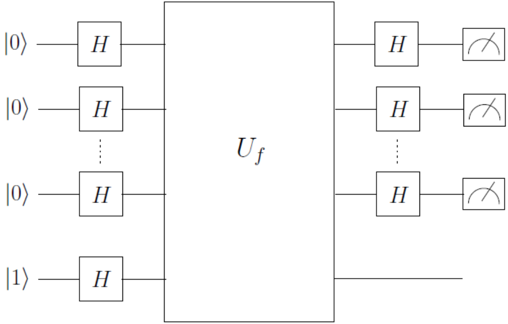
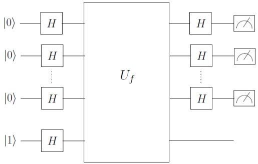
The Deutsch-Jozsa Algorithm was proposed by Deutsch and Jozsa in 1992 to solve the Deutsch-Jozsa problem. It solves the problem using only one query.

Quantum circuit for the Deutsch-Jozsa algorithm:



Analysis of the Deutsch-Jozsa Algorithm



Summary

- Query Model of Computation
- Deutsch's Algorithm
- Deutsch-Jozsa Algorithm
  The Deutsch-Jozsa highlights the potential for quantum algorithms to outperform classical counterparts in specific tasks.

References

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3. Lecture Notes : CMSC 491: Introduction to Quantum Computation by Prof. Sevag Gharibian.
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Thank You