Quantum Teleportation

Comparing Units of Computation

Classical Computing

Bits

0 1

$b = \{0, 1\}$

States

State is Deterministic

Quantum Computing

Qubits

Complex Numbers

$|0\rangle$ $|1\rangle$ $\alpha|0\rangle + \beta|1\rangle$

$$|q\rangle = \alpha|0\rangle + \beta|1\rangle$$

States (Unit Vectors)

Superposition (Linear Combination)

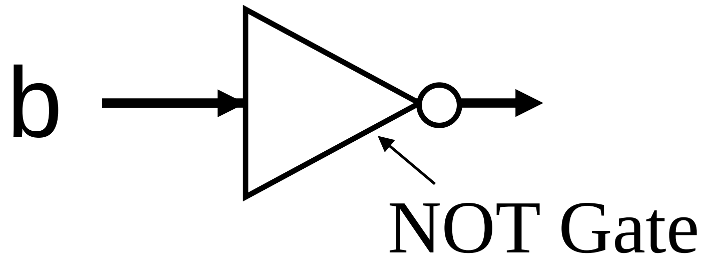
State is Probabilistic

$|0\rangle: |\alpha|^2$

$|1\rangle: |\beta|^2$

Comparing Gates and Computational Methods

Classical Computing

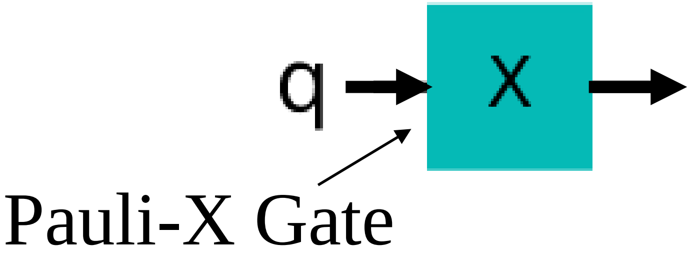


Boolean Algebra

$$NOT\ 0 = 1$$

$$NOT\ 1 = 0$$

Quantum Computing



$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Linear Algebra

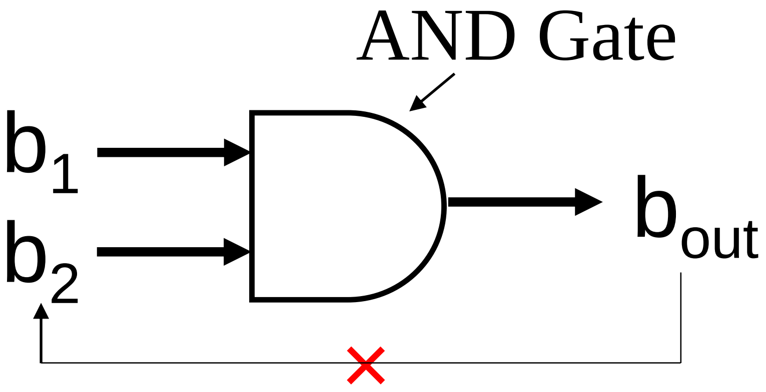
$$X|0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$$

$$X|1\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$$

$$X(\alpha|0\rangle + \beta|1\rangle) = \beta|0\rangle + \alpha|1\rangle$$

Comparing Gates and Computational Methods

Classical Computing

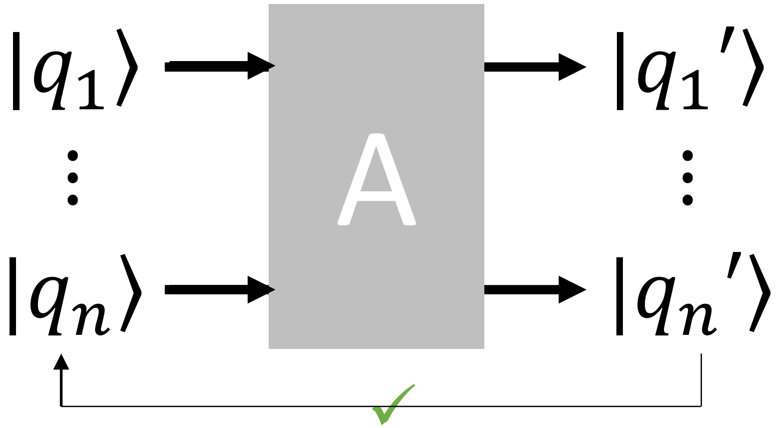


Operation irreversible

State Space Size: *N*

Linear in size of input

Quantum Computing



Operation reversible

State Space Size: *2^N*

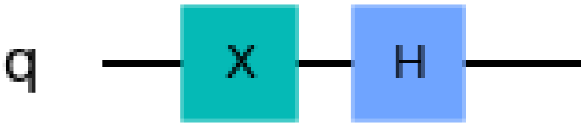
Exponential in size of input

Quantum Circuit

Sequence of building blocks (gates)

Gates carry out elementary computations

Circuits carry out complex computations

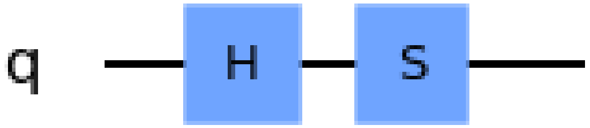


$$H\sigma_X = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}$$

Matrix multiplication

Unitary matrix (combination of linear transformations is a linear transformation)

$$H\sigma_X|0\rangle = |-\rangle \qquad H\sigma_X|1\rangle = |+\rangle$$



$$SH = \begin{bmatrix}1 & 0\\ 0 & i\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$

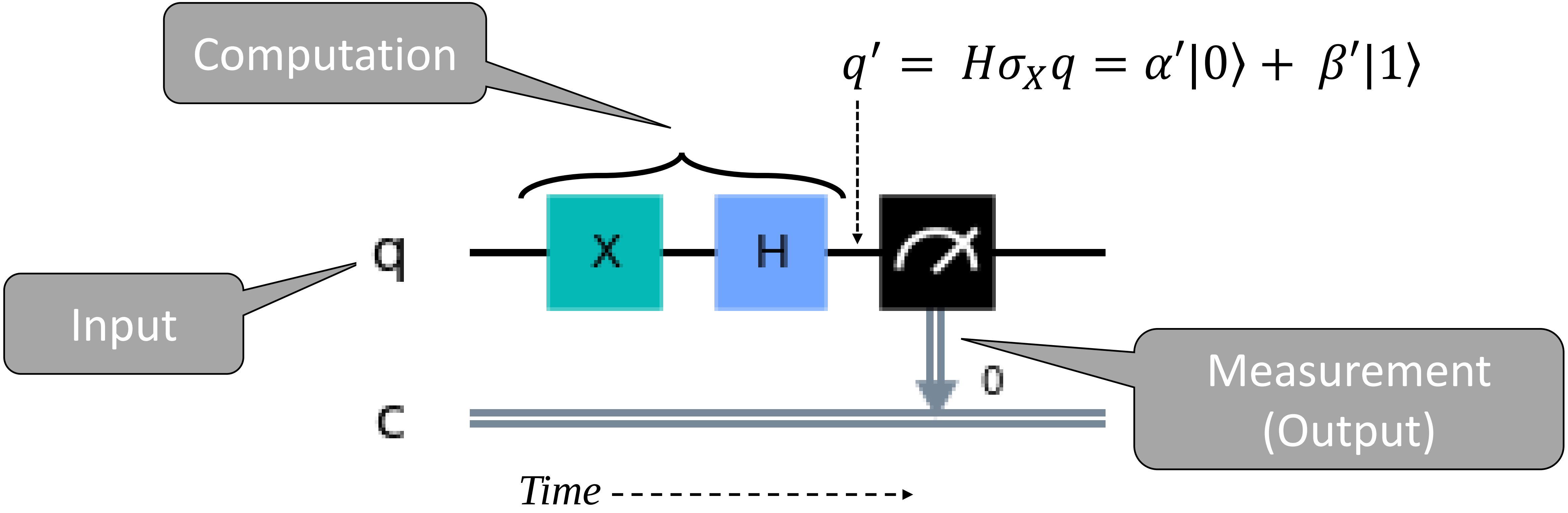
Change between Y and Z bases

$$SH|0\rangle = |+i\rangle \qquad SH|1\rangle = |-i\rangle$$

Quantum Circuit

Measurement following input and computation

$$q' = H\sigma_X q = \alpha'|0\rangle + \beta'|1\rangle$$



$c = |0\rangle$ *with probability* $|\alpha'|^2$ $\quad$ $c = |1\rangle$ *with probability* $|\beta'|^2$

Multiple Qubits and Multipartite Quantum States

Scale up state space by combining single qubits

With n qubits, we can represent 2^n states (vectors)

Tensor product

2^n-dim vector (multipartite state)

2-qubit state $|q_1\rangle \otimes |q_2\rangle = \begin{bmatrix} q_{11} \\ q_{12} \end{bmatrix} \otimes \begin{bmatrix} q_{21} \\ q_{22} \end{bmatrix} = \begin{bmatrix} q_{11}q_{21} \\ q_{11}q_{22} \\ q_{12}q_{21} \\ q_{12}q_{22} \end{bmatrix}$

4-dim vector (bipartite state)

n -qubit state $|q_1\rangle \otimes |q_2\rangle \otimes \cdots \otimes |q_n\rangle = \begin{bmatrix} q_{11} \\ q_{12} \end{bmatrix} \otimes \begin{bmatrix} q_{21} \\ q_{22} \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} q_{n1} \\ q_{n2} \end{bmatrix} = \begin{bmatrix} q_{11}q_{21} \cdots q_{n1} \\ \vdots \\ q_{12}q_{22} \cdots q_{n2} \end{bmatrix}$

$|01\rangle = |0\rangle \otimes |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$|0\rangle^{\otimes n} = |00\cdots0\rangle = |0\rangle \otimes |0\rangle \otimes \cdots \otimes |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \cdots$

$\begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix}$

Dirac notation

Probabilities: $\{|q_{11}q_{21}|^2,$

4 states possi

$q_{11}q_{21}|0$

$q_{11} \cdots q_{n1}|0\rangle^{\otimes n} + \cdots + q_{1n} \cdots q_{nn}\,|1\rangle^{\otimes n}$

State Classification

Product State: state that can be expressed as a tensor product of two states

$$|01\rangle = |0\rangle \otimes |1\rangle$$

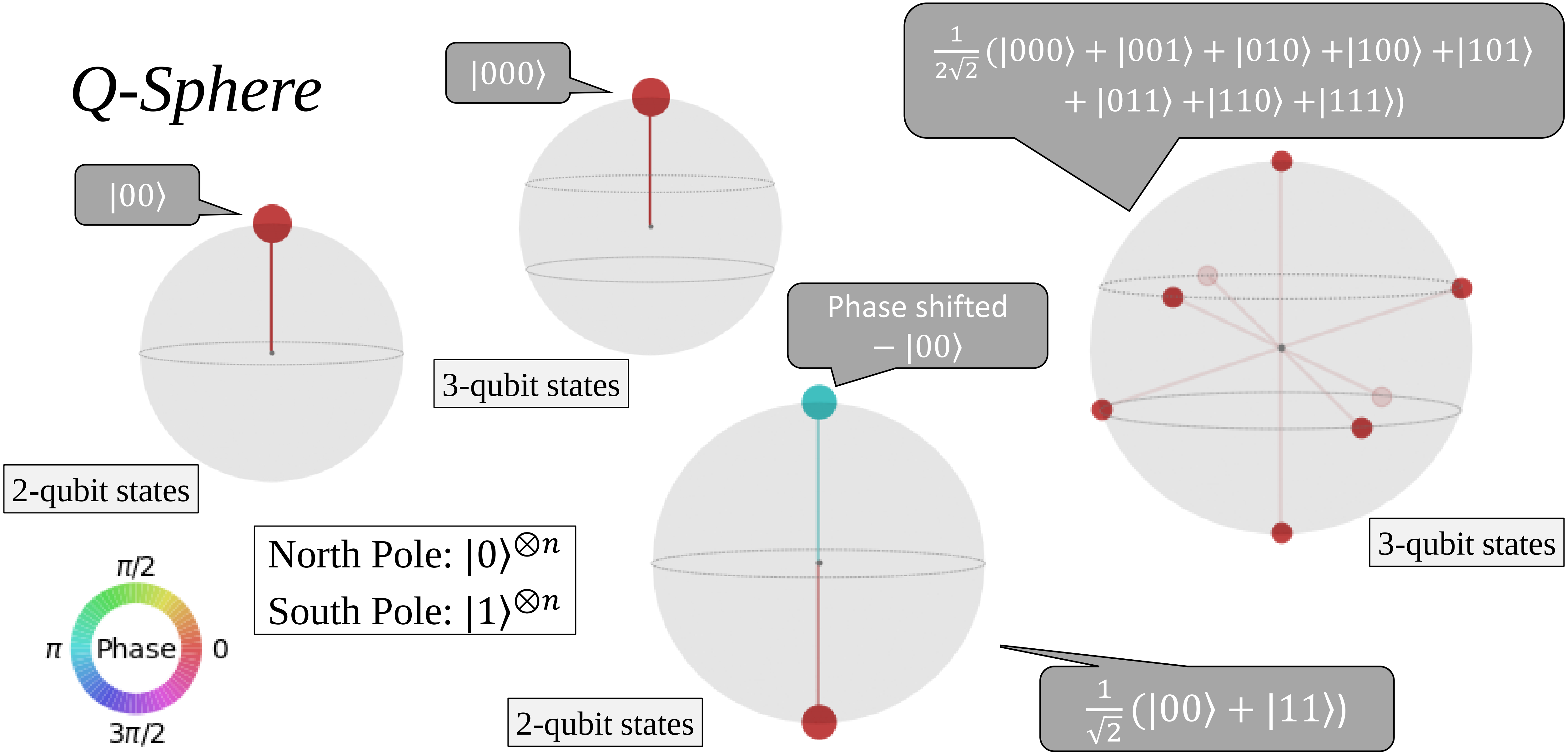
$$\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

Entangled State: state that cannot be expressed as a tensor product of two states

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Visual Representation of Multipartite State

Q-Sphere



Multiple Qubit Gates

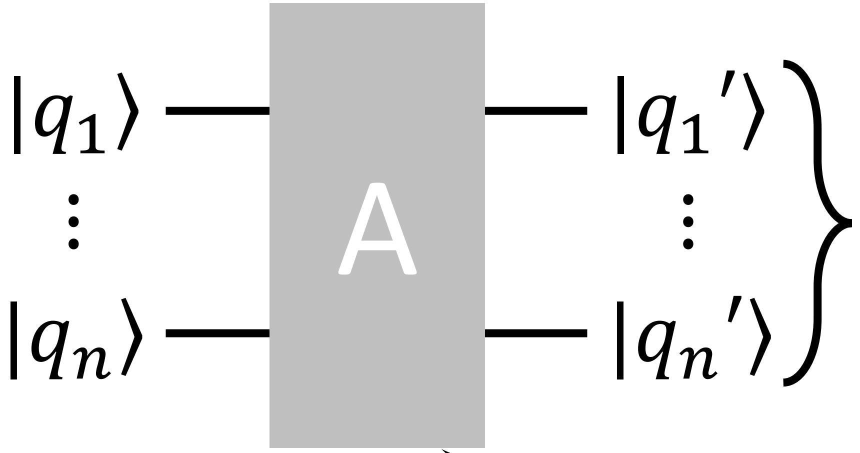
Transforms multipartite quantum state

Quantum theory is unitary

Matrices representing gates are unitary: $AA^{\dagger} = I$

Which implies that matrices representing gates are invertible: $A^{\dagger} = A^{-1}$

Which means that quantum gates (and circuits) must be *reversible*

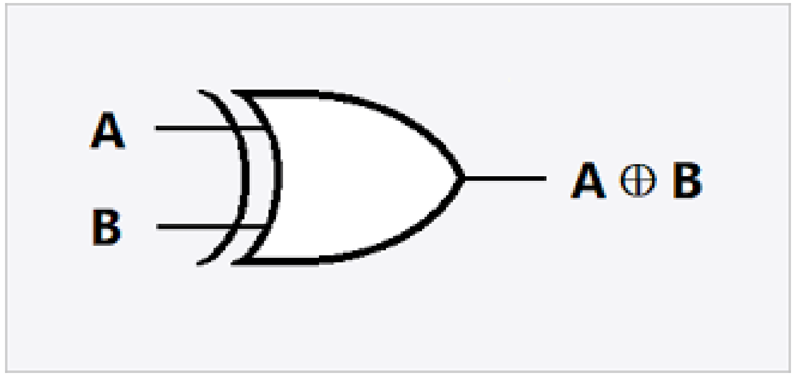


Extra outputs for recovery of inputs: *reversibility*

Knowing A (function), $|q_1\rangle \cdots |q_n\rangle$ can be recovered from $|q_1{}'\rangle \cdots |q_n{}'\rangle$

The CNOT Gate

Equivalent of the XOR gate in classical computing



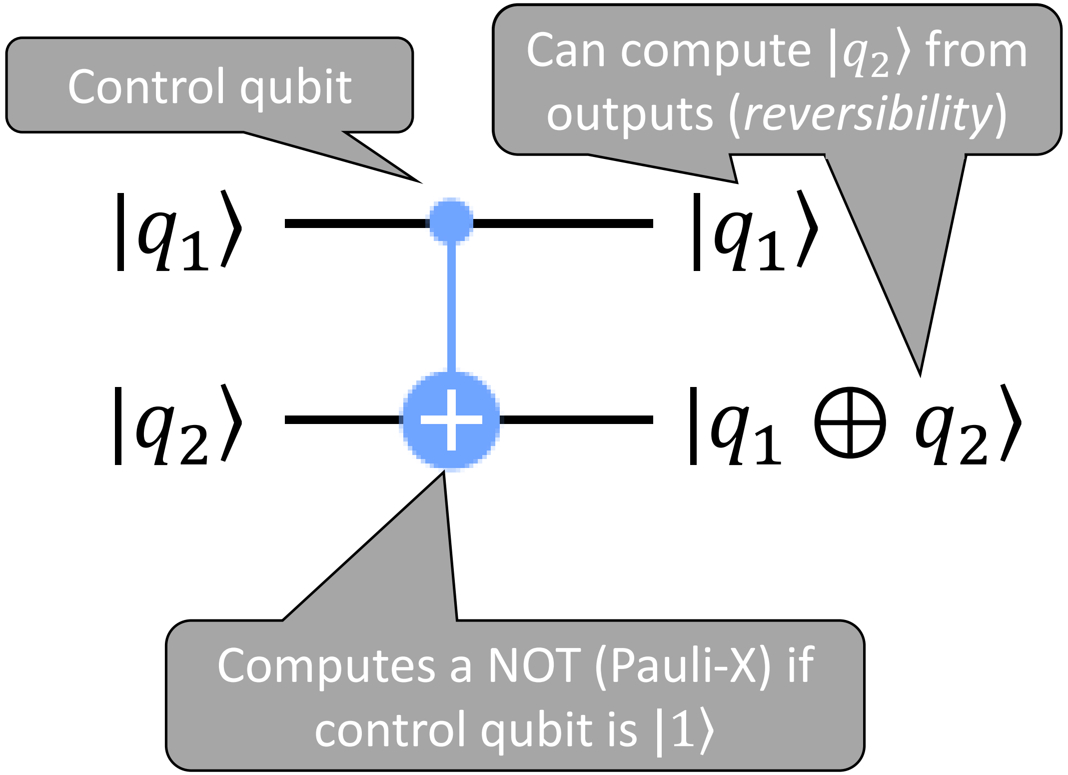
Controlled NOT (or Controlled-X)

$$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10|$$

$\|q_1 q_2\rangle$	**Output**
$\|00\rangle$	$\|00\rangle$
$\|01\rangle$	$\|01\rangle$
$\|10\rangle$	$\|11\rangle$
$\|11\rangle$	$\|10\rangle$

$$CNOT|00\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = |00\rangle$$



Other Multi-Qubit Gates

CNOT is also called CX (Controlled-X or Controlled-Pauli-X)

Similarly, there are 2-qubit gates called CY and CZ, which preserve the state of a qubit if the control qubit is $|0\rangle$ and transform it if the control qubit is $|1\rangle$

$$CPHASE = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\varphi} \end{bmatrix}$$

Shifts phase by φ only if state is $|11\rangle$

$$SWAP = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Swaps two qubits

$$Toffoli\ (CCNOT) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Flips third qubit (NOT) only if first two (control) are $|1\rangle$ and $|1\rangle$

Universality

Universal Gate: Single gate or set of gates that can compute any function through some combination

Classical Computing

- {AND, OR, NOT}
- NAND
- NOR

Implement any possible boolean function (i.e., logical expression)

Quantum Computing

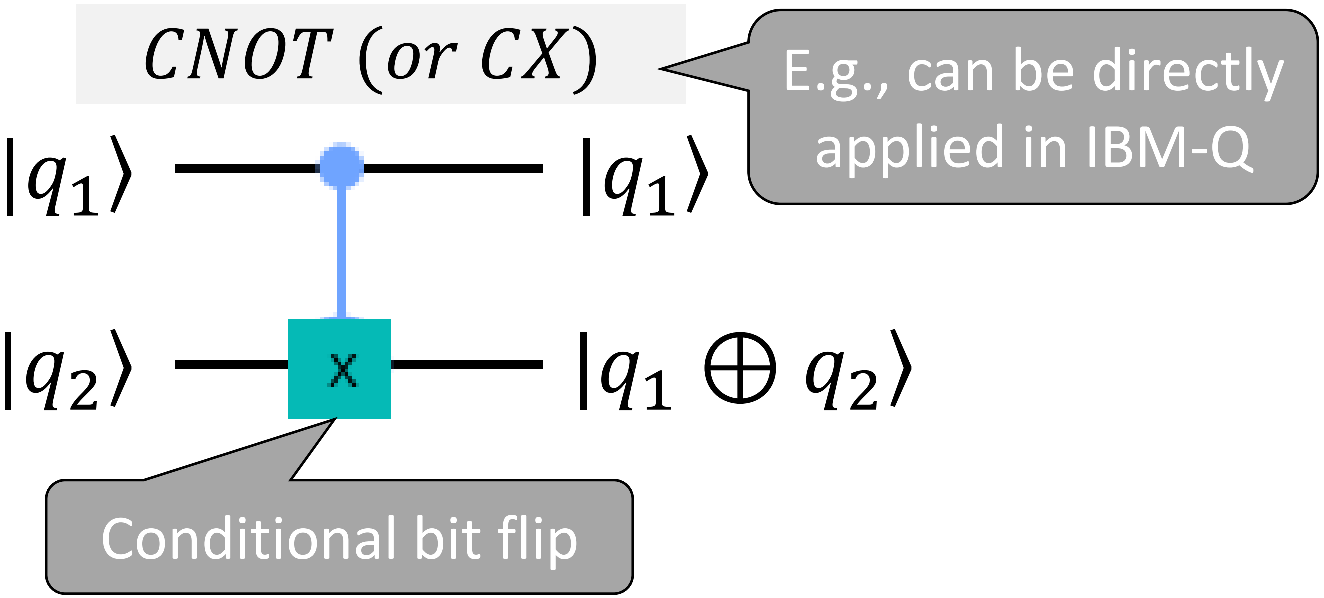
- {H, T, CNOT}: 2-qubit
- {H, Toffoli}

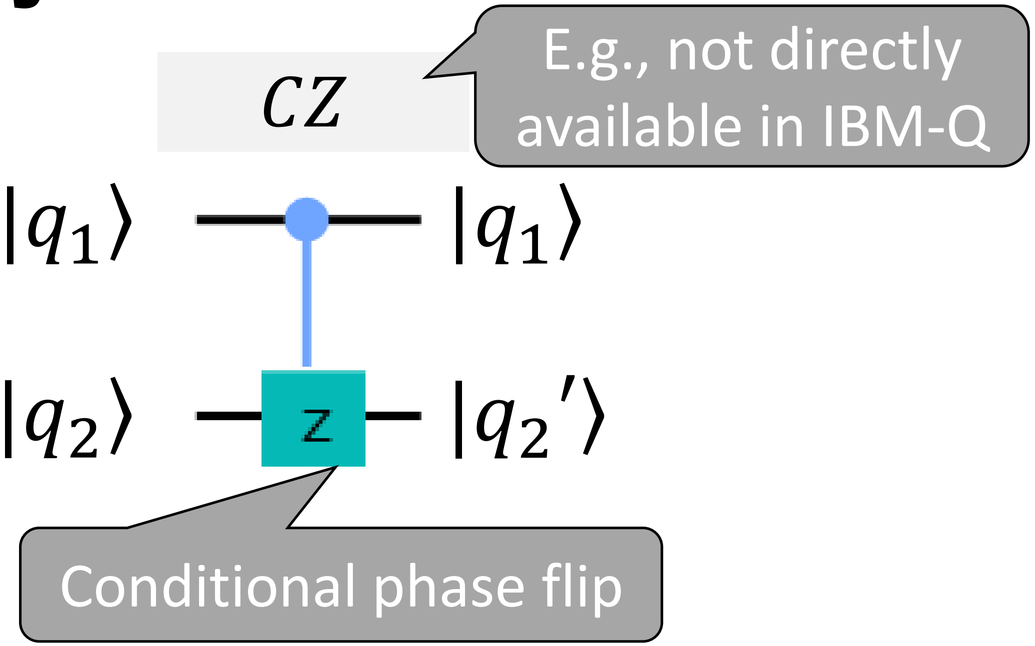
Implement any possible unitary function (i.e., matrix)

Circuit Identities

Not all important gates can be directly applied by hardware

Instead, we can derive certain gates using a combination of other gates supported by the hardware





Derive an identity to realize CZ gate using the CX gate!

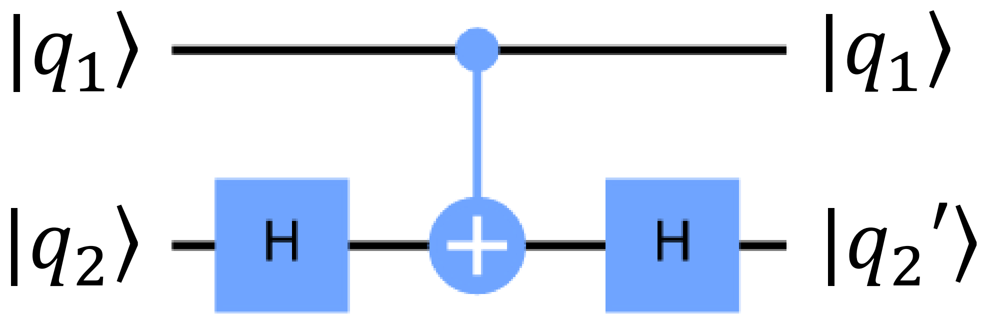
Circuit Identities

Use Hadamard gates to switch X,Z bases

$H|0\rangle = |+\rangle$ $\quad H|1\rangle = |-\rangle$ $\quad H|+\rangle = |0\rangle$ $\quad H|-\rangle = |1\rangle$

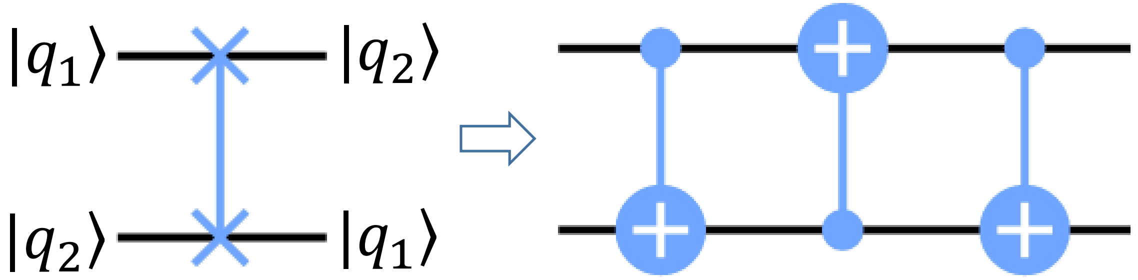
Using matrix multiplication, we can derive: $HXH = Z$ $\quad HZH = X$

CZ



$H - CNOT - H$

Another example $SWAP$



$CNOT - CNOT - CNOT$

Quantum Computing: Entanglement and Teleportation

Entanglement: Bell States

Entangled state is a state $|\psi\rangle$ that cannot be expressed as a tensor product $|q_1\rangle \otimes |q_2\rangle$

Bell States: four 2-qubit states that are maximally entangled

$$|\psi_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Measurement collapses state to $|00\rangle$ or $|11\rangle$

If we measure the first qubit, we automatically know the state of the second

$$|\psi_{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

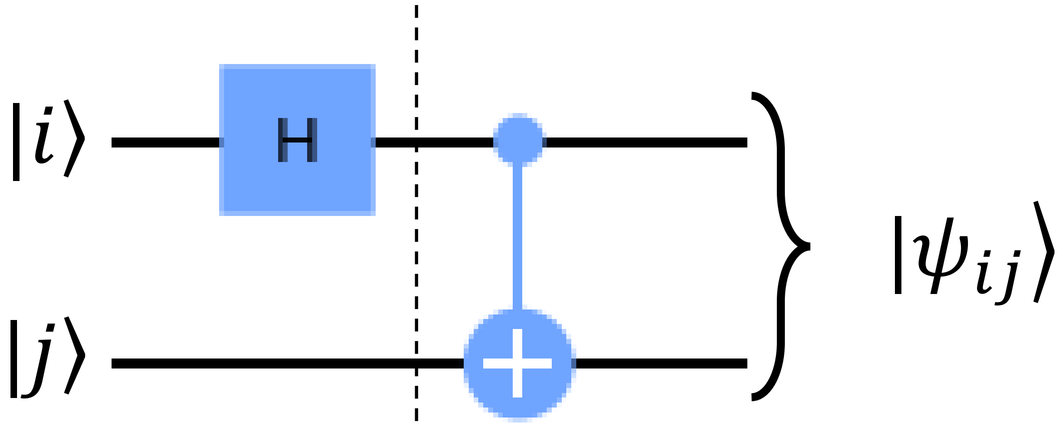
$$|\psi_{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

$$|\psi_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

High amount of correlation, regardless of distance: *useful in quantum computations*

Entanglement: Circuit to Generate Bell States

Input $\lvert ij\rangle$	**Output** $\lvert\psi_{ij}\rangle$
$\lvert 00\rangle$	$\frac{1}{\sqrt{2}}(\lvert 00\rangle + \lvert 11\rangle)$
$\lvert 01\rangle$	$\frac{1}{\sqrt{2}}(\lvert 01\rangle + \lvert 10\rangle)$
$\lvert 10\rangle$	$\frac{1}{\sqrt{2}}(\lvert 00\rangle - \lvert 11\rangle)$
$\lvert 11\rangle$	$\frac{1}{\sqrt{2}}(\lvert 01\rangle - \lvert 10\rangle)$



$\lvert 00\rangle$

$\lvert i\rangle = \lvert 0\rangle$
$\lvert j\rangle = \lvert 0\rangle$

$$\frac{1}{\sqrt{2}}(\lvert 0\rangle + \lvert 1\rangle)\lvert 0\rangle = \frac{1}{\sqrt{2}}(\lvert 00\rangle + \lvert 10\rangle)$$

$$= \frac{1}{\sqrt{2}}(\lvert 00\rangle + \lvert 11\rangle)$$

$\lvert 10\rangle$

$\lvert i\rangle = \lvert 1\rangle$
$\lvert j\rangle = \lvert 0\rangle$

$$\frac{1}{\sqrt{2}}(\lvert 0\rangle - \lvert 1\rangle)\lvert 0\rangle = \frac{1}{\sqrt{2}}(\lvert 00\rangle - \lvert 10\rangle)$$

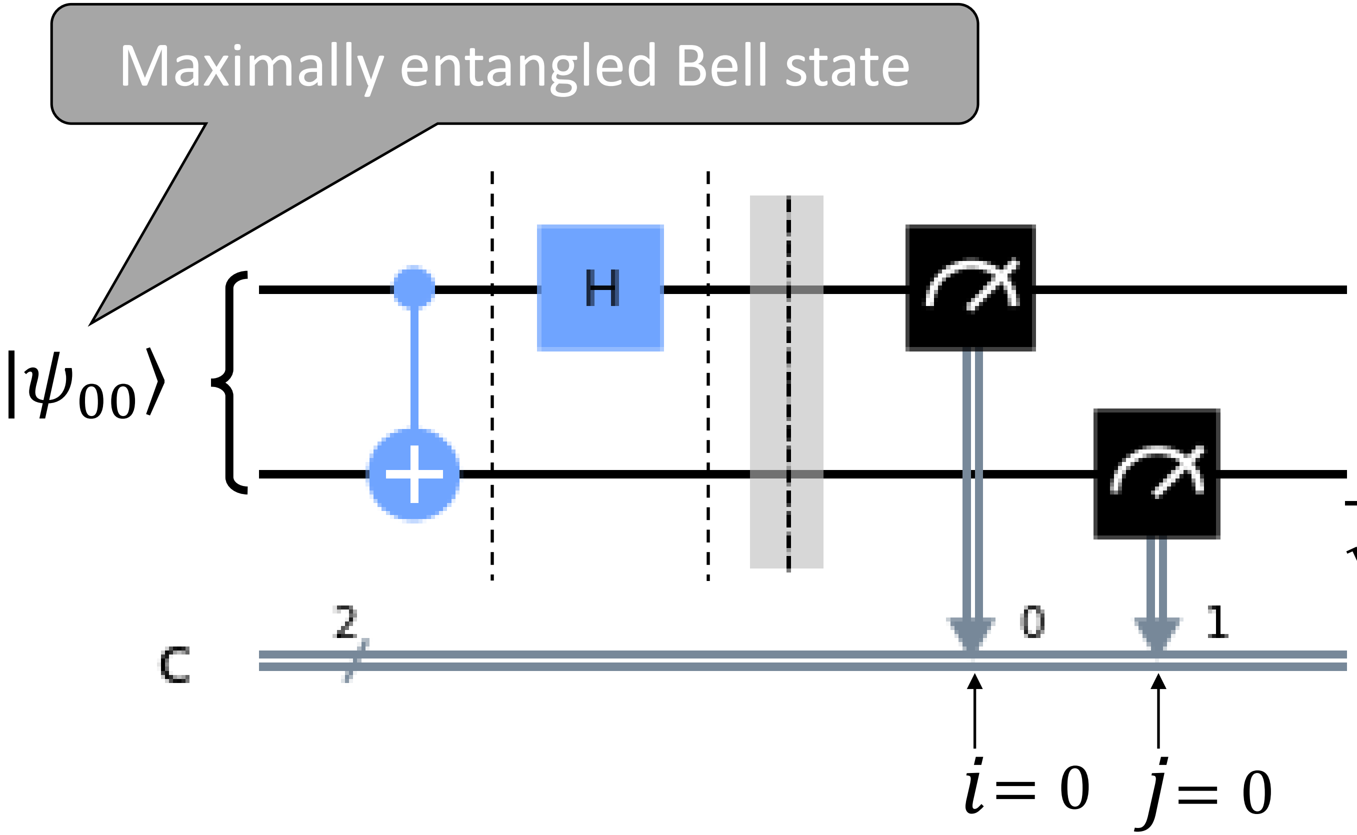
$$= \frac{1}{\sqrt{2}}(\lvert 00\rangle - \lvert 11\rangle)$$

Try deriving the other two Bell states as exercises!

Entanglement: Bell Measurement

The reverse problem: given $|\psi_{ij}\rangle$ find i and j

Solution: reverse the circuit



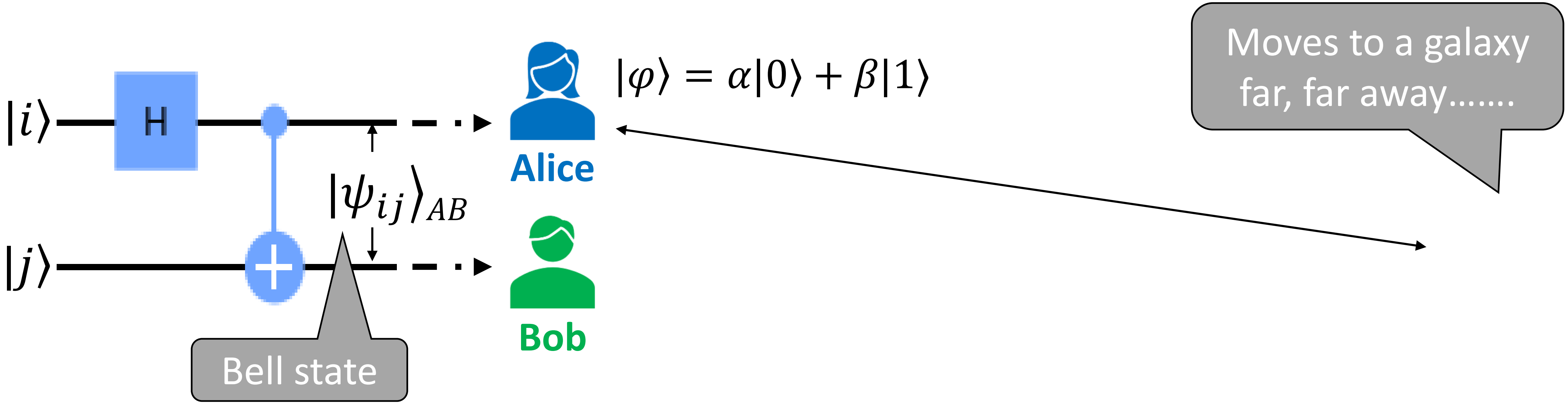
Input (Bell State) $\lvert\psi_{ij}\rangle$	Output ij
$\frac{1}{\sqrt{2}}(\lvert 00\rangle + \lvert 11\rangle)$	00
$\frac{1}{\sqrt{2}}(\lvert 01\rangle + \lvert 10\rangle)$	01
$\frac{1}{\sqrt{2}}(\lvert 00\rangle - \lvert 11\rangle)$	10
$\frac{1}{\sqrt{2}}(\lvert 01\rangle - \lvert 10\rangle)$	11

Quantum Teleportation

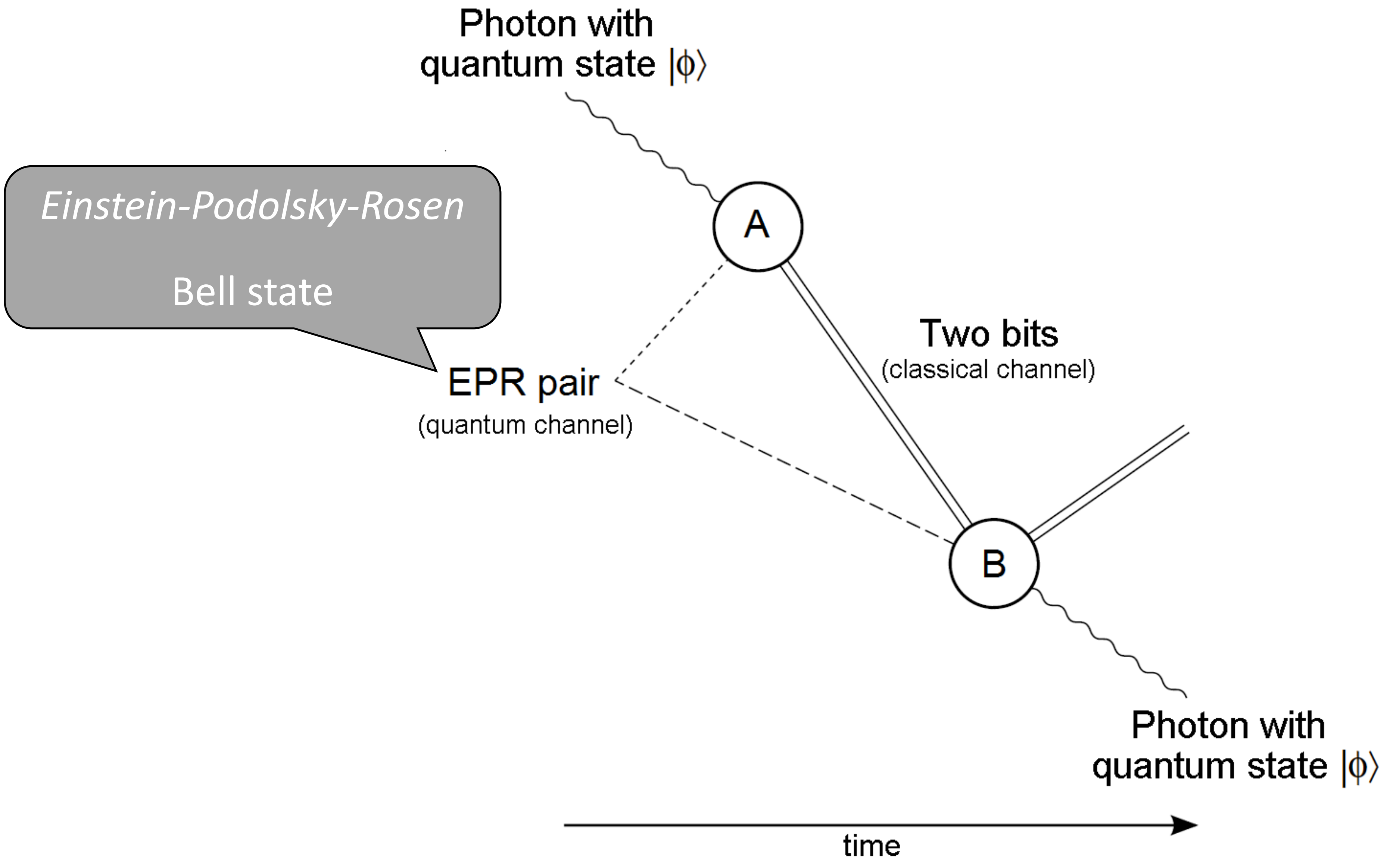
No, this is not Sci-Fi!

It's about communicating information over arbitrarily long distances using the power of quantum entanglement

Consider the following scenario

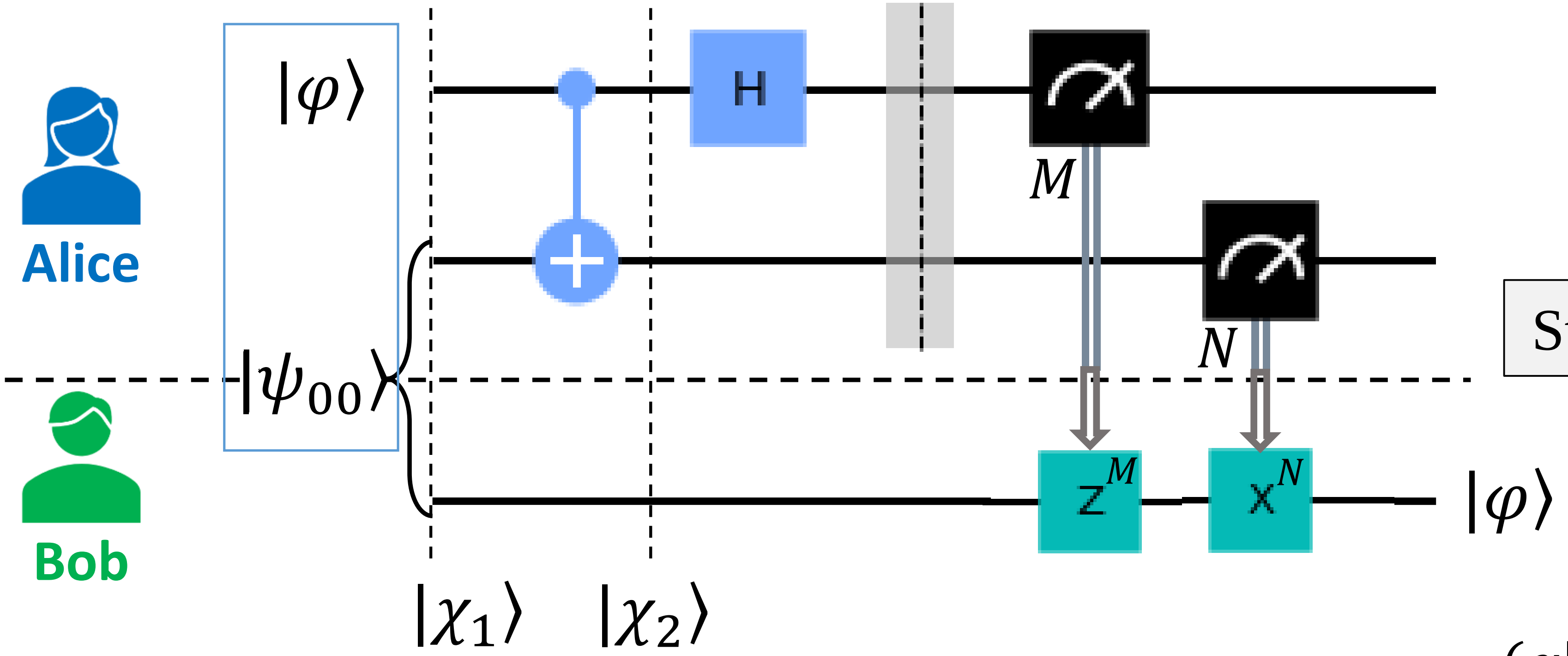


Quantum Teleportation: Protocol Overview



Teleportation Protocol

Solution: use Bell Measurement



$$|\psi_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Start state $|\chi_1\rangle = |\varphi\rangle \otimes |\psi_{00}\rangle$

$$= (\alpha|0\rangle + \beta|1\rangle)\,|\psi_{00}\rangle$$

$$\Downarrow$$

$$= (\alpha|0\rangle + \beta|1\rangle)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

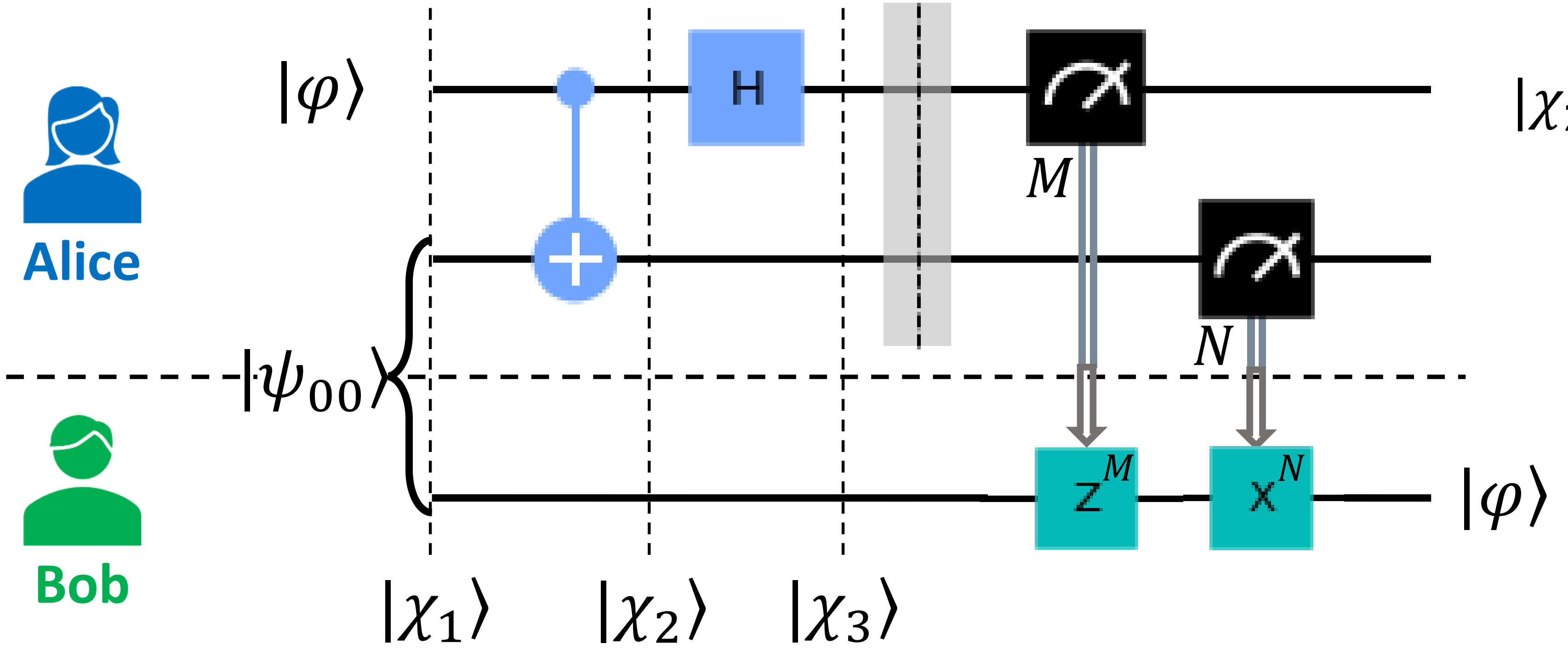
$$= \frac{1}{\sqrt{2}}(\,\alpha|0\rangle\,(|00\rangle + |11\rangle) + \beta|1\rangle\,(|00\rangle + |11\rangle)\,)$$

CNOT $\Leftarrow$

$$|\chi_2\rangle = \frac{1}{\sqrt{2}}(\,\alpha|0\rangle\,(|00\rangle + |11\rangle) + \beta|1\rangle\,(|10\rangle + |01\rangle)\,)$$

Teleportation Protocol

Solution: use Bell Measurement



$$|\chi_2\rangle = \frac{1}{\sqrt{2}}\big(\,\alpha|0\rangle\,(|00\rangle + |11\rangle) + \beta|1\rangle\,(|10\rangle + |01\rangle)\,\big)$$

$\Downarrow$

Hadamard

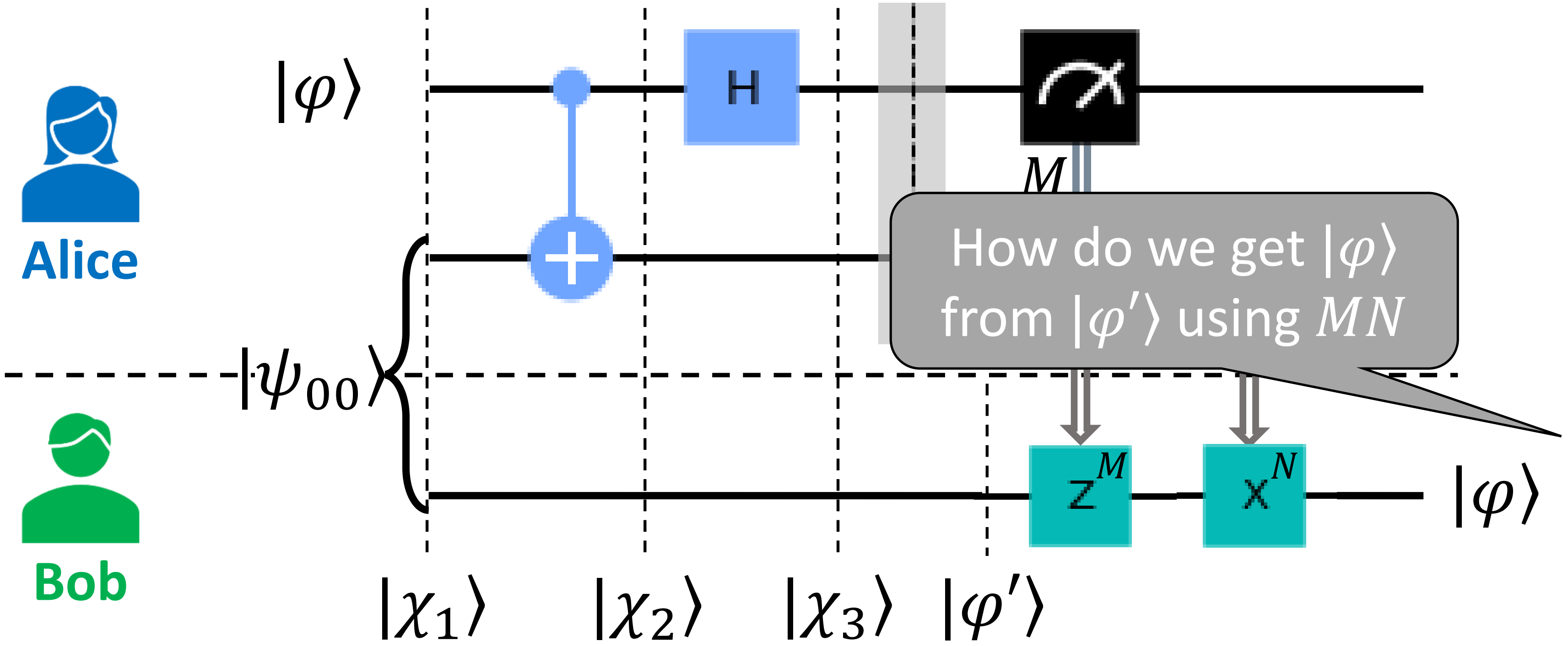
$$|\chi_3\rangle = \frac{1}{\sqrt{2}}\Big(\,\alpha\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)(|00\rangle + |11\rangle) + \beta\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)(|10\rangle + |01\rangle)\,\Big)$$

$$= \frac{1}{2}\big(\,\alpha(|000\rangle + |011\rangle + |100\rangle + |111\rangle) + \beta(|010\rangle + |001\rangle - |110\rangle - |101\rangle)\,\big)$$

$$= \frac{1}{2}\big(\,|00\rangle\,(\alpha|0\rangle + \beta|1\rangle) + |01\rangle\,(\alpha|1\rangle + \beta|0\rangle) + |10\rangle\,(\alpha|0\rangle - \beta|1\rangle) + |11\rangle\,(\alpha|1\rangle - \beta|0\rangle)\,\big)$$

Teleportation Protocol

Solution: use Bell Measurement



Alice's Measurement MN	Bob's State $\|\varphi'\rangle$
00	$\alpha\|0\rangle + \beta\|1\rangle$
01	$\alpha\|1\rangle + \beta\|0\rangle$
10	$\alpha\|0\rangle - \beta\|1\rangle$
11	$\alpha\|1\rangle - \beta\|0\rangle$

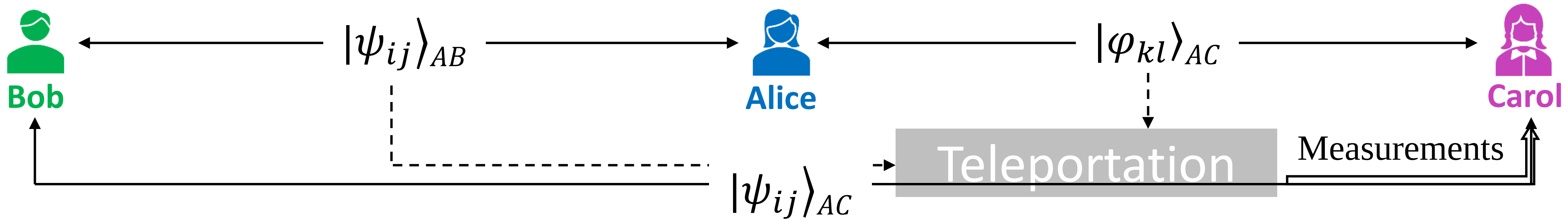
$\|\chi_3\rangle = \frac{1}{2}(\|00\rangle(\alpha\|0\rangle + \beta\|1\rangle)$ $MN = 00 \Longrightarrow$ Do nothing Z^0 X^0	$+ \beta\|0\rangle$ $MN = 10 \Longrightarrow$ Phase flip (Z gate) $\|11\rangle(\alpha\|0\rangle - \beta\|1\rangle)$ Z^1 X^0 $)$
$MN = 01 \Longrightarrow$ Bit flip (X gate) Z^0 X^1	$MN = 11 \Longrightarrow$ Bit and phase flip (X and Z gates) Z^1 X^1

More About Teleportation

This it not just theory. It has been experimentally verified.

Teleportation protocol demonstrated in China between ground and satellite (~1400 km)

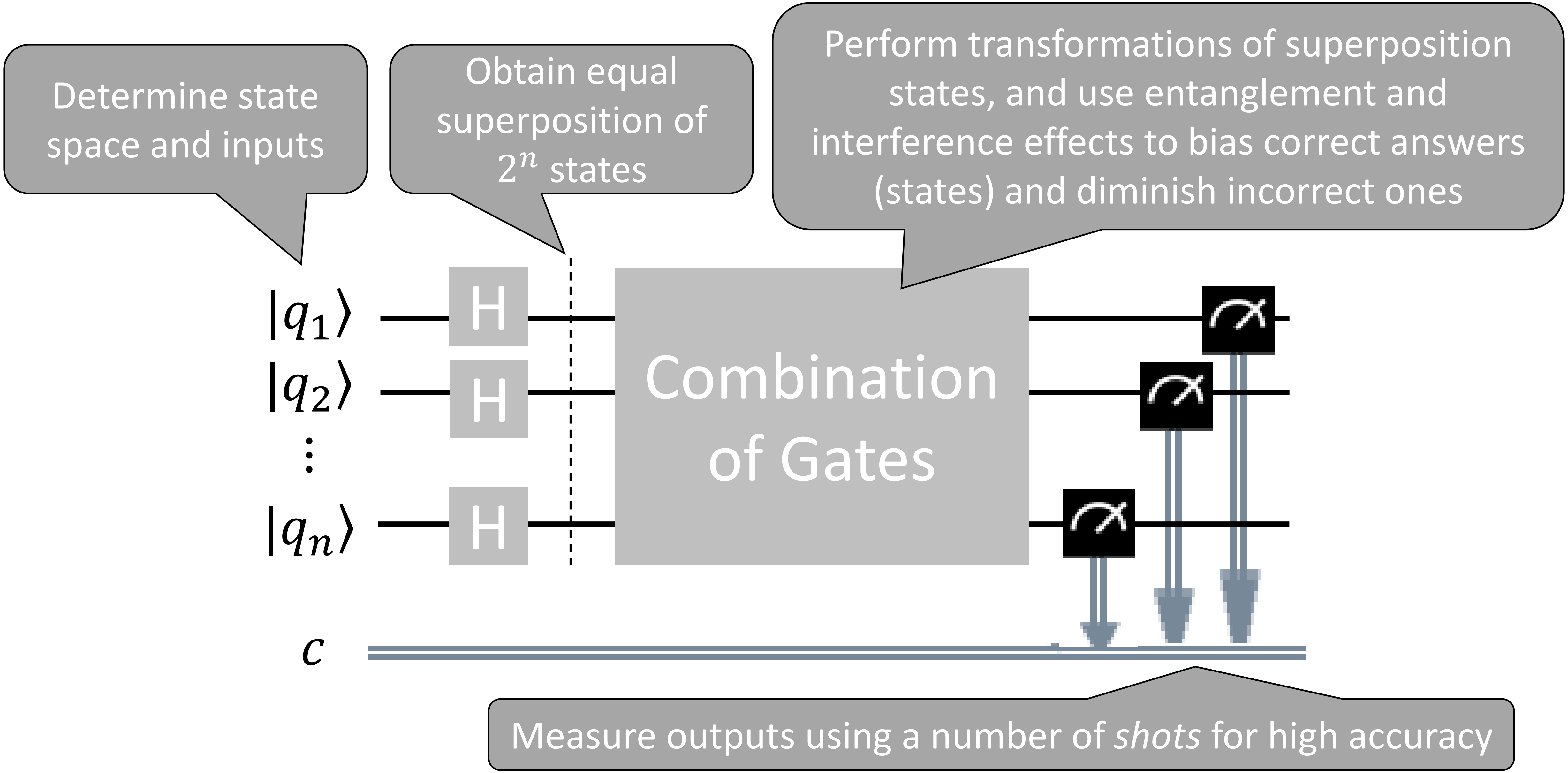
Entanglement swapping demonstrated between two of the Canary Islands (~143 km)



So does this mean we can communicate information infinitely fast over long distances?

No, because the Bell measurement results can only be communicated over classical communication channels, which cannot exceed the speed of light.

The Intuition Behind Quantum Algorithm Construction



Knowledge is like money: to be of value it must circulate, and in circulating it can increase in quantity and, hopefully, in value.

– Louis L'Amour

Thanks !