Geometric representation of the action of the Apollonian group on Descartes quadruples

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1 Introduction

Our main goal, at the start of our collaboration, was to find a more geometric representation for the action of the Apollonian group on the set of all Descartes quadruples in a particular packing. We have accomplished that goal wholeheartedly, as will be shown in section 5. Since Descartes quadruples are four dimensional in nature, we begin by moving the vectors to a three dimensional setting – making sure that no information has been lost in the process. To do so, we take advantage of Descartes' equation. The whole process, including motivation behind the process itself, is outlined in sections 2 and 3. Once the quadruples have been moved to three space, we represent an isomorphic copy of the Apollonian group within our new framework. This is explained in section 4. Finally, using Mathematica, we look at the orbit of a Descartes quadruple under the action of the Apollonian group in this new setting. We end with further questions and possible approaches that can be taken in this study.

2 Scaling down quadruples to norm $\frac{1}{\sqrt{2}}$

The set of all Descartes quadruples is defined as follows:

$$\mathcal{D} = \left\{(a,b,c,d) \in \mathbb{R}^4 - \{0\} : 2(a^2+b^2+c^2+d^2) = (a+b+c+d)^2\right\}$$

We are going to consider a subset of these quadruples called *primitive* Descartes quadruples:

$$\mathcal{D}_P = \{(a,b,c,d) \in \mathcal{D} : a,b,c,d \in \mathbb{Z},\ \gcd(a,b,c,d) = 1,\ a+b+c+d \neq 0\}$$

These quadruples live in $\mathbb{R}^4$. To accomplish our geometric goals, we will need to move every primitive Descartes quadruple into $\mathbb{R}^3$, while making sure that no information is lost. The process, including motivation, is outlined below.

2.1 Scaling

We first show that every $(a,b,c,d) \in \mathcal{D}$ is such that $|d| = \frac{a+b+c+d}{\sqrt{2}}$, where $|,|$ is the standard Euclidean norm. This is done through Descartes' equation:

$$\begin{aligned}
(a+b+c+d)^2 &= 2(a^2+b^2+c^2+d^2)\\
(a+b+c+d) &= \sqrt{2}\sqrt{a^2+b^2+c^2+d^2}\\
\frac{a+b+c+d}{\sqrt{2}} &= \sqrt{a^2+b^2+c^2+d^2}\\
\frac{a+b+c+d}{\sqrt{2}} &= |d|,
\end{aligned}$$

as desired.
It follows that scaling every $d \in \mathcal{D}$ to norm 1 yields

$$\frac{d}{|d|} = \frac{1}{|d|}d = \frac{\sqrt{2}}{a+b+c+d}(a,b,c,d) = (\frac{\sqrt{2}a}{a+b+c+d}, \frac{\sqrt{2}b}{a+b+c+d}, \frac{\sqrt{2}c}{a+b+c+d}, \frac{\sqrt{2}d}{a+b+c+d})$$

A simple inspection shows that $\frac{d}{|d|} \in \mathcal{D}$, and that any non-zero scaled Descartes quadruple remains a Descartes quadruple. For purposes of equational comfort, we would like to get rid of that factor of $\sqrt{2}$, and so instead, we consider $\frac{d}{\sqrt{2}|d|}$. Since we will only be interested in primitive Descartes

quadruples, we obtain a map from the set of all primitive Descartes quadruples to the set of all Descartes quadruples of norm $\frac{1}{\sqrt{2}}$.
Let ${\mathbb{Z}_{\mathcal{D}}}^4, {\mathbb{Q}_{\mathcal{D}_{\frac{1}{\sqrt{2}}}}}^4$, be the set of all primitive Descartes quadruples, and the set of all Descartes quadruples of norm $\frac{1}{\sqrt{2}}$, respectively. Our map is defined by

$$f : {\mathbb{Z}_{\mathcal{D}}}^4 \to {\mathbb{Q}_{\mathcal{D}_{\frac{1}{\sqrt{2}}}}}^4$$

$$(a, b, c, d) \mapsto (\frac{a}{a+b+c+d}, \frac{b}{a+b+c+d}, \frac{c}{a+b+c+d}, \frac{d}{a+b+c+d})$$

We show that f is bijective by offering a two-sided inverse f^{-1} defined as:

$$f^{-1} : {\mathbb{Q}_{\mathcal{D}_{\frac{1}{\sqrt{2}}}}}^4 \to {\mathbb{Z}_{\mathcal{D}}}^4$$

$$(a, b, c, d) = (\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}, \frac{p_4}{q_4}) \mapsto \frac{\text{lcm}(q_1, ..., q_4)}{\text{gcd}(p_1, ..., p_4)}(a, b, c, d)$$

Suppose that $(a, b, c, d) \in {\mathbb{Z}_{\mathcal{D}}}^4$.

$$\begin{aligned}
f^{-1}(f((a, b, c, d))) &= f^{-1}(\frac{a}{a+b+c+d}, \frac{b}{a+b+c+d}, \frac{c}{a+b+c+d}, \frac{d}{a+b+c+d}) \\
&= (a+b+c+d)(\frac{a}{a+b+c+d}, \frac{b}{a+b+c+d}, \frac{c}{a+b+c+d}, \frac{d}{a+b+c+d}) \\
&= (a, b, c, d) \to f^{-1}f = id_{{\mathbb{Z}_{\mathcal{D}}}^4}
\end{aligned}$$

Suppose that $(a, b, c, d) \in {\mathbb{Q}_{\mathcal{D}_{\frac{1}{\sqrt{2}}}}}^4$.
Let $\frac{l}{g} = \frac{\text{lcm}(q_1, ..., q_4)}{\text{gcd}(p_1, ..., p_4)}$.

$$\begin{aligned}
f(f^{-1}((a, b, c, d))) &= f(f^{-1}((\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}, \frac{p_4}{q_4}))) \\
&= f((\frac{l}{g}\frac{p_1}{q_1}, \frac{l}{g}\frac{p_2}{q_2}, \frac{l}{g}\frac{p_3}{q_3}, \frac{l}{g}\frac{p_4}{q_4})) \\
&= (\frac{\frac{l}{g}\frac{p_1}{q_1}}{\frac{l}{g}\sum\frac{p_i}{q_i}}, \frac{\frac{l}{g}\frac{p_2}{q_2}}{\frac{l}{g}\sum\frac{p_i}{q_i}}, \frac{\frac{l}{g}\frac{p_3}{q_3}}{\frac{l}{g}\sum\frac{p_i}{q_i}}, \frac{\frac{l}{g}\frac{p_4}{q_4}}{\frac{l}{g}\sum\frac{p_i}{q_i}}) \\
&= \frac{1}{\frac{p_1}{q_1} + \frac{p_2}{q_2} + \frac{p_3}{q_3} + \frac{p_4}{q_4}}(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}, \frac{p_4}{q_4}) \\
&= \frac{1}{1}(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}, \frac{p_4}{q_4}) \\
&= (a, b, c, d) \to ff^{-1} = id_{{\mathbb{Q}_{\mathcal{D}_{\frac{1}{\sqrt{2}}}}}^4},
\end{aligned}$$

where the second-to-last equality follows because $\frac{p_1}{q_1} + \frac{p_2}{q_2} + \frac{p_3}{q_3} + \frac{p_4}{q_4} = a+b+c+d = \sqrt{2}\sqrt{a^2+b^2+c^2+d^2} = \sqrt{2}\frac{1}{\sqrt{2}} = 1$, via Descartes' equation.
Therefore f is a bijection, as desired.

2.2 Why bother scaling integral quadruples?

What we have accomplished above is a one-to-one correspondence between the set of all primitive Descartes quadruples and the set of all Descartes quadruples of norm $\frac{1}{\sqrt{2}}$. But why would we want to scale our quadruples to such a particular norm? And why only scale *primitive* Descartes quadruples. First, we find it necessary to describe why *integral*, let alone primitive, Descartes quadruples are under consideration.

All Descartes quadruples in $\mathbb{R}^4$ are equivalent up to Möbius transformation. And it seems that, a priori, integral packings seem to contain a lot of number theoretic information. Note, for example, how the Apollonian group has integer coordinates as well. It makes sense, then, to study packings with an integrality condition. The fact that there is no harm in doing so is also displayed via our initial observation that all quaruples are equivalent up to Möbius transformation.

Every non-primitive Descartes quadruple is equivalent to a primitive after scaling by the greatest common divisor. Thus, in order to simplify the framework within which we work, we might as well work with primitive quadruples.

Lastly, we have chosen norm $\frac{1}{\sqrt{2}}$ for the reasons mentioned in section 2.1 – for equational convenience. The main point that should be driven home is that scaling does not change anything essential. Our one-to-one correspondence is evidence of this. Scaling can thus only help be to our advantage.

3 Representing scaled quadruples in $\mathbb{R}^3$

Now that we have scaled our quadruples, we want to pursue our ultimate goal of understanding, geometrically, the action of the Apollonian group on Descartes quadruples. To do so, we must leave the four dimensional realm that the quadruples live in. And we must leave that realm without any loss of information. The main idea is to get rid of one fixed coordinate from every Descartes quadruple, in a way such that the coordinate can be retrieved at any time from the new form. In fact, due to the relationship induced from Descartes' equation, we can simply, without loss of generality, exclude the first coordinate.

3.1 Getting rid of a Descartes coordinate

Let ${\mathbb{Q}_{\mathcal{D}_{\frac{1}{\sqrt{2}}}}}^3$ be the set of Descartes quadruples of norm $\frac{1}{\sqrt{2}}$ projected down to three dimensions. Our map $\tilde{f}$ is defined as follows:

$$\tilde{f} : {\mathbb{Q}_{\mathcal{D}_{\frac{1}{\sqrt{2}}}}}^4 \to {\mathbb{Q}_{\mathcal{D}_{\frac{1}{\sqrt{2}}}}}^3$$

$$(w, x, y, z) \mapsto (x, y, z)$$

Clearly $\tilde{f}$ is a bijection, and given any $(x, y, z) \in {\mathbb{Q}_{\mathcal{D}_{\frac{1}{\sqrt{2}}}}}^3$, one can easily find the corresponding $(w, x, y, z) \in {\mathbb{Q}_{\mathcal{D}_{\frac{1}{\sqrt{2}}}}}^4$ via Descartes' equation.

3.2 Descartes ellipsoid

One can see that the map f defined in section 2 gives rise to two new equations via Descartes' equation for quadruples $(w, x, y, z) \in \mathbb{Q}_{\mathcal{D}_{\frac{1}{\sqrt{2}}}}{}^4$, as follows:

$$\begin{aligned} w^2 + x^2 + y^2 + z^2 &= \frac{1}{2} && (1) \\ w + x + y + z &= 1 && (2) \end{aligned}$$

The first equation represents a 3-sphere, whilst the second represents a hyperplane. The intersection of these two equations gives us a sphere in 3-space, or more specifically an ellipsoid. Every point on this ellipsoid represents, via our maps, a primitive Descartes quadruple. And since our maps are bijective, every primitive Descartes quadruple represents a point on the ellipsoid. Thus, we get a very nice geometric representation of the set of all primitive Descartes quadruples. Since our map $\tilde{f}$ gets rid of the first coordinate, we can solve for w above and get an equation representing the ellipsoid via substitution into the first equation. The equation is as follows:

$$x^2 + y^2 + z^2 - (x + y + z) + xy + yz + zy = \frac{-1}{4} \qquad (3)$$

4 Apollonian group in new framework

Now that we have a more tractable representation of all the primitive Descartes quadruples, we would like to study the action of the Apollonian group within this new framework. We proceed by noting that if (x_1, x_2, x_3, x_4) is a Descartes quadruple corresponding to circles C_1, C_2, C_3, C_4, respectively, then the curvature, $d^{'}$ of the circle $C_4{}^{'}$ that is mutually tangent to C_1, C_2, C_3 yet different from C_4 is given by the equation

$$d^{'} = 2a + 2b + 2c - d \qquad (4)$$

The main idea is that we use formula (4) to move between Descartes quadruples in a particular packing with respect to the ellipsoid.

4.1 Construction

Let $\mathcal{A}$ be the Apollonian group. We construct four maps $\tilde{f}_1, \tilde{f}_2, \tilde{f}_3, \tilde{f}_4$ such that
$f_i : \mathbb{Q}_{\mathcal{D}_{\frac{1}{\sqrt{2}}}}{}^3 \to \mathbb{Q}_{\mathcal{D}_{\frac{1}{\sqrt{2}}}}{}^3$, for $i = 1, 2, 3, 4$, and $< \tilde{f}_1, \tilde{f}_2, \tilde{f}_3, \tilde{f}_4 > \cong A$ Our maps are defined as follows:

$$\begin{aligned} \tilde{f}_1(x, y, z) &= \left(\frac{2 - 3x}{3 - 4x}, \frac{y}{3 - 4x}, \frac{z}{3 - 4x}\right) \\ \tilde{f}_2(x, y, z) &= \left(\frac{x}{3 - 4y}, \frac{2 - 3y}{3 - 4y}, \frac{z}{3 - 4y}\right) \\ \tilde{f}_3(x, y, z) &= \left(\frac{x}{3 - 4z}, \frac{y}{3 - 4z}, \frac{2 - 3z}{3 - 4z}\right) \\ \tilde{f}_4(x, y, z) &= \left(\frac{x}{4(x + y + z) - 1}, \frac{y}{4(x + y + z) - 1}, \frac{z}{4(x + y + z) - 1}\right) \end{aligned}$$

We do not go in much detail, but rather explain how the maps arise. We construct the maps in such a way so as to have, say, $\tilde{f}_1$ correspond to the map that takes a $(w, x, y, z) \in \mathbb{Q}_{\mathcal{D}_{\frac{1}{\sqrt{2}}}}{}^4$ and sends it to $(w^{'}, x, y, z)$ – scaling down afterwards to make sure it remains of norm $\frac{1}{\sqrt{2}}$; where $w^{'}$ is

described in the beginning of section 4. These functions correspond to the canonical generators of the Apollonian group as a subgroup of $GL(4, \mathbb{R})$.
More specifically, we derive $\tilde{f}_1$ as follows: Take any $(w, x, y, z) \in \mathbb{Q}_{\mathcal{D}_{\frac{1}{\sqrt{2}}}}{}^4$. First note that by equation (4) we have $w^{'} = 2x + 2y + 2z - w = 2(x + y + z) - w = 2(w + x + y + z) - 3w = 2 - 3w$. Since we get rid of the first coordinate which is w, we would like to solve for w. By equation (2) we get that $w = 1 - (x + y + z)$. This gives $w^{'} = 2 - 3(1 - (x + y + z)) = 3(x + y + z) - 1$. Now map (w, x, y, z) to $(w^{'}, x, y, z) = (3(x + y + z) - 1, x, y, z)$. To keep the vector in $\mathbb{Q}_{\mathcal{D}_{\frac{1}{\sqrt{2}}}}{}^4$, we apply f to it and get that $f((w^{'}, x, y, z)) = (\frac{3(x+y+z)-1}{4(x+y+z)-1}, \frac{x}{4(x+y+z)-1}, \frac{y}{4(x+y+z)-1}, \frac{z}{4(x+y+z)-1})$, which is our desired Descartes quadruple. Now applying $\tilde{f}$ to this in order for desired quadruple to live in $\mathbb{R}^3$, we get $\tilde{f}(f(w^{'}, x, y, z)) = (\frac{x}{4(x+y+z)-1}, \frac{y}{4(x+y+z)-1}, \frac{z}{4(x+y+z)-1})$. The other functions are derived in a similar manner.

4.2 Remark

We initially wanted to understand how the Apollonian group acts on the ellipsoid as a whole. We thought that the Apollonian group acts on the ellipsoid by isometries. We attempted to show this by showing that under each function geodesics on the ellipsoid are sent to geodesics. It turns out that the Apollonian group *does not* act on the ellipsoid by Apollonian group.

5 Orbit of a Descartes quadruples under Apollonian group

Now that we had set up the geometric framework necessary to study the action of the Apollonian group on the Descartes quadruples, we acquired the appropriate Mathematica code necessary to plot the ellipsoid, and also make visible the orbit of a Descartes quadruple on the ellipsoid. For the sake of completeness, we provide the code in this section, and offer some nice pictures showing the orbit of a Descartes quadruple on the ellipsoid – fulfilling our main goal.

5.1 *Mathematica* code

After dabbling with some settings, and even consulting some experienced Mathematica users for help at times, we were able to assimilate the following file:

5.2 Some packings

6 Further questions