SCHOOL OF MECHANICAL AND AEROSPACE ENGINEERING

MAE 6263 Course Project II

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1 INTRODUCTION

The purpose of this course project is to document the development of a numerical routine and its solution for the 2-D viscous Burgers equation for a plane channel flow. Through the course of this document, iterative numerical schemes shall be devised and the results of numerical experiments shall be analysed with the view of understanding the effect of the viscosity and grid discretizations through the global and cell Peclet number.

1.1 PROBLEM STATEMENT

The 2-D viscous Burgers equation can be written as:

$$\frac{\partial \bar{u}}{\partial t} + \bar{u}.\Delta \bar{u} = \nu \Delta^2 \bar{u}, \tag{1.1}$$

for a rectangular domain (Ω) of unit width and length two units. The initial conditions for this problem -

$$u(x,y) = 0 \text{ for } 0 < x < 1, \;\; 0 < y < 1, \tag{1.2}$$

and constant boundary conditions given by

$$\begin{aligned} u(x,y) &= 1 \;\text{ for }\; y = 1; \\ u(x,y) &= 0 \;\text{ for }\; y = 0; \\ u(x,y) &= u(x-2,y) \;\text{ for }\; x = 2; \\ u(x,y) &= u(x+2,y) \;\text{ for }\; x = 2; \end{aligned} \tag{1.3}$$

2 CODE DEVELOPMENT

2.1 DEVELOPMENT OF EXPLICIT ITERATIVE SCHEME

2.1.1 SCHEME DEVELOPMENT

An explicit discretization of the 2-D Burgers equation can be devised using a forward in time discretization for the unsteady derivative in Eq. (1.1). The second order spatial derivative is approximated in the usual central difference sense and the first order advective derivative is approximated using both the second order central and the first order upwind discretizations to give us equations for the time evolution of velocity in both the x and y directions.

Let us first assume an upwind discretization for the advective term. The following explicit scheme can be devised for the x component of the fluid velocity u-

$$\begin{aligned}&\frac{\bar{u}^{n+1}_{i,j}-\bar{u}^{n}_{i,j}}{\Delta t}+\bar{u}^{n}_{i,j}\frac{\bar{u}^{n}_{i,j}-\bar{u}^{n}_{i-1,j}}{\Delta x}+\bar{v}^{n}_{i,j}\frac{\bar{u}^{n}_{i,j}-\bar{u}^{n}_{i,j-1}}{\Delta y}=\\&\nu\left(\frac{\bar{u}^{n}_{i+1,j}+\bar{u}^{n}_{i-1,j}-2\bar{u}^{n}_{i,j}}{\Delta x^2}+\frac{\bar{u}^{n}_{i,j+1}+\bar{u}^{n}_{i,j-1}-2\bar{u}^{n}_{i,j}}{\Delta y^2}\right);\end{aligned} \tag{2.1}$$

which may be written as the following algebraic system after some rearrangement-

$$\bar{u}^{n+1}_{i,j}=\bar{u}^{n}_{i,j}+\Delta t\,(T_3+T_4-T_1-T_2) \tag{2.2}$$

where -

$$\begin{aligned}T_1&=\bar{u}^{n}_{i,j}\frac{\bar{u}^{n}_{i,j}-\bar{u}^{n}_{i-1,j}}{\Delta x}\\T_2&=\bar{v}^{n}_{i,j}\frac{\bar{u}^{n}_{i,j}-\bar{u}^{n}_{i,j-1}}{\Delta y}\\T_3&=\nu\frac{\bar{u}^{n}_{i+1,j}+\bar{u}^{n}_{i-1,j}-2\bar{u}^{n}_{i,j}}{\Delta x^2}\\T_4&=\nu\frac{\bar{u}^{n}_{i,j+1}+\bar{u}^{n}_{i,j-1}-2\bar{u}^{n}_{i,j}}{\Delta y^2}\end{aligned} \tag{2.3}$$

Similarly, the y component of the fluid velocity v evolves explicitly according to the following evolution equation -

$$\begin{aligned}&\frac{\bar{v}^{n+1}_{i,j}-\bar{v}^{n}_{i,j}}{\Delta t}+\bar{u}^{n}_{i,j}\frac{\bar{v}^{n}_{i,j}-\bar{v}^{n}_{i-1,j}}{\Delta x}+\bar{v}^{n}_{i,j}\frac{\bar{v}^{n}_{i,j}-\bar{v}^{n}_{i,j-1}}{\Delta y}=\\&\nu\left(\frac{\bar{v}^{n}_{i+1,j}+\bar{v}^{n}_{i-1,j}-2\bar{v}^{n}_{i,j}}{\Delta x^2}+\frac{\bar{v}^{n}_{i,j+1}+\bar{v}^{n}_{i,j-1}-2\bar{v}^{n}_{i,j}}{\Delta y^2}\right)\end{aligned} \tag{2.4}$$

which may be written as -

$$\bar{v}^{n+1}_{i,j}=\bar{v}^{n}_{i,j}+\Delta t\,(T_3+T_4-T_1-T_2) \tag{2.5}$$

where -

$$
\begin{aligned}
T_1 &= \bar{u}^n_{i,j}\frac{\bar{v}^n_{i,j}-\bar{v}^n_{i-1,j}}{\Delta x}\\
T_2 &= \bar{v}^n_{i,j}\frac{\bar{v}^n_{i,j}-\bar{v}^n_{i,j-1}}{\Delta y}\\
T_3 &= \nu\frac{\bar{v}^n_{i+1,j}+\bar{v}^n_{i-1,j}-2\bar{v}^n_{i,j}}{\Delta x^2}\\
T_4 &= \nu\frac{\bar{v}^n_{i,j+1}+\bar{v}^n_{i,j-1}-2\bar{v}^n_{i,j}}{\Delta y^2}
\end{aligned}
\tag{2.6}
$$

This concludes the iterative expressions for the explicit scheme for a first order upwind discretization of the advective derivative. A similar set of expressions can also be derived for a second order central difference scheme for the advective derivative. This may be represented as follows -

$$
\begin{aligned}
&\frac{\bar{u}^{n+1}_{i,j}-\bar{u}^n_{i,j}}{\Delta t}+\bar{u}^n_{i,j}\frac{\bar{u}^n_{i+1,j}-\bar{u}^n_{i-1,j}}{2\Delta x}+\bar{v}^n_{i,j}\frac{\bar{u}^n_{i,j+1}-\bar{u}^n_{i,j-1}}{2\Delta y}=\\
&\nu\left(\frac{\bar{u}^n_{i+1,j}+\bar{u}^n_{i-1,j}-2\bar{u}^n_{i,j}}{\Delta x^2}+\frac{\bar{u}^n_{i,j+1}+\bar{u}^n_{i,j-1}-2\bar{u}^n_{i,j}}{\Delta y^2}\right);
\end{aligned}
\tag{2.7}
$$

which can be set up for recursive computation of the solution field as follows -

$$
\bar{u}^{n+1}_{i,j}=\bar{u}^n_{i,j}+\Delta t\,(T_3+T_4-T_1-T_2)
\tag{2.8}
$$

and where -

$$
\begin{aligned}
T_1 &= \bar{u}^n_{i,j}\frac{\bar{u}^n_{i+1,j}-\bar{u}^n_{i-1,j}}{2\Delta x}\\
T_2 &= \bar{v}^n_{i,j}\frac{\bar{u}^n_{i,j+1}-\bar{u}^n_{i,j-1}}{2\Delta y}\\
T_3 &= \nu\frac{\bar{u}^n_{i+1,j}+\bar{u}^n_{i-1,j}-2\bar{u}^n_{i,j}}{\Delta x^2}\\
T_4 &= \nu\frac{\bar{u}^n_{i,j+1}+\bar{u}^n_{i,j-1}-2\bar{u}^n_{i,j}}{\Delta y^2}
\end{aligned}
\tag{2.9}
$$

Similarly, the y component of the fluid velocity v evolves explicitly according to the following evolution equation -

$$
\begin{aligned}
&\frac{\bar{v}^{n+1}_{i,j}-\bar{v}^n_{i,j}}{\Delta t}+\bar{u}^n_{i,j}\frac{\bar{v}^n_{i+1,j}-\bar{v}^n_{i-1,j}}{2\Delta x}+\bar{v}^n_{i,j}\frac{\bar{v}^n_{i,j+1}-\bar{v}^n_{i,j-1}}{2\Delta y}=\\
&\nu\left(\frac{\bar{v}^n_{i+1,j}+\bar{v}^n_{i-1,j}-2\bar{v}^n_{i,j}}{\Delta x^2}+\frac{\bar{v}^n_{i,j+1}+\bar{v}^n_{i,j-1}-2\bar{v}^n_{i,j}}{\Delta y^2}\right)
\end{aligned}
\tag{2.10}
$$

which may be written as -

$$\bar{v}^{n+1}_{i,j} = \bar{v}^{n}_{i,j} + \Delta t\,(T_3 + T_4 - T_1 - T_2) \tag{2.11}$$

where -

$$\begin{aligned}
T_1 &= \bar{u}^n_{i,j}\frac{\bar{v}^n_{i,j} - \bar{v}^n_{i-1,j}}{2\Delta x}\\
T_2 &= \bar{v}^n_{i,j}\frac{\bar{v}^n_{i,j} - \bar{v}^n_{i,j-1}}{2\Delta y}\\
T_3 &= \nu\frac{\bar{v}^n_{i+1,j} + \bar{v}^n_{i-1,j} - 2\bar{v}^n_{i,j}}{\Delta x^2}\\
T_4 &= \nu\frac{\bar{v}^n_{i,j+1} + \bar{v}^n_{i,j-1} - 2\bar{v}^n_{i,j}}{\Delta y^2}
\end{aligned} \tag{2.12}$$

The explicit iterative method may be summarized in the following pseudocode -

Algorithm 1 Explicit solver for 2D Viscous Burgers Equation

Require: $u^n_{i,j}$,$v^n_{i,j}$ & Stability
  while $t < N_s$(Number of timesteps) **do**
    while $j > 0$ & $j < N_y$ **do**
      while $i \geq 0$ & $i \leq N_x$ **do**
        UPDATE $u^{n+1}_{i,j}$ from $u^n_{i,j}$ & $v^n_{i,j}$
        UPDATE $v^{n+1}_{i,j}$ from $u^n_{i,j}$ & $u^n_{i,j}$
        $i \leftarrow i+1$
      end while
      $j \leftarrow j+1$
    end while
  end while

2.1.2 Von-Neumann Analysis for the Explicit Iterative Scheme

We first carry out a Von-Neumann stability analysis for the explicit iterative scheme assuming a first order upwind discretization for the advective derivative. A Fourier discretization of the error in the solution and subsequent substitution into the explicitly discretized governing PDE gives us -

$$E^{a\Delta t}(G-1) + UE^{a\Delta t}(1-e^{-\Delta x}) + VE^{a\Delta t}(1-e^{-\Delta x}) = \gamma E^{a\Delta t}\left(2(e^{\Delta x} + e^{-\Delta x}) - 4\right); \tag{2.13}$$

where we have assumed $\Delta x = \Delta y$, $U = V = \frac{u^n_{i,j}\Delta t}{\Delta x}$ and $\gamma = \Delta t/\Delta x^2$. Rearranging terms to give an expression for the amplification factor G gives us -

$$G = 1 + 4\gamma(\cos\phi_x - 1) - (U+V)(1-\cos\phi_x - i\sin\phi_x). \tag{2.14}$$

It is required for a stable solution that the magnitude of the amplification factor $|G|$ remains less than 1. Thus we get -

$$|G| = ((1 + (4\gamma + U + V)(\cos\phi_x - 1))^2 + (U + V)^2 \sin^2\phi_x)^{1/2} \tag{2.15}$$

Let $4\gamma + U + V = G_m$ and $U + V = C_m$. We then get -

$$(1 - 2G_m \sin^2\frac{\phi_x}{2})^2 + C_m^2 \sin^2\phi_k \leq 1 \tag{2.16}$$

which can further be expanded to get -

$$4G_m^2 \sin^4\frac{\phi_x}{2} - 4G_m \sin^2\frac{\phi_x}{2} + C_m^2 \sin^2\phi_k \leq 0 \tag{2.17}$$

which becomes

$$4G_m^2 \sin^2\frac{\phi_x}{2} - 4G_m + 4C_m^2(1 - \sin^2\frac{\phi_k}{2}) \leq 0 \tag{2.18}$$

and

$$\sin^2\frac{\phi_x}{2}(4G_m^2 - 4C_m^2) - 4G_m + 4C_m^2 \leq 0 \tag{2.19}$$

From the above expression we can obtain two conservative relations for the stability of the explicit iterative scheme -

$$\begin{gathered} G_m \leq 1 \\ C \leq \sqrt{G_m} \end{gathered} \tag{2.20}$$

We then use the central difference scheme for the advective derivative to determine the new stability criterion. We have, using an explicit discretization -

$$E^{a\Delta t}(G-1) + UE^{a\Delta t}(\frac{e^{\Delta x} - e^{-\Delta x}}{2}) + VE^{a\Delta t}(\frac{e^{\Delta x} - e^{-\Delta x}}{2}) = \gamma E^{a\Delta t}\big(2(e^{\Delta x} + e^{-\Delta x}) - 4\big); \tag{2.21}$$

which simplifies to -

$$G = 1 - (U + V)\, i \sin\phi_x - 8\gamma \sin^2\frac{\phi_x}{2}. \tag{2.22}$$

Two stability criterion for this case can be devised (in a similar manner to the previous scheme) as -

$$\begin{gathered} \gamma \leq \frac{1}{4} \\ (U + V) \leq \sqrt{4\gamma} \end{gathered} \tag{2.23}$$

2.2 Implicit Iterative Scheme

2.2.1 Scheme development

An implicit iterative scheme can be generated for the 2D viscous Burgers equation using a backward in time discretization. The spatial derivatives are determined in a manner similar to the explicit method. We first describe the the first order upwind scheme for the advective derivative -

$$\begin{aligned}\frac{\bar{u}_{i,j}^{n+1}-\bar{u}_{i,j}^{n}}{\Delta t}+\bar{u}_{i,j}^{n+1}\frac{\bar{u}_{i,j}^{n+1}-\bar{u}_{i-1,j}^{n+1}}{\Delta x}+\bar{v}_{i,j}^{n+1}\frac{\bar{u}_{i,j}^{n+1}-\bar{u}_{i,j-1}^{n+1}}{\Delta y}=\\ \nu\left(\frac{\bar{u}_{i+1,j}^{n+1}+\bar{u}_{i-1,j}^{n+1}-2\bar{u}_{i,j}^{n+1}}{\Delta x^2}+\frac{\bar{u}_{i,j+1}^{n+1}+\bar{u}_{i,j-1}^{n+1}-2\bar{u}_{i,j}^{n+1}}{\Delta y^2}\right);\end{aligned} \tag{2.24}$$

A rearrangement of terms can be used to give us an iterative scheme as follows -

$$\bar{u}_{i,j}^{n+1}=\frac{\bar{u}_{i,j}^{n}+T_2+T_3+T_4}{T_1} \tag{2.25}$$

where -

$$\begin{aligned}T_1&=1+\Delta t\left(\frac{\bar{u}_{i,j}^{n+1}}{\Delta x}-\frac{\bar{u}_{i-1,j}^{n+1}}{\Delta x}+\frac{\bar{v}_{i,j}^{n+1}}{\Delta y}+\frac{2\nu}{\Delta x^2}+\frac{2\nu}{\Delta y^2}\right)\\ T_2&=\Delta t\frac{\bar{v}_{i,j}^{n+1}\bar{u}_{i,j-1}^{n+1}}{\Delta y}\\ T_3&=\nu\Delta t\frac{\bar{u}_{i+1,j}^{n+1}+\bar{u}_{i-1,j}^{n+1}}{\Delta x^2}\\ T_4&=\nu\Delta t\frac{\bar{u}_{i,j+1}^{n+1}+\bar{u}_{i,j-1}^{n+1}}{\Delta y^2}\end{aligned} \tag{2.26}$$

If k denotes the iteration number within the Gauss-Seidel method and a domain sweep is carried out from 0 to N_x and N_y, the following iterative scheme can be determined for the fully implicit discretization of our governing equation -

$$\begin{aligned}T_1&=1+\Delta t\left(\frac{{}_k\bar{u}_{i,j}^{n+1}}{\Delta x}-\frac{{}_{k+1}\bar{u}_{i-1,j}^{n+1}}{\Delta x}+\frac{{}_k\bar{v}_{i,j}^{n+1}}{\Delta y}+\frac{2\nu}{\Delta x^2}+\frac{2\nu}{\Delta y^2}\right)\\ T_2&=\Delta t\frac{{}_k\bar{v}_{i,j}^{n+1}\,{}_{k+1}\bar{u}_{i,j-1}^{n+1}}{\Delta y}\\ T_3&=\nu\Delta t\frac{{}_k\bar{u}_{i+1,j}^{n+1}+{}_{k+1}\bar{u}_{i-1,j}^{n+1}}{\Delta x^2}\\ T_4&=\nu\Delta t\frac{{}_k\bar{u}_{i,j+1}^{n+1}+{}_{k+1}\bar{u}_{i,j-1}^{n+1}}{\Delta y^2}\end{aligned} \tag{2.27}$$

An implicit scheme for the governing partial differential equation can be devised in exactly the same manner but with a central difference approximation used for the advective derivative and is shown below. Firstly, our PDE is discretized to give -

$$
\begin{aligned}
&\frac{\bar{u}_{i,j}^{n+1}-\bar{u}_{i,j}^{n}}{\Delta t}+\bar{u}_{i,j}^{n+1}\frac{\bar{u}_{i+1,j}^{n+1}-\bar{u}_{i-1,j}^{n+1}}{2\Delta x}+\bar{v}_{i,j}^{n+1}\frac{\bar{u}_{i,j+1}^{n+1}-\bar{u}_{i,j-1}^{n+1}}{2\Delta y}= \\
&\qquad \nu\left(\frac{\bar{u}_{i+1,j}^{n+1}+\bar{u}_{i-1,j}^{n+1}-2\bar{u}_{i,j}^{n+1}}{\Delta x^2}+\frac{\bar{u}_{i,j+1}^{n+1}+\bar{u}_{i,j-1}^{n+1}-2\bar{u}_{i,j}^{n+1}}{\Delta y^2}\right);
\end{aligned} \tag{2.28}
$$

A rearrangement of terms can be used to give us an iterative scheme as follows -

$$
\bar{u}_{i,j}^{n+1}=\frac{\bar{u}_{i,j}^{n}+T_2+T_3+T_4}{T_1} \tag{2.29}
$$

where -

$$
\begin{aligned}
T_1 &= 1+\Delta t\left(\frac{\bar{u}_{i+1,j}^{n+1}}{2\Delta x}-\frac{\bar{u}_{i-1,j}^{n}}{2\Delta x}+\frac{2\Delta\nu}{\Delta x^2}+\frac{2\Delta\nu}{\Delta y^2}\right)\\
T_2 &= -\Delta t\,\bar{v}_{i,j}^{n+1}\frac{\bar{u}_{i,j+1}^{n+1}-\bar{u}_{i,j-1}^{n+1}}{2\Delta y}\\
T_3 &= \nu\Delta t\frac{\bar{u}_{i+1,j}^{n+1}+\bar{u}_{i-1,j}^{n+1}}{\Delta x^2}\\
T_4 &= \nu\Delta t\frac{\bar{u}_{i,j+1}^{n+1}+\bar{u}_{i,j-1}^{n+1}}{\Delta y^2}
\end{aligned} \tag{2.30}
$$

If k denotes the iteration number within the Gauss-Seidel method and a domain sweep is carried out from 0 to N_x and N_y, the following iterative scheme can be determined for the fully implicit discretization of our governing equation -

$$
\begin{aligned}
T_1 &= 1+\Delta t\left(\frac{{}_k\bar{u}_{i+1,j}^{n+1}}{2\Delta x}-\frac{{}_{k+1}\bar{u}_{i-1,j}^{n}}{2\Delta x}+\frac{2\Delta\nu}{\Delta x^2}+\frac{2\Delta\nu}{\Delta y^2}\right)\\
T_2 &= -\Delta t_k\,\bar{v}_{i,j}^{n+1}{}_k\frac{\bar{u}_{i,j+1}^{n+1}-{}_{k+1}\bar{u}_{i,j-1}^{n+1}}{2\Delta y}\\
T_3 &= \nu\Delta t_k\frac{\bar{u}_{i+1,j}^{n+1}+{}_{k+1}\bar{u}_{i-1,j}^{n+1}}{\Delta x^2}\\
T_4 &= \nu\Delta t_k\frac{\bar{u}_{i,j+1}^{n+1}+{}_{k+1}\bar{u}_{i,j-1}^{n+1}}{\Delta y^2}
\end{aligned} \tag{2.31}
$$

An indicative pseudocode for the implicit scheme can be shown in algorithm 2. It must be noted here that tol stands for the tolerance used for terminating Gauss-Seidel iterations. This study uses a $tol = 10^{-4}$.

Algorithm 2 Implicit solver for 2D Viscous Burgers Equation

Require: $u^n_{i,j}$,$v^n_{i,j}$

  while $t < N_s$(Number of timesteps) **do**
    while $j > 0$ & $j < N_y$ **do**
      while $i \geq 0$ & $i \leq N_x$ **do**
        while E1 and E2 > tol **do**
          UPDATE $_{k+1}\bar{u}^{ij}_{n+1}$ from $_{k+1}\bar{u}^{i-1,j}_{n+1}$,$_{k+1}\bar{u}^{i,j-1}_{n+1}$,$_{k}\bar{u}^{i+1,j}_{n+1}$,$_{k}\bar{u}^{i,j+1}_{n+1}$,$_{k}\bar{u}^{i,j}_{n+1}$
          UPDATE $_{k+1}\bar{v}^{ij}_{n+1}$ from $_{k+1}\bar{v}^{i-1,j}_{n+1}$,$_{k+1}\bar{v}^{i,j-1}_{n+1}$,$_{k}\bar{v}^{i+1,j}_{n+1}$,$_{k}\bar{v}^{i,j+1}_{n+1}$,$_{k}\bar{v}^{i,j}_{n+1}$
          E1 $\leftarrow {}_{k+1}\bar{u}^{ij}_{n+1} - {}_{k}\bar{u}^{ij}_{n+1}$
          E2 $\leftarrow {}_{k+1}\bar{v}^{ij}_{n+1} - {}_{k}\bar{v}^{ij}_{n+1}$
          $k \leftarrow k+1$
        end while
        $i \leftarrow i+1$
      end while
      $j \leftarrow j+1$
    end while
  end while

2.2.2 Von-Neumann analysis for the Implicit Iterative Scheme

We now develop the stability criterion for the implicit iterative scheme using the Von-Neumann stability analysis. Using a process similar to the one described in section 2.1.2, we can first discretize the first order upwind version of the implicit scheme to get -

$$E^{a\Delta t}(1-\frac{1}{G}) + UE^{a\Delta t}(1-e^{-\Delta x}) + VE^{a\Delta t}(1-e^{-\Delta x}) = E^{a\Delta t}\gamma(\cos\phi_x - 1) \tag{2.32}$$

where the usual definitions for various terms are in use. Since $1/|G| \geq 1$, we obtain -

$$(1+(U+V+4\gamma)(1-\cos\phi_x))^2 + (U+V)^2\sin^2\phi_x \geq 1 \tag{2.33}$$

If we assume $U+V+4\gamma = G_m$ and $U+V = C_m$ we can get a stability criterion -

$$\begin{aligned} G_m &\geq -1 \\ G_m &\geq -C_m^2 \end{aligned} \tag{2.34}$$

which is an unconditionally stable scheme. Using a similar analysis on the central difference discretization gives us -

$$E^{a\Delta t}(1-\frac{1}{G}) + UE^{a\Delta t}(\frac{e^{\Delta x}-e^{-\Delta x}}{2}) + VE^{a\Delta t}(\frac{e^{\Delta x}-e^{-\Delta x}}{2}) = E^{a\Delta t}\gamma(\cos\phi_x - 1) \tag{2.35}$$

which gives us the conditions for stability as -

$$\gamma \geq -\frac{1}{4}$$
$$\gamma \geq \frac{(U+V)^2}{4} \tag{2.36}$$

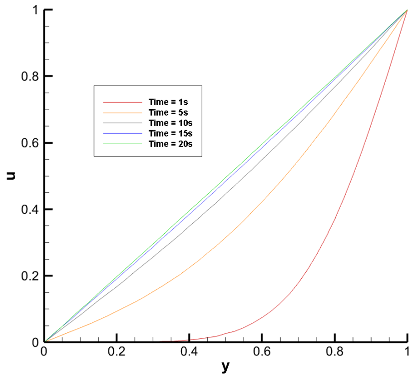
Provided we have real solution fields for $u_{i,j}$ and $v_{i,j}$, the above two statements shall always be true. So this scheme is unconditionally stable as well.

3 Numerical Experimentation & Results

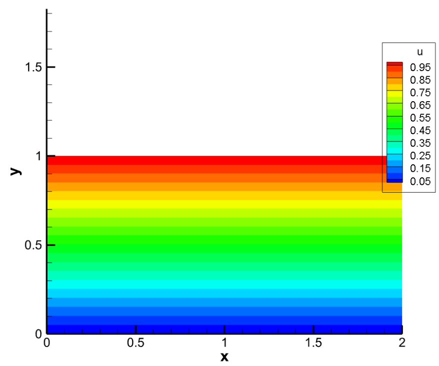
3.1 Comparison of explicit and implicit solvers

The first order upwind scheme (for the convective derivative) and the second order central difference scheme (for the Laplacian) were used to perform 4 numerical runs upto steady state (which was seen to be attained at 20 seconds of physical time). Two cell Peclet ($Pe_{\Delta x}$)numbers were also simulated to quantify their effect on both schemes.

Steady state contours for both schemes at different Peclet numbers are plotted alongwith lineplots for the x component of velocity u against y, at different time instances. For $Pe_{\Delta x} = 1.0$, the following images describe the development of the boundary layer to steady state.



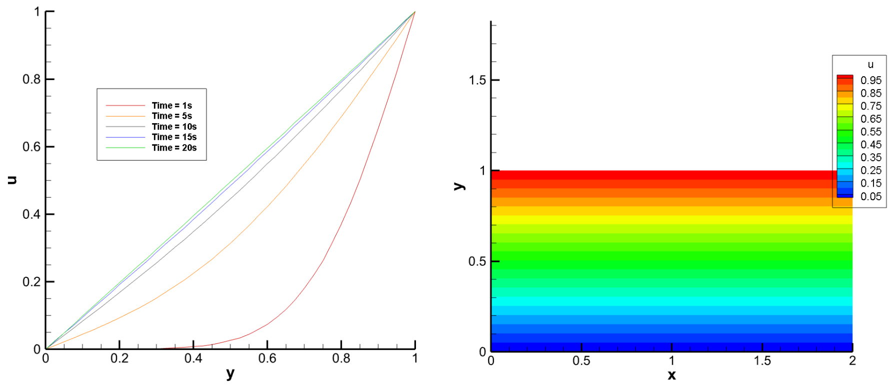
(a) u versus y at $y = 0.5$



(b) Steady State Contour

Figure 3.1: Explicit scheme with $Pe_{\Delta x} = 1.0, \gamma = 0.1$

The implicit method at the same Peclet number (1.0) gave us the following results -

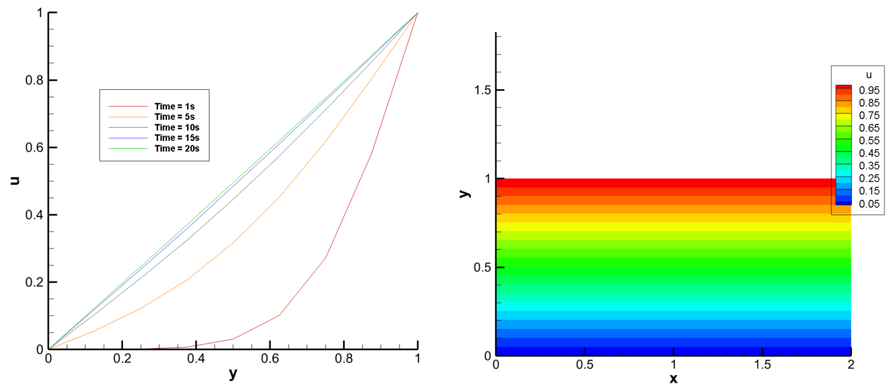


(a) u versus y at $y = 0.5$ (b) Steady State Contour

Figure 3.2: Implicit scheme with $Pe_{\Delta x} = 1.0, \gamma = 0.1$

It would appear from these results that at $Pe_{\Delta x} = 1.0$ and $\gamma = 0.1$, both methods are equally accurate. For the purpose of computational expense reduction, it would be profitable to go for the explicit method in this case. We now look at the effect of a higher cell peclet number ($Pe_{\Delta x} = 5.0$). It is expected that the higher value of $Pe_{\Delta x}$ increases the grid size and thus reduces the discretization.

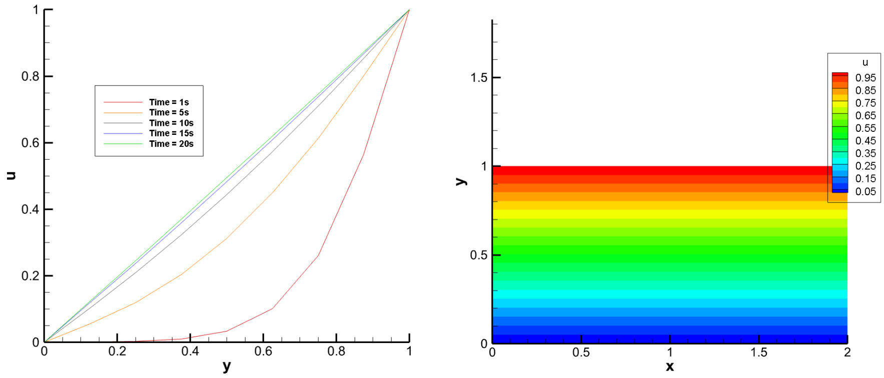
The explicit scheme for $Pe_{\Delta x} = 5.0$ is shown as -



(a) u versus y at $y = 0.5$ (b) Steady State Contour

Figure 3.3: Explicit scheme with $Pe_{\Delta x} = 5.0, \gamma = 0.1$

The impicit scheme for $Pe_{\Delta x} = 5.0$ is shown as -



(a) u versus y at $y = 0.5$ (b) Steady State Contour

Figure 3.4: Implicit scheme with $Pe_{\Delta x} = 5.0, \gamma = 0.1$

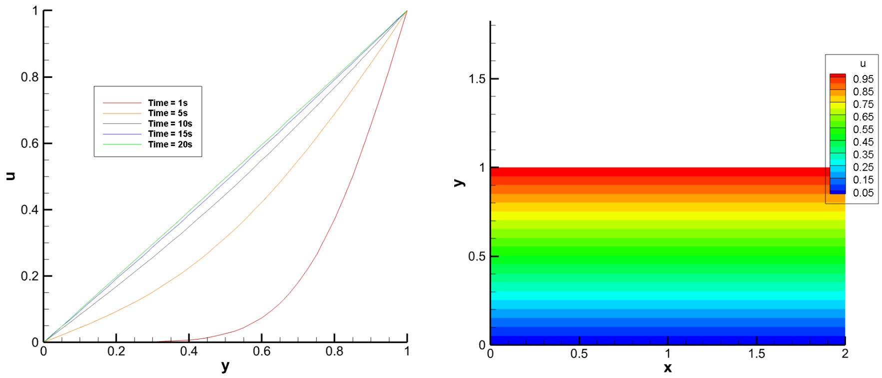
In light of the results above, it would seem that there is no particular advantage in choosing an implicit solver over the explicit one since stability is not an issue for the parameters under investigation. To avoide Gauss-Seidel iterations, it would be beneficial to choose the explicit solver.

3.2 Effect of advection scheme

A prior examination of the flow configuration of the problem tells us that the choice of advection scheme should not affect the outcome of the numerical solution. This is because the direction of flow developed in the solution will only be in the positive x direction and the velocity profile will only be a function of the wall normal distance. In this case -

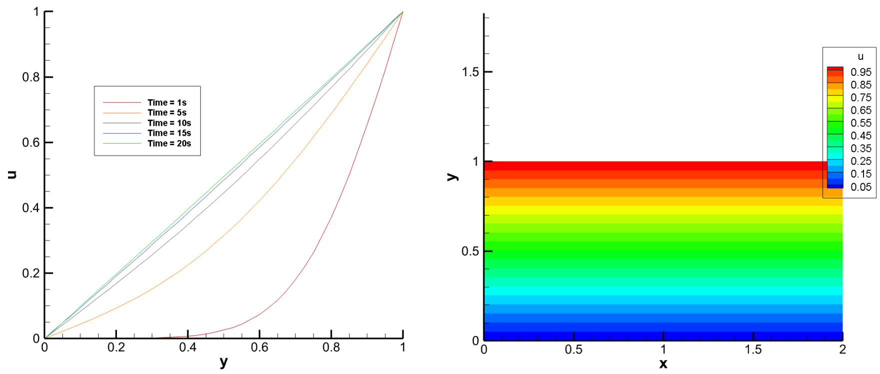
$$\frac{\partial u}{\partial x} = \frac{\bar{u}^n_{i,j} - \bar{u}^n_{i-1,j}}{\Delta x} = \frac{\bar{u}^n_{i+1,j} - \bar{u}^n_{i-1,j}}{2\Delta x} \tag{3.1}$$

In order to confirm this assessment, numerical simulations were carried out for $\gamma = 0.1$ and $Pe_{\Delta x} = 1.0, 5.0$ using an explicit solver. For the case of $Pe_{\Delta x} = 1.0$, the following results were obtained -



(a) u versus y at $y = 0.5$ (b) Steady State Contour

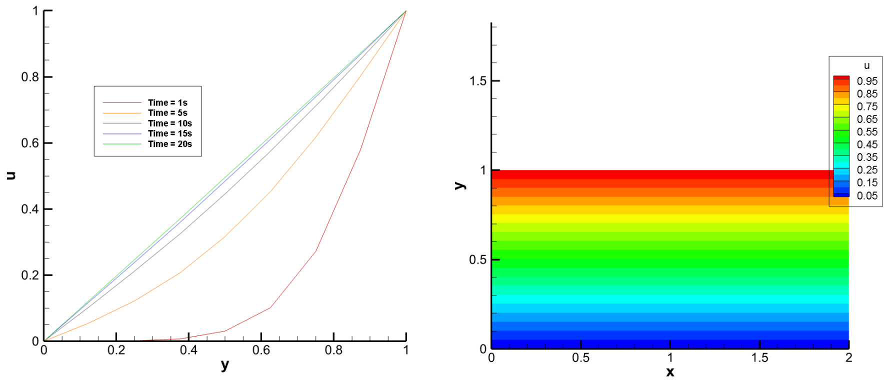
Figure 3.5: Explicit scheme with $Pe_{\Delta x} = 1.0, \gamma = 0.1$ and central difference scheme for advective term



(a) u versus y at $y = 0.5$ (b) Steady State Contour

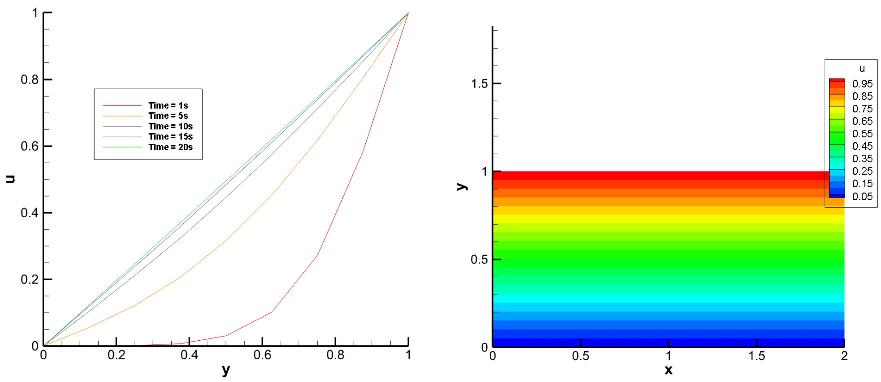
Figure 3.6: Explicit scheme with $Pe_{\Delta x} = 1.0, \gamma = 0.1$ and first order upwind scheme for advective term

A similar trend was seen for $Pe_{\Delta x} = 5.0$ as shown below -



(a) u versus y at $y = 0.5$ (b) Steady State Contour

Figure 3.7: Explicit scheme with $Pe_{\Delta x} = 5.0, \gamma = 0.1$ and central difference scheme for advective term



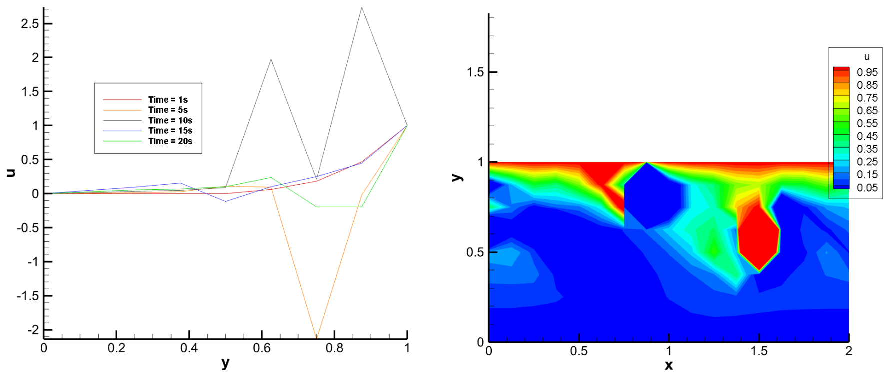
(a) u versus y at $y = 0.5$ (b) Steady State Contour

Figure 3.8: Explicit scheme with $Pe_{\Delta x} = 5.0, \gamma = 0.1$ and first order upwind scheme for advective term

It would seem that the choice of the local Peclet number does not affect the general trend of the progressing solution. However, it would naturally lead to less accurate solutions between nodes due to linear interpolation.

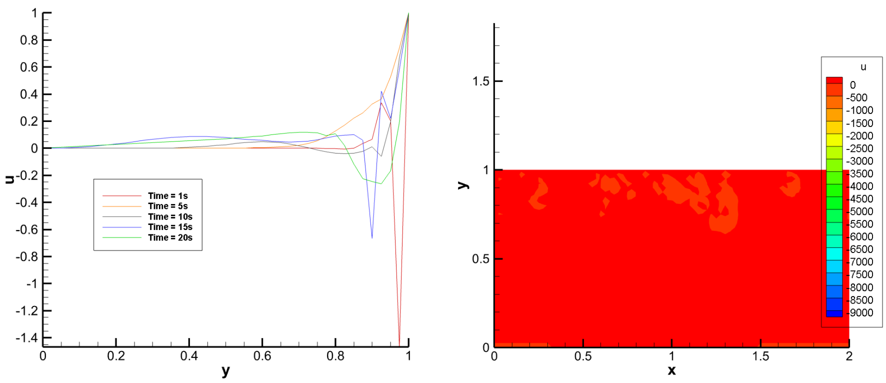
3.3 Effect of Re on Boundary layer

A first order advective scheme and second order central derivative scheme was used with $\gamma = 1.0$ and $Pe_{\Delta x} = 5.0$ to test the effect of the Reynolds numbers 40 and 200. However, it was observed that for the increased value of γ, the errors were too high and any meaningful analysis could not be concluded. After careful thought, a potential error causing term was pin-pointed to be T_1 in equation 2.26. In particular the $\Delta t \frac{\bar{u}^{n+1}_{i-1,j}}{\Delta x}$ term which causes addition of error. I have tried to rigorously identify what the mechanics of this error are but unfortunately could not determine it with confidence.



(a) u versus y at $y = 0.5$ (b) Steady State Contour

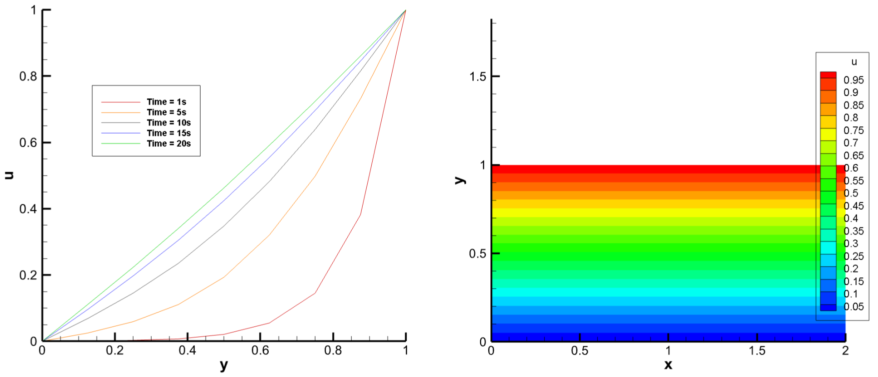
Figure 3.9: Implicit scheme with $Pe_{\Delta x} = 5.0, \gamma = 1.0$ and first order upwind scheme for advective term. Note error.



(a) u versus y at $y = 0.5$ (b) Steady State Contour

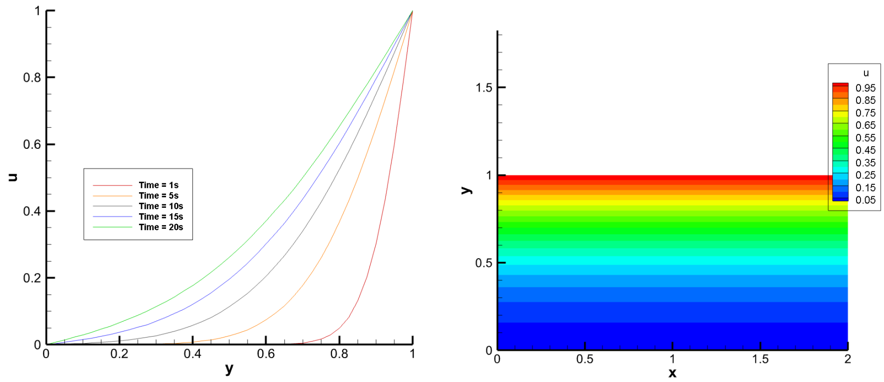
Figure 3.10: Implicit scheme with $Pe_{\Delta x} = 5.0, \gamma = 1.0$ and first order upwind scheme for advective term.

In order to complete the investigation demanded, the central difference discretization scheme was used for the advective derivative where it was seen that no errors were developed. The following results were generated -



(a) u versus y at $y = 0.5$ (b) Steady State Contour

Figure 3.11: Implicit scheme with $Pe_{\Delta x} = 5.0, \gamma = 1.0$ and second order central difference scheme for advective term.



(a) u versus y at $y = 0.5$ (b) Steady State Contour

Figure 3.12: Implicit scheme with $Pe_{\Delta x} = 5.0, \gamma = 1.0$ and second order central difference scheme for advective term. Note curvature in steady state line plot and incomplete boundary layer formation.

From these images it was concluded that a higher Reynolds number caused a slower transition to steady state (as at our defined steady state time of 20s, a curvature in the boundary layer profile is seen).

4 Conclusions & Comments

For this geometry and flow configuration, it would seem that the choice of the discretization of the advective derivative is not of significant importance. Also, runs with lower viscosity being used tend to show a slower convergence to steady state due to a longer duration required for momentum transfer through diffusion. Also, a detailed study needs to be carried out to quantify the reason the first order advective derivative introduces error at higher values of γ.