# Steady state 2-D heat transfer and Finite Difference Model (FDM)

# Introduction

- How to get temperature distribution in 2D and 3D objects?
  - Experiment
  - Analytical solutions
  - Numerical simulations
    - Finite Difference Method
    - Finite Element Method
    - Finite Volume Method
    - Spectral Element Method

# Finite Difference Method (FDM):

Derivation from Taylor's Polynomial:

$$f(x_0 + h) = f(x_0) + \frac{f'(x_0)}{1!}h + \frac{f''(x_0)}{2!}h^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}h^n + R_n(x_0)$$

Approximating for first derivative:

$$f(x_0 + h) = f(x_0) + \frac{f'(x_0)}{1!}h + R_1(x_0)$$

Truncating $R_1(x_0)$ and solving for $f'(x_0)$:

$$f'(x_0) = \frac{f(x_0 + h) - f(x_0)}{h}$$

This is first order accurate for first derivative.

- Higher order derivatives and higher order accuracy can be obtained.

- https://en.wikipedia.org/wiki/Finite_difference_coefficient

# Finite Difference Method (FDM):

- Forward and backward finite difference:
  - **Forward** first order accurate first derivative:

$$f'(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} + O(h)$$

  - **Backward** first order accurate first derivative:

$$f'(x_0) = \frac{f(x_0) - f(x_0 - h)}{h} + O(h)$$

- **Central** finite difference
  - Second order accuracy for first derivative:

$$f'(x_0) = \frac{f(x_0 + h) - f(x_0 - h)}{2h} + O(h^2)$$

  - Second order accuracy for second derivative:

$$f''(x_0) = \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h)}{h^2} + O(h^2)$$

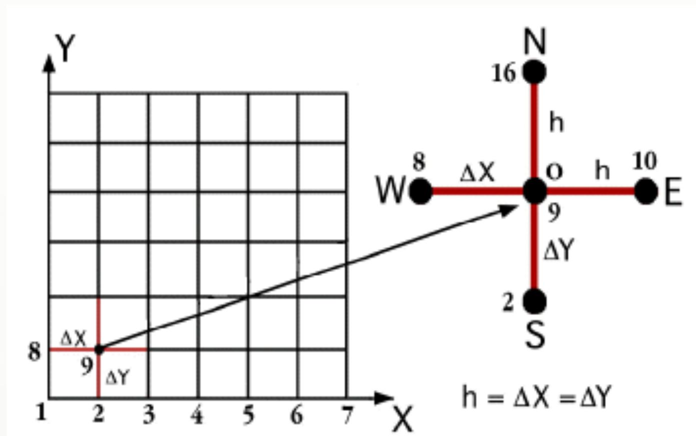
# Discretization:

- Governing equations:

  Heat diffusion equation in 2-D, no heat generation and steady state:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

- Discretization:




$$\frac{T(x+\Delta x, y) - 2T(x,y) + T(x-\Delta x, y)}{(\Delta x)^2} + \frac{T(x, y+\Delta y) - 2T(x,y) + T(x, y-\Delta y)}{(\Delta y)^2} = 0$$

If $\Delta x = \Delta y$

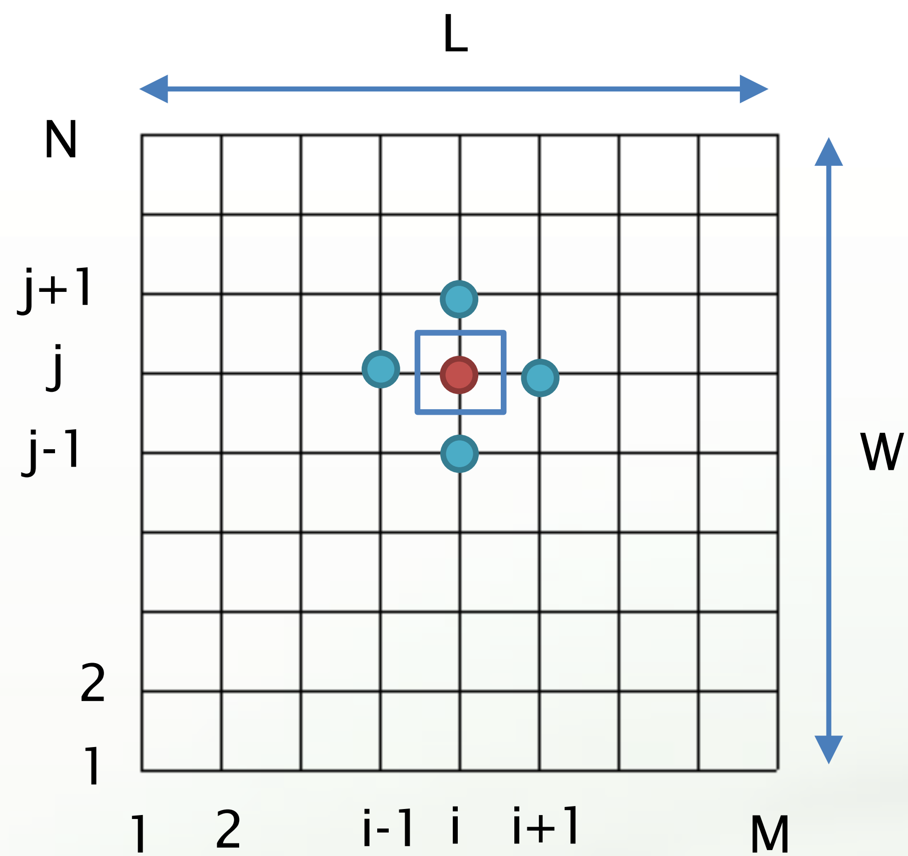
$$T(x+\Delta x, y) + T(x-\Delta x, y) + T(x, y+\Delta y) + T(x, y-\Delta y) - 4T(x,y) = 0$$

# Discretization:

- Discretized heat diffusion equation:

$$T(i+1,j) + T(i-1,j) + T(i,j+1) + T(i,j-1) - 4T(i,j) = 0$$

- Using energy balance approach, we can get the same equation:



# Discretization:

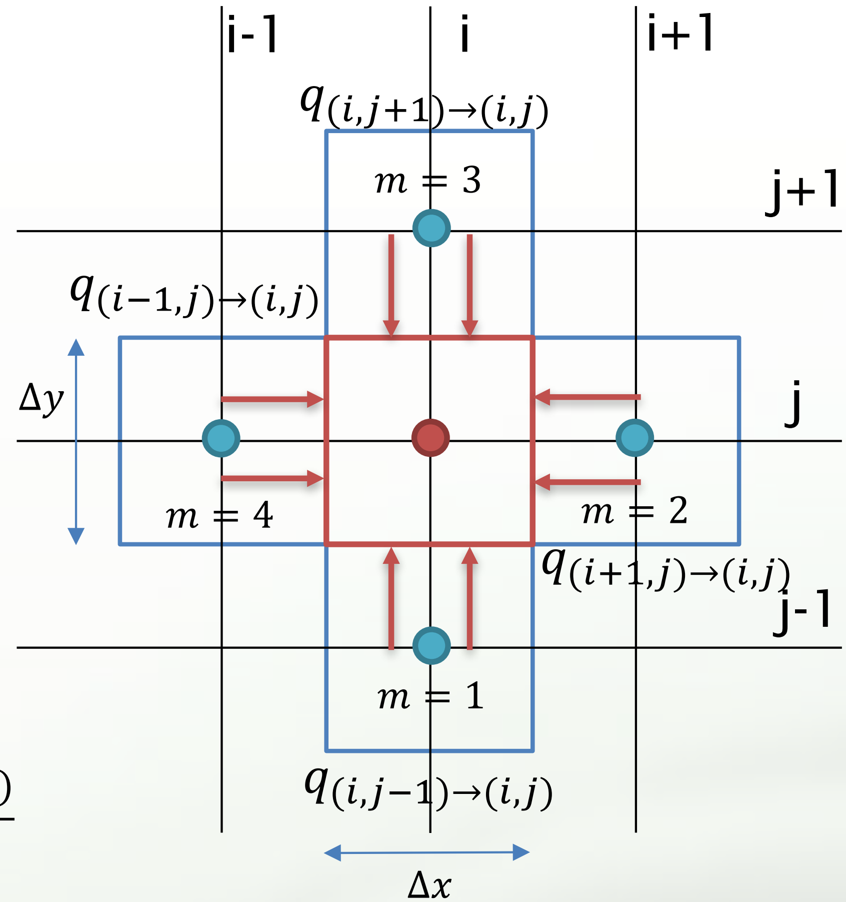
- Energy balance approach:

$$\sum_{m=1}^{4} q_{m\rightarrow(i,j)} = 0$$

  - For example:

$$q_{(i,j-1)\rightarrow(i,j)} = -k\frac{\partial T}{\partial y}\Delta x = -k\frac{T(i,j)-T(i,j-1)}{\Delta y}$$

$$-k\frac{T(i,j)-T(i,j-1)}{\Delta y}\Delta x - k\frac{T(i,j)-T(i,j+1)}{\Delta y}\Delta x$$
$$-k\frac{T(i,j)-T(i-1,j)}{\Delta x}\Delta y - k\frac{T(i,j)-T(i+1,j)}{\Delta x}\Delta y = 0$$




Divide by $\Delta x.\Delta y$

$$-k\frac{T(i,j)-T(i,j-1)}{\Delta y^2} - k\frac{T(i,j)-T(i,j+1)}{\Delta y^2} - k\frac{T(i,j)-T(i-1,j)}{\Delta x^2}$$
$$-k\frac{T(i,j)-T(i+1,j)}{\Delta x^2} = 0$$

Assuming $\Delta x = \Delta y$

$$T(i+1,j) + T(i-1,j) + T(i,j+1) + T(i,j-1) - 4T(i,j)$$
$$= 0$$

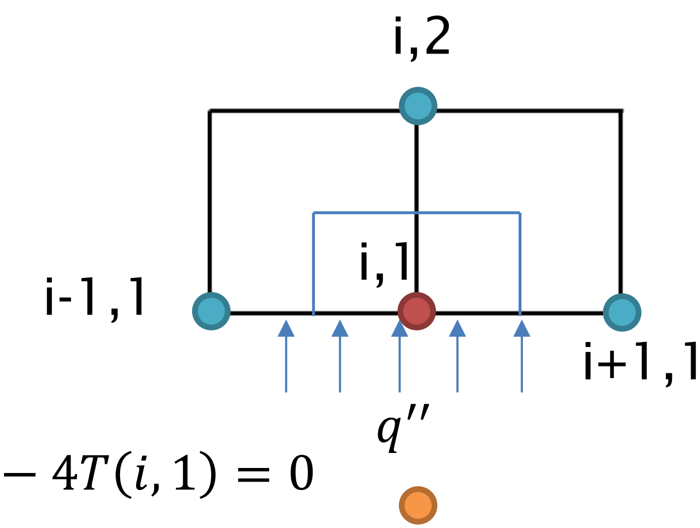
# Boundary conditions

- Constant temperature:

$$T(M, j) = T_{bc}$$

- Constant heat flux boundary condition:

  Using energy balance approach:




$$2T(i,2) + T(i+1,1) + T(i-1,1) + \frac{2q''\Delta y}{K} - 4T(i,1) = 0$$

- Adiabatic B.C:

$$2T(i,2) + T(i+1,1) + T(i-1,1) - 4T(i,1) = 0$$

  It's similar to having node (i,0) which is always equal to node (i,2)

- How about other boundary conditions? (Convection, radiation, etc)

# How to solve: System of equations

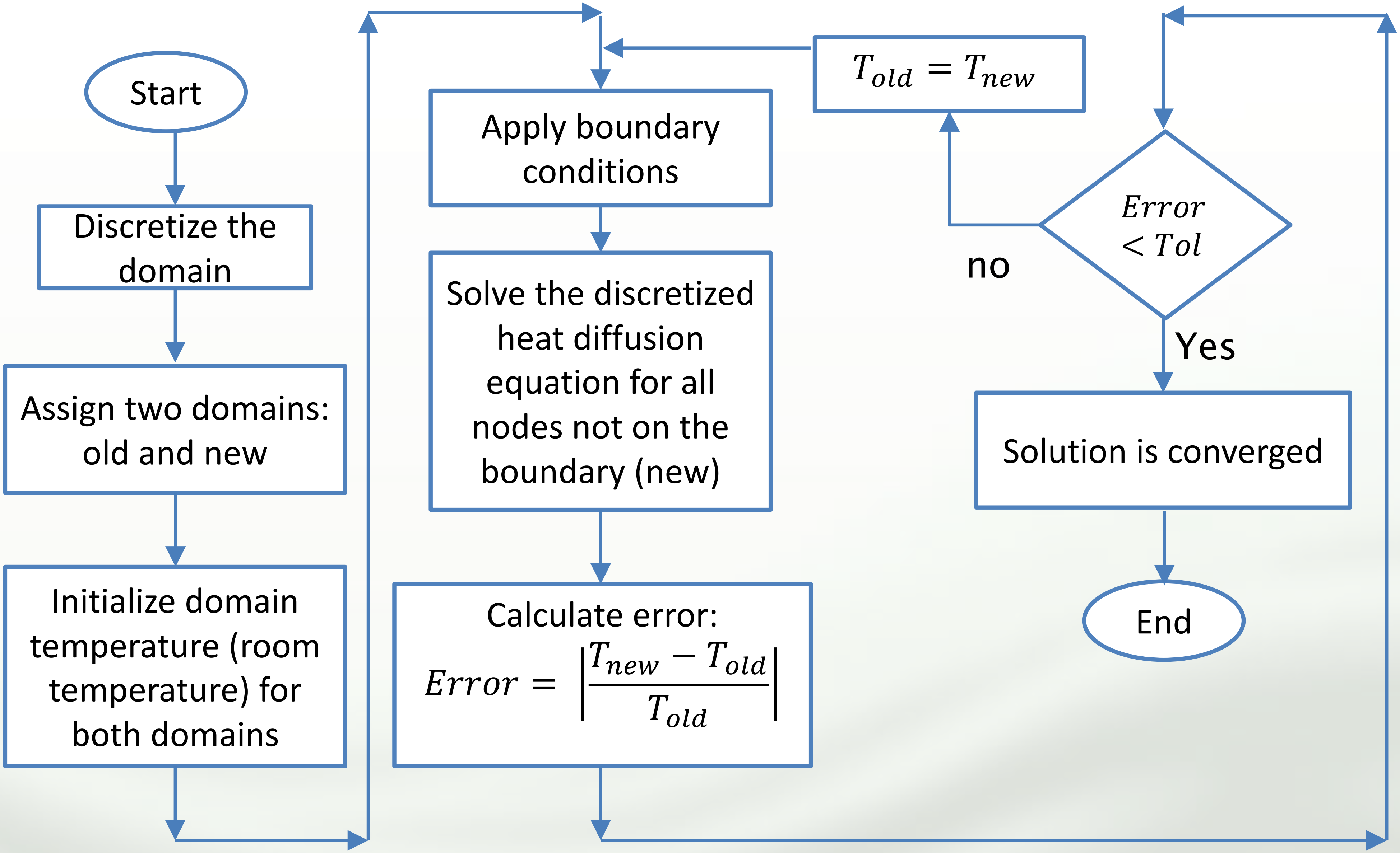
- By writing the equations for all nodes and boundary nodes, we can make a system of equations that looks like this:
- $$\begin{bmatrix} & & & & \cdots & & & & \\ & \ddots & & & & & & & \\ & & & & & & & & \\ 0 & & 2 & -4 & 1 & 0 & 0 & & \vdots \\ & 1 & 0 & 1 & -4 & 1 & 0 & 1 & \\ & & 1 & 0 & 1 & -4 & 1 & 0 & 1 \\ \vdots & & & & & & \ddots & & \\ & & & & & & & 1 & \\ & & & & \cdots & & & & 1 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{bmatrix} = \begin{bmatrix} \\ \vdots \\ \\ 0 \\ 0 \\ \\ \vdots \\ T_b \\ T_b \end{bmatrix}$$

  Adiabatic BC.

  General diffusion nodes

  Constant T
- Try making this matrix for your case.
- Solving this matrix equation gives temperatures for each nodes.
  - How to solve this matrix equation?

# How to solve: iterative technique



# Report:

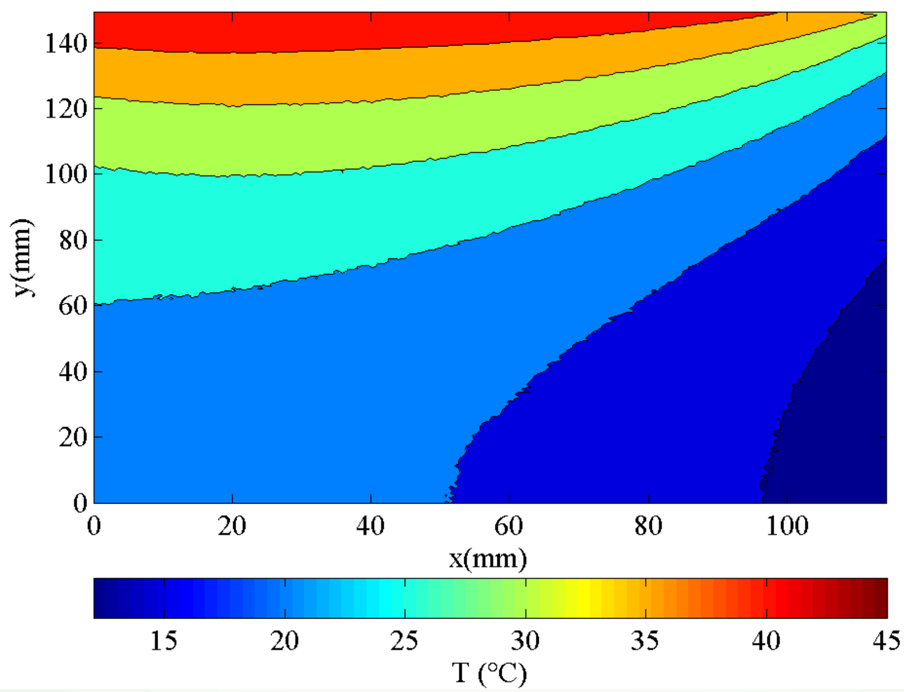
- In your report:
  - Obtain the equations for heat diffusion equation
  - Obtain the equations for boundary conditions

- Boundary condition assumptions:
  - Temperature on each edge read from camera

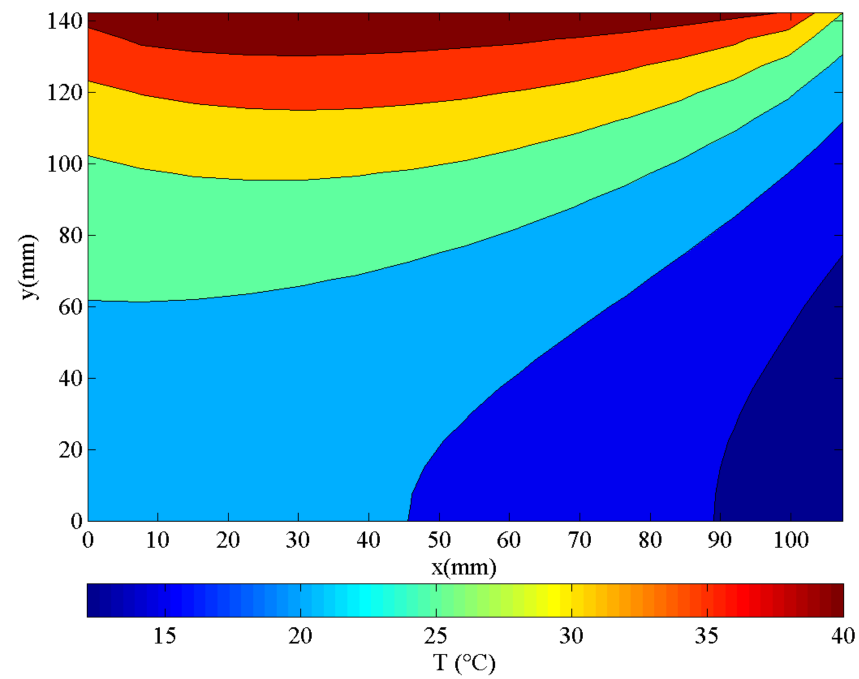
- Presenting results:
  - Filled contours plots or measured and computer temperature
  - Line contour plot of isotherms for measured and computer temperature
  - Filled contour plot of difference between experiment and numerical solution

# Sample results:

## Experimental data




## Numerical solution




Using edge temperatures from experiment in numerical simulation