# Introduction to cryptology (GBIN8U16)

✧

# Finite fields, block ciphers

Pierre Karpman
pierre.karpman@univ-grenoble-alpes.fr
https://www-ljk.imag.fr/membres/Pierre.Karpman/tea.html

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$\mathbb{F}_2$ primer

Symmetric cryptography

BC: First definitions

Symmetric encryption schemes

# Bits as field elements

- Digital processing of information $\rightsquigarrow$ dealing with bits
- Error-correcting codes, crypto $\rightsquigarrow$ need analysis $\rightsquigarrow$ maths
- $\Rightarrow$ bits (no structure) $\mapsto$ field elements (math object)

- "Natural" match: $\{0, 1\} \cong \mathbb{F}_2 \equiv \mathbb{Z}/2\mathbb{Z} \equiv$ "(natural) integers modulo 2"
- $\mathbb{F}_2$: two elements (0, 1), two operations ($+$, $\times$)

# What's $\mathbb{F}_2$ like?

- Addition $\equiv$ exclusive or (XOR ($\oplus$))
- Multiplication $\equiv$ logical and ($\wedge$)
- $\Rightarrow$ "Boolean" arithmetic

- Fact 1: any Boolean function $f : \{0,1\}^n \to \{0,1\}$ can be computed using only $\oplus$ and $\wedge$
- Fact 2: ditto, $g : \{0,1\}^n \to \{0,1\}^m$
- Fact 3: ditto, using NAND ($\neg \circ \wedge$)

# One bit is nice, but...

- We rather need bit strings $\{0,1\}^n$ than single bits
- Now two "natural" matches:

- $\mathbb{F}_2^n$ (vectors over $\mathbb{F}_2$)
  - Can add two vectors
  - Cannot multiply "internally" (but there's a dot/scalar product)

- $\mathbb{Z}/2^n\mathbb{Z}$ (natural integers modulo $2^n$)
  - Can add, multiply
  - Not all elements are invertible (e.g. 2) $\Rightarrow$ only a ring

Exercise: How do you implement operations in $\mathbb{F}_2^{64}$, $\mathbb{Z}/2^{64}\mathbb{Z}$ in C?

# A third way

- Also possible: $\mathbb{F}_{2^n}$: an *extension* field
  - Addition "like in $\mathbb{F}_2^n$"
  - Plus an internal multiplication
  - All elements (except zero) are invertible
- Not for today!

# Why are these useful?

- Allows to perform operations on inputs
  - E.g. adding two messages together
- Vector spaces ⇒ linear algebra (matrices)
  - Powerful tools to solve “easy” problems
  - (Intuition: crypto shouldn’t be linear)
- Fields ⇒ polynomials
  - With one or more variable
  - ⇒ Can compute differentials

# Recall that...

- Cryptography: we want to hide stuff (e.g., messages to be sent over an insecure channel)
- Symmetric: we only do that assuming a preexisting shared secret

- A major question: when is the hiding "good enough"?
  - "HELLO" $\mapsto$ "HULLO": not great
  - "HELLO" $\mapsto$ "ZNPQE": maybe better
  - "HELLO" $\mapsto$ "ZNPQE"; "HELLO" $\mapsto$ "ZNPQE"; "HELLO" $\mapsto$ "ZNPQE"...: (Okay, those same 5 letters at the start of your messages probably always mean "hello")

# The problem with deterministic encryption


Figure: XKCD #257

# So...

- Encryption MUST be non-deterministic
- Also (a bit harder to see): messages MUST *always* be authenticated to prevent tampering if the adversary is active (even if only "confidentiality" is a concern)

Now our main concerns:

- How do we formalise what we want to achieve?
- How do we actually build schemes that work?

# Block ciphers: for what?

Ultimate goal: symmetric encryption (and more!)

- plaintext + key $\mapsto$ ciphertextS
- ciphertext + key $\mapsto$ plaintext
- ciphertexts $\mapsto$ ???

With *arbitrary* plaintexts $\in \{0, 1\}^*$

Block ciphers: do that *one-to-one* (for a fixed key) for plaintexts $\in \{0, 1\}^n$

- (Very) small example: 32 randomly shuffled cards = 5-bit block cipher
- Typical block sizes = "what's easy to implement"
- Mostly useless in isolation (e.g. they're deterministic) but very useful when plugged into suitable higher-level schemes

# Block ciphers as a figure

$\rightsquigarrow$ on the board

# A main alternative: stream ciphers, still as a figure

$\rightsquigarrow$ still on the board

# Block ciphers: “simple” binary mappings

## Block cipher

A block cipher is a mapping $\mathcal{E} : \mathcal{K} \times \mathcal{M} \to \mathcal{M}'$ s.t. $\forall k \in \mathcal{K}$, $\mathcal{E}(k, \cdot)$ is invertible

In practice, most of the time:

- Keys $\mathcal{K} = \{0,1\}^\kappa$, with $\kappa \in \{\cancel{64}, \cancel{80}, \cancel{96}, \cancel{112}, 128, 192, 256\}$
- Plaintexts/ciphertexts $\mathcal{M} = \mathcal{M}' = \{0,1\}^n$, with $n \in \{64, 128, 256\}$

$\Rightarrow$ BCs are *families of permutations* over binary domains

- Exception: *Format Preserving Encryption* (FPE)

# What's a good block cipher?

One that's:

- "Efficient"
  - Fast (e.g. a few *cycles per byte* on modern high-end CPUs)
  - $\wedge/\vee$ Compact (small code, circuit size)
  - $\wedge/\vee$ Easy to implement "securely" (e.g. to prevent side-channel attacks)
  - Etc.
- "Secure"
  - Large security parameters (key, block size)
  - $\wedge$ No (known) dedicated attacks.

# What's a secure block cipher?

What do you think?

# What's a secure block cipher?

Expected behaviour:

- Given *oracle access* to $\mathcal{E}(k,\cdot)$, with a secret $k \twoheadleftarrow \mathcal{K}$, it is "hard" to find $k$
- (Same with oracle access to $\mathcal{E}^{\pm}(k,\cdot) \coloneqq \{\mathcal{E}(k,\cdot), \mathcal{E}^{-1}(k,\cdot)\}$)
- Given $c = \mathcal{E}(k,m)$, it is "hard" to find $m$ (when $k$'s unknown)
- Given $m$, it is "hard" to find $c = \mathcal{E}(k,m)$ (idem)

But that's not enough!

# We need more

Define $\mathcal{E}_k : x_L||x_R \mapsto x_L||\,\mathcal{E}'_k(x_R)$ for some $\mathcal{E}'$

- If $\mathcal{E}'$ verifies all props. from the previous slide, then so does $\mathcal{E}$
- But $\mathcal{E}$ is obviously not so nice

$\Rightarrow$ need a better way to formulate expectations

# Ideal block ciphers

## Ideal block cipher

Let $\mathrm{Perm}(\mathcal{M})$ be the set of the $(\#\mathcal{M})!$ permutations of $\mathcal{M}$; an *ideal block cipher* $\mathcal{E} : \mathcal{K} \times \mathcal{M} \to \mathcal{M}$ is s.t. $\forall k \in \mathcal{K}$, $\mathcal{E}(k, \cdot) \twoheadleftarrow \mathrm{Perm}(\mathcal{M})$

- "Maximally random"
- All keys yield truly independent permutations
- Quite costly to implement
  - Say $\mathcal{M} = \{0,1\}^{32} \rightsquigarrow 2^{32}! < (2^{32})^{2^{32}}$ permutations
  - So about $32 \times 2^{32} = 2^{37}$ *bits* to describe one ($\leftsquigarrow$ key size)

$\Rightarrow$ Not very practical

# (S)PRP security

Most of the time, good enough if $\mathcal{E}$ is a "good" *pseudo-random permutation* (PRP):

- An adversary has access to an oracle $\mathbb{O}$
- In one world, $\mathbb{O} \twoheadleftarrow \mathrm{Perm}(\mathcal{M})$
- In another, $k \twoheadleftarrow \mathcal{K}$, $\mathbb{O} = \mathcal{E}(k, \cdot)$
- It is "hard" for the adversary to tell in which world s/he lives
- ("Strong/Super" variant: give oracle access to $\mathbb{O}^{\pm}$)

$\Rightarrow$ *Stronger* requirement than key recovery (is implied by it, converse is not true)

# (S)PRP security: why it makes sense

It's easy to distinguish the two worlds if:

- It's easy to recover the key of $\mathcal{E}(k, \cdot)$ (try and see)
- It's easy to predict what $\mathcal{E}(k, m)$ will be (ditto)
- $\mathcal{E}_k : x_L \| x_R \mapsto x_L \| \mathcal{E}'_k(x_R)$ (random permutations usually don't do that)
- $\mathcal{E}$ is $\mathbb{F}_2$-linear (say), or even "close to"
- Etc.

$\Rightarrow$ Don't have to explicitly define all the "bad cases"

Plus:

- Can't do better than a random permutation anyways
- If it looks like one, either it's fine, or BCs are useless ($\leftarrow$ "true" most of the time but not always)

# (S)PRP: it's not everything

- Sometimes a PRP is not enough and one needs a stronger/different model such as the *ideal block cipher* model
- For instance when the adversary has access to the key ($\rightsquigarrow$ considering a uniform choice doesn't make sense anymore)
- Example: when using block ciphers to build compression functions (cf. the hash function lecture)

# Complexity issues

We still need to define what means "hard" $\Rightarrow$ relevant metrics:

- Time (T) ("how much computation")
- Memory (M) ("how much storage")
  - Memory type (sequential access (cheap tape), RAM (costly))
- Data (D) ("how many oracle queries")
  - Query type (to $\mathcal{E}$, to $\mathcal{E}^{-1}$, *adaptive* or not, etc.)
- Success probability (p)

# Generic attack examples

Take $\mathcal{E} : \{0,1\}^\kappa \times \{0,1\}^n \to \{0,1\}^n$

- Can find an unknown key with $T = 2^\kappa$, $M = O(\kappa)$, $D = O(\kappa)$, $p = 1$
- Can find an unknown key with $T = 1$, $M = 0$, $D = 0$, $p = 2^{-\kappa}$
- In general, can find an unknown key with $T = t$, $M = O(\kappa)$, $D = O(\kappa)$, $p = t/2^\kappa$

We have "small" secrets $\Rightarrow$ attacks always possible $=$ *computational security*

# A "single" measure

Define *advantage* functions associated w/ the security properties. For instance:

**$\mathbf{Adv}^{\mathsf{PRP}}$**

$\mathbf{Adv}_{\mathcal{E}}^{\mathsf{PRP}}(q,t) =$

$$\max_{A_{q,t}} \Big| \Pr[A_{q,t}^{\mathbb{O}}() = 1 : \mathbb{O} \twoheadleftarrow \mathrm{Perm}(\mathcal{M})] - \Pr[A_{q,t}^{\mathbb{O}}() = 1 : \mathbb{O} = \mathcal{E}(k,\cdot), k \twoheadleftarrow \mathcal{K}] \Big|$$

$A_{q,t}^{\mathbb{O}}$: An algorithm running in time $\leq t$, making $\leq q$ queries to $\mathbb{O}$

# “Good PRPs”

There is no formal definition of what a “good” PRP $\mathcal{E}$ is, but one can expect in that case that:

$$\mathbf{Adv}_{\mathcal{E}}^{\mathrm{PRP}}(q, t) \approx t/2^{\kappa}$$

(As long as $q \geq D \approx \lceil \kappa/n \rceil$)

- Matched by a generic attack (i.e. key guessing)
- “Equality” if $\mathcal{E}$ is ideal
- Anything that’s (sensibly) better is a *dedicated* attack

# Parameters choice

Even a good PRP is useless if its keyspace is too small

- E.g. if $\kappa = 32$, $t = 2^\kappa = 2^{32}$ is small
- But when do you know $\kappa$'s large enough?
- Look at the time/energy/infrastructure to count up to $2^\kappa$

Some examples

- $\approx 40 \rightsquigarrow$ breakable w/ a small Raspberry Pi cluster
- $\approx 60 \rightsquigarrow$ breakable w/ a large CPU/GPU cluster
  - Already done (equivalently) several times in the academia:
  - Ex. RSA-768 (Kleinjung et al., 2010), 2000 core-years ($\equiv 2^{67}$ bit operations)
  - Ex. DL-768 (Kleinjung et al., 2016), 5300 core-years
  - Ex. SHA-1 collision (Stevens et al., and me!, 2017), 6500 core-years + 100 GPU-year ($\equiv 2^{63}$ hash computations)
- $\approx 80 \rightsquigarrow$ breakable w/ an ASIC cluster (cf. Bitcoin mining)

# Parameters choice (cont.)

Two caveats:

1. Careful about multiuser security
   - If a single user changes keys *a lot* and breaking one is enough
   - If targeting one random user among many
   - A mix of the two (best!)
   - $\rightsquigarrow$ have to account for that
2. Should we care about quantum computers??
   - Would gain a $\sqrt{\cdot}$ factor
   - "128-bit classical" $\Rightarrow$ "64-bit quantum"
   - (But a direct comparison is not so meaningful, actually)

In case of doubt, 256 bits?

# Parameters choice (cont.)

What about block size?

- Security not (directly) related to computational power
- Dictated by the volume encrypted with a single key (cf. next)

In the end, it's always a cost/security tradeoff

(If you need a conventional BC with ridiculously large params, SHACAL-2, w/ $n = 256$, $\kappa = 512$ is a good choice!)

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# Block ciphers are not enough

What block ciphers do:

- One-to-one encryption of fixed-size messages

What do we want:

- One-to-many encryption of variable-size messages
- Why?
  - Variable-size → kind of obvious?
  - One-to-many → necessary for *semantic security* → cannot tell if two ciphertexts are of the same message or not

# Enter modes of operation

- A *mode of operation* transforms a block cipher into a *symmetric encryption scheme*
- $\approx \mathcal{E} \rightsquigarrow \mathsf{Enc} : \{0,1\}^\kappa \times \{0,1\}^r \times \{0,1\}^* \to \{0,1\}^*$
- For all $k \in \{0,1\}^\kappa$, $r \in \{0,1\}^r$, $\mathsf{Enc}(k, r, \cdot)$ is invertible
- $\{0,1\}^r$, $r \geq 0$ is used to make encryption non-deterministic
- A mode is "good" if it gives "good encryption schemes" when used with "good BCs"
- So what's a good encryption scheme?

# IND-CPA for Symmetric encryption

IND-CPA for Enc: An adversary cannot distinguish $\mathsf{Enc}(k, m_0)$ from $\mathsf{Enc}(k, m_1)$ for an unknown key $k$ and equal-length messages $m_0$, $m_1$ when given oracle access to an $\mathsf{Enc}(k, \cdot)$ oracle:

1. The Challenger chooses a key $k \twoheadleftarrow \{0, 1\}^\kappa$
2. The Adversary may repeatedly submit queries $x_i$ to the Challenger
3. The Challenger answers a query with $\mathsf{Enc}(k, r_i, x_i)$
4. The Adversary now submits $m_0$, $m_1$ of equal length
5. The Challenger draws $b \twoheadleftarrow \{0, 1\}$, answers with $\mathsf{Enc}(k, r', m_b)$
6. The Adversary tries to guess $b$

- The choice of $r_i$, $r'$ is defined by the mode (made explicit here, may be omitted)

# IND-CPA comments

- A random adversary succeeds w/ prob. $1/2 \rightarrow$ the correct success measure is (again) the *advantage* over this
  - (Same as for PRP security)
- An adversary may always succeed w/ advantage 1 given enough ressources
  - Find the key spending time $t \leq 2^{\kappa}$ and a few oracle queries
- What matters (again) is the "best possible" advantage in function of the attack complexity

# First (non-) mode example: ECB

- ECB: just concatenate independent calls to $\mathcal{E}$

**Electronic Code Book mode**

$m_0||m_1||\ldots \mapsto \mathcal{E}(k, m_0)||\mathcal{E}(k, m_1)||\ldots$

- No security
  - Exercise: give a simple attack on ECB for the IND-CPA security notion w/ advantage 1, low complexity

# Second (actual) mode example: CBC

- Cipher Block Chaining: Chain blocks together (duh)

**Cipher Block Chaining mode**

$r \times m_0||m_1||\ldots \mapsto c_0 := \mathcal{E}(k, m_0 \oplus r)||c1 := \mathcal{E}(k, m_1 \oplus c_0)||\ldots$

- Output block $i$ (ciphtertext) added (XORed) to input block $i+1$ (plaintext)
- For first ($m_0$) block: use random IV $r$
- Okay security in theory $\rightsquigarrow$ okay security in practice *if used properly*

# CBC IVs

CBC has bad IND-CPA security if the IVs are not random

- Consider an IND-CPA adversary who asks an oracle query CBC-ENC$(m)$, gets $r, c = \mathcal{E}(k, m \oplus r)$ (where $\mathcal{E}$ is the cipher used in CBC-ENC)
- Assume the adversary knows that for the next IV $r'$, $\Pr[r' = x]$ is large
- Sends two challenges $m_0 = m \oplus r \oplus x$, $m_1 = m_0 \oplus 1$
- Gets $c_b = \text{CBC-ENC}(m_b)$, $b \twoheadleftarrow \{0, 1\}$
- If $c_b = c$, guess $b = 0$, else $b = 1$

# Generic CBC collision attack

Even with random IVs, CBC has some drawbacks

An observation:

- In CBC, inputs to $\mathcal{E}$ are of the form $x \oplus y$ where $x$ is a message block and $y$ an IV or a ciphertext block
- If $x \oplus y = x' \oplus y'$, then $\mathcal{E}(k, x \oplus y) = \mathcal{E}(k, x' \oplus y')$

A consequence:

- If $c_i = \mathcal{E}(k, m_i \oplus c_{i-1}) = c'_j = \mathcal{E}(k, m'_j \oplus c'_{j-1})$, then $c_{i-1} \oplus c'_{j-1} = m_i \oplus m'_j$
- $\rightsquigarrow$ knowing identical ciphertext blocks reveals information about the message blocks
- $\Rightarrow$ breaks IND-CPA security
- Regardless of the security of $\mathcal{E}$ (i.e. even if it is ideal)!

# CBC collisions: how likely?

How soon does a collision happen?

- Assumption: the distribution of the $(x \oplus y)$ is $\approx$ uniform
  - If $y$ is an IV it has to be (close to) uniformly random, otherwise we have an attack (two slides ago)
  - If $y = \mathcal{E}(k, z)$ is a ciphertext block, ditto for $y$ knowing $z$, otherwise we have an attack on $\mathcal{E}$
- $\Rightarrow$ A collision occurs w.h.p. after $\sqrt{\#\{0,1\}^n} = 2^{n/2}$ blocks are observed (with identical key $k$) $\leftarrow$ *The birthday bound*
- (Slightly more precisely, w/ prob. $\approx q^2/2^n, q \leq 2^{n/2}$ after $q$ blocks)

# Some CBC recap

A decent mode, but

- Must use uniformly random IVs
- Must change key *much* before encrypting $2^{n/2}$ blocks when using an $n$-bit block cipher
- And this *regardless of the key size* $\kappa$
- Only "birthday bound" security: this is a common restriction for modes of operation (cf. next slide)

# Another classical mode: CTR

**Counter mode**

$m_0||m_1||\ldots \mapsto \mathcal{E}(k, \mathtt{s{+}{+}}) \oplus m_0||\mathcal{E}(k, \mathtt{s{+}{+}}) \oplus m_1||\ldots$

- This uses a global state $\mathtt{s}$ for the *counter*, with C-like semantics for $\mathtt{s{+}{+}}$
- Encrypts a public counter $\rightsquigarrow$ pseudo-random keystream $\rightsquigarrow$ (perfect) one-time-pad approximation (i.e. a *stream cipher*)
- Like CBC, must change key *much* before encrypting $2^{n/2}$ blocks when using an $n$-bit block cipher

# Security reduction

- For good modes such as CBC, CTR, one can prove statements of the form: “if [the mode] is instantiated with a ‘good PRP’, then this gives a ‘good IND-CPA encryption scheme’ ”
- This is an example of *security reduction* (here of the encryption scheme to the block cipher)
- Quite common & useful in crypto $\rightsquigarrow$ modular designs are nice