ISC 4220
Algorithms 1
Lab 5
Due: March 24, 2016

Conjugate Gradient Method and Interpolation

1. Write a conjugate gradient program to solve the linear system $\mathbf{Ax} = \mathbf{b}$ with

$$A = \begin{bmatrix} 38 & -18 & 0 \\ -18 & 24 & 18 \\ 0 & 18 & 21 \end{bmatrix}$$

and

$$\mathbf{b} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}.$$

The solution is

$$x^* = \begin{bmatrix} -57.500 \\ -121.500 \\ 104.333 \end{bmatrix}$$

and your program should reach this solution with three steps, since this is a 3-dimensional problem.
(a) What is the condition number of matrix **A**? Use the cond() function in MATLAB/Octave for this task. [5 points]
(b) Submit your Matla/Octave program. Report the norm of the residual at each iteration and your solution. Your program may be based on the class notes. [15 points]

2. Consider the data points

x_i	$f(x_i)$
1	1
2	2
3	5
4	45
5	12

- Use Newton's divided differences to determine a polynomial of degree 4 that interpolates through the 5 points. Submit the divided differences table (no coding is needed) [5 points].
- Use the Lagrange interpolation to determine the same polynomial. (no coding is needed) [5 points]
- Use the in-built cubic spline in Matlab/Octave (for example, `interp1` with splines) to interpolate the same data. Report the Matlab/Octave commands that are used for the task. [5 points]

- Use the in-built shape-preserving piecewise cubic interpolation in Matlab/Octave (for example, interp1 with "pchip") to interpolate the same data. Report the Matlab/Octave commands that are used for the task. Compared to the cubic spline, do you observe overshoot? [10 points]

- Plot and briefly discuss all of the interpolated polynomials. [5 points]

3. The maximum of a quadratic polynomial $p_2(x)$ passing through the points (x_0, f_0), (x_1, f_1), and (x_2, f_2) is given by:

$$x_{max} = \frac{f_0(x_1^2 - x_2^2) + f_1(x_2^2 - x_0^2) + f_2(x_0^2 - x_1^2)}{2f_0(x_1 - x_2) + 2f_1(x_2 - x_0) + 2f_2(x_0 - x_1)}.$$

Let us test this formula numerically.

- Use Newton's divided differences method to find the quadratic interpolating polynomial, $p_2(x)$, which passes through the three points (1,3), (2,5), (3,3). Report the divided differences table. [10 points]

- Find the maximum of $p_2(x)$ by solving $p_2'(x) = 0$. How does this compare with the result obtained using the above formula? [5 points]

Let us test this formula analytically.

- Use the Lagrange interpolation to write down the quadratic interpolating polynomial, $p_2(x)$, which passes through (x_0, f_0), (x_1, f_1), and (x_2, f_2). [5 points]

- By setting $p_2'(x) = 0$, solve for the x_{max}. Can you reproduce the equation above? [10 points]

No coding is needed for the problem 3.