Assignment 1

ISC 4232/5935

Due: September 16 at 4:00pm

Programming Languate: any sensible language
Submit write-up, code, and any auxilary files to Lukas Bystricky, lb13f@my.fsu.edu
Undergraduates: Submit all questions except 2d and 5
Graduates: Submit all questions

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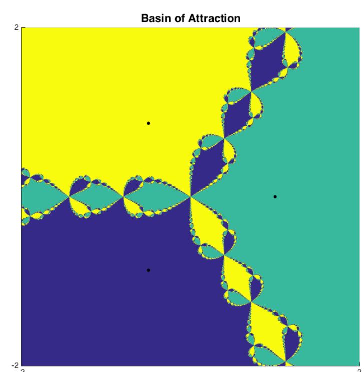
1. Construct the interpolation polynomial for a function $f(x)$ using the nodes $-h$, 0, and h. Use this polynomial to construct a finite difference method for approximating $f'(0)$. Use Taylor's theorem to find the order of convergence. Using the function $f(x) = e^x$, perform a convergence study to verify the expected convergence rate.

2. Newton's method can also be used for complex-valued functions. Consider the complex polynomial $f(z) = z^3 - 1$. This function has three roots in the complex plane $\mathbb{C}$: $z_1 = 1$, $z_2 = e^{\frac{2\pi i}{3}} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$, and $z_3 = e^{\frac{4\pi i}{3}} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$. These points are the black dots in the Figure.

   When using Newton's method on functions with multiple roots we often want to know the root that an initial guess will result in. The set of initial guesses in $\mathbb{C}$ that lead to a particular root z_k is called the basin of attraction of z_k. To visualize the basins of attraction, we run Newton's method at various points in the complex plane and see which root they converge to. We can then group all the points based on which root they go to and plot the result. These plots often form interesting patterns and are called Newton fractals.



   (a) Explain why each point on the real axis converges to z_1.

   (b) Describe what happens when an initial guess of $z^{[0]} = 0$ is used.

   (c) Reproduce the basins of attraction for $f(z) = z^3 - 1$ in the included plot. Here green points map to z_1, yellow points map to z_2, and blue points map to z_3. The plots is generated by first making a grid of initial guesses in the complex plane using `meshgrid`, looping over every point in that grid, using Newton's method with this initial guess, and computing the root that Newton converges towards. After you have a grid of points in the complex plane representing the root found, you can use the following code to generate the plot:

```
surf(real(z),imag(z),angle(root))
view(2)
shading interp
axis equal
axis([-2 2 -2 2])
```

(d) (**Grad Students Only**) With only small modifications to your code, plot the basins of attraction for $f(z) = z^6 + z^3 - 1$. The six roots are $z_1 = 0.851799642$, $z_2 = 0.5869924982 + 1.016700831i$, $z_3 = -0.425899821 + 0.737680129i$, $z_4 = -1.173984996$, $z_5 = \overline{z_4}$, and $z_6 = \overline{z_2}$. Note that the angle of the six roots are equally distributed in $[0, 2\pi]$.

3. Suppose we are given a twice continuously differentiable closed curve in $\mathbb{R}^2$ parameterized by $\mathbf{r}(\theta) = (r_1(\theta), r_2(\theta))$, $\theta \in [0, 2\pi]$ and an arbitrary point $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$. The goal is to find the closest point $\mathbf{r}(\theta_0)$ to $\mathbf{x}$.

   (a) Use the Euclidean distance to write a function that should be minimized to compute θ_0. The calculation will be much cleaner if you avoid square roots.

   (b) Using the first derivative, recast this minimization problem as a root finding problem. Interpret the resulting equation geometrically.

   (c) Solve this equation analytically when $\mathbf{r}(\theta)$ is a parameterization of the unit circle. How many solutions are there? If there are more than one, explain how this is possible.

   (d) Implement Newton's method for finding the closest point between the helix $\mathbf{r}(\theta) = (\cos(\theta), \sin(\theta), \theta)$, $\theta \in \mathbb{R}$, and an arbitrary point $\mathbf{x} \in \mathbb{R}^3$. Use a plot to show the Newton iterates for $\mathbf{x} = (2, 1, 2)$ with an initial guess of $\theta = 0$.

   (e) Use your code to find the closest point to $\mathbf{x} = (2, 1, 2)$ with 1000 initial guesses that are evenly distributed in $[-4, 4]$. Plot the number of iterations required to reach a tolerance of 10^{-5}. Stop the method if it has not converged after 1000 iterations. The exact solution is given by $\theta = 0.9516304422052698$.

   (f) The basin of attraction is defined to be the set of initial guesses that converge to the desired solution. Use the result from the previous question to estimate an interval around the exact solution that is contained in the basin of attraction.

4. Write a code to implement the forward Euler method for the prototype initial value problem $y'(t) = f(t, y)$, $y(t_0) = y_0$. As input you need to specify the initial and final times, the time step, and the initial condition. You should have separate functions for the given derivative in the problem (i.e., $f(t, y)$) and the exact solution. Test your code on each of the following problems by calculating the solution for $\Delta t = 1/4, 1/8, \ldots, 1/64$ and calculating the numerical rate of convergence. Tabulate the solution and the error at the final time for each value of Δt. Does it agree with the expected rate of convergence?

   (a) $y'(t) = t^3 y, \quad 0 < t \leq 1, \quad y(0) = 0.5.$

   (b) $y''(t) + 3y'(t) + 2y(t) = 0, \quad 0 < t \leq 1, \quad y(0) = 3, \quad y'(0) = 4.$

5. (**Grad Students Only**) Consider the motion of an ideal spring governed by $x''(t) = x(t)$ with initial conditions $x(0) = x_0$, $v(0) = v_0$.

   (a) Introduce a new variable $v(t) := x'(t)$ and recast this second-order initial value problem as a system of first-order initial value problems.

   (b) Consider the function $H = \frac{1}{2}(x^2 - v^2)$. Show that H is independent of time.

   (c) Use forward Euler to solve the system of initial value problems for $0 < t \leq 4\pi$. Plot the error of H as a function of time for a particular time step size.

(d) Perform a convergence study of the error of H at the time horizon 4π.

(e) Making only a small modification to your code, implement the semi-implicit symplectic Euler method. Repeat the two previous questions with this new time integrator and discuss the differences and similarities between the results.