# Assignment 4

ISC 4232/5935

**Due**: November 4 at 4:00pm

---

**Programming Language:** any sensible language
Submit write-up, code, and any auxiliary files to bquaife@fsu.edu
**Undergraduates:** Submit questions except 2d, 2e, and 2f
**Graduates:** Submit all questions

---

1. We have been solving the linear system

$$a_i x_{i-1} + b_i x_i + c_i x_{i+1} = d_i, \quad i = 1, \ldots, N,$$

where $a_0 = 0$ and $c_N = 0$. This linear system can be solved with the Thomas algorithm. This is done by first manipulating the vectors $c$ and $d$ with

$$c_i' = \begin{cases} \frac{c_i}{b_i}, & i = 1, \\ \frac{c_i}{b_i - a_i c_{i-1}'}, & i = 2, \ldots, N-1, \end{cases}$$

and

$$d_i' = \begin{cases} \frac{d_i}{b_i}, & i = 1, \\ \frac{d_i - a_i d_{i-1}'}{b_i - a_i c_{i-1}'}, & i = 2, \ldots, N, \end{cases}$$

where the primes denote the new coefficients. Then, the coefficients $x$ are back solved using

$$\begin{aligned} x_N &= d_N' \\ x_i &= d_i' - c_i' x_{i+1}, \quad i = N-1, N-2, \ldots, 1. \end{aligned}$$

   (a) Does the Thomas algorithm require introducing the new coefficients $c_i'$ and $d_i'$, or can these values be overwritten as they are computed? Explain.

   (b) Implement the Thomas algorithm. The inputs should be the four vectors $a, b, c, d$, and the output is $x$. Test your code on the example $a_i = 1$, $b_i = -2$, $c_i = 1$, and $d$ is a vector of zeros except $d_1 = -1$ and $d_N = -1$. The exact answer should be a vector of ones.

   (c) What is the complexity of the Thomas algorithm? Verify this complexity by running the example from part (1b) with $N = 2^{10}, 2^{11}, \ldots, 2^{20}$ and timing each of these runs. Plot the CPU time versus $N$ on a log-log scale. What do you observe? Does this make sense?

2. Consider the one-dimensional BVP with periodic boundary conditions

$$\begin{aligned} -u''(x) + u(x) &= (4\pi^2 + 1)\sin(2\pi x), \quad x \in (0,1), \\ u(0) &= u(1), \\ u'(0) &= u'(1). \end{aligned} \tag{1}$$

The unique exact solution is $u(x) = \sin(2\pi x)$.

(a) This problem is easiest to discretize with the discretization points $x_i = i\Delta x$, $i = 0, \dots, N-1$, where $\Delta x = 1/N$. Discretizing with the second centered difference and using the periodic boundary conditions, write down the corresponding linear system in matrix form.

(b) Write code that constructs the matrix from part (2a) in sparse form. Use this code to solve the linear system.

(c) Perform a convergence study. What order of convergence is observed?

(d) (**Grad Students Only**) Show that the matrix $B$ from part (2a) can be decomposed as

$$B = A + UV,$$

where $A$ is tridiagonal, $U$ is a $N \times 2$ matrix, and $V$ is a $2 \times N$ matrix

(e) (**Grad Students Only**) We can not apply the Thomas algorithm directly to $B$ since it is not tridiagonal. However, the Sherman-Morrison identity

$$B^{-1} = (A + UV)^{-1} = A^{-1} - A^{-1}U(I + VA^{-1}U)^{-1}VA^{-1}$$

can be used to write the inverse of $B$ in terms of inverses of $A$ and the inverse of a $2 \times 2$ matrix. How many evaluations of $A^{-1}$ will be required to apply Sherman-Morrison?

(f) (**Grad Students Only**) Use your Thomas algorithm code and the Sherman-Morrison identity to write a linear complexity solver for the BVP (1). Report timings to demonstrate that you are achieving linear complexity.

3. Consider the BVP

$$\begin{aligned} -u''(x) + q(x)u(x) &= f(x), \quad x \in (a, b), \\ u(a) &= A, \\ u(b) &= B. \end{aligned}$$

(a) Develop a scheme by using the centered fourth-order accurate stencil (5 point stencil) for discretization points that are at least two points away from the boundary, and using the centered second-order accurate stencil (3 point stencil) for discretization points that are next to a boundary. What is the resulting linear system? Can the Thomas algorithm be applied? Why or why not?

(b) Write a function that takes in as its inputs the number of discretization points, function handles for $q$ and $f$, boundary locations $a$ and $b$, boundary conditions $A$ and $B$, and outputs the discretization points and an approximate solution of $u$ at the discretization points.

(c) Apply your scheme to the BVP with $q(x) = x$, $f(x) = (x-1)e^x$, $A = 1$, and $B = e^1$. The exact solution is $u(x) = e^x$. Perform a convergence study. What order of convergence do you observe?