ISC 4220
Algorithms 1
Due on Feb. 18, 2016 in the Lab

LU Decomposition and Gauss-Siedel

1. From an algorithmic standpoint, it is advisable to avoid computing the inverse of a matrix. However, LU decomposition can be used if you *have* to compute $\mathbf{A}^{-1}$, by using the property $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$. For a $n \times n$ matrix $\mathbf{A}$, one can set up the following system:

$$\begin{aligned} \mathbf{A}\mathbf{A}^{-1} &= \mathbf{I} \\ \mathbf{A}\,[\mathbf{x}_1 \;\; \mathbf{x}_2 \;\; \ldots \;\; \mathbf{x}_n] &= [\mathbf{b}_1 \;\; \mathbf{b}_2 \;\; \ldots \;\; \mathbf{b}_n] \\ \mathbf{A}\mathbf{X} &= \mathbf{B} \end{aligned}$$

where $\mathbf{x}_i$ and $\mathbf{b}_i$ are $n \times 1$ vectors, $\mathbf{b}_i$ is the i^{th} column of the $n \times n$ identity matrix, and $\mathbf{A}^{-1} = [\mathbf{x}_1 \;\; \mathbf{x}_2 \;\; \ldots \;\; \mathbf{x}_n]$. Thus, we have to repeatedly solve for $\mathbf{A}\mathbf{x}_i = \mathbf{b}_i$.

Task: use the Matlab intrinsic function `[L, U, P] = lu(A)` to compute the matrix inverse of

$$\mathbf{A} = \begin{bmatrix} 6 & 15 & 55 \\ 15 & 55 & 225 \\ 55 & 225 & 979 \end{bmatrix}.$$

Verify your answer by computing $\mathbf{A}\mathbf{A}^{-1}$. *Hint*: (a) Note, $\mathbf{PAX} = \mathbf{PB} \implies \mathbf{LUX} = \mathbf{PB}$. (b) You need to write two routines to perform forward and backward substitutions for solving $\mathbf{LU}\mathbf{x}_i = \mathbf{b}'_i$, where $\mathbf{b}'_i$ is the ith column of the matrix $\mathbf{PB}$.

2. Write a program to perform Gauss-Siedel to solve the following tridiagonal system until the 2-norm of the residual is less than 0.01. Compare it with the "true" solution obtained by using Matlab's backslash operator.

$$\begin{bmatrix} 0.80 & -0.40 & 0.00 \\ -0.40 & 0.80 & -0.40 \\ 0.00 & -0.40 & 0.80 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 41 \\ 25 \\ 105 \end{bmatrix}$$

Hint: Look up the documentation on the Matlab functions `tril` and `triu` to extract the upper and lower triangular parts of a matrix.