Bonus Assignment 6:

ISC 4232/5935

Due: December 16 at 11:59pm (no extensions possible)

Programming Language: any sensible language
Submit write-up, code, and any auxiliary files to bquaife@fsu.edu
Undergraduates: Submit questions 1 and 2
Graduates: Submit all questions

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1. Consider the one-dimensional heat equation with $\nu = 1$ and Dirichlet boundary conditions

$$
\begin{aligned}
u_t &= u_{xx}, & x \in (0,1),\ t \in (0,T], \\
u(x,0) &= u_0(x), & x \in (0,1), \\
u(0,t) &= A, & t \in (0,T], \\
u(1,t) &= B, & t \in (0,T].
\end{aligned} \tag{1}
$$

   We showed in class that if centered difference is applied in space and backward Euler is applied in time, then the resulting scheme is

$$
-\lambda U_{i+1}^n + (1+2\lambda)U_i^n - \lambda U_{i-1}^n = U_i^{n-1},
$$

   where $\lambda = \Delta t / \Delta x^2$.

   (a) Write down the scheme when centered difference is applied in space and Crank-Nicolson is applied in time. Write down, in matrix form, the linear system that must be inverted at each time step. The matrix should be tridiagonal. What order of convergence do you expect in space and time? Explain.

   (b) Use the Thomas algorithm from Assignment 4 to implement a solver for the heat equation (1) using centered differences and Crank-Nicolson. Your code should take as an input the time step size, the spatial grid spacing, the time horizon T, the time-independent boundary conditions A and B, and a function handle for u_0. Be sure to handle the boundary conditions correctly.

   (c) Perform a convergence study of your code with the initial condition $u_0(x) = \sin(2\pi x)$, $A = 0$, $B = 0$ whose exact solution is $u(x,t) = \exp(-4\pi^2 t)\sin(2\pi x)$. Refine Δt and Δx at the same rate by setting $\Delta t = \Delta x$. Quantify the error as

$$
\frac{\max |U_i^n - u(x_i, T)|}{\max |u(x_i, T)|},
$$

   where $t_n = T$. Do you observe the order of convergence you were expecting?

2. Consider the heat equation with Dirichlet boundary conditions in equation (1).

   (a) Use your code from Question 1 to solve the heat equation with $u_0(x) = x^5 + 1$, $A = 1$, $B = 2$, and $T = 1$. Discretize using $\Delta t = 0.01$ and $\Delta x = 0.01$. Create and attach plots of the solution at the times $t = 0, 0.25, 0.5, 0.75, 1$.

(b) As time evolves, you should notice that the solution is tending towards a particular function. This limiting function is known as the steady-state solution. Write down a boundary value problem that the steady-state solution satisfies by taking the limit of the heat equation as $t \to \infty$.

(c) Solve the BVP for the steady-state solution. Does it agree with what you are observing in part 2a? Explain.

3. (**Grad Students Only**) Repeat Question 1 for the periodic heat equation

$$\begin{aligned} u_t &= u_{xx}, && x \in (0,1),\ t \in (0,T], \\ u(x,0) &= u_0(x), && x \in (0,1), \\ u(0,t) &= u(1,t), && t \in (0,T], \\ u_x(0,t) &= u_x(1,t), && t \in (0,T], \end{aligned}$$

using centered differences in space and your two-stage DIRK time integrator from Assignment 2. The exact solution is the same as for equation (1). You will have to use your Thomas algorithm coupled with the Sherman-Morrison formula to apply centered differences in space. In time, use your three-stage DIRK time integrator from Assignment 2. Perform a convergence study by refining Δt and Δx at the same rate.

4. (**Grad Students Only**) Consider the non-linear heat equation with Dirichlet boundary conditions

$$\begin{aligned} u_t - \epsilon u_{xx} &= u(1-u^2), && x \in (0,1),\ t \in (0,T], \\ u(x,0) &= u_0(x), && x \in (0,1), \\ u(0,t) &= -1, && t \in (0,T], \\ u(1,t) &= 1, && t \in (0,T], \end{aligned} \tag{2}$$

where ϵ is a small parameter. One way to discretize in time is to use an implicit-explicit (IMEX) method where the linear term u_{xx} is treated implicitly and the non-linear term $u(1-u^2)$ is treated explicitly.

(a) Write down a first-order scheme where backward Euler is applied to the linear term, forward Euler is applied to the non-linear term, and the spatial derivative is discretized with centered differences. This method is known as IMEX Euler.

(b) Modify your code from Question 1 to solve equation (2) using IMEX Euler. Make sure to use the Thomas algorithm so that no sparse matrices are computed.

(c) Using the time horizon $T = 50$, and the discretization $\Delta x = 10^{-3}$ and $\Delta t = 0.02$, solve equation (2) for values of ϵ in the interval $[10^{-8}, 10^{-3}]$. For an initial condition, use a random function uniformly distributed in $[-1, 1]$. The steady-state solution should be a function that is near the constant values -1 or 1 in subintervals of $[-1, 1]$, and transitions between these intervals. Describe what you observe as ϵ decreases. Include plots.

(d) The transition width can be estimated numerically by looking at a single transition, and determining how quickly it transitions from $[-1, 1]$ (or some nearby interval like $[-0.950.95]$. Estimate the transition width for several values of ϵ and plot ϵ versus the transition width. Do you observe any scaling law?