Lecture 16. Nested Quantifiers (Section 1.5)

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$$\forall x \forall y(x + y = y + x)$$

says that $x + y = y + x$ for all real numbers x and y. This is the commutative law for addition of real numbers. Likewise, the statement

$$\forall x \exists y(x + y = 0)$$

says that [1] for every real number x there is a real number y such that $x + y = 0$. This states that every real number has an additive inverse.

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Example 16.2. Translate predicate logic into English:

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- **Solution.** This statement says that for every real number x and for every real number y, if $x > 0$ and $y < 0$, then $xy < 0$. That is, for real numbers x and y, if x is positive and y is negative, then xy is negative. This can be stated more clearly as "The product of a positive real number and a negative real number is always a negative real number."

The Order of Quantifiers

Example 16.3. Let $P(x, y)$ be the statement "$x + y = y + x$." What are the truth values of the quantifications

$$\forall x \forall y\, P(x, y) \quad \text{and} \quad \forall y \forall x\, P(x, y)$$

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Example 16.4. Let $Q(x, y)$ denote "$x + y = 0$." What are the truth values of the quantifications

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Remark. Example 16.4 illustrates that the order in which quantifiers appear makes a difference. The statements $\exists y \forall x Q(x, y)$ and $\forall x \exists y Q(x, y)$ are not logically equivalent.

Rules of Inference for Quantified Statements

Name	Rule	Example
Universal Instantiation	$\forall x P(x)$ $\therefore P(c)$	"All women are brave." "Therefore, Lily is brave."
Universal Generalization	$P(c)$ *for an arbitrary c* $\therefore \forall x P(x)$	"Lily is brave." "Therefore, all women are brave."
Existential Instantiation	$\exists x P(x)$ $\therefore P(c)$ *for some element c*	"There is someone who ran a mile in 4 minutes." "Let's call him Sparky and say that Sparky ran a mile in 4 minutes."
Existential Generalization	$P(c)$ *for some element c* $\therefore \exists x P(x)$	"Sparky ran a mile in 4 minutes." "Therefore, someone ran a mile in 4 minutes."

Example 16.5. Translate the statement

$$\forall x \, [C(x) \vee \exists y \, (C(y) \wedge F(x, y))]$$

into English, where $C(x)$ is "x has a computer," $F(x, y)$ is "x and y are friends," and the domain for both x and y consists of all students in your school.

Example 16.5. Translate the statement

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into English, where $C(x)$ is "x has a computer," $F(x,y)$ is "x and y are friends," and the domain for both x and y consists of all students in your school.

Solution.

- The statement says that for every student x in your school, x has a computer or there is a student y such that y has a computer and x and y are friends.
- In other words, every student in your school has a computer or has a friend who has a computer.

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- Next, we introduce the variables x and y to obtain "For all positive integers x and y, $x + y$ is positive."

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- Next, we introduce the variables x and y to obtain "For all positive integers x and y, $x + y$ is positive."
- Consequently, we can express this statement as

$$\forall x \forall y[(x > 0) \wedge (y > 0) \to (x + y > 0)]$$

where the domain for both variables consists of all integers.

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- Next, we introduce the variables x and y to obtain "For all positive integers x and y, $x + y$ is positive."
- Consequently, we can express this statement as

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  where the domain for both variables consists of all integers.
- **Remark**. If we restrict the domain to all positive integers only, then it becomes $\forall x \forall y\, (x + y > 0)$.

Example 16.7. Use quantifiers to express the statement "There is a woman who has taken a flight on every airline in the world."

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- Let $P(w, f)$ be "w has taken f" and $Q(f, a)$ be "f is a flight on a."
- We can express the statement as

$$\exists w \forall a \exists f \left(P(w, f) \wedge Q(f, a) \right),$$

where the domains for w, f, and a, consist of all women, all airplane flights, and all airlines, respectively.

Activity 16.8. Express the following system specification using predicates, quantifiers and logical connectives.

> *No directories in the file system can be opened and no files can be closed when system errors have been detected.*