Lecture 20. Recurrence Relations

Recurrence Relations (or Recursion)

Suppose we want to define a function $P : \mathbb{N} \to \mathbb{Z}$

1. The easiest way is to give an explicit formula [1]:

$$P(n) = \frac{n(n+1)}{2}$$

2. Another way is to define recursively

$$P(n) = \begin{cases} 1 & \text{if} \quad n = 1 \\ n + P(n-1) & \text{if} \quad n > 1. \end{cases}$$

This is a recurrence relation; that is, $P(n)$ expressed in terms of one or more of the previous terms of the rule, namely, $P(0), P(1), ..., P(n-1)$.

[1] also, called a closed formula or closed-form solution to a recurrence relation

Three Laws of Recursion

A well-defined recurrence relation (recursion) obeys three important laws:

1. A recursion must have a nonrecursive base case that gives at least one value of the function explicitly.
2. A recursion must call itself, repeatedly.
3. A recursion must change its state and move toward the base case.

For example,

$$P(n) = \begin{cases} 1 & \text{if} \quad n = 1 \\ n + P(n-1) & \text{if} \quad n > 1. \end{cases}$$

satisfies all three laws.

Example 20.1. Find a recurrence relation $P(n)$ that yields the following sequence:

$$5, 11, 18, 26, 35, 45, \ldots$$

Then, find the closed formula for $P(n)$.

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- Answer:

$$P(n) = \begin{cases} 5 & \text{if} \quad n = 0 \\ P(n-1) + 5 + n & \text{if} \quad n \geq 1 \end{cases}$$

$$P(n) = 5 + 5n + \frac{n(n+1)}{2} \quad \text{for } n = 0, 1, 2, \ldots$$

Practice 20.2 Find a recursive definition and an explicit formula for the sequence below. Assume the first term listed is a_0:

$$50, 43, 36, 29, \ldots.$$

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- Answer:

$$A(n) = \begin{cases} 50 & \text{if} \quad n = 0 \\ A(n-1) - 7 & \text{if} \quad n \geq 1 \end{cases}$$

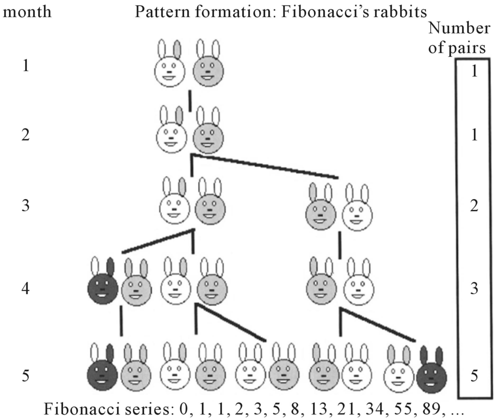
$$A(n) = 50 - 7n \text{ for } n = 0, 1, 2, \ldots$$

The Fibonacci Sequence

- In the early thirteenth century, the Italian mathematician Leonardo Pisano Fibonacci proposed the following problem.

A certain person put **a pair of rabbits** in a place surrounded by a wall.

If every month each pair begets a new pair from the second month, how many pairs of rabbits can be produced from that pair in a year?



The Fibonacci numbers $F(n)$ satisfy the following recurrence relation:

$$F(n) = \begin{cases} 1 & \text{if} \quad n = 1 \text{ or } n = 2 \\ F(n-1) + F(n-2) & \text{if} \quad n > 2. \end{cases}$$

Example 20.3. Use the above function to list the Fibonacci numbers.

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Example 20.3. Use the above function to list the Fibonacci numbers.

The Fibonacci numbers: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

Recursive Implementation of Fibonacci numbers in Python

```python
def rfib(n):
    if n<2:
        return 1
    else:
        return rfib(n-1) + rfib(n-2)
```

Practice 20.4. Suppose we model the spread of a virus in a certain population as follows. On day 1, one person is infected. On each subsequent day, each infected person gives the cold to two others.

(a) Write down a recurrence relation for this model.

(b) What are some of the limitations of this model? How does it fail to be realistic?

Practice 20.4. Suppose we model the spread of a virus in a certain population as follows. On day 1, one person is infected. On each subsequent day, each infected person gives the cold to two others.

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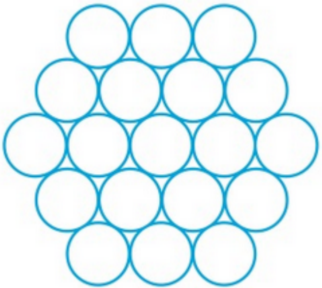
Answer: The recurrence relation $V(n)$ for the number of infected persons is

$$V(n) = \begin{cases} 1 & \text{if} \quad n = 1 \\ V(n-1) + 2V(n-1) & \text{if} \quad n > 1 \end{cases}$$

There are several unrealistic aspects: For example, nobody ever gets better, and there is no limit on the who get infected, etc.

Example 20.5. Suppose that you are collecting coins to make hexagons in a natural way by packing circles as tightly as possible.



The figure shows how 19 circles fit into a hexagonal shape with 3 circles on each edge. Let $H(n)$ be the number of circles you need to form a hexagon with n circles on each edge. From the figure it is clear that $H(2) = 7$ and $H(3) = 19$. Find a recurrence relation for $H(n)$.

Answer:

$$H(n) = \begin{cases} 7 & \text{if} \quad n = 2 \\ H(n-1) + 6n - 6 & \text{if} \quad n \geq 3. \end{cases}$$