Lecture 3. Matrix Representation of Graphs: Adjacency Matrix

What is a matrix?

A matrix is a rectangular array of elements arranged in horizontal rows and vertical columns, and usually enclosed in brackets.

$$\begin{bmatrix} 9 & 13 & 5 & 2 \\ 1 & 11 & 7 & 6 \\ 3 & 7 & 4 & 1 \\ 6 & 0 & 7 & 10 \end{bmatrix}$$

The above is a 4×4 matrix (4 rows and 4 columns).

$$\mathrm{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

A is an $m \times n$ matrix.
(m rows and n columns.)

Adjacency Matrix

If an unweighted simple graph G contains a total of n vertices, we can define an $n \times n$ matrix A by

$$a_{ij} = \begin{cases} 1 & \text{if } [v_i, v_j] \text{ is an edge of } G \\ 0 & \text{if there is no edge joining } v_i \text{ and } v_j \end{cases}$$

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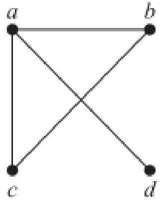
We order the vertices as a, b, c, d. That is, $v_1 = a, v_2 = b, v_3 = c, v_4 = d$. The matrix representing this graph is



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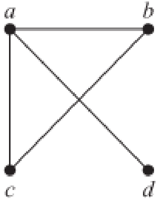

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

For a weighted simple graph G

$$a_{ij} = \begin{cases} w_{ij} & \text{if the length of the edge } [v_i, v_j] = w_{ij} \\ 0 & \text{if there is no edge joining } v_i \text{ and } v_j \end{cases}$$

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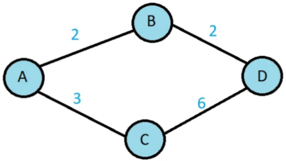

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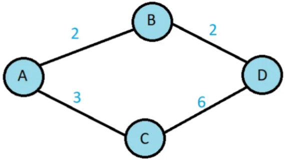

$$\begin{array}{c c} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 2 & 3 & 0 \\ 2 & 0 & 0 & 2 \\ 3 & 0 & 0 & 6 \\ 0 & 2 & 6 & 0 \end{bmatrix} \end{array}$$

Practice 3.3. Draw an unweighted simple graph with the adjacency matrix

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

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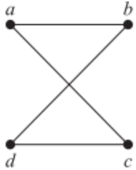

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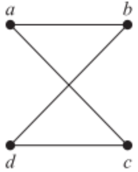

Note that an adjacency matrix of a graph is based on the ordering chosen for the vertices. Hence, there may be as many as n! different adjacency matrices for a graph with n vertices, because there are n! different orderings of n vertices.

Also, note that adjacency matrices of undirected graphs are symmetric.

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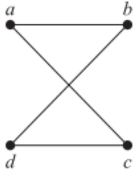


Note that an adjacency matrix of a graph is based on the ordering chosen for the vertices. Hence, there may be as many as $n!$ different adjacency matrices for a graph with n vertices, because there are $n!$ different orderings of n vertices.

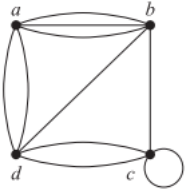
Also, note that adjacency matrices of undirected graphs are symmetric.

Question. What is the sum of the entries in a row of the adjacency matrix for an undirected simple graph?

Adjacency matrices can also be used to represent undirected graphs with loops and with multiple edges.

- A loop at the vertex v_i is represented by a 1 at the (i, i)th position of the adjacency matrix. [1]
- When multiple edges connecting the same pair of vertices v_i and v_j, or multiple loops at the same vertex, are present, the (i, j)th entry of the adjacency matrix equals the number of edges that are associated to $[v_i, v_j]$.

Example 3.4. Use an adjacency matrix to represent the graph



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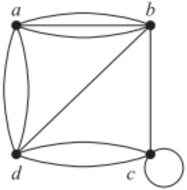

$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

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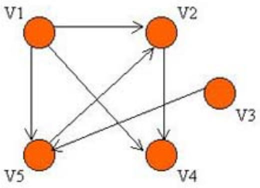
The adjacency matrix for a directed graph does not have to be symmetric, because there may not be an edge from v_j to v_i when there is an edge from v_i to v_j.

Practice 3.5. Use an adjacency matrix to represent the directed graph



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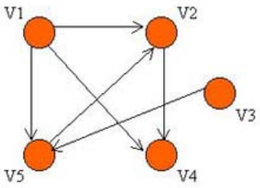

	v1	v2	v3	v4	v5
v1	0	1	0	1	1
v2	0	0	0	1	0
v3	0	0	0	0	1
v4	0	0	0	0	0
v5	0	1	0	0	0

Question. What is the sum of the entries in a row of the adjacency matrix for a directed graph?