# Lecture 19. Rules of Inference for Quantified Statements

Review from Lecture 18:

- **Universal instantiation** is the rule of inference used to conclude that $P(c)$ is true, where $c$ is a particular member of the domain, given the premise $\forall x\, P(x)$.
- **Universal Generalization** is the rule of inference that states that $\forall x\, P(x)$ is true, given the premise that $P(c)$ is true for all elements $c$ in the domain. (The element $c$ that we select must be an arbitrary, and not a specific element of the domain.)
- **Existential instantiation** is the rule saying that there is an element $c$ in the domain for which $P(c)$ is true if we know that $\exists x\, P(x)$ is true. (Usually we have no knowledge of what $c$ is, only that it exists.)
- **Existential generalization** is the rule of inference stating that $\exists x\, P(x)$ is true when $P(c)$ is true for a particular element $c$. (That is, if we know one element $c$ in the domain for which $P(c)$ is true, then we know that $\exists x\, P(x)$ is true.)

**Example 19.1.** Show that two premises "ChatGPT is an artificial-intelligence chatbot" and "Every artificial-intelligence chatbot can summarize a newsfeed article" imply the conclusion "ChatGPT can summarize a newsfeed article". Here the domain contains all chatbots.

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| Step | Reasoning |
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| (1) $\forall x(A(x) \to S(x))$ | Premise |

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| (2) $A(ChatGPT) \to S(ChatGPT)$ | Universal instantiation from (1) |

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| (1) $\forall x(A(x) \to S(x))$ | Premise |
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| (3) $A(ChatGPT)$ | Premise |

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|---|---|
| (1) $\forall x(A(x) \rightarrow S(x))$ | Premise |
| (2) $A(ChatGPT) \rightarrow S(ChatGPT)$ | Universal instantiation from (1) |
| (3) $A(ChatGPT)$ | Premise |
| (4) $S(ChatGPT)$ | Modus ponens from (2) and (3) |

**Example 19.2.** A logical proof that uses the laws of inference for quantified statements: Assume the domain is all integers.

q $\forall x\,(P(x) \vee Q(x))$

3 is an integer

$\neg P(3)$

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$\therefore Q(3)$

**Example 19.2.** A logical proof that uses the laws of inference for quantified statements: Assume the domain is all integers.

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$$\begin{array}{l} \forall x\,(P(x) \vee Q(x)) \\ 3 \text{ is an integer} \\ \neg P(3) \\ \hline \therefore Q(3) \end{array}$$

| | | |
|---|---|---|
| 1 | $\forall x\,(P(x) \vee Q(x))$ | Premise |
| 2 | 3 is an integer | Premise |

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| 2 | 3 is an integer | Premise |
| 3 | $(P(3) \vee Q(3))$ | Universal instantiation, 1, 2 |

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| 1 | $\forall x\,(P(x) \vee Q(x))$ | Premise |
| 2 | 3 is an integer | Premise |
| 3 | $(P(3) \vee Q(3))$ | Universal instantiation, 1, 2 |
| 4 | $\neg P(3)$ | Premise |

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| 1 | $\forall x\,(P(x) \vee Q(x))$ | Premise |
| 2 | 3 is an integer | Premise |
| 3 | $(P(3) \vee Q(3))$ | Universal instantiation, 1, 2 |
| 4 | $\neg P(3)$ | Premise |
| 5 | $Q(3)$ | Disjunctive syllogism, 3, 4 |

**Example 19.3**. Indicate whether the proof fragment is a correct or incorrect use of the rule of inference.

(a) True or False?

| | | |
|---|---|---|
| 1. | c is an element | Hypothesis |
| 2. | P(c) | Hypothesis |
| 3. | $\forall x\, P(x)$ | Universal generalization, 1, 2 |

(b) True or False?

| | | |
|---|---|---|
| 1. | $\exists x\, P(x)$ | Hypothesis |
| 2. | (c is a particular element) $\wedge$ P(c) | Existential instantiation, 1 |
| 3. | $\exists x\, Q(x)$ | Hypothesis |
| 4. | (c is a particular element) $\wedge$ Q(c) | Existential instantiation, 3 |

(c) True or False?

| | | |
|---|---|---|
| 1. | c is an element | Hypothesis |
| 2. | ∀x P(x) | Hypothesis |
| 3. | P(c) | Universal instantiation, 1, 2 |

(d) True or False?

| | | |
|---|---|---|
| 1. | c is an element | Hypothesis |
| 2. | P(c) | Hypothesis |
| 3. | d, an element | Hypothesis |
| 4. | Q(d) | Hypothesis |
| 5. | P(c) ∧ Q(d) | Conjunction, 2, 3 |
| 6. | ∃x (P(x) ∧ Q(x)) | Existential generalization, 1, 3, 5 |

**Example 19.4** Show that the premises "A student in this class has not read the book," and "Everyone in this class passed the first exam" imply the conclusion "Someone who passed the first exam has not read the book."

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Let $C(x)$ be "$x$ is in this class," $B(x)$ be "$x$ has read the book," and $P(x)$ be "$x$ passed the first exam."

The premises are $\exists x(C(x) \wedge \neg B(x))$ and $\forall x(C(x) \to P(x))$. The conclusion is $\exists x(P(x) \wedge \neg B(x))$.

| **Step** | **Reasoning** |
| --- | --- |

**Example 19.4** Show that the premises "A student in this class has not read the book," and "Everyone in this class passed the first exam" imply the conclusion "Someone who passed the first exam has not read the book."

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| **Step** | **Reasoning** |
|---|---|
| (1) $\exists x(C(x) \wedge \neg B(x))$ | Premise |

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| **Step** | **Reasoning** |
|---|---|
| (1) $\exists x(C(x) \wedge \neg B(x))$ | Premise |
| (2) $C(a) \wedge \neg B(a)$ | Existential instantiation from (1) |

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| **Step** | **Reasoning** |
| --- | --- |
| (1) $\exists x(C(x) \wedge \neg B(x))$ | Premise |
| (2) $C(a) \wedge \neg B(a)$ | Existential instantiation from (1) |
| (3) $C(a)$ | Simplification from (2) |

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Let $C(x)$ be "$x$ is in this class," $B(x)$ be "$x$ has read the book," and $P(x)$ be "$x$ passed the first exam."

The premises are $\exists x(C(x) \land \neg B(x))$ and $\forall x(C(x) \to P(x))$. The conclusion is $\exists x(P(x) \land \neg B(x))$.

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| (1) $\exists x(C(x) \land \neg B(x))$ | Premise |
| (2) $C(a) \land \neg B(a)$ | Existential instantiation from (1) |
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| (4) $\forall x(C(x) \to P(x))$ | Premise |

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| (1) $\exists x(C(x) \wedge \neg B(x))$ | Premise |
| (2) $C(a) \wedge \neg B(a)$ | Existential instantiation from (1) |
| (3) $C(a)$ | Simplification from (2) |
| (4) $\forall x(C(x) \rightarrow P(x))$ | Premise |
| (5) $C(a) \rightarrow P(a)$ | Universal instantiation from (4) |

**Example 19.4** Show that the premises "A student in this class has not read the book," and "Everyone in this class passed the first exam" imply the conclusion "Someone who passed the first exam has not read the book."

Let $C(x)$ be "$x$ is in this class," $B(x)$ be "$x$ has read the book," and $P(x)$ be "$x$ passed the first exam."

The premises are $\exists x(C(x) \land \neg B(x))$ and $\forall x(C(x) \to P(x))$. The conclusion is $\exists x(P(x) \land \neg B(x))$.

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| (1) $\exists x(C(x) \land \neg B(x))$ | Premise |
| (2) $C(a) \land \neg B(a)$ | Existential instantiation from (1) |
| (3) $C(a)$ | Simplification from (2) |
| (4) $\forall x(C(x) \to P(x))$ | Premise |
| (5) $C(a) \to P(a)$ | Universal instantiation from (4) |
| (6) $P(a)$ | Modus ponens from (3) and (5) |

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| (1) $\exists x(C(x) \wedge \neg B(x))$ | Premise |
| (2) $C(a) \wedge \neg B(a)$ | Existential instantiation from (1) |
| (3) $C(a)$ | Simplification from (2) |
| (4) $\forall x(C(x) \rightarrow P(x))$ | Premise |
| (5) $C(a) \rightarrow P(a)$ | Universal instantiation from (4) |
| (6) $P(a)$ | Modus ponens from (3) and (5) |
| (7) $\neg B(a)$ | Simplification from (2) |

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The premises are $\exists x(C(x) \wedge \neg B(x))$ and $\forall x(C(x) \rightarrow P(x))$. The conclusion is $\exists x(P(x) \wedge \neg B(x))$.

| **Step** | **Reasoning** |
|---|---|
| (1) $\exists x(C(x) \wedge \neg B(x))$ | Premise |
| (2) $C(a) \wedge \neg B(a)$ | Existential instantiation from (1) |
| (3) $C(a)$ | Simplification from (2) |
| (4) $\forall x(C(x) \rightarrow P(x))$ | Premise |
| (5) $C(a) \rightarrow P(a)$ | Universal instantiation from (4) |
| (6) $P(a)$ | Modus ponens from (3) and (5) |
| (7) $\neg B(a)$ | Simplification from (2) |
| (8) $P(a) \wedge \neg B(a)$ | Conjunction from (6) and (7) |

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| (1) $\exists x(C(x) \wedge \neg B(x))$ | Premise |
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| (3) $C(a)$ | Simplification from (2) |
| (4) $\forall x(C(x) \rightarrow P(x))$ | Premise |
| (5) $C(a) \rightarrow P(a)$ | Universal instantiation from (4) |
| (6) $P(a)$ | Modus ponens from (3) and (5) |
| (7) $\neg B(a)$ | Simplification from (2) |
| (8) $P(a) \wedge \neg B(a)$ | Conjunction from (6) and (7) |
| (9) $\exists x(P(x) \wedge \neg B(x))$ | Existential generalization from (8) |

**Practice 19.5**. Consider the argument

> *"There is a self-driving car that has been involved in a fatal accident. All self-driving cars are designed to prioritize passenger safety. Therefore, there is a self-driving car designed to prioritize passenger safety that has been involved in a fatal accident.*

Determine whether the argument is valid. Justify your answer by explaining which rules of inference are used for each step or by finding a counterexample.

**MSC 19.6** Consider the following argument based on Asimov's Three Laws of Robotics:

*1. "All robots are programmed to prioritize preventing harm to humans above all else."*

*2. "All robots are programmed to obey human orders unless those orders conflict with preventing harm to humans."*

*3. "All robots are programmed to protect their own existence only when it does not conflict with preventing harm to humans or obeying human orders."*

*4. "A robot has been ordered to save a child from a collapsing building on the other side of the street."*

*5. "Therefore, there is a robot that will risk its own destruction to save the child."*

Formalize this argument in predicate logic and determine its validity by providing a step-by-step proof sequence.