Lecture 23. Linear Nonhomogeneous Recurrence Relations

Definition 23.1. A linear nonhomogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_k a_{n-k} + F(n)$$

where $r_1, r_2, ..., r_k$ are real numbers $(r_k \neq 0)$ with $k < n$ and $F(n)$ is a function not identically zero depending only on n. The same recurrence with $F(n)$ omitted is called the associated homogeneous recurrence relation.

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Algorithm of solving a nonhomogeneous recurrence relation:

1. Find the solution to the associated homogeneous recurrence relation $a_n^{(h)}$.
2. Find a particular solution of the nonhomogeneous linear recurrence relation $a_n^{(p)}$.
3. $a_n^{(h)} + a_n^{(p)}$ forms the solution to the nonhomogeneous recurrence relation:

$$a_n = a_n^{(h)} + a_n^{(p)}.$$

Example 23.2. Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$ with $a_1 = 3$.

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Answer: $a_n = -n - \frac{3}{2} + \frac{11}{6} 3^n$.

Problem 23.3. Find all solutions of the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$.

Theorem 23.4. Suppose that a_n satisfies the linear nonhomogeneous recurrence

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_k a_{n-k} + F(n),$$

where $r_1, r_2, \ldots, r_k$ are real numbers and

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0) s^n,$$

where $b_0, \ldots b_t$ and s are real numbers.

(a) When s is not a root of the characteristic equation of the associated linear homogeneous recurrence, there is a particular solution of the form

$$a_n^{(p)} = (p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n,$$

(b) When s is a root of this characteristic equation and its multiplicity is m, there is a particular solution of the form

$$a_n^{(p)} = n^m (p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n,$$

Example 23.5. What form does a particular solution of the linear nonhomogeneous recurrence

$$a_n = 6a_{n-1} - 9a_{n-2} + F(n)$$

have when $F(n) = 3^n$, $F(n) = n3^n$, $F(n) = n^2 2^n$, and $F(n) = (n^2 + 1)3^n$?

$F(n)$	$a_n^{(p)}$
3^n	$n^2 p_0 3^n$
$n3^n$	$n^2(p_1 n + p_0)3^n$
$n^2 2^n$	$(p_2 n^2 + p_1 n + p_0)2^n$
$n^2 3^n$	$n^2(p_2 n^2 + p_1 n + p_0)3^n$

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Problem 23.6 on WeBWork. Solve the linear nonhomogeneous recurrence relation

$$a_n = 2a_{n-1} - a_{n-2} + 2^n + 2$$

with initial conditions $a_1 = 7$ and $a_2 = 19$.
(Test Time Limit on WeBWork: 15 min)