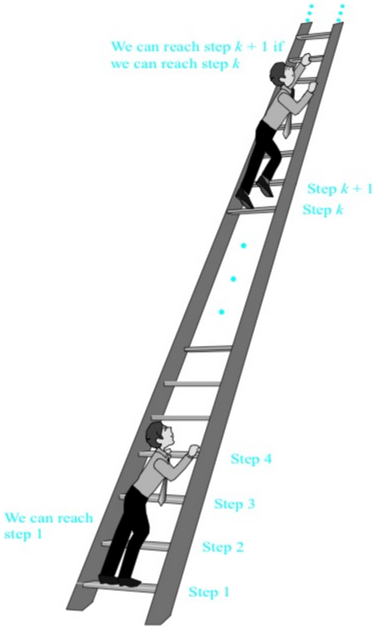
Lecture 20. Mathematical Induction



Motivation: Climbing an Infinite Ladder

Suppose we have an infinite ladder and the following capabilities:

1. We can reach the first rung of the ladder.
2. If we can reach a particular rung of the ladder, then we can reach the next rung.

Principle of Mathematical Induction

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  2. **Inductive Step**: Let $k > 1$ be some (unspecified) arbitrary integer. If $P(k) = f(k)$, show that $P(k+1) = f(k+1)$. (In other wors, show that $P(k) \to P(k+1)$ is true for all $k \in \mathbb{Z}^+$.)

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- **Remark:** Mathematical induction can be expressed as the rule of inference (here the domain is the set of all positive integers)

$$[P(1) \wedge \forall k [P(k) \to P(k+1)]] \to \forall n\, P(n)$$

Example 20.1. Show that if n is a positive integer, then

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Solution to Example 1

Solution: Let $P(n)$ be the proposition that the sum of the first n positive integers, $1 + 2 + \cdots n = \frac{n(n+1)}{2}$, is $n(n+1)/2$. We must do two things to prove that $P(n)$ is true for $n = 1, 2, 3, \ldots$. Namely, we must show that $P(1)$ is true and that the conditional statement $P(k)$ implies $P(k+1)$ is true for $k = 1, 2, 3, \ldots$.

BASIS STEP: $P(1)$ is true, because $1 = \dfrac{1(1+1)}{2}$. (The left-hand side of this equation is 1 because 1 is the sum of the first positive integer. The right-hand side is found by substituting 1 for n in $n(n+1)/2$.)

INDUCTIVE STEP: For the inductive hypothesis we assume that $P(k)$ holds for an arbitrary positive integer k. That is, we assume that

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}.$$

Under this assumption, it must be shown that $P(k+1)$ is true, namely, that

$$1 + 2 + \cdots + k + (k+1) = \frac{(k+1)[(k+1)+1]}{2} = \frac{(k+1)(k+2)}{2}$$

is also true. When we add $k+1$ to both sides of the equation in $P(k)$, we obtain

$$\begin{aligned} 1 + 2 + \cdots + k + (k+1) &\overset{\text{IH}}{=} \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2}. \end{aligned}$$

This last equation shows that $P(k+1)$ is true under the assumption that $P(k)$ is true. This completes the inductive step.

We have completed the basis step and the inductive step, so by mathematical induction we know that $P(n)$ is true for all positive integers n. That is, we have proven that $1 + 2 + \cdots + n = n(n+1)/2$ for all positive integers n. ◀

Practice 20.2. Conjecture a formula for the sum of the first n positive odd integers. Then prove your conjecture using mathematical induction.

Example 20.3. The closed-form solution $f(n)$ for the recurrence relation

$$H(n) = \begin{cases} 1 & \text{if} \quad n = 1 \\ H(n-1) + 6n - 6 & \text{if} \quad n > 1 \end{cases}$$

was

$$f(n) = 3n^2 - 3n + 1, \ \forall n = 1, 2, ...$$

Use the mathematical induction to prove that $H(n) = f(n)$ for all $n \geq 1$.