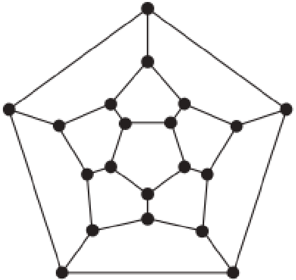
# Lecture 6. Hamilton Paths and Circuits [1]

[1] This terminology comes from a game, called the Icosian puzzle, invented in 1857 by the Irish mathematician Sir William Rowan Hamilton.

Does a path or circuit exist that uses every vertex exactly once?



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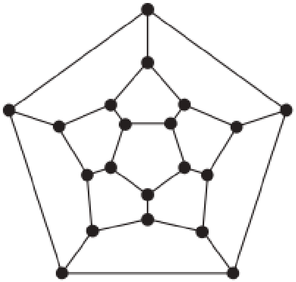
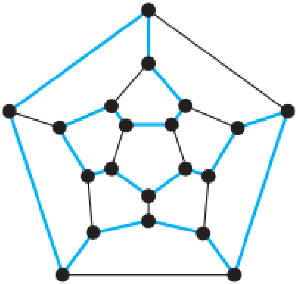
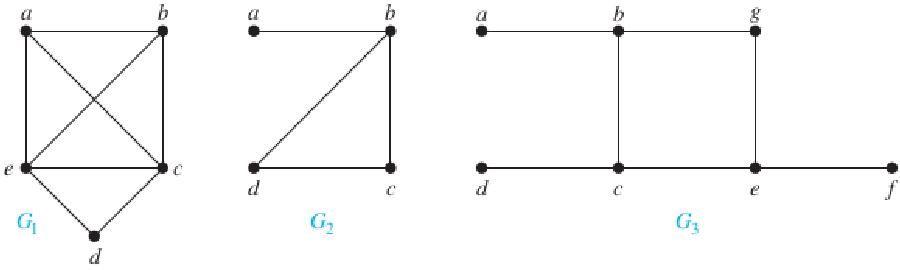




- A Hamilton path in a graph $G$ is a simple path **containing every vertex of $G$ exactly once**. That is, a Hamilton path is a path that visits every vertex exactly once (allowing for revisiting edges).
- A Hamilton circuit in a graph $G$ is a Hamilton path that starts and ends on the same vertex.

**Example 6.1.** Which graphs have a Hamilton circuit or, if not, a Hamilton path?



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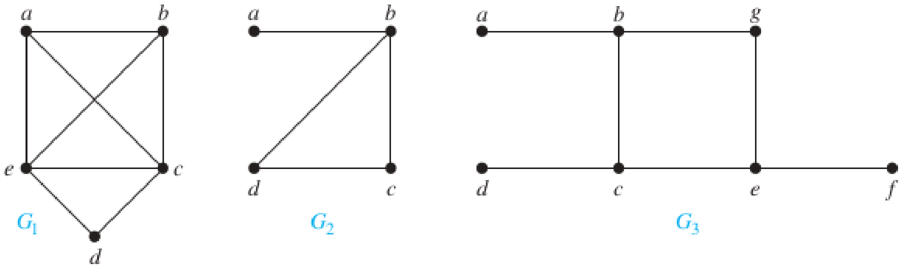


**Solution:** $G_1$ has a Hamilton circuit: $a, b, c, d, e, a$. There is no Hamilton circuit in $G_2$ (this can be seen by noting that any circuit containing every vertex must contain the edge $\{a, b\}$ twice), but $G_2$ does have a Hamilton path, namely, $a, b, c, d$. $G_3$ has neither a Hamilton circuit nor a Hamilton path, because any path containing all vertices must contain one of the edges $\{a, b\}$, $\{e, f\}$, and $\{c, d\}$ more than once.

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> *Finding efficient algorithm = ultimate computer science glory.*

The best algorithms known so far for finding a Hamilton circuit in a graph, or determining that no such circuit exists, have exponential worst-case time complexity (in the number of vertices of the graph), which is incredibly slow for sufficiently large graphs.

Finding an algorithm that solves this problem with polynomial worst-case time complexity would be a major accomplishment, and you would probably be given every single computer science award in existence.

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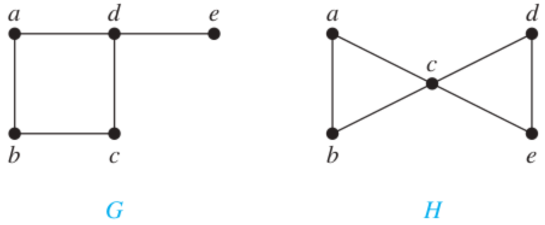
**Example 6.2.** Use the above properties to explain why the following graphs don't have Hamilton circuits.



**Activity 6.3**. For what values of $m$ and $n$ does the complete bipartite graph $K_{m,n}$ have a Hamiltonian circuit? Explain your reasoning. Also, how many different Hamilton circuits exist in the graph?