Lecture 9. Introduction to Trees

Trees

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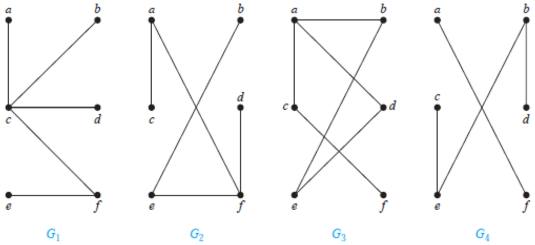

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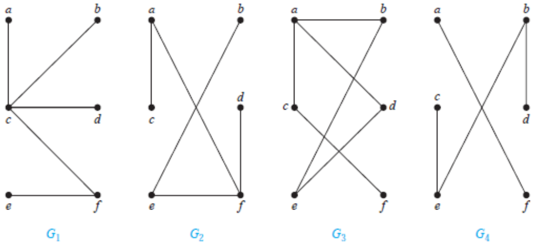


Theorem 9.3. An undirected graph is a tree if and only if there is a unique simple path between any two of its vertices.

Definition 9.4. A forest is a graph with no simple circuit but is not connected. Each of the connected components in a forest is a tree.

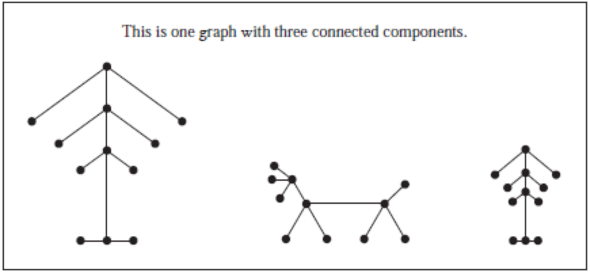
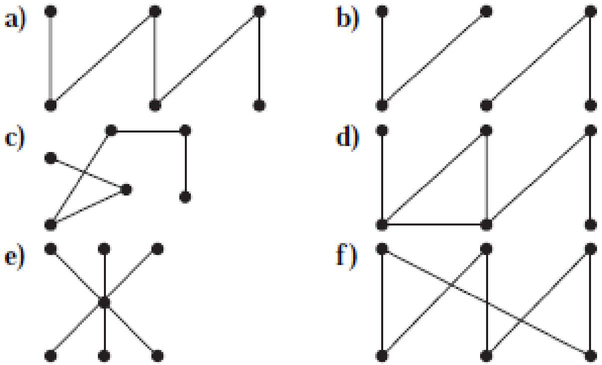


FIGURE 3 **Example of a Forest.**

Example 9.5. Which of these are trees? Any forest?



Definition 9.6. A rooted tree is a tree in which one vertex has been designated as the root and every edge is directed away from the root.

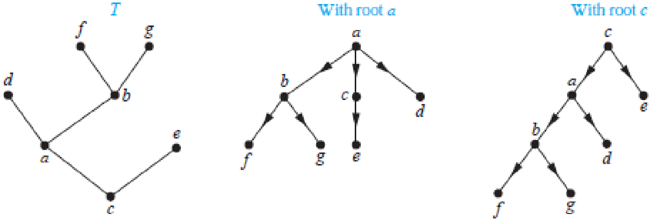
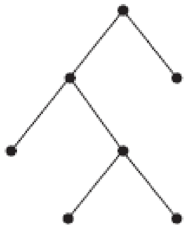


FIGURE 4 A Tree and Rooted Trees Formed by Designating Two Different Roots.

An unrooted tree can be converted into different rooted trees when different vertices are chosen as the root.

Terminology for Rooted Trees [1]

If v is a vertex of a rooted tree other than the root, the parent of v is the unique vertex u such that there is a directed edge from u to v.



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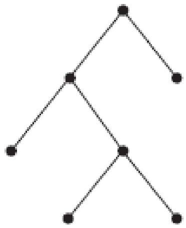

When u is a parent of v, v is called a child of u. Vertices with the same parent are called siblings.

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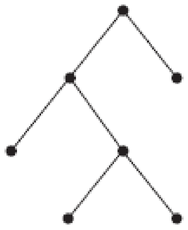
The ancestors of a vertex are the vertices in the path from the root to this vertex, excluding the vertex itself and including the root.



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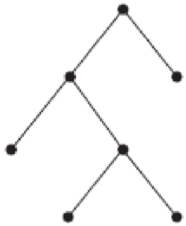
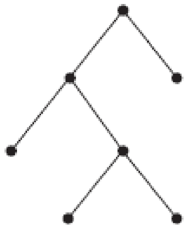


The descendants of a vertex v are those vertices that have v as an ancestor.

A vertex of a rooted tree with no children is called a leaf. Vertices that have children are called internal vertices.



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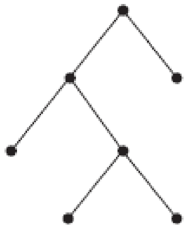
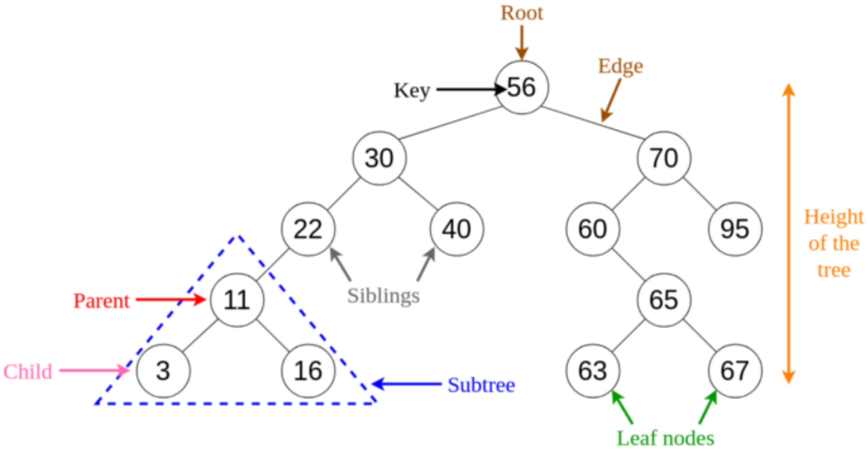
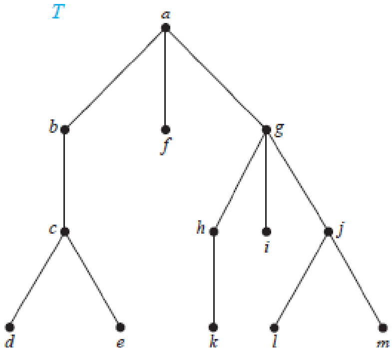


If a is a vertex in a tree, the subtree with a as its root is the subgraph of the tree consisting of a and its descendants and all edges incident to these descendants.


Root
Key
56
Edge
30
70
22
40
60
95
Height of the tree
Parent
11
Siblings
65
Child
3
16
Subtree
63
67
Leaf nodes

Practice 9.7. In the rooted tree T (with root a):

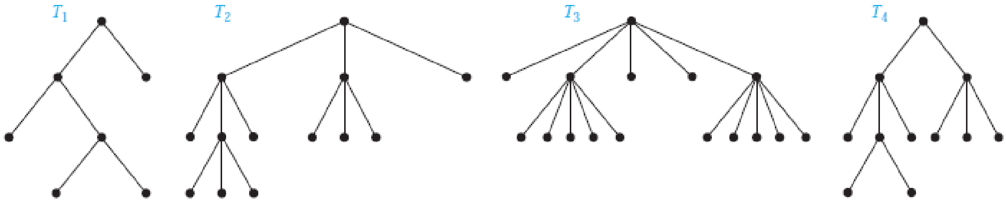
1. Find the parent of c, the children of g, the siblings of h, the ancestors of e, and the descendants of b.
2. Find all internal vertices and all leaves.
3. What is the subtree rooted at g?



m-ary Rooted Trees

Definition 9.8. A rooted tree is called an m-ary tree if every internal vertex has no more than m children. The tree is called a full m-ary tree if every internal vertex has exactly m children. An m-ary tree with $m = 2$ is called a binary tree.

Example 9.9. Are the following rooted trees full m-ary trees for some positive integer m?



Properties of Trees

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1. A *full m-ary tree* **with n vertices** has $i = (n-1)/m$ internal vertices and $\ell = [(m-1)n+1]/m$ leaves,
2. A *full m-ary tree* **with i internal vertices** has $n = mi + 1$ vertices and $\ell = (m-1)i + 1$ leaves,
3. A *full m-ary tree* **with ℓ leaves** has $n = (m\ell - 1)/(m-1)$ vertices and $i = (\ell - 1)/(m-1)$ internal vertices.

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Example 9.13. How many edges and leaves does a full binary tree with 2000 internal vertices have?

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Practice 9.14. Suppose that someone starts a chain letter. Each person who receives the letter is asked to send it on to four other people. Some people do this, but others do not send any letters. How many people have seen the letter, including the first person, if no one receives more than one letter and if the chain letter ends after there have been 100 people who read it but did not send it out? How many people sent out the letter?

Depth and Height

The depth $d(v)$ of a node v in a rooted tree is the number of edges in the path from the root to v. The height of a rooted tree is the maximum value of $d(v)$ over all the nodes in the tree. In other words, the height of a rooted tree is the length of the longest path from the root to any vertex.

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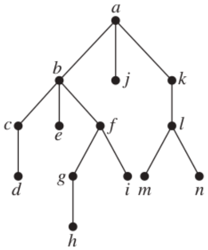
Example 9.15. Find the depth of each vertex in the rooted tree shown below. What is the height of this tree?



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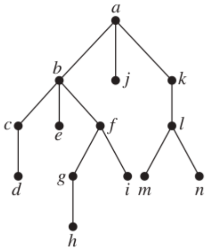
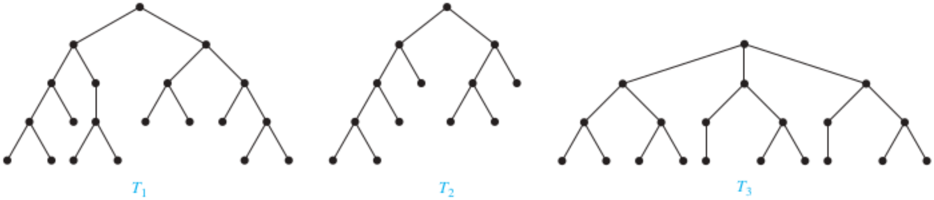


Theorem 9.16. There are at most m^h leaves in an m-ary tree of height h; $\ell \leq m^h$.

A rooted m-ary tree of height h is balanced if all leaves are at depths h or $h-1$.

Example 9.17. Which of the rooted trees are balanced?



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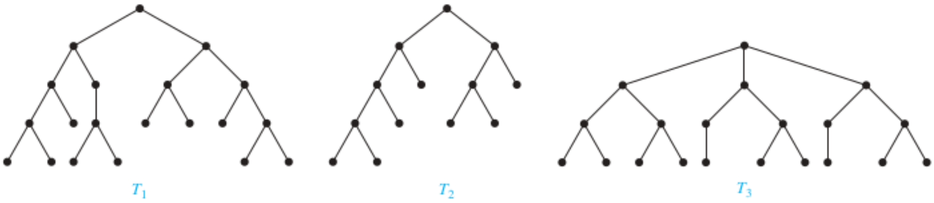

Corollary 9.18. If an m-ary tree of height h has ℓ leaves, then $h \geq \lceil \log_m \ell \rceil$.
If the m-ary tree is full and balanced, then $h = \lceil \log_m \ell \rceil$.

(We are using the ceiling function here. Recall that $\lceil x \rceil$ is the smallest integer greater than or equal to x.)

Activity 9.19. A full m-ary tree T has 81 leaves and height 4.

(a) Give the upper and lower bounds for m.

(b) What is m if T is also balanced?