Lecture 19. Solving Recurrence Relations

Recursive Thinking

In the previous lecture note, we have seen some examples how to think recursively about a problem by describing it with a recurrence relation. Remember that any recurrence relation has two parts: a **base case** that describes some initial conditions, and a **recursive case** that describes a **future value in terms of previous values**. Armed with this way of thinking, we can model other problems using recurrence relations.

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Example 19.1. Ahmed lends money at outrageous rates of interest. He demands to be paid 10% interest *per week* on a loan, compounded weekly. Suppose you borrow 500 Dhs from him. Let $M(n) =$ the money you owed at n-th week.

(a) Find the recurrence relation for $M(n)$.

(b) If you wait four weeks to pay him back, how much will you owe?

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Answer:

$$M(n) = \begin{cases} 500 & \text{if} \quad n = 0 \\ 1.10\, M(n-1) & \text{if} \quad n > 0 \end{cases} \qquad M(4) = \$732.05$$

Practice 19.2. Find a closed-form solution for the recurrence relation from

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Answer: $M(n) = 500(1.10)^n$

Polynomial sequences: Using differences

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then we can detect such a sequence by looking at the differences between terms. Given any sequence,

$$a_0, a_1, a_2, ..., a_{n-1}, a_n$$

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form another sequence, called the sequence of differences.

- A linear sequence $a_n = An + B$ will have a constant sequence of differences (because a line has constant slop).
- A quadratic sequence $a_n = An^2 + Bn + C$ will have a linear sequence of differences.
- a cubic sequence $a_n = An^3 + Bn^2 + Cn + D$ will have a quadratic sequence of differences, etc.
- If we eventually end up with a constant sequence, then the original sequence is given by a polynomial function.
- The degree of the conjectured polynomial is the number of times we had to calculate the sequence of differences.

Example 19.3. Find a closed-form solution $f(n)$ for the recurrence relation

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Answer. $H(n) = 3n^2 - 3n + 1$ is a good candidate for a closed-form solution.

Remark. The result of these procedures is still only a guess. To be sure that our guess is right, we need to prove that the formula matches the recurrence relation for all n.

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Answer. $a_n = \frac{n(n+1)}{2} + 4$ for $n \geq 0$.

Linear Homogeneous Recurrence Relations

Definition 19.5. A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_k a_{n-k}$$

where $r_1, r_2, \ldots, r_k$ are real numbers and $r_k \neq 0$ with $k < n$. This recurrence includes k initial conditions, $a_0 = \alpha_0, a_1 = \alpha_1, \ldots, a_k = \alpha_k$.

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Example 19.6. Linear and/or Homogeneous?

$$(a)\ M_n = (1.1)M_{n-1} \quad (b)\ F_n = F_{n-1} + F_{n-2} \quad (c)\ a_n = a_{n-1} + a_{n-2}^2 \quad (d)\ h_n = 2h_{n-1} + 1$$

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(a) linear homogeneous recurrence relation of degree 1, (b) linear homogeneous recurrence relation of degree 2, (c) not linear, (d) not homogeneous,

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- Rearranging terms leads to the characteristic equation:

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Theorem 19.7. The characteristic equation of the recurrence relation $a_n = r_1 a_{n-1} + r_2 a_{n-2}$ is

$$x^2 - r_1 x - r_2 = 0.$$

If the characteristic equation has two distinct roots, x_1 and x_2, then

$$a_n = p x_1^n + q x_2^n \text{ for some } p, q$$

is the explicit formula for the sequence. Here, p and q depend on the initial conditions.

Example 19.8. Find an explicit formula for the sequence defined by $a_n = 7a_{n-1} - 10a_{n-2}$ with $a_0 = 2$ and $a_1 = 3$.

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Answer: $a_n = \frac{7}{3}2^n - \frac{1}{3}5^n$.

Theorem 19.7. If the characteristic equation

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of the recurrence relation $a_n = r_1 a_{n-1} + r_2 a_{n-2}$ has two distinct roots, x_1 and x_2, then

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where p and q depend on the initial conditions, is the explicit formula for the sequence.

Practice 19.9. (Fibonacci Sequence) Solve the recurreence relation $f_n = f_{n-1} + f_{n-2}$ with the initial conditions $f_0 = 0$ and $f_1 = 1$.

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Answer: $f_n = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}}$, where $\phi = \frac{1+\sqrt{5}}{2}$. (Note that $\phi \approx 1.618\ 033\ 988\ 749...$ is a golden ratio.)

Theorem 19.10. If the characteristic equation

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Practice 19.11. Solve the recurrence relation $b_n = 6b_{n-1} - 9b_{n-2}$ with the initial conditions $b_0 = 1$ and $b_1 = 4$.

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Practice 19.11. Solve the recurrence relation $b_n = 6b_{n-1} - 9b_{n-2}$ with the initial conditions $b_0 = 1$ and $b_1 = 4$.

Answer: $b_n = 3^n + \frac{1}{3} n 3^n$

Although we will not consider examples more complicated than these, this characteristic root technique can be applied to much more complicated recurrence relations.