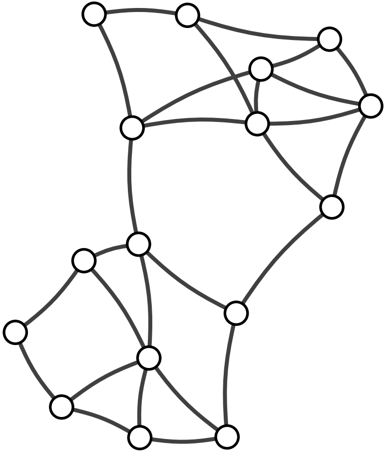
Lecture 4. Graph Coloring Problem

Given a set of k colors, can we assign colors to each vertex so that no two neighbors are assigned the same color?

We may revise the question as "Given a graph, what is the minimum number of colors that can be assigned to each vertex so that no two neighbors have the same color?"



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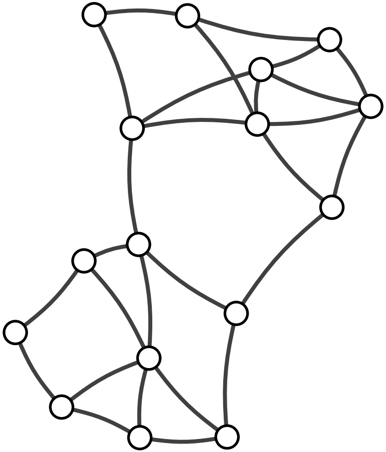

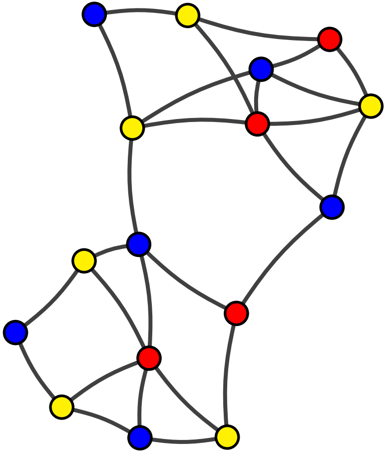

Definition 4.1. The chromatic number of a graph is the minimum number of colors needed for a coloring of the graph. The chromatic number of a graph G is denoted by $\chi(G)$.

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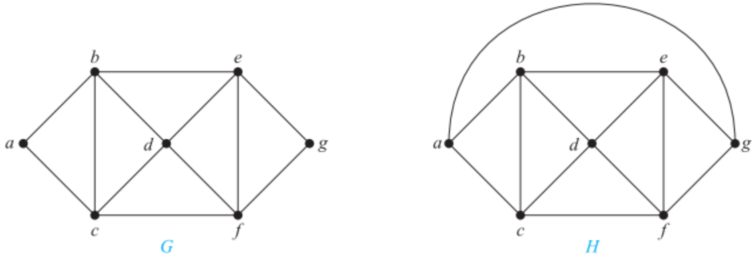
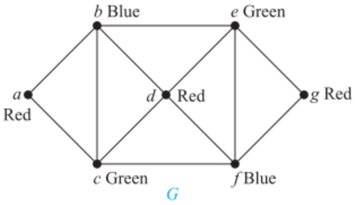
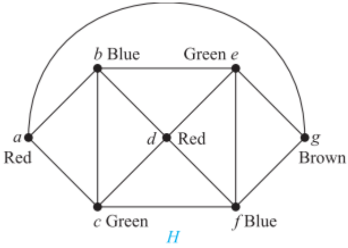
Example 4.2. What are the chromatic numbers of the graphs G and H?



Answer:

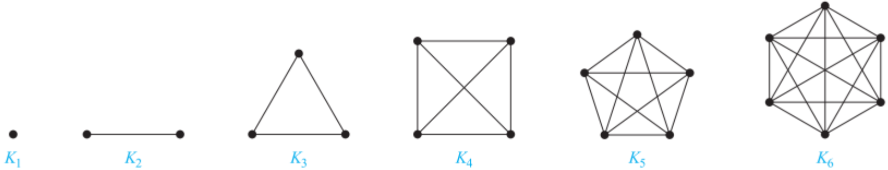


b Blue
e Green
a
Red
d
Red
g Red
c Green
f Blue
G



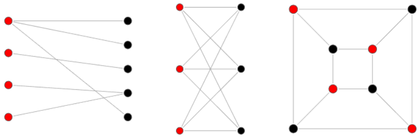
b Blue
Green e
a
Red
d
Red
g
Brown
c Green
f Blue
H

A complete graph on n vertices, denoted by K_n, is a *simple graph* that contains exactly one edge between each pair of distinct vertices



Example 4.3. What is $\chi(K_n)$?

A bipartite graph (also called a bigraph) is a graph whose vertices can be partitioned into two parts, say $\{v_1, v_2, \ldots, v_n\}$ and $\{w_1, w_2, \ldots, w_m\}$ so that all edges join some v_i to some w_j; no two vertices v_i and v_j are adjacent, nor are any vertices w_i and w_j.[1]



[1] That is, vertices are decomposed into two disjoint subsets such that no two vertices within the same subset are adjacent.

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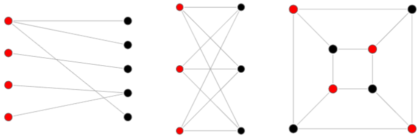
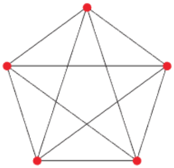


Example 4.4. Remove 4 edges from the complete graph K_5 to to make a bipartite graph.

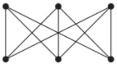


K_5

[1] That is, vertices are decomposed into two disjoint subsets such that no two vertices within the same subset are adjacent.

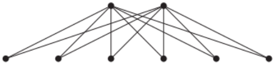
A complete bipartite graph $\mathbf{K_{m,n}}$ is a special kind of bipartite graph where every vertex of the first subset of m vertices is connected to every vertex of the second subset of n vertices.

$K_{2,3}$



$K_{3,3}$

$K_{3,5}$



$K_{2,6}$

Example 4.5 What is $\chi(K_{m,n})$?

Example 4.6 (Scheduling). A college registra would like to schedule the following courses in as few time slots as possible: **Physics**, **Computer Science**, **Chemistry**, **Calculus**, **Discrete Math**, **Biology**, and **Psychology**. However, from previous experience, the following pairs of classes always have students in common, so they can't be scheduled in the same time slot:

Physics	Computer Science, Chemistry
Calculus	Chemistry, Physics, Compute Science, Discrete Math, Biology
Discrete math	computer Science, Biology
Psychology	Biology, Chemistry

What is the fewest number of time slots needed to schedule all these classes without conflicts?

- The natural way to model this problem with a graph is to make each course into a vertex and connect any two vertices (representing courses) that cannot be scheduled in the same time slot:

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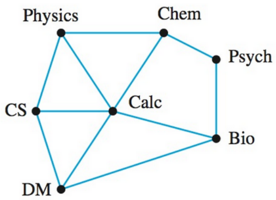

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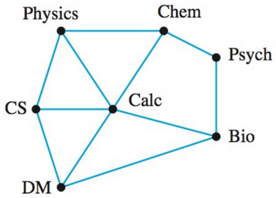

- Then, what is the fewest number of time slots needed to schedule all these classes without conflicts? That is, how many colors are needed for the graph coloring?
- The minimum number of colors for the graph coloring is the fewest number of time slots needed for the courses.

Activity 4.7. (Allocating the radio frequencies to the tower in a location) Suppose some of the transmitters are located so close that they can overlap and there will be a disturbance. We need to allocate different frequencies to the towers which are too close to each other in a location. How many different frequencies are needed for six transmitter towers located at the distances shown in the table, if two towers cannot use the same frequency when they are within 150 km of each other?

	1	*2*	*3*	*4*	*5*	*6*
1	—	85	175	200	50	100
2	85	—	125	175	100	160
3	175	125	—	100	200	250
4	200	175	100	—	210	220
5	50	100	200	210	—	100
6	100	160	250	220	100	—