Lecture 6. Matrix Representation of Graphs: Adjacency Matrix (Section 10.3 and 10.4)

Adjacency Matrix

If an unweighted simple graph G contains a total of n vertices, we can define an $n \times n$ matrix A by

$$a_{ij} = \begin{cases} 1 & \text{if } [v_i, v_j] \text{ is an edge of } G \\ 0 & \text{if there is no edge joining } v_i \text{ and } v_j \end{cases}$$

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For a weighted simple graph G

$$a_{ij} = \begin{cases} w_{ij} & \text{if the length of the edge } [v_i, v_j] = w_{ij} \\ 0 & \text{if there is no edge joining } v_i \text{ and } v_j \end{cases}$$

The resulting matrix $A = [a_{ij}]$ is called the adjacency matrix of the graph G.

Practice 6.1. Draw an unweighted simple graph with the adjacency matrix

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

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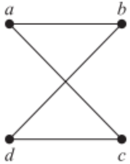

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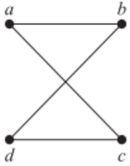

Note that an adjacency matrix of a graph is based on the ordering chosen for the vertices. Hence, there may be as many as n! different adjacency matrices for a graph with n vertices, because there are n! different orderings of n vertices.

Also, note that adjacency matrices of undirected graphs are symmetric.

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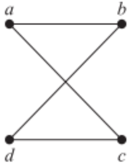


Note that an adjacency matrix of a graph is based on the ordering chosen for the vertices. Hence, there may be as many as $n!$ different adjacency matrices for a graph with n vertices, because there are $n!$ different orderings of n vertices.

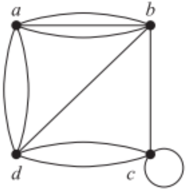
Also, note that adjacency matrices of undirected graphs are symmetric.

Question. What is the sum of the entries in a row of the adjacency matrix for an undirected graph?

Adjacency matrices can also be used to represent undirected graphs with loops and with multiple edges.

- A loop at the vertex v_i is represented by a 1 at the (i, i)th position of the adjacency matrix.
- When multiple edges connecting the same pair of vertices v_i and v_j, or multiple loops at the same vertex, are present, the (i, j)th entry of the adjacency matrix equals the number of edges that are associated to $[v_i, v_j]$.

Example 6.2. Use an adjacency matrix to represent the graph



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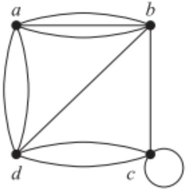
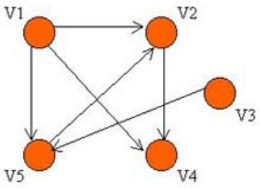


$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

The adjacency matrix for a directed graph does not have to be symmetric, because there may not be an edge from v_j to v_i when there is an edge from v_i to v_j.

Practice 6.3. Use an adjacency matrix to represent the directed graph



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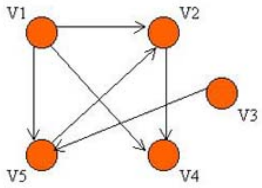
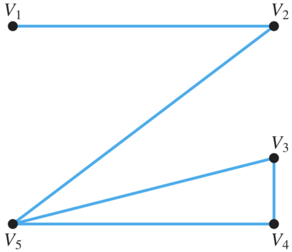
Practice 6.3. Use an adjacency matrix to represent the directed graph



	v1	v2	v3	v4	v5
v1	0	1	0	1	1
v2	0	0	0	1	0
v3	0	0	0	0	1
v4	0	0	0	0	0
v5	0	1	0	0	0

Question. What is the sum of the entries in a row of the adjacency matrix for a directed graph?

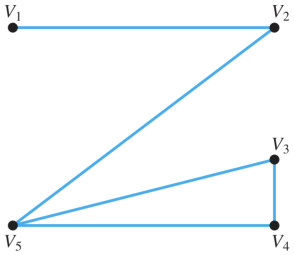
Powers of A Matrix



$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

In general, by taking **powers** of the adjacency matrix, we can determine the number of paths of any specified length between two vertices.

Theorem 6.4. Let A be an $n \times n$ adjacency matrix of a directed or undirected graph G. If $a_{ij}^{(k)}$ represents the (i, j) entry of A^k, then $a_{ij}^{(k)}$ is equal to the number of paths of length k from V_i to V_j. (Remark: A^k is called the path matrix of length k.) If n is the smallest nonnegative integer, such that $a_{ij}^{(n)} > 0$, then n is the length of a shortest path between V_i and V_j.



$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Example 6.5. Determine the number of paths of length 3 between any two vertices of the graph above.

We need only compute A^3, where

$$A^2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} =$$

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Thus, the number of paths of length 3 from V_3 to V_5 is $a_{35}^{(3)} = 4$, etc.

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Thus, the number of paths of length 3 from V_3 to V_5 is $a_{35}^{(3)} = 4$, etc.

Practice 6.6. Consider the matrix

$$B = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

(a) Draw an undirected graph that has B as its adjacency matrix. Be sure to label the vertices of the graph.

(b) By inspecting the graph, list the paths of length 3 from V_2 to V_3.

(c) Compute B^3 to determine the number of paths of length 3 from V_2 to V_3 and from V_1 to V_4.

(d) Find the length of the shortest path from V_2 to V_4. How many of them?

$$B^2 = \begin{bmatrix} 5 & 0 & 3 & 0 \\ 0 & 5 & 0 & 3 \\ 3 & 0 & 2 & 0 \\ 0 & 3 & 0 & 2 \end{bmatrix} \qquad B^3 = \begin{bmatrix} 0 & 13 & 0 & 8 \\ 13 & 0 & 8 & 0 \\ 0 & 8 & 0 & 5 \\ 8 & 0 & 5 & 0 \end{bmatrix}$$