Lecture 14. Logical Arguments (Section 1.6)

"My theory says: if P, then Q. I design an experiment to see if Q obtains. It does. Therefore, P is true."

Is this argument valid?

What is a logical argument?

Logical argument. A sequence of statements aimed at demonstrating the truth of an assertion

- Premise (or, assumption, hypothesis): A statement in an argument that provides reason or support for the conclusion. There can be one or many premises in a single argument.
- Conclusion: A statement in an argument that indicates of what the arguer is trying to convince the reader/listener. There can be only one conclusion in a single argument.

Proof/Derivation/Logical deduction. A valid sequence of statements used for establishing new mathematical truths (or conclusions or propositions or theorems) from acceptable/established mathematical truths (premises or axioms or assumptions)

Argument

Premise$_1$
Premise$_2$
$\vdots$
Premise$_m$

$\therefore$ Conclusion

If Premise$_1$ and Premise$_2$ and $\cdots$ and Premise$_m$, then Conclusion.

For example,

> Every employee who received a large bonus works hard.
> Aisha is an employee at the company.
> Aisha received a large bonus.
>
> ---
>
> $\therefore$ Some employee works hard.

Example 14.1. Rewrite the following arguments listing the **premise(s)** first and the **conclusion** last. Each line should be a single statement written as a complete sentence. Feel free to modify the sentences as you deem necessary, without changing their basic meaning (not writing a new argument!) Label the premise(s) as P_1, P_2, etc. and the conclusion C. Leave out any indicator words and any fluff (i.e., sentences which are neither the conclusion nor a premise).

"Cats with long hair shed all over the house so you should not get a long-haired cat. I have heard that they also have lots of fleas."

Example 14.1. Rewrite the following arguments listing the **premise(s)** first and the **conclusion** last. Each line should be a single statement written as a complete sentence. Feel free to modify the sentences as you deem necessary, without changing their basic meaning (not writing a new argument!) Label the premise(s) as P_1, P_2, etc. and the conclusion C. Leave out any indicator words and any fluff (i.e., sentences which are neither the conclusion nor a premise).

"Cats with long hair shed all over the house so you should not get a long-haired cat. I have heard that they also have lots of fleas."

- P_1: Long-haired cats shed all over the house.
- P_2: Long-haired cats have a lot of fleas.
- C: You should not get a long haired cat.

Validity of an Argument

An argument is valid if the conclusion must be true whenever the premises are all true..

- That is, a valid argument means that no matter what particular statements are substituted for the statement variables in its premises, if the resulting premises are all true, then the conclusion is also true

Example 14.2. Valid or invalid?

(a) If it is raining, then it is cloudy.
It is raining.
Therefore, it is cloudy.

(b) If it is raining, then it is cloudy.
It is not raining.
Therefore, it is not cloudy.

(c) If $x > 2$, then $x^2 > 4$.
$x \leq 2$.
Therefore, $x^2 \leq 4$.

How to determine if an argument is valid/invalid?

Method 1: Construct a truth table

1. Identify the premises and conclusion.

2. Construct a truth table for premises and conclusion.

3. A row of the truth table in which all the premises are true is called a critical row.

- If there is a critical row in which the conclusion is false, then the argument is invalid.
- If the conclusion in every critical row is true, then the argument is valid

Method 2: Find a counterexample

1. If there is a counterexample that has all premises true and false conclusion, then the argument is invalid.

2. Failing to find a counterexample does not prove that the argument is valid.

Method 3: Apply the rules of inference

Example 14.3. Use a truth table to determine the validity of the argument:

$$\begin{array}{l} p \to q \vee \neg r \\ q \to p \wedge r \\ \hline \therefore\ p \to r \end{array}$$

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p	q	r	$\sim r$	$q \vee \sim r$	$p \wedge r$	$p \to q \vee \sim r$	$q \to p \wedge r$	$p \to r$
T	T	T	F	T	T	T	T	T
T	T	F	T	T	F	T	F	
T	F	T	F	F	T	F	T	
T	F	F	T	T	F	T	T	F
F	T	T	F	T	F	T	F	
F	T	F	T	T	F	T	F	
F	F	T	F	F	F	T	T	T
F	F	F	T	T	F	T	T	T

4th row (a critical row) has false conclusion.
Invalid argument.

Practice 14.4. Consider the following argument:

"Graphs with high vertex degrees often have high edge weights."
"Graphs with high edge weights are computationally expensive to traverse."
Therefore,
"Graphs with high vertex degrees are computationally expensive to traverse."

Write the argument in a logical form and use a truth table to determine the validity of the argument.

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"Graphs with high edge weights are computationally expensive to traverse."
Therefore,
"Graphs with high vertex degrees are computationally expensive to traverse."

Write the argument in a logical form and use a truth table to determine the validity of the argument.

- Premise 1: $X \to W$ ("If a graph has high vertex degrees, then it often has high edge weights.")
- Premise 2: $W \to Y$ ("If a graph has high edge weights, then it is computationally expensive to traverse.")
- Conclusion: $X \to Y$ ("If a graph has high vertex degrees, then it is computationally expensive to traverse.")

X	W	Y	$X \to W$ (P1)	$W \to Y$ (P2)	$X \to Y$ (Conclusion)	Critical Row?
T	**T**	**T**	**T**	**T**	**T**	**Yes**
T	T	F	T	F	F	No
T	F	T	F	T	T	No
T	F	F	F	T	F	No
F	**T**	**T**	**T**	**T**	**T**	**Yes**
F	T	F	T	F	T	No
F	**F**	**T**	**T**	**T**	**T**	**Yes**
F	**F**	**F**	**T**	**T**	**T**	**Yes**

This shows that whenever the premises are true, the conclusion is true, confirming the validity of the argument.

A rule of inference is a valid argument form that can be used to establish logical deductions

Rule	Tautology	Name
$p \rightarrow q$ p $\therefore q$	$((p \rightarrow q) \wedge p) \Rightarrow q$	Modus Ponens (Law of Detachment)
$p \rightarrow q$ $\neg q$ $\therefore \neg p$	$((p \rightarrow q) \wedge \neg q) \Rightarrow \neg q$	Modus Tollens
$p \rightarrow q$ $q \rightarrow r$ $\therefore p \rightarrow r$	$((p \rightarrow q) \wedge (q \rightarrow r)) \Rightarrow (p \rightarrow r)$	Hypothetical Syllogism (Transitivity)
$p \vee q$ $\neg p$ $\therefore q$	$((p \vee q) \wedge \neg p) \Rightarrow q$	Disjunctive Syllogism
p $\therefore p \vee q$	$p \Rightarrow p \vee q$	Addition
$p \wedge q$ $\therefore p$	$(p \wedge q) \Rightarrow p$	Simplification
p q $\therefore p \wedge q$	$(p) \wedge (q) \Rightarrow (p \wedge q)$	Conjunction
$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	$((p \vee q) \wedge (\neg p \vee r)) \Rightarrow (q \vee r)$	Resolution

- Note that all rules of inference are tautology.
- **Notation:** When a tautology is of the form $(C \wedge D) \Rightarrow E$, we prefer to write as

$$\left.\begin{array}{l} C \\ D \end{array}\right\} \Rightarrow E$$

This notation highlights the fact that if you know both C and D, then you can conclude E.

Modus ponens.

$$\left.\begin{array}{c} p \to q \\ p \end{array}\right\} \Rightarrow q$$

Example 14.5. Justify the conclusion (i.e., determine whether the given argument is valid): "If the self-drive car detects an obstacle in its path, it will apply the brakes. The self-drive car has detected an obstacle in its path. Therefore, the self-drive car will apply the brakes."

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p	q	$p \to q$	p	q
T	T	T	T	T
T	F	F	T	
F	T	T	F	
F	F	T	F	

Modus Tollens

Modus tollens (= denial mode).

$$\left.\begin{array}{c} p \to q \\ \neg q \end{array}\right\} \Rightarrow \neg p$$

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Example 14.6. Justify the conclusion: "If the self-drive car detects an obstacle in its path, it will apply the brakes. The self-drive car has not applied the brakes. Therefore, the self-drive car has not detected an obstacle in its path."

Modus tollens (= denial mode).

$$\left.\begin{array}{c} p \to q \\ \neg q \end{array}\right\} \Rightarrow \neg p$$

Example 14.6. Justify the conclusion: "If the self-drive car detects an obstacle in its path, it will apply the brakes. The self-drive car has not applied the brakes. Therefore, the self-drive car has not detected an obstacle in its path."

p	q	$p \to q$	$\sim q$	$\sim p$
T	T	T	F	
T	F	F	T	
F	T	T	F	
F	F	T	T	T

Proof Sequences

We now have enough tools to derive some new tautologies from old ones. A proof sequence is a sequence of statements with reasons [1] to justify an assertion of the form

$$[A_1 \wedge A_2 \wedge \cdots A_n] \Rightarrow C$$

[1] logical equivalences and inference rules

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Example 14.7. Write a proof of sequence for the assertion

$$\left.\begin{array}{c} p \vee q \\ \neg p \end{array}\right\} \Rightarrow q$$

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Solution

Statements	**Reasonings**
1. $p \vee q$	given

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Statements	**Reasonings**
1. $p \vee q$	given
2. $\neg p$	given

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Solution

Statements	**Reasonings**
1. $p \vee q$	given
2. $\neg p$	given
3. $\neg(\neg p) \vee q$	Double negation with 1

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Solution

Statements	**Reasonings**
1. $p \vee q$	given
2. $\neg p$	given
3. $\neg(\neg p) \vee q$	Double negation with 1
4. $\neg p \rightarrow q$	Conditional identity with 3

[1] logical equivalences and inference rules

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Solution

Statements	**Reasonings**
1. $p \vee q$	given
2. $\neg p$	given
3. $\neg(\neg p) \vee q$	Double negation with 1
4. $\neg p \rightarrow q$	Conditional identity with 3
5. q	Modus ponens using 4 and 2

[1] logical equivalences and inference rules

Practice 14.8. Premises:

"It is not sunny this afternoon, and it is colder than yesterday."
"We will go swimming only if it is sunny."
"If we do not go swimming, then we will take a canoe trip."
"If we take a canoe trip, then we will be home by sunset."

Use the inference rules to construct a valid argument for the conclusion [2]:
"We will be home by sunset."

[2] that is, justify the conclusion by a proof sequence.

Example 14.9. You are about to leave for school in the morning and discover that you don't have your glasses. You know the following statements are true:

(a) If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table.

(b) If my glasses are on the kitchen table, then I saw them at breakfast.

(c) I did not see my glasses at breakfast.

(d) I was reading the newspaper in the living room or I was reading the newspaper in the kitchen.

(e) If I was reading the newspaper in the living room then my glasses are on the coffee table.

Where are the glasses?

Solution

Let:

- RK = I was reading the newspaper in the kitchen.
- GK = My glasses are on the kitchen table.
- SB = I saw my glasses at breakfast.
- RL = I was reading the newspaper in the living room.
- GC = My glasses are on the coffee table.

Given statements:

(a) $RK \to GK$
(b) $GK \to SB$
(c) $\sim SB$
(d) $RL \vee RK$
(e) $RL \to GC$

Solution

Sequence of steps to draw the conclusion:

1. $RK \to GK$ ▷ by (a)
   $GK \to SB$ ▷ by (b)
   $\therefore RK \to SB$ Hypothetical syllogism ▷ by transitivity
2. $RK \to SB$ ▷ by the conclusion of (1)
   $\sim SB$ ▷ by (c)
   $\therefore \sim RK$ ▷ by modus tollens
3. $RL \vee RK$ ▷ by (d)
   $\sim RK$ ▷ by the conclusion of (2)
   $\therefore RL$ Disjunctive syllogism ▷ by elimination
4. $RL \to GC$ ▷ by (e)
   RL ▷ by the conclusion of (3)
   $\therefore GC$ ▷ by modus ponens

Thus, the glasses are on the coffee table.

Validity $\neq$ Truthfulness; Invalid $\neq$ Falsity

- A valid argument can have a false conclusion.
- An invalid argument can have a true conclusion.

Example 14.10. Valid argument with false conclusion (Modus ponens):
If Isaac Newton was a scientist, then Albert Einstein was not a scientist.
Isaac Newton was a scientist.
Therefore, Albert Einstein was not a scientist.

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Example 14.10. Valid argument with false conclusion (Modus ponens):
If Isaac Newton was a scientist, then Albert Einstein was not a scientist.
Isaac Newton was a scientist.
Therefore, Albert Einstein was not a scientist.

Example 14.11. Invalid argument with true conclusion (Converse error):
If New York is a big city, then New York has tall buildings.
New York has tall buildings.
Therefore, New York is a big city.

What is a sound argument?

A sound argument is an argument that is valid and has true premises.

- Validity shows that an argument is logical.
- Soundness shows that an argument is truthful

Example 14.12. Valid argument with true premises (Modus ponens):
If Isaac Newton was a scientist, then Albert Einstein was a scientist.
Isaac Newton was a scientist.
Therefore, Albert Einstein was a scientist.