Lecture 13. Tautology and Equivalence Rules (Section 1.3.1-1.3.4)

When truth tables become impractical.

- There is a limitation in using the truth table. Each time you add a new statement to a truth table, you must double the number of rows. This makes truth table analysis unwieldy for all but the simplest examples.
- So, we will develop a system of rules for manipulating propositional logic.
- Those rules will be useful for analyzing complex logical problems, especially where truth tables are impractical.

Practice 13.1. Here's a question about playing Monopoly:

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True or false?

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The statement about monopoly is an example of a tautology, a statement which is true on the basis of its logical form alone. Tautologies are always true but they don't tell us much about the world. No knowledge about monopoly was required to determine that the statement was true. In fact, it is equally true that "If the moon is made of cheese, then Elvis is still alive, or if Elvis is still alive, then unicorns have five legs."

Tautology and Contradiction

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- A tautology is a statement that is **always true** regardless of the truth values of the individual statements substituted for its statement variables.
  - **Example** $p \vee \neg p$ (I will get A or not A in this course)

  A tautology is also called a valid statement.
- A contradiction is a statement that is **always false** regardless of the truth values of the individual statements substituted for its statement variables.
  - **Example** $p \wedge \neg p$ (I will get A and not A in this course)

Logical Equivalence again

In Lecture 12, we defined that two or more compound statements are logically equivalent if they have the same truth values in all possible cases.

Theorem 13.2. The compound statements P and Q are logically equivalent if $P \leftrightarrow Q$ is a tautology.

- Why does this Theorem make sense? What if the truth values of P and Q are different?

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Theorem 13.2. The compound statements P and Q are logically equivalent if $P \leftrightarrow Q$ is a tautology.

- Why does this Theorem make sense? What if the truth values of P and Q are different?
- **Notation** We write $P \leftrightarrow Q$ as $P \Leftrightarrow Q$ to indicate the statement is tautology.

Key Logical Equivalences

De Morgan's Laws

1. $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$
2. $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$

[1] It is not true that the patient has migraines or high blood pressure.

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Example 13.3.

- p: The patient has migraines
- q: The patient has high blood pressure

De Morgan's law says that the following two English statements are logically equivalent:

- The patient does not have migraines or high blood pressure. [1]
- The patient does not have migraines and does not have high blood pressure.

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Practice 13.4. Are the statements, "it will not rain and snow" and "it will not rain and it will not snow" logically equivalent?

[1] It is not true that the patient has migraines or high blood pressure.

TABLE 6 Logical Equivalences.

Equivalence	*Name*
$p \land \mathbf{T} \equiv p$ $p \lor \mathbf{F} \equiv p$	Identity laws
$p \lor \mathbf{T} \equiv \mathbf{T}$ $p \land \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \lor p \equiv p$ $p \land p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \lor q \equiv q \lor p$ $p \land q \equiv q \land p$	Commutative laws
$(p \lor q) \lor r \equiv p \lor (q \lor r)$ $(p \land q) \land r \equiv p \land (q \land r)$	Associative laws
$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$ $p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$	Distributive laws
$\neg(p \land q) \equiv \neg p \lor \neg q$ $\neg(p \lor q) \equiv \neg p \land \neg q$	De Morgan's laws
$p \lor (p \land q) \equiv p$ $p \land (p \lor q) \equiv p$	Absorption laws
$p \lor \neg p \equiv \mathbf{T}$ $p \land \neg p \equiv \mathbf{F}$	Negation laws

Practice 13.5. What is the output of this computer program? Explain.

```
if ((a < b or c == d) and (a >= b or c == d) and (a < b or c != d) and (a >= b or c != d)):
    print("Hi")
else:
    print("Hey")
```

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Solution

- Answer. Hey
- Reason. Let p : `(a < b)` and q : `(c == d)`
  $((p \vee q) \wedge (\sim p \vee q) \wedge (p \vee \sim q) \wedge (\sim p \vee \sim q)) \equiv p \wedge \sim p \equiv \mathbf{c}$

TABLE 7 Logical Equivalences Involving Conditional Statements.

$p \to q \equiv \neg p \vee q$
$p \to q \equiv \neg q \to \neg p$
$p \vee q \equiv \neg p \to q$
$p \wedge q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \wedge \neg q$
$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

TABLE 8 Logical Equivalences Involving Biconditional Statements.

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Example 13.6. Rewrite the statement "If a number is a multiple of 4, then it is even" equivalently without using the logical connective "if".

Constructing New Logical Equivalences

The logical equivalences on the previous page can be used to construct additional logical equivalences.

Example 13.7. Show that $\neg(p \to q)$ and $p \wedge \neg q$ are logically equivalent without using a truth table.

Constructing New Logical Equivalences

The logical equivalences on the previous page can be used to construct additional logical equivalences.

Example 13.7. Show that $\neg(p \rightarrow q)$ and $p \wedge \neg q$ are logically equivalent without using a truth table.

- **Solution**

$$\begin{aligned}
\neg(p \rightarrow q) &\Leftrightarrow \neg(\neg p \vee q) && \text{by Conditional identities} \\
&\Leftrightarrow \neg(\neg p) \wedge \neg q && \text{by De Morgan's laws} \\
&\Leftrightarrow p \wedge \neg q && \text{by Double negation}
\end{aligned}$$

Practice 13.8. Fill in the reasons in the following proof sequence. Make sure you indicate which equivalence rule is used.

Statements	Reasons
1. $p \to (q \to r)$	given
2. $\neg p \vee (q \to r)$	
3. $\neg p \vee (\neg q \vee r)$	
4. $(\neg p \vee \neg q) \vee r$	
5. $\neg(p \wedge q) \vee r$	
6. $(p \wedge q) \to r$	

Practice 13.9. Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalence.