Lecture 15. Predicate Logic and Quantifiers (§1.5)

Predicates

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  - For example, the sentence "**The number** $x + 2$ **is an even integer**" is not necessary true or false unless we know what value is substituted for x.
  - If we restrict our choices to integers, then when x is replaced by -7, 1, or 5, for instance, the resulting statement is false.
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- We can denote the sentence "The number $x+2$ is an even integer" by $P(x)$, which is called a propositional function at x.
- **Definition** A predicate logic [1] is a declarative sentence whose truth value depends on one or more variables that becomes a statement when the variables in it are replaced by certain allowable choices. We will say **predicate logic** as just **predicate**.
- In the example above, $P(3)$ is false and $P(2)$ is true.

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Example 15.2. Equations are predicates: If $E(x)$ stands for the equation

$$x^2 - x - 6 = 0,$$

then $E(3)$ is true and $E(4)$ is false.

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- $\exists x \in D,\ P(x) =$ "There exists an x in the domain D such that $P(x)$ is true"; i.e., $P(x)$ is true for **some** x in D.

Universal Quantifier: $\forall x \in D,\ P(x)$ is read as "For all x in the domain D, $P(x)$ is true"

Example 15.3.

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3. If D consists of $3, 4$, and 5, and $P(x)$ is the statement "$x > 2$", then both $\exists x \in D,\ P(x)$ and $\forall x \in D,\ P(x)$ are true. But if $P(x)$ is the statement "$x < 2$", then both $\exists x \in D,\ P(x)$ and $\forall x \in D,\ P(x)$ are false.

Practice 15.6 Consider the predicate $R(x, y) : 2x + y = 0$, where the domain of x and y is all rational numbers. True or False?

$$(a)\ R(0,0) \quad (b)\ R(2,-1) \quad (c)\ R(\frac{1}{5}, -\frac{2}{5}) \quad (d)\ \exists y, R(0.2, y) \quad (e)\ \forall y, R(7, y)$$

Precedence of Quantifiers

The quantifiers $\forall$ and $\exists$ have higher precedence than all the logical operators.

- For example, $\forall x\, P(x) \vee Q(x)$ means $(\forall x\, P(x)) \vee Q(x)$.
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- $\forall x\, (P(x) \vee Q(x))$ means something different.
- To avoid any confusion just put brackets right after every quantifier you use; i.e., $\forall x \in D,\ [P(x) \vee Q(x)]$ or $\forall x\, [P(x) \vee Q(x)]$.

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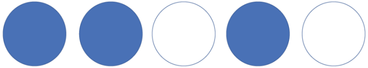
Example 15.7. Translate the following sentence into predicate logic:

"Every student in this class has taken a course in Java."

- **Solution 1** If D is all students in this class, define a predicate $J(x)$ denoting "x has taken a course in Java" and translate as $\forall x \in D,\ J(x)$.
- **Solution 2** But if D is all people, also define a predicate $S(x)$ denoting "x is a student in this class" and translate as $\forall x \in D,\ [S(x) \rightarrow J(x)]$.

Negating Quantified Expressions



Using the five circles, as seen above, we make the following statement:

"Some circles are shaded."

What would the negation of this statement be?

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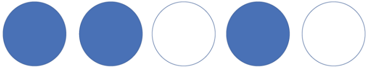

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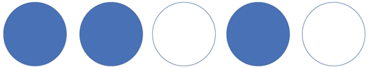

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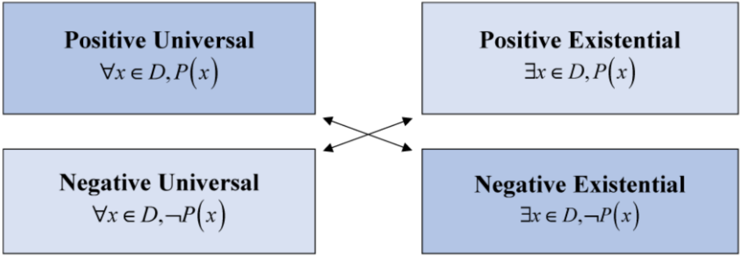
The trick is to change from a universal quantifier to an existential quantifier or vice versa, adding a “not” to say,

“All circles are not shaded.”

Therefore, the negation of quantification has the following properties:



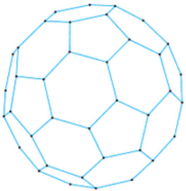
$$\neg(\forall x \in D, P(x)) \equiv \exists x \in D, \neg P(x)$$

$$\neg(\exists x \in D, P(x)) \equiv \forall x \in D, \neg P(x)$$

Example 15.8. Express the statement in predicate logic and find its negation.

"There does not exist anyone who likes skiing over magma."

Practice 15.10. Let the domain be all faces of the following truncated **icosahedron**. [2]



Consider the following predicates:

- $P(x)$ = "x is a pentagon"
- $H(x)$ = "x is a hexagon"
- $B(x, y)$ = "x borders y"

Here we say that two polygons border each other if they share an edge. Confirm that the following observations are true for any truncated icosahedron.

- No two pentagons border each other.
- Every pentagon borders some hexagon.
- Every hexagon borders another hexagon.

Write these statements in predicate logic, and negate them. Simplify the negated statements so that no quantifier or connective lies within the scope of a negation. Translate your negated statement back into English.

[2] A solid figure having 20 faces.