Lecture 21. Solving Recurrence Relations

Recursive Thinking

In the previous lecture note, we have seen some examples how to think recursively about a problem by describing it with a recurrence relation. Remember that any recurrence relation has two parts: a **base case** that describes some initial conditions, and a **recursive case** that describes a **future value in terms of previous values**. Armed with this way of thinking, we can model other problems using recurrence relations.

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Example 20.1. Ahmed lends money at outrageous rates of interest. He demands to be paid 10% interest *per week* on a loan, compounded weekly. Suppose you borrow 500 Dhs from him. Let $M(n) =$ the money you owed at n-th week.

(a) Find the recurrence relation for $M(n)$.

(b) If you wait four weeks to pay him back, how much will you owe?

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Answer:

$$M(n) = \begin{cases} 500 & \text{if} \quad n = 0 \\ 1.1\, M(n-1) & \text{if} \quad n > 0 \end{cases} \qquad M(4) = \$732.05$$

Practice 20.2. Find a closed-form solution for the recurrence relation from

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Answer: $M(n) = 500(1.1)^n$

Polynomial sequences: Using differences

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then we can detect such a sequence by looking at the differences between terms. Given any sequence,

$$a_0, a_1, a_2, ..., a_{n-1}, a_n$$

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form another sequence, called the sequence of differences.

- A linear sequence $a_n = An + B$ will have a constant sequence of differences (because a line has constant slop).
- A quadratic sequence $a_n = An^2 + Bn + C$ will have a linear sequence of differences.
- a cubic sequence $a_n = An^3 + Bn^2 + Cn + D$ will have a quadratic sequence of differences, etc.
- If we eventually end up with a constant sequence, then the original sequence is given by a polynomial function.
- The degree of the conjectured polynomial is the number of times we had to calculate the sequence of differences.

Example 20.3. Find a closed-form solution $f(n)$ for the recurrence relation

$$H(n) = \begin{cases} 1 & \text{if} \quad n = 1 \\ H(n-1) + 6n - 6 & \text{if} \quad n > 1 \end{cases}$$

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Answer. $H(n) = 3n^2 - 3n + 1$ is a good candidate for a closed-form solution.

Remark. The result of these procedures is still only a guess. To be sure that our guess is right, we need to prove that the formula matches the recurrence relation for all n.

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Answer. $a_n = \frac{n(n+1)}{2} + 4$ for $n \geq 0$.