Lecture 22. Linear Homogeneous Recurrence Relations

Definition 22.1. A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_k a_{n-k}$$

where $r_1, r_2, ..., r_k$ are real numbers and $r_k \neq 0$ with $k < n$. This recurrence includes k initial conditions, $a_0 = \alpha_0, a_1 = \alpha_1, ..., a_k = \alpha_k$.

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Example 22.2. Linear and/or Homogeneous?

(a) $M_n = (1.1)M_{n-1}$ (b) $F_n = F_{n-1} + F_{n-2}$ (c) $a_n = a_{n-1} + a_{n-2}^2$ (d) $h_n = 2h_{n-1} + 1$

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(a) linear homogeneous recurrence relation of degree 1, (b) linear homogeneous recurrence relation of degree 2, (c) not linear, (d) not homogeneous,

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- Rearranging terms leads to the characteristic equation:

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Theorem 22.3. The characteristic equation of the recurrence relation $a_n = r_1 a_{n-1} + r_2 a_{n-2}$ is

$$x^2 - r_1 x - r_2 = 0.$$

If the characteristic equation has two distinct roots, x_1 and x_2, then

$$a_n = p x_1^n + q x_2^n \text{ for some } p, q$$

is the explicit formula for the sequence. Here, p and q depend on the initial conditions.

Example 22.4. Find an explicit formula for the sequence defined by $a_n = 7a_{n-1} - 10a_{n-2}$ with $a_0 = 2$ and $a_1 = 3$.

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Answer: $a_n = \frac{7}{3} 2^n - \frac{1}{3} 5^n$.

Theorem 22.5. If the characteristic equation

$$x^2 - r_1 x - r_2 = 0$$

of the recurrence relation $a_n = r_1 a_{n-1} + r_2 a_{n-2}$ has two distinct roots, x_1 and x_2, then

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where p and q depend on the initial conditions, is the explicit formula for the sequence.

Practice 22.6. (Fibonacci Sequence) Solve the recurreence relation $f_n = f_{n-1} + f_{n-2}$ with the initial conditions $f_0 = 0$ and $f_1 = 1$.

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Practice 22.6. (Fibonacci Sequence) Solve the recurreence relation $f_n = f_{n-1} + f_{n-2}$ with the initial conditions $f_0 = 0$ and $f_1 = 1$.

Answer: $f_n = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}}$, where $\phi = \frac{1+\sqrt{5}}{2}$. (Note that $\phi \approx 1.618\ 033\ 988\ 749...$ is a golden ratio.)

Theorem 22.7. If the characteristic equation

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of the recurrence relation $a_n = r_1 a_{n-1} + r_2 a_{n-2}$ has a single root, x, then

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Practice 22.8. Solve the recurrence relation $b_n = 6b_{n-1} - 9b_{n-2}$ with the initial conditions $b_0 = 1$ and $b_1 = 4$.

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Practice 22.8. Solve the recurrence relation $b_n = 6b_{n-1} - 9b_{n-2}$ with the initial conditions $b_0 = 1$ and $b_1 = 4$.

Answer: $b_n = 3^n + \frac{1}{3} n 3^n$

Although we will not consider examples more complicated than these, this characteristic root technique can be applied to much more complicated recurrence relations.