Lecture 17. Predicate Logic and Quantifiers (§1.5)

Predicates

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  - For example, the sentence "**The number** $x + 2$ **is an even integer**" is not necessary true or false unless we know what value is substituted for x.
  - If we restrict our choices to integers, then when x is replaced by -7, 1, or 5, for instance, the resulting statement is false.
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- We can denote the sentence "The number $x + 2$ is an even integer" by $P(x)$, which is called a propositional function at x.
- **Definition** A predicate logic [1] is a declarative sentence whose truth value depends on one or more variables that becomes a statement when the variables in it are replaced by certain allowable choices. We will say **predicate logic** as just **predicate**.
- In the example above, $P(3)$ is false and $P(2)$ is true.

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Example 17.2. Equations are predicates: If $E(x)$ stands for the equation

$$x^2 - x - 6 = 0,$$

then $E(3)$ is true and $E(4)$ is false.

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Universal Quantifier: $\forall x \in D,\ P(x)$ is read as "For all x in the domain D, $P(x)$ is true"

Example 17.3.

1. If $P(x)$ denotes "$x > 0$" and its domain D is the integers, then the statement $[\forall x \in D,\ P(x)]$ is false.

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3. If D consists of $3, 4$, and 5, and $P(x)$ is the statement "$x > 2$", then both $\exists x \in D,\ P(x)$ and $\forall x \in D,\ P(x)$ are true. But if $P(x)$ is the statement "$x < 2$", then both $\exists x \in D,\ P(x)$ and $\forall x \in D,\ P(x)$ are false.

Practice 17.6 Consider the predicate $R(x, y) : 2x + y = 0$, where the domain of x and y is all rational numbers. True or False?

$$(a)\ R(0,0) \quad (b)\ R(2,-1) \quad (c)\ R(\frac{1}{5},-\frac{2}{5}) \quad (d)\ \exists y, R(0.2,y) \quad (e)\ \forall y, R(7,y)$$

Precedence of Quantifiers

The quantifiers $\forall$ and $\exists$ have higher precedence than all the logical operators.

- For example, $\forall x\, P(x) \vee Q(x)$ means $(\forall x\, P(x)) \vee Q(x)$.
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- $\forall x\, (P(x) \vee Q(x))$ means something different.
- To avoid any confusion just put brackets right after every quantifier you use; i.e., $\forall x \in D,\, [P(x) \vee Q(x)]$ or $\forall x\, [P(x) \vee Q(x)]$.

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Example 17.7. Translate the following sentence into predicate logic:

"Every student in this class has taken a course in Java."

- **Solution 1** If D is all students in this class, define a predicate $J(x)$ denoting "x has taken a course in Java" and translate as $\forall x \in D,\ J(x)$.
- **Solution 2** But if D is all people, also define a predicate $S(x)$ denoting "x is a student in this class" and translate as $\forall x \in D,\ [S(x) \rightarrow J(x)]$.

Negating Quantified Expressions

- Consider $\forall x\, J(x)$:
  "Every student in your class has taken a course in Java."
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- Now consider $\exists x\, J(x)$
  "There is a student in this class who has taken a course in Java."
  where $J(x)$ is "x has taken a course in Java" and the domain is students in this class.
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Negation Rules for Quantifiers

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\begin{aligned}
\neg\,[(\forall x)\,P(x)] \quad &\Leftrightarrow \quad (\exists x)(\neg P(x)) \quad \text{(universal negation)} \\
\neg\,[(\exists x)\,P(x)] \quad &\Leftrightarrow \quad (\forall x)(\neg P(x)) \quad \text{(existential negation)}
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Example 17.8. Express the statement in predicate logic and find its negation.

"There does not exist anyone who likes skiing over magma."

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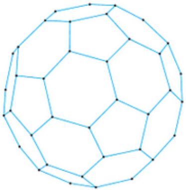
Practice 17.9 Show that $\neg\,(\forall x[P(x) \to Q(x)])$ and $\exists x[P(x) \wedge \neg Q(x)]$ are logically equivalent.

Practice 17.10. Let the domain be all faces of the following truncated **icosahedron**. [2]



Consider the following predicates:

- $P(x)$ = "x is a pentagon"
- $H(x)$ = "x is a hexagon"
- $B(x, y)$ = "x borders y"

Here we say that two polygons border each other if they share an edge. Confirm that the following observations are true for any truncated icosahedron.

- No two pentagons border each other.
- Every pentagon borders some hexagon.
- Every hexagon borders another hexagon.

Write these statements in predicate logic, and negate them. Simplify the negated statements so that no quantifier or connective lies within the scope of a negation. Translate your negated statement back into English.

[2] A solid figure having 20 faces.