Lecture 17. Rules of Inference for Quantified Statements (§1.6.7)

TABLE 2 **Rules of Inference for Quantified Statements.**

Rule of Inference	*Name*
$\dfrac{\forall x P(x)}{\therefore\ P(c)}$	Universal instantiation
$\dfrac{P(c) \text{ for an arbitrary } c}{\therefore\ \forall x P(x)}$	Universal generalization
$\dfrac{\exists x P(x)}{\therefore\ P(c) \text{ for some element } c}$	Existential instantiation
$\dfrac{P(c) \text{ for some element } c}{\therefore\ \exists x P(x)}$	Existential generalization

- **Universal instantiation** is the rule of inference used to conclude that $P(c)$ is true, where c is a particular member of the domain, given the premise $\forall x\, P(x)$.
- **Universal Generalization** is the rule of inference that states that $\forall x\, P(x)$ is true, given the premise that $P(c)$ is true for all elements c in the domain. (The element c that we select must be an arbitrary, and not a specific element of the domain.)
- **Existential instantiation** is the rule saying that there is an element c in the domain for which $P(c)$ is true if we know that $\exists x\, P(x)$ is true. (Usually we have no knowledge of what c is, only that it exists.)
- **Existential generalization** is the rule of inference stating that $\exists x\, P(x)$ is true when $P(c)$ is true for a particular element c. (That is, if we know one element c in the domain for which $P(c)$ is true, then we know that $\exists x\, P(x)$ is true.)

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Solution. Let $A(x)$ denote "x is an artificial-intelligence chatbot" and $S(x)$ denote "x can summarize a newsfeed article". Then the premises are $\forall x(A(x) \rightarrow S(x))$ and $A(ChatGPT)$. The following steps can be used to establish the conclusion from the premises.

Step	**Reasoning**

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Step	**Reasoning**
(1) $\forall x(A(x) \to S(x))$	Premise

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Step	**Reasoning**
(1) $\forall x(A(x) \rightarrow S(x))$	Premise
(2) $A(ChatGPT) \rightarrow S(ChatGPT)$	Universal instantiation from (1)

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Step	**Reasoning**
(1) $\forall x(A(x) \to S(x))$	Premise
(2) $A(ChatGPT) \to S(ChatGPT)$	Universal instantiation from (1)
(3) $A(ChatGPT)$	Premise

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Step	**Reasoning**
(1) $\forall x(A(x) \to S(x))$	Premise
(2) $A(ChatGPT) \to S(ChatGPT)$	Universal instantiation from (1)
(3) $A(ChatGPT)$	Premise
(4) $S(ChatGPT)$	Modus ponens from (2) and (3)

Example 17.2. A logical proof that uses the laws of inference for quantified statements: Assume the domain is all integers.

$\forall x\,(P(x) \vee Q(x))$
3 is an integer
$\neg P(3)$

$\therefore Q(3)$

Example 17.2. A logical proof that uses the laws of inference for quantified statements: Assume the domain is all integers.

$$\begin{array}{l} \forall x\,(P(x) \vee Q(x)) \\ 3 \text{ is an integer} \\ \neg P(3) \\ \hline \therefore Q(3) \end{array}$$

1. $\forall x\,(P(x) \vee Q(x))$ Premise
2. 3 is an integer Premise

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1	$\forall x\,(P(x) \vee Q(x))$	Premise
2	3 is an integer	Premise
3	$(P(3) \vee Q(3))$	Universal instantiation, 1, 2

Example 17.2. A logical proof that uses the laws of inference for quantified statements: Assume the domain is all integers.

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1	$\forall x\,(P(x) \vee Q(x))$	Premise
2	3 is an integer	Premise
3	$(P(3) \vee Q(3))$	Universal instantiation, 1, 2
4	$\neg P(3)$	Premise

Example 17.2. A logical proof that uses the laws of inference for quantified statements: Assume the domain is all integers.

$$\begin{array}{l} \forall x\,(P(x) \lor Q(x)) \\ 3 \text{ is an integer} \\ \neg P(3) \\ \hline \therefore Q(3) \end{array}$$

1	$\forall x\,(P(x) \lor Q(x))$	Premise
2	3 is an integer	Premise
3	$(P(3) \lor Q(3))$	Universal instantiation, 1, 2
4	$\neg P(3)$	Premise
5	$Q(3)$	Disjunctive syllogism, 3, 4

Example 17.3. Indicate whether the proof fragment is a correct or incorrect use of the rule of inference.

(a) True or False?

1.	c is an element	Hypothesis
2.	P(c)	Hypothesis
3.	$\forall x\, P(x)$	Universal generalization, 1, 2

(b) True or False?

1.	$\exists x\, P(x)$	Hypothesis
2.	(c is a particular element) $\wedge$ P(c)	Existential instantiation, 1
3.	$\exists x\, Q(x)$	Hypothesis
4.	(c is a particular element) $\wedge$ Q(c)	Existential instantiation, 3

(c) True or False?

1.	c is an element	Hypothesis
2.	$\forall x\, P(x)$	Hypothesis
3.	P(c)	Universal instantiation, 1, 2

(d) True or False?

1.	c is an element	Hypothesis
2.	P(c)	Hypothesis
3.	d, an element	Hypothesis
4.	Q(d)	Hypothesis
5.	$P(c) \wedge Q(d)$	Conjunction, 2, 3
6.	$\exists x\, (P(x) \wedge Q(x))$	Existential generalization, 1, 3, 5

Example 17.4 Show that the premises "A student in this class has not read the book," and "Everyone in this class passed the first exam" imply the conclusion "Someone who passed the first exam has not read the book."

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Let $C(x)$ be "x is in this class," $B(x)$ be "x has read the book," and $P(x)$ be "x passed the first exam."

The premises are $\exists x(C(x) \wedge \neg B(x))$ and $\forall x(C(x) \rightarrow P(x))$. The conclusion is $\exists x(P(x) \wedge \neg B(x))$.

Step	**Reasoning**

Example 17.4 Show that the premises "A student in this class has not read the book," and "Everyone in this class passed the first exam" imply the conclusion "Someone who passed the first exam has not read the book."

Let $C(x)$ be "x is in this class," $B(x)$ be "x has read the book," and $P(x)$ be "x passed the first exam."

The premises are $\exists x(C(x) \wedge \neg B(x))$ and $\forall x(C(x) \rightarrow P(x))$. The conclusion is $\exists x(P(x) \wedge \neg B(x))$.

Step	**Reasoning**
(1) $\exists x(C(x) \wedge \neg B(x))$	Premise

Example 17.4 Show that the premises "A student in this class has not read the book," and "Everyone in this class passed the first exam" imply the conclusion "Someone who passed the first exam has not read the book."

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Step	**Reasoning**
(1) $\exists x(C(x) \wedge \neg B(x))$	Premise
(2) $C(a) \wedge \neg B(a)$	Existential instantiation from (1)

Example 17.4 Show that the premises "A student in this class has not read the book," and "Everyone in this class passed the first exam" imply the conclusion "Someone who passed the first exam has not read the book."

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Step	**Reasoning**
(1) $\exists x(C(x) \wedge \neg B(x))$	Premise
(2) $C(a) \wedge \neg B(a)$	Existential instantiation from (1)
(3) $C(a)$	Simplification from (2)

Example 17.4 Show that the premises "A student in this class has not read the book," and "Everyone in this class passed the first exam" imply the conclusion "Someone who passed the first exam has not read the book."

Let $C(x)$ be "x is in this class," $B(x)$ be "x has read the book," and $P(x)$ be "x passed the first exam."

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Step	**Reasoning**
(1) $\exists x(C(x) \wedge \neg B(x))$	Premise
(2) $C(a) \wedge \neg B(a)$	Existential instantiation from (1)
(3) $C(a)$	Simplification from (2)
(4) $\forall x(C(x) \rightarrow P(x))$	Premise

Example 17.4 Show that the premises "A student in this class has not read the book," and "Everyone in this class passed the first exam" imply the conclusion "Someone who passed the first exam has not read the book."

Let $C(x)$ be "x is in this class," $B(x)$ be "x has read the book," and $P(x)$ be "x passed the first exam."

The premises are $\exists x(C(x) \land \neg B(x))$ and $\forall x(C(x) \to P(x))$. The conclusion is $\exists x(P(x) \land \neg B(x))$.

Step	**Reasoning**
(1) $\exists x(C(x) \land \neg B(x))$	Premise
(2) $C(a) \land \neg B(a)$	Existential instantiation from (1)
(3) $C(a)$	Simplification from (2)
(4) $\forall x(C(x) \to P(x))$	Premise
(5) $C(a) \to P(a)$	Universal instantiation from (4)

Example 17.4 Show that the premises "A student in this class has not read the book," and "Everyone in this class passed the first exam" imply the conclusion "Someone who passed the first exam has not read the book."

Let $C(x)$ be "x is in this class," $B(x)$ be "x has read the book," and $P(x)$ be "x passed the first exam."

The premises are $\exists x(C(x) \wedge \neg B(x))$ and $\forall x(C(x) \rightarrow P(x))$. The conclusion is $\exists x(P(x) \wedge \neg B(x))$.

Step	**Reasoning**
(1) $\exists x(C(x) \wedge \neg B(x))$	Premise
(2) $C(a) \wedge \neg B(a)$	Existential instantiation from (1)
(3) $C(a)$	Simplification from (2)
(4) $\forall x(C(x) \rightarrow P(x))$	Premise
(5) $C(a) \rightarrow P(a)$	Universal instantiation from (4)
(6) $P(a)$	Modus ponens from (3) and (5)

Example 17.4 Show that the premises "A student in this class has not read the book," and "Everyone in this class passed the first exam" imply the conclusion "Someone who passed the first exam has not read the book."

Let $C(x)$ be "x is in this class," $B(x)$ be "x has read the book," and $P(x)$ be "x passed the first exam."

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Step	**Reasoning**
(1) $\exists x(C(x) \wedge \neg B(x))$	Premise
(2) $C(a) \wedge \neg B(a)$	Existential instantiation from (1)
(3) $C(a)$	Simplification from (2)
(4) $\forall x(C(x) \to P(x))$	Premise
(5) $C(a) \to P(a)$	Universal instantiation from (4)
(6) $P(a)$	Modus ponens from (3) and (5)
(7) $\neg B(a)$	Simplification from (2)

Example 17.4 Show that the premises "A student in this class has not read the book," and "Everyone in this class passed the first exam" imply the conclusion "Someone who passed the first exam has not read the book."

Let $C(x)$ be "x is in this class," $B(x)$ be "x has read the book," and $P(x)$ be "x passed the first exam."

The premises are $\exists x(C(x) \wedge \neg B(x))$ and $\forall x(C(x) \rightarrow P(x))$. The conclusion is $\exists x(P(x) \wedge \neg B(x))$.

Step	**Reasoning**
(1) $\exists x(C(x) \wedge \neg B(x))$	Premise
(2) $C(a) \wedge \neg B(a)$	Existential instantiation from (1)
(3) $C(a)$	Simplification from (2)
(4) $\forall x(C(x) \rightarrow P(x))$	Premise
(5) $C(a) \rightarrow P(a)$	Universal instantiation from (4)
(6) $P(a)$	Modus ponens from (3) and (5)
(7) $\neg B(a)$	Simplification from (2)
(8) $P(a) \wedge \neg B(a)$	Conjunction from (6) and (7)

Example 17.4 Show that the premises "A student in this class has not read the book," and "Everyone in this class passed the first exam" imply the conclusion "Someone who passed the first exam has not read the book."

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The premises are $\exists x(C(x) \land \neg B(x))$ and $\forall x(C(x) \to P(x))$. The conclusion is $\exists x(P(x) \land \neg B(x))$.

Step	**Reasoning**
(1) $\exists x(C(x) \land \neg B(x))$	Premise
(2) $C(a) \land \neg B(a)$	Existential instantiation from (1)
(3) $C(a)$	Simplification from (2)
(4) $\forall x(C(x) \to P(x))$	Premise
(5) $C(a) \to P(a)$	Universal instantiation from (4)
(6) $P(a)$	Modus ponens from (3) and (5)
(7) $\neg B(a)$	Simplification from (2)
(8) $P(a) \land \neg B(a)$	Conjunction from (6) and (7)
(9) $\exists x(P(x) \land \neg B(x))$	Existential generalization from (8)

Activity 17.5. Consider the argument

> *"Ahmad is a student in this class who owns a yellow Lamborghini. Everyone who owns a yellow Lamborghini has gotten at least one speeding ticket. Therefore, someone in this class has gotten a speeding ticket."*

Determine whether the argument is valid. Justify your answer by explaining which rules of inference are used for each step or by finding a counterexample.