Lecture 11. Introduction to Logic

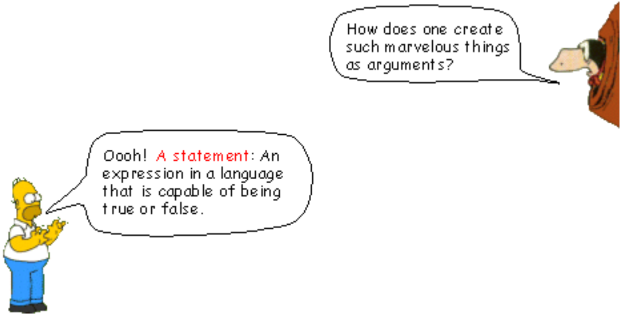

Figure Source: http://web.csulb.edu/∼cwallis/100/slides/toolkit/toolkit.html

The following two **inquiry problems** are designed to help you begin thinking about the ideas in the topic of logic. Think about them on your own and discuss your thoughts, conclusions, and questions with your classmates.

1. Maryam knows whether or not Ahmed is lying. She promises that if Ahmed is lying, she will give you a cookie. Maryam always keeps her promises. Suppose she does not give you a cookie; what can you conclude? Suppose she gives you a cookie; what can you conclude?

2. Camp Halcyon and Camp Placid are two summer camps with the following daily policies on pool use and cleanup duties.

   - Camp Halcyon's Policy: If you used the pool in the afternoon and you didn't clean up after lunch, then you must clean up after dinner.
   - Camp Placid's Policy: You must do at least one of the following: (a) Stay out of the pool in the afternoon, (b) clean up after lunch, or (c) clean up after dinner.

   How do these policies differ? Explain your reasoning.

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  - The assertion at the end of the sequence is called the conclusion, and the preceding statements are called premises.
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- Uses and Applications in Computer Science
  - To prove correctness of software/hardware.
  - Used in computer circuit design.
  - Used in modeling programming languages.
  - Used in the design of expert systems, robots, and artificial intelligence.

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- 42 is a perfect square.

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- 42 is a perfect square.
- If it is raining, then the ground is wet.

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- Abu Dhabi is the capital of United Arab Emirates.
- x is even.

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- The sum of two squares.
- Would you like some cake?
- Read this carefully.

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We can use statement variables (or propositional variables) to represent a simple statement. For a statement variable, a lowercase letter is usually used, for example: $p, q, r, \ldots,$ and so on. The truth value of a statement variable is **True** or **False**.

Example 11.2.

- p: January has 31 days.
- q: February has 33 days.

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A compound statement consists of several simple statements joined together by words such as "**and**", "**or**", "**if ... then**", etc. These **connecting words** are represented by the five logical connectives

Logical operator	Notation	Read as
Negation	$\sim p$	not p
Conjunction	$p \wedge q$	p and q
Disjunction	$p \vee q$	p or q
Conditional	$p \rightarrow q$	p implies q if p, then q p only if q q if p q, provided that p
Biconditional	$p \leftrightarrow q$	p if and only if q

Negation

- Consider the **statement**
  - p: Discrete Math is a required course for sophomores.
- The negation of p is denoted by $\neg p$ and is read " *not p*".
  - $\neg p$: "Discrete Math is not a required course for sophomores" or "It is not the case that Discrete Math is a required course for sophomores."

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Example 11.4. Let p : represent "x is a real number such that $x < 4$." Then $\neg p$: "x is a real number such that $x \geq 4$."

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Conjunction: The conjunction of the statements p, q is denoted by $p \wedge q$, which is read "**p and q**."

Consider the statements:

- p: Sam is poor.
- q: Sam is happy.

There are many ways to express the proposition $p \wedge q$ in English:

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- $p \wedge q =$ Sam is poor and he is happy.
- $p \wedge q =$ Sam is poor, but he is happy.
- $p \wedge q =$ Despite the fact that he is poor, Sam is happy.
- $p \wedge q =$ Although Sam is poor, he is happy.

What are the truth values of $p \wedge q$?

Truth Table of $p \wedge q$:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
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Example 11.5. Write $0 \leq x \leq 1$ using conjunction:

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Example 11.5. Write $0 \leq x \leq 1$ using conjunction: $(x \geq 0) \wedge (x \leq 1)$.

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- **Example 11.6.** $p \oplus q$: "Students who have taken calculus or computer science, but not both, can enroll in this class."

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- If you manage to get a 100% on the final, then you would expect to receive an A. If you do not get 100% you may or may not receive an A depending on other factors. However, if you do get 100%, but the professor does not give you an A, you will feel cheated.

- Because conditional statements play such an essential role in mathematical reasoning, a variety of terminology is used to express $p \to q$.
  1. "if p, then q"; "p implies q"; "q if p"; "q whenever p"; "q when p"
  2. "p is sufficient for q"; "a sufficient condition for q is p"
  3. "q is necessary for p"; "a necessary condition for p is q"; "q follows from p"
  4. "p only if q" $=$ p cannot be true when q is not true; that is, the statement is false if p is true but q is false. [2] When p is false, q may be either true or false, because the statement says nothing about the truth value of q.

[2] In other words, p only if q means that the truth of q is necessary, or required, in order for p to be true. That is, p only if q rules out just one possibility: that p is true and q is false. But that is exactly what $p \to q$ rules out. So it's obviously correct to read $p \to q$ as p only if q.

Example 11.8. To get really clear on the difference between **if** and **only if**, consider the following sentences:

(a) a and b are the same size if $a = b$.

$$a = b \rightarrow \mathrm{SameSize}(a, b)$$

(b) a and b are the same size only if $a = b$.

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(a) is a logical truth: if a and b are one and the same object, then there is no difference between a and b in size, shape, location, or anything else.

(b) But (b) makes a substantive claim that could well be false: it is possible for a and b to be the same size but be two different objects. a and b might be a pair of large cubes, or a might be a large cube and b a large tetrahedron.

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  - p: Maria learns discrete math.
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2. Maria will find a good job when she learns discrete math.
3. For Maria to get a good job, it is sufficient for her to learn discrete mathematics.
4. Maria will find a good job unless [3] she does not learn discrete math.

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- **Example 11.10.** $p \leftrightarrow q =$ "You can take the flight if and only if you buy a ticket."
- Other expression: "p is necessary and sufficient for q"; "if p then q, and conversely"; "p iff q."

Summery of logical connectives

P	Q	$\sim P$	$P \wedge Q$	$P \vee Q$	$P \rightarrow Q$	$P \leftrightarrow Q$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

- We can use these connectives to build up complicated compound statements involving any number of propositional variables.
- Then, we can use truth tables to determine the truth values of these compound statements.
- **Practice 11.11** Construct the truth table of the compound statement

$$(p \wedge \neg q) \rightarrow (p \vee q).$$