

## Lecture 18. Recurrence Relations

## Recurrence Relations (or Recursion)

Suppose we want to define a function  $P : \mathbb{N} \rightarrow \mathbb{Z}$

- 1 The easiest way is to give an **explicit** formula <sup>1</sup>:

$$P(n) = \frac{n(n+1)}{2}$$

- 2 Another way is to **define recursively**

$$P(n) = \begin{cases} 1 & \text{if } n = 1 \\ n + P(n-1) & \text{if } n > 1. \end{cases}$$

This is a **recurrence relation**; that is,  $P(n)$  expressed in terms of one or more of the previous terms of the rule, namely,  $P(0), P(1), \dots, P(n-1)$ .

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<sup>1</sup>also, called a **closed formula** or **closed-form solution to a recurrence relation**

## Three Laws of Recursion

A **well-defined** recurrence relation (recursion) obeys three important laws:

- ① A recursion must have a nonrecursive **base case** that gives at least one value of the function explicitly.
- ② A recursion must call itself, repeatedly.
- ③ A recursion must change its state and move toward the base case.

For example,

$$P(n) = \begin{cases} 1 & \text{if } n = 1 \\ n + P(n-1) & \text{if } n > 1. \end{cases}$$

satisfies all three laws.

**Example 18.1.** Find a recurrence relation  $P(n)$  that yields the following sequence:

$$5, 11, 18, 26, 35, 45, \dots$$

Then, find the closed formula for  $P(n)$ .

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- Answer:

$$P(n) = \begin{cases} 5 & \text{if } n = 0 \\ P(n-1) + 5 + n & \text{if } n \geq 1 \end{cases}$$
$$P(n) = 5 + 5n + \frac{n(n+1)}{2} \quad \text{for } n = 0, 1, 2, \dots$$

**Practice 18.2** Find a recursive definition and an explicit formula for the sequence below. Assume the first term listed is  $a_0$ :

50, 43, 36, 29, ....

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- Answer:

$$A(n) = \begin{cases} 50 & \text{if } n = 0 \\ A(n-1) - 7 & \text{if } n \geq 1 \end{cases}$$

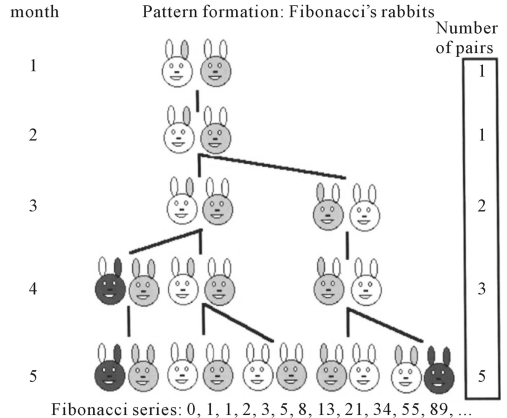
$$A(n) = 50 - 7n \text{ for } n = 0, 1, 2, \dots$$

# The Fibonacci Sequence

- In the early thirteenth century, the Italian mathematician Leonardo Pisano Fibonacci proposed the following problem.

A certain person put **a pair of rabbits** in a place surrounded by a wall.

If **every month** each pair begets a new pair **from the second month**, how many pairs of rabbits can be produced from that pair in a year?





The Fibonacci numbers  $F(n)$  satisfy the following recurrence relation:

$$F(n) = \begin{cases} 1 & \text{if } n = 1 \text{ or } n = 2 \\ F(n-1) + F(n-2) & \text{if } n > 2. \end{cases}$$

**Example 18.3.** Use the above function to list the Fibonacci numbers.

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**Example 18.3.** Use the above function to list the Fibonacci numbers.

The **Fibonacci numbers**: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

Recursive Implementation of Fibonacci numbers in Python

```
def rfib(n):  
    if n<2:  
        return 1  
    else:  
        return rfib(n-1) + rfib(n-2)
```

**Activity 18.4.** Suppose we model the spread of a virus in a certain population as follows. On day 1, one person is infected. On each subsequent day, each infected person gives the cold to two others.

- (a) Write down a recurrence relation for this model.
- (b) What are some of the limitations of this model? How does it fail to be realistic?

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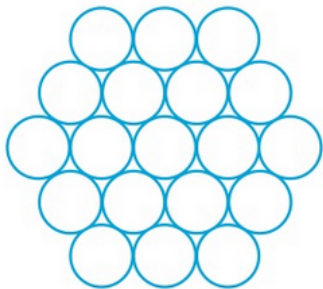
- (a) Write down a recurrence relation for this model.
- (b) What are some of the limitations of this model? How does it fail to be realistic?

Answer: The recurrence relation  $V(n)$  for the number of infected persons is

$$V(n) = \begin{cases} 1 & \text{if } n = 1 \\ V(n-1) + 2V(n-1) & \text{if } n > 1 \end{cases}$$

There are several unrealistic aspects: For example, nobody ever gets better, and there is no limit on the who get infected, etc.

**Activity 18.5.** Suppose that you are collecting coins to make hexagons in a natural way by packing circles as tightly as possible.



The figure shows how 19 circles fit into a hexagonal shape with 3 circles on each edge. Let  $H(n)$  be the number of circles you need to form a hexagon with  $n$  circles on each edge. From the figure it is clear that  $H(2) = 7$  and  $H(3) = 19$ . Find a recurrence relation for  $H(n)$ .

Answer:

$$H(n) = \begin{cases} 7 & \text{if } n = 2 \\ H(n-1) + 6n - 6 & \text{if } n \geq 3. \end{cases}$$