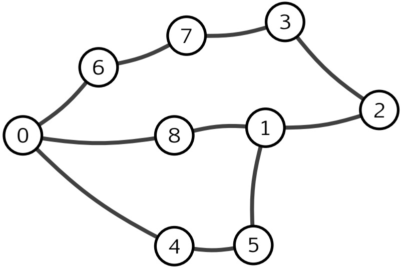
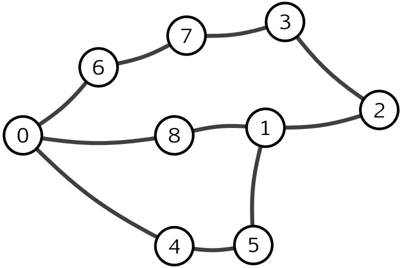
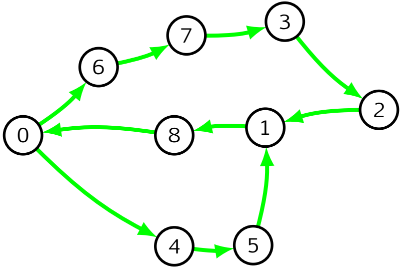
# Lecture 5. Euler Paths and Circuits

Does a path or circuit exist that uses every edge exactly once?



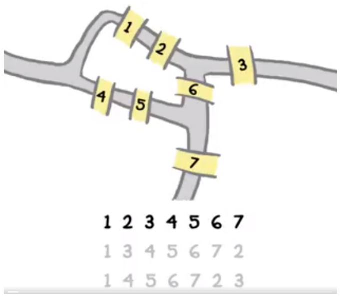
Does a path or circuit exist that uses every edge exactly once?



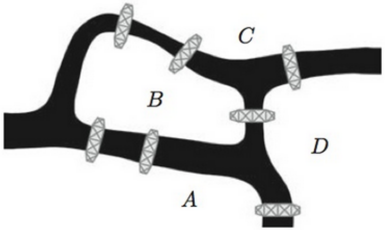

- An Euler path in a graph $G$ is a simple path **containing every edge of** $G$. That is, an Euler path is a path that visits every edge exactly once (allowing for revisiting vertices).
- An Euler circuit in a graph $G$ is an Euler path that starts and ends on the same vertex.

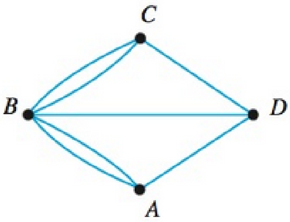
[The Bridges of Königsberg ]. Königsberg is the name for the historic Prussian city that is now Kaliningrad, Russia. The town had a river with two islands. The islands were connected to the river banks by seven bridges (see below). There was an entertaining or interesting exercise for the citizens of Konigsberg. Start from any land regions and come back to the starting point after crossing each of the seven bridges exactly once without repeating same path. Is it possible? The Swiss mathematician Leonhard Euler solved this problem [1] in 1736. Can you solve it?




[1] Leonard Euler's Solution to the Konigsberg Bridge Problem

The problem of traveling across every bridge without crossing any bridge more than once can be rephrased in terms of this graph model.

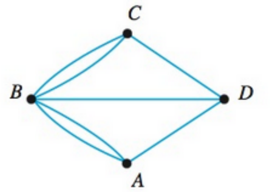



We can rephrase the question like:

Is there a simple circuit in this graph that contains every edge?

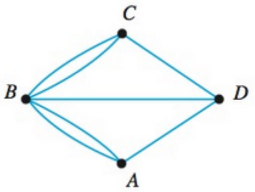
Equivalently, does this graph contain an Euler circuit?




1. In any graph $G = (V, E)$, the sum of the *degree of the vertices* equals twice the *number of of edges*, because each edge contributes $2$ to the sum of the degrees:

$$\sum_{v \in V} deg(v) = 2|E|$$

where $|E|$ = the number of edges in $G$.

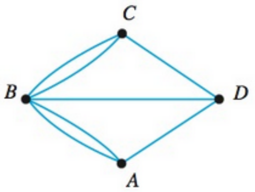



1. In any graph $G = (V, E)$, the sum of the *degree of the vertices* equals twice the *number of of edges*, because each edge contributes 2 to the sum of the degrees:

$$\sum_{v \in V} deg(v) = 2|E|$$

where $|E|$ = the number of edges in $G$.

2. If all the vertices of a connected graph have even degree, then the graph has an Euler circuit.



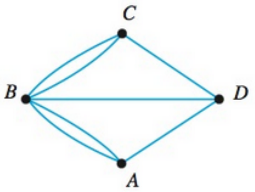

1. In any graph $G = (V, E)$, the sum of the *degree of the vertices* equals twice the *number of of edges*, because each edge contributes 2 to the sum of the degrees:

$$\sum_{v \in V} deg(v) = 2|E|$$

   where $|E|$ = the number of edges in $G$.

2. If all the vertices of a connected graph have even degree, then the graph has an Euler circuit.

3. If a connected graph has exactly two vertices, $v$ and $w$, of odd degree, then there is an Euler path from $v$ to $w$.




1. In any graph $G = (V, E)$, the sum of the *degree of the vertices* equals twice the *number of of edges*, because each edge contributes 2 to the sum of the degrees:

$$\sum_{v \in V} deg(v) = 2|E|$$

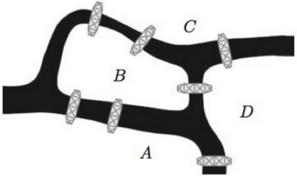
where $|E|$ = the number of edges in $G$.

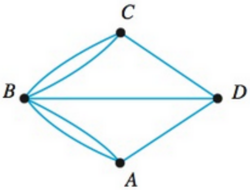
2. If all the vertices of a connected graph have even degree, then the graph has an Euler circuit.

3. If a connected graph has exactly two vertices, $v$ and $w$, of odd degree, then there is an Euler path from $v$ to $w$.

4. If a graph has more than two *vertices* of odd degree, it does not have an Euler path.

# Answer: The Seven Bridges of Königsberg



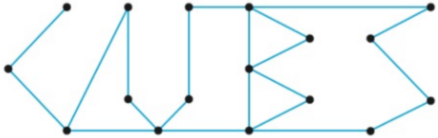




The vertices $A, B, C$, and $D$ of the graph have degrees 3, 5, 3, and 3, respectively. Therefore, this graph does not have an Euler path. In the language of bridges, there is no way a connected walk can cross each bridge exactly once.

# NECESSARY AND SUFFICIENT CONDITIONS FOR EULER CIRCUITS AND PATHS

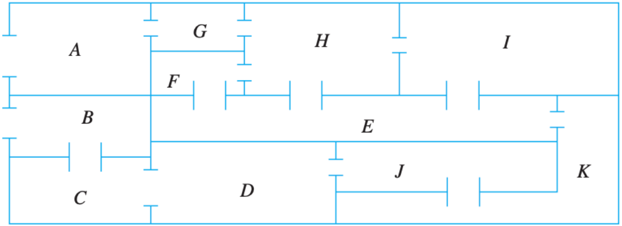
- **Theorem 5.1.** A connected graph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.
- **Theorem 5.2.** A connected graph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree.

**Example 5.3.** Does the following graph have an Euler path? Why or why not?



How many different Euler paths are there in the graph?

**Practice 5.4.** The floor plan shown below is for a house open for public viewing. Is it possible to find a path that starts in room $A$, ends in room $B$, and passes through every *interior* doorway of the house exactly once? Illustrate the graph that represents the question and determine whether such a path exists on the graph.




How many different Euler paths are there in the graph?

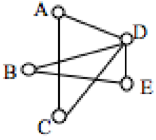
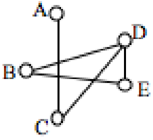
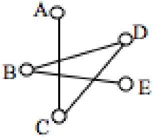
# Fleury's Algorithm

Now we know how to determine if a graph has an Euler circuit, but if it does, how do we find one?

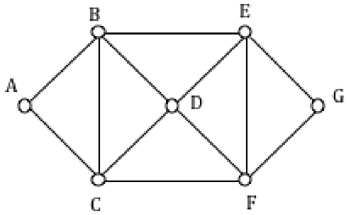
<u>FLEURY'S ALGORITHM</u>

1. Start at any vertex if finding an Euler circuit. If finding an Euler path, start at one of the two vertices with odd degree.
2. Choose any edge leaving your current vertex, provided deleting that edge will not separate the graph into two disconnected sets of edges.
3. Add that edge to your circuit, and delete it from the graph.
4. Continue until you're done.

**Example 5.5.** Find an Euler Circuit on this graph using Fleury's algorithm, starting at vertex $A$.

| | | |
|---|---|---|
| Original Graph.<br>Choosing edge AD.<br><br>Circuit so far: AD | AD deleted. D is current.<br>Can't choose DC since that would disconnect graph.<br>Choosing DE<br><br>Circuit so far: ADE | E is current.<br>From here, there is only one option, so the rest of the circuit is determined.<br><br>Circuit: ADEBDCA |

**Activity 5.6.** Consider the graph




(a) Does the graph have an Euler Circuit? If so, find one using Fleury's algorithm. (Visualize each step of solving.)

(b) How many different Euler circuits are there in this graph?