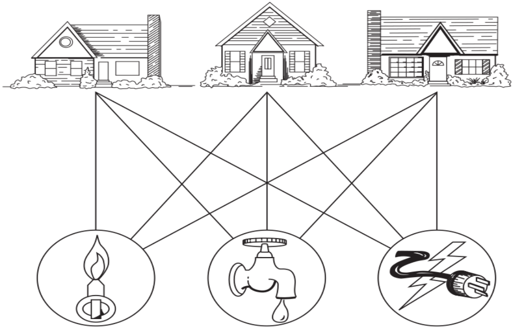
Lecture 7. Planar Graphs (Section 10.7)

Consider the problem of joining three houses to each of three separate utilities. Is it possible to join these houses and utilities such that none of the connections cross?



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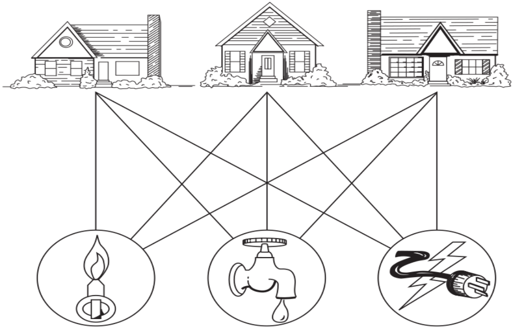
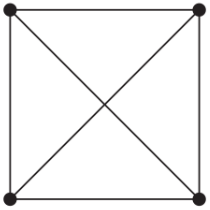
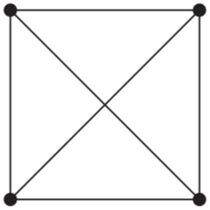
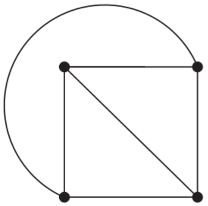
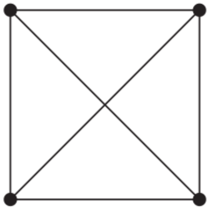


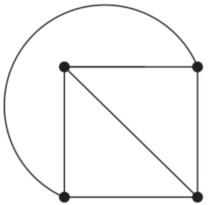
Definition. A graph is called planar if it can be drawn on the plane in such a way that no edges cross each other.

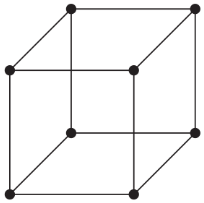


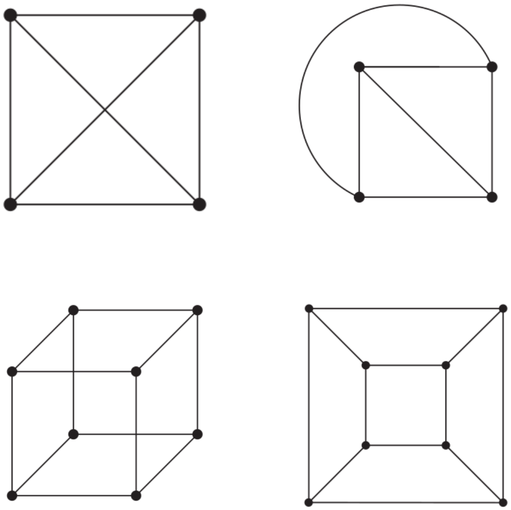




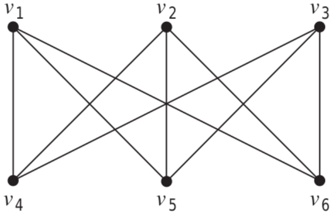








Example 7.1. Show that a complete bipartite graph $K_{3,3}$ is not a planar graph.



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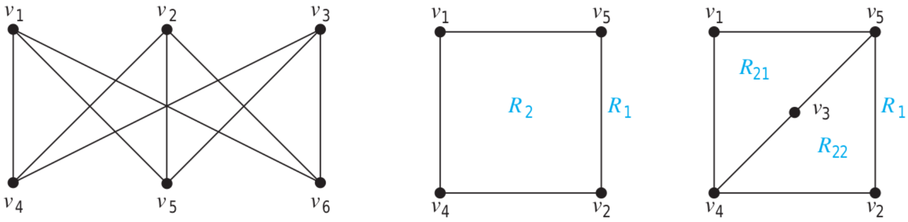
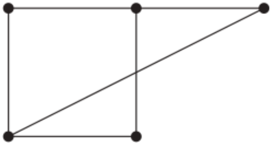


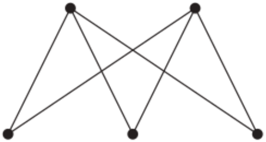
Where to put v_6?

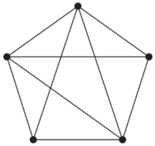
Solution: Any attempt to draw $K_{3,3}$ in the plane with no edges crossing is doomed. We now show why. In any planar representation of $K_{3,3}$, the vertices v_1 and v_2 must be connected to both v_4 and v_5. These four edges form a closed curve that splits the plane into two regions, R_1 and R_2, as shown in Figure 7(a). The vertex v_3 is in either R_1 or R_2. When v_3 is in R_2, the inside of the closed curve, the edges between v_3 and v_4 and between v_3 and v_5 separate R_2 into two subregions, R_{21} and R_{22}, as shown in Figure 7(b).

Next, note that there is no way to place the final vertex v_6 without forcing a crossing. For if v_6 is in R_1, then the edge between v_6 and v_3 cannot be drawn without a crossing. If v_6 is in R_{21}, then the edge between v_2 and v_6 cannot be drawn without a crossing. If v_6 is in R_{22}, then the edge between v_1 and v_6 cannot be drawn without a crossing.

Practice 7.2. Determine if the given graph is planner by drawing the graph without any crossings.

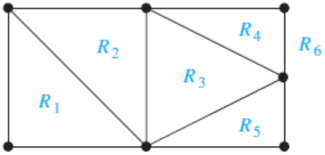






Euler Characteristic of A Planar Graph

A planar representation of a graph splits the plane into regions, including an unbounded region. For instance, the planar representation of the graph shown in Figure splits the plane into six regions.



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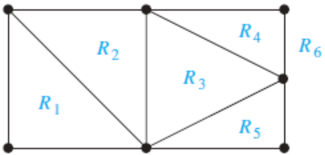
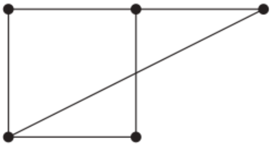


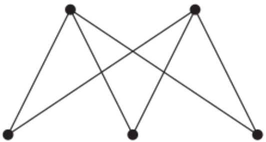
Definition. Let G be a planar graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. The quantity $v - e + r$ is called the Euler characteristic of the graph G.

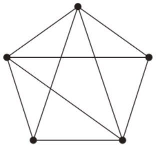
Example 7.3. What is the Euler characteristic of the graph in the above picture?

Definition. Let G be a planar graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. The quantity $v - e + r$ is called the Euler characteristic of the graph G.

Example 7.4. Find the Euler characteristic of the graphs below, if possible.







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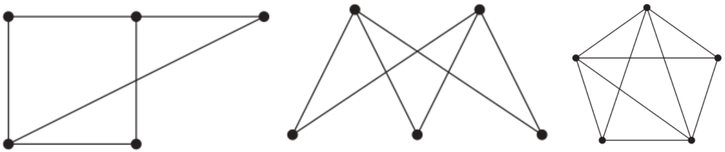

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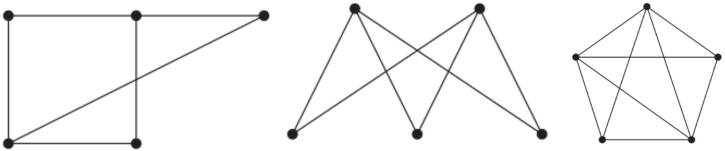

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Theorem 7.5 (Euler's Formula). For any connected planar graph, $v - e + r = 2$.

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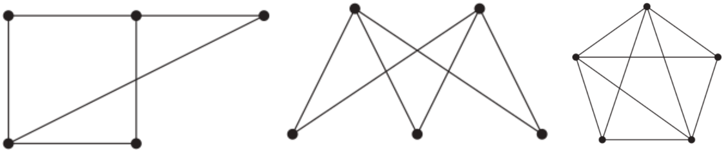

Question. Any observation from Examples 7.3 and 7.4?

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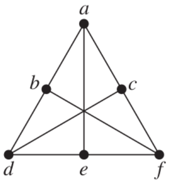
Example 7.6. Suppose that a connected planar simple graph has 20 vertices, each of degree 3. Into how many regions does a representation of this planar graph split the plane?

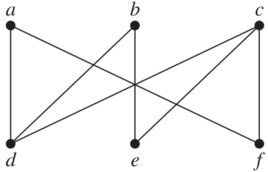
Theorem 7.7 If a connected planar simple graph has e edges and v vertices with $v \geq 3$, then $e \leq 3v - 6$.

Theorem 7.8. If a connected planar simple graph has e edges, v vertices with $v \geq 3$ and no circuits of length three, then $e \leq 2v - 4$.

Example 7.9. $K_{3,3}$ satisfies Theorem 7.7 but not Theorem 8.8. Thus, $K_{3,3}$ is nonplanar.

Practice 7.10. Determine whether the given graph is planar. If so, draw it so that no edges cross.





Activity 7.11. Show that a complete graph K_5 is not a planar graph using a similar argument to that given in Example 7.1.