Cardiovascular Model

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1 Schematics

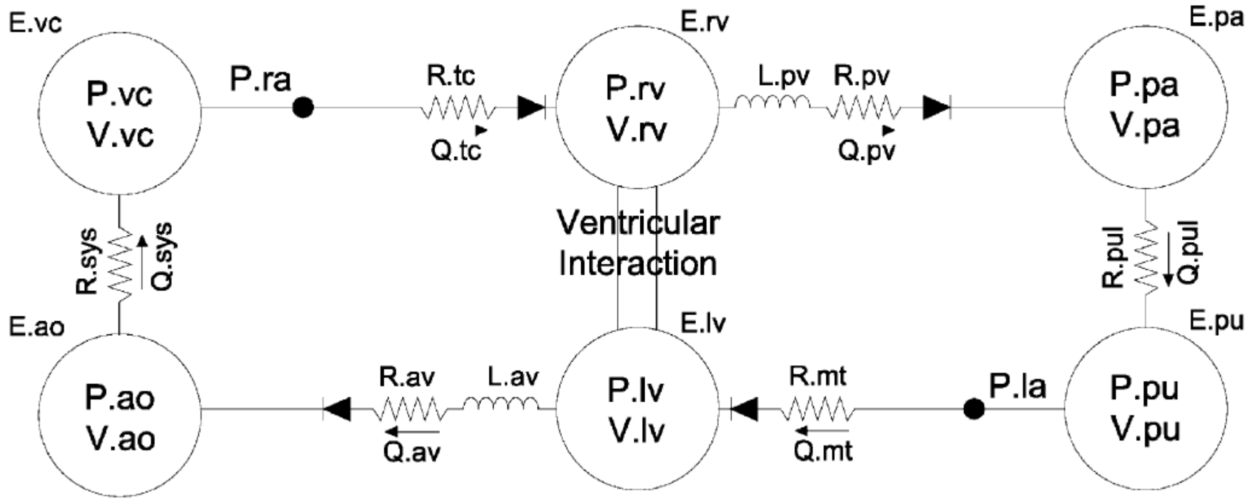


Fig. 1. The presented closed loop model of the cardiovascular system.

2 Mathematical Model

For a summary jump forward to 2.4.

2.1 Model Assemblies

The closed loop is made of resistors, diodes, inductors and elastic chambers. The model is excited by an 'external' driver function $e(t)$.

Resistors - simulate pressure drops of blood flow;

Diodes - simulate valves at the entrance and exit of each ventricle;

Inductors - simulate inertia effects of blood flow;

Elastic chambers - simulate a relation of blood volume to blood pressure. The relations at the passive chambers (vena-cava, aorta, pulmunary vein, pulmunary artery) are linear while the relations at the heart ventricles are exponential. In addition, the heart ventricles have internal interactions which are also modeled.

Driver function - simulates the heartbeats by setting the elasticity of the ventricles. The function is given by $e(t) = \sum_{i=1}^{N} A_i e^{-B_i \left(t - C_i\right)^2}$. Substituting $A = 1, B = 80, C = 0.27, N = 1$ produces the following profile:

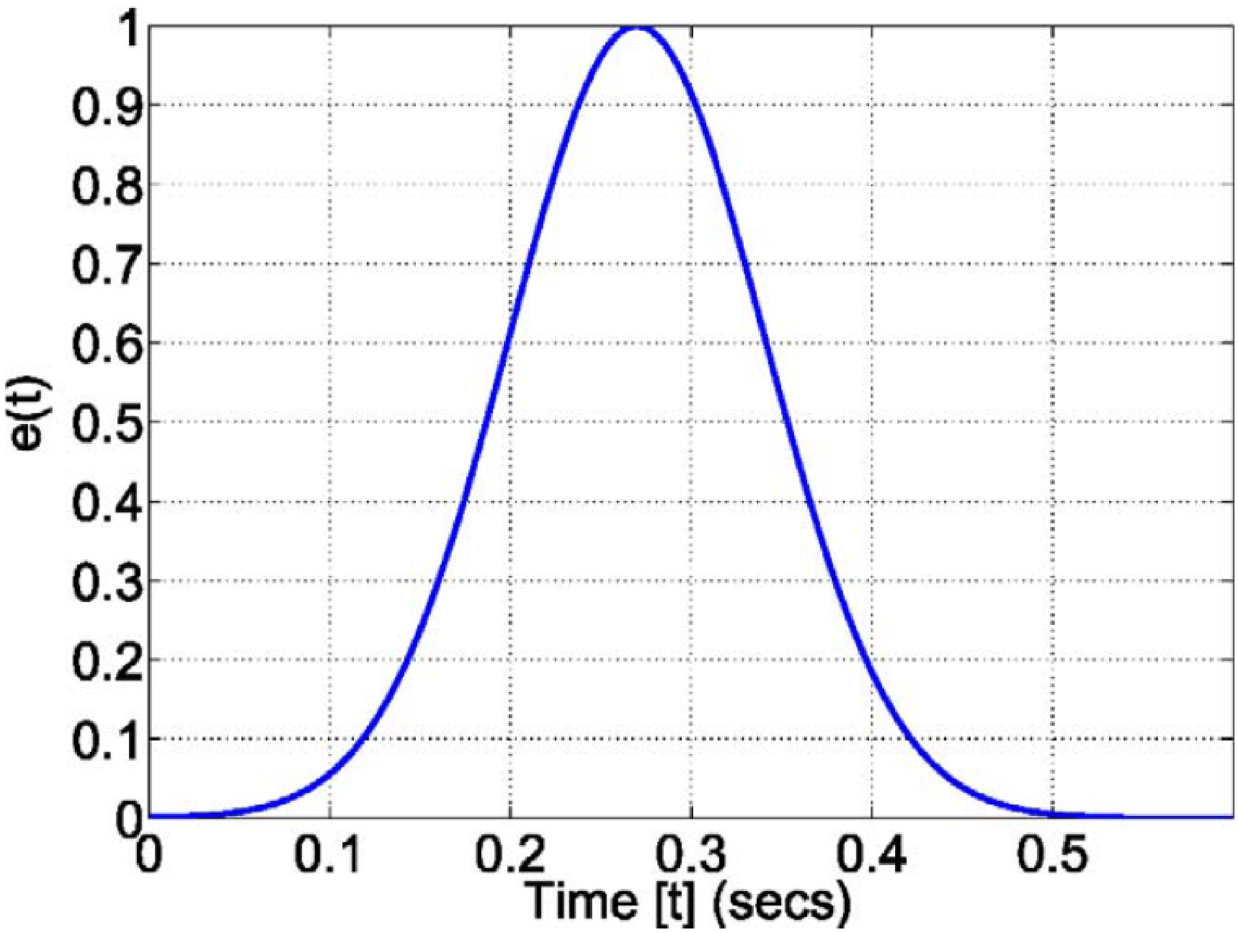


Fig. 3. The model driver ($e(t)$).

2.2 Use-full definitions

Some use-full definitions for later compact model presentation:

$$
\begin{aligned}
V =& VolumesVec = [V_{pa}, V_{pu}, V_{lv}, V_{ao}, V_{vc}, V_{rv}]^T \\
VS =& ValveStateVec = [tricuspid, pulmunary, mitral, aortic]^T \\
IF =& InertialFlowsVec = [Q_{av}, Q_{pv}]^T \\
Q =& [Q_{pv}, Q_{pul}, Q_{mt}, Q_{av}, Q_{sys}, Q_{tc}]^T \\
f_{\langle x\rangle}(V) =& P_{0,\langle x\rangle}\left(e^{\lambda_{\langle x\rangle}\left(V - V_{0,\langle x\rangle}\right)} - 1\right) \\
g_{\langle x\rangle}(V) =& e(t)\, E_{es}\left(V - V_{d,\langle x\rangle}\right) + (1 - e(t))\, f_{\langle x\rangle}(V) \\
h_{\langle x\rangle}(P_1, P_2) =& e^{-\frac{R_{\langle x\rangle}}{L_{\langle x\rangle}} t_s} Q_{\langle x\rangle} + \left(1 - e^{-\frac{R_{\langle x\rangle}}{L_{\langle x\rangle}} t_s}\right) \frac{P_1 - P_2}{R_{\langle x\rangle}} \\
A =& \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
\tag{1}
$$

2.3 Model Equations

We note that the Model cannot be written in the vector form of $\dot{x} = f(x)$ since it has no simple analytic expression, it is an implicit model. The state vector is formed of all the volumes of the different chambers, the boolean state of all four valves and of two flows in which inertial effects are included:

$$X = StateVec = [V; VS; IF] \tag{2}$$

All elements of the state vector X are identifiable in Fig 1.

The model is presented by the hierarchical dependence order. When implementing, the calculations should off-course be made at a reverse order as specified in 2.4.
Volumes are a function of flows as explained in 2.3.2.
Flows are a function of pressures and valve states as explained in 2.3.3.
Pressures are a function of volumes as explained in 2.3.4.

2.3.1 $\dot{x} = f(x)$ Equations form

The difficulty in writing the model in the form $\dot{x} = f(x)$ is due to the following similar example:

$$\begin{aligned} \dot{x} =& e^{(x-y)} - 1 \\ y =& log\left(e^{\lambda(x-y)} - 1\right) + x \end{aligned} \tag{3}$$

And in general:

$$\begin{aligned} \dot{x} =& f(x, y) \\ 0 =& h(x, y) \end{aligned} \tag{4}$$

where $h(x, y) = 0$ is an implicit function.
Conclusion: the model can be written as a set of differential equations and implicit algebraic equations.

2.3.2 Volume Equations

The change of volume of every elastic chamber is due to a difference between input and output flows.

$$\dot{V} = AQ \tag{5}$$

Propagating the model in a time-step method:

$$V \Leftarrow V + t_s AQ \tag{6}$$

where t_s $[sec]$ is the time-step.

2.3.3 Flow Equations and valve states

General concept of flow equations:

The flow between two chambers is determined by the pressure drop between the chambers and the previous flow when inertia effects are included.

The flow from chamber with pressure P_1 to chamber with pressure P_2 with no inertial effects (i.e no inductor is present) is given by:

$$Q = \frac{P_1 - P_2}{R} \tag{7}$$

The flow from chamber with pressure P_1 to chamber with pressure P_2 including inertial effects (i.e an inductor is present) is given by the differential equation:

$$\dot{Q} = \frac{P_1 - P_2 - QR}{L} \tag{8}$$

whose solution is given by:

$$Q = e^{-\frac{R}{L}t_s} Q_0 + \left(1 - e^{-\frac{R}{L}t_s}\right) \frac{P_1 - P_2}{R} \tag{9}$$

where Q_0 is the previous flow, P_1 and P_2 are the current pressures, t_s $[sec]$ is the time-step. We note that the flow is a weighted average between the previous flow and the new flow were there not an inertia effect present.

The closed loop has four valves. The valves operate by the law: "close on flow, open on pressure" - the valve opens on a negative pressure gradient, but is delayed from closing on a positive pressure gradient due to the inertia of the blood, matching known physiological response.

Flows and valves update equations:

Using the flow equations and the valve law we can write the time-update equations for flows and valves states. We remind the reader that the valve-state vector has boolean values.

$$\begin{aligned} Q_{sys} &\Leftarrow \frac{P_{ao} - P_{vc}}{R_{sys}} \\ Q_{pul} &\Leftarrow \frac{P_{pa} - P_{pu}}{R_{pul}} \end{aligned} \tag{10}$$

$$\begin{aligned} Q_{av} &\Leftarrow max\left(h_{av}\left(P_{lv}, P_{ao}\right)\left[aortic + \left(not\left(aortic\right) and\left(P_{lv} > P_{ao}\right)\right)\right], 0\right) \\ Q_{pv} &\Leftarrow max\left(h_{pv}\left(P_{rv}, P_{pa}\right)\left[pulmunary + \left(not\left(pulmunary\right) and\left(P_{rv} > P_{pa}\right)\right)\right], 0\right) \\ Q_{mt} &\Leftarrow max\left(\frac{P_{pu} - P_{lv}}{R_{mt}}\left[mitral + \left(not\left(mitral\right) and\left(P_{pu} > P_{lv}\right)\right)\right], 0\right) \\ Q_{tc} &\Leftarrow max\left(\frac{P_{vc} - P_{rv}}{R_{tc}}\left[tricuspid + \left(not\left(tricuspid\right) and\left(P_{vc} > P_{rv}\right)\right)\right], 0\right) \end{aligned} \tag{11}$$

Now, using the new flow values we update the valve states:

$$\begin{aligned} aortic &\Leftarrow \left(Q_{av} > 0\right) \\ pulmunary &\Leftarrow \left(Q_{pv} > 0\right) \\ mitral &\Leftarrow \left(Q_{mt} > 0\right) \\ tricuspid &\Leftarrow \left(Q_{tc} > 0\right) \end{aligned} \tag{12}$$

2.3.4 Pressure Equations

General concept of pressure equations:

End systolic pressure has a linear pressure-volume relation:

$$P_{es}\left(V\right) = E_{es}\left(V - V_d\right) \tag{13}$$

where E_{es} is an elastance parameter and V_d is the volume at zero pressure.

End diastolic pressure has an exponential pressure-volume relation:

$$P_{ed}\left(V\right) = A\left(e^{\lambda(V - V_0)} - 1\right) \tag{14}$$

where A, λ, V_0 are parameters.

For the passive chambers we are using the simple linear pressure-volume relation. For the ventricles, which change their elastance during the coarse of a heartbeat, we describe the pressure-volume relation as a weighted average between end-diastolic pressure and end-systolic pressure

$$P\left(t\right) = e(t)P_{es} + \left(1 - e(t)\right)P_{ed} \tag{15}$$

Ventricular interaction is also modeled by introducing some non-physical intermediate volumes and pressures:
P_{spt}, V_{spt} - septum wall pressure and volume;
P_{lvf}, V_{lvf} - left ventricle free wall pressure and volume;
P_{rvf}, V_{rvf} - left ventricle free wall pressure and volume;

The internal pressure in a chamber is the sum of the external pressure and the pressure across the chamber's wall. Defining the pressure in the thoracic cavity as P_{pl} and the pressure across the pericardium (the heart) as P_{pcd} we can write the pericardium internal pressure as $P_{peri} = P_{pcd} + P_{pl}$; The internal pressure at the ventricles as $P_{lv} = P_{lvf} + P_{peri}$ and $P_{rv} = P_{rvf} + P_{peri}$.

The pressure across the septum wall is defined as the pressure difference between the ventricles pressures: $P_{spt} = P_{lv} - P_{rv}$.
We define the (non-physical) volume of the septum wall as $V_{spt} = V_{lvf} - V_{lv} = V_{rvf} + V_{rv}$.

The total volume of the pericardium is the sum of the ventricles volumes: $V_{pcd} = V_{lv} + V_{rv}$.

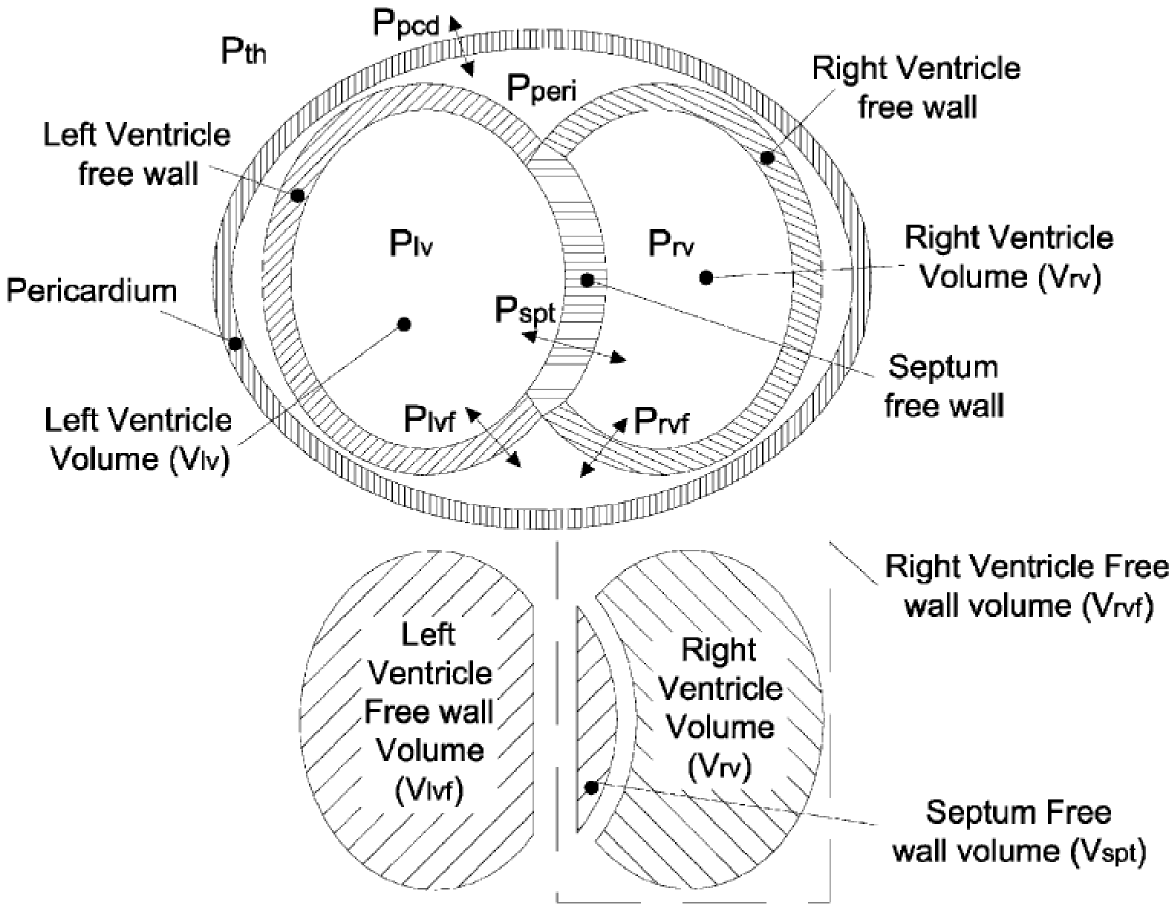


Fig. 4. Pressure and volume definitions in the heart.

Pressure update equations:

The pressures in the peripheral chambers - aorta, vena-cava, pulmunary artery and pulmunary vein is given by:

$$\begin{aligned} P_{ao} =& E_{es,ao} \left(V_{ao} - V_{d,ao}\right) \\ P_{vc} =& E_{es,vc} \left(V_{vc} - V_{d,vc}\right) \\ P_{pa} =& E_{es,pa} \left(V_{pa} - V_{d,pa}\right) \\ P_{pu} =& E_{es,pu} \left(V_{pu} - V_{d,pu}\right) \end{aligned} \tag{16}$$

The ventricles pressures are calculated the following way:
1. Calculate P_{peri}:

$$P_{peri} = f_{pcd} \left(V_{lv} + V_{rv}\right) + P_{pl} \tag{17}$$

2. Calculate V_{spt} by solving the implicit function given by $P_{spt} = P_{lvf} - P_{rvf}$:

$$g_{spt} \left(V_{spt}\right) = g_{lvf} \left(V_{lv} + V_{spt}\right) - g_{rvf} \left(V_{rv} - V_{spt}\right) \tag{18}$$

3. Calculating P_{lv}, P_{rv} by substituting all values at their equations:

$$\begin{aligned} P_{lvf} =& g_{lvf}\left(V_{lv} + V_{spt}\right) \\ P_{rvf} =& g_{rvf}\left(V_{rv} - V_{spt}\right) \\ P_{lv} =& P_{lvf} + P_{peri} \\ P_{rv} =& P_{rvf} + P_{peri} \end{aligned} \tag{19}$$

2.4 State time-update Summary

Given the current state vector $X = [V; VS; IF]$ do:

1. Calculate P_{lv}, P_{rv}, the pressures at the ventricles, as function of the ventricles volumes and the driver function $e(t)$:

$$\begin{aligned} P_{peri} =& f_{pcd}\left(V_{lv} + V_{rv}\right) + P_{pl} \\ g_{spt}\left(V_{spt}\right) =& g_{lvf}\left(V_{lv} + V_{spt}\right) - g_{rvf}\left(V_{rv} - V_{spt}\right) \Rightarrow V_{spt} \\ P_{lvf} =& g_{lvf}\left(V_{lv} + V_{spt}\right) \\ P_{rvf} =& g_{rvf}\left(V_{rv} - V_{spt}\right) \\ P_{lv} =& P_{lvf} + P_{peri} \\ P_{rv} =& P_{rvf} + P_{peri} \end{aligned} \tag{20}$$

2. Calculate the pressure at peripheral chambers as function of the peripheral chambers volumes:

$$\begin{aligned} P_{ao} =& E_{es,ao}\left(V_{ao} - V_{d,ao}\right) \\ P_{vc} =& E_{es,vc}\left(V_{vc} - V_{d,vc}\right) \\ P_{pa} =& E_{es,pa}\left(V_{pa} - V_{d,pa}\right) \\ P_{pu} =& E_{es,pu}\left(V_{pu} - V_{d,pu}\right) \end{aligned} \tag{21}$$

3. Calculate the flows as function of the calculated pressures, previous flow values and valve states VS:

$$\begin{aligned} Q_{sys} \Leftarrow& \frac{P_{ao} - P_{vc}}{R_{sys}} \\ Q_{pul} \Leftarrow& \frac{P_{pa} - P_{pu}}{R_{pul}} \\ Q_{av} \Leftarrow& max\left(h_{av}\left(P_{lv}, P_{ao}\right)\left[aortic + \left(not\left(aortic\right) and \left(P_{lv} > P_{ao}\right)\right)\right], 0\right) \\ Q_{pv} \Leftarrow& max\left(h_{pv}\left(P_{rv}, P_{pa}\right)\left[pulmunary + \left(not\left(pulmunary\right) and \left(P_{rv} > P_{pa}\right)\right)\right], 0\right) \\ Q_{mt} \Leftarrow& max\left(\frac{P_{pu} - P_{lv}}{R_{mt}}\left[mitral + \left(not\left(mitral\right) and \left(P_{pu} > P_{lv}\right)\right)\right], 0\right) \\ Q_{tc} \Leftarrow& max\left(\frac{P_{vc} - P_{rv}}{R_{tc}}\left[tricuspid + \left(not\left(tricuspid\right) and \left(P_{vc} > P_{rv}\right)\right)\right], 0\right) \end{aligned} \tag{22}$$

4. Update the valve states as function of the calculated flows:

$$\begin{aligned} aortic \Leftarrow& \left(Q_{av} > 0\right) \\ pulmunary \Leftarrow& \left(Q_{pv} > 0\right) \\ mitral \Leftarrow& \left(Q_{mt} > 0\right) \\ tricuspid \Leftarrow& \left(Q_{tc} > 0\right) \end{aligned} \tag{23}$$

5. Calculate the new chambers volumes:

$$V \Leftarrow V + t_s A Q \tag{24}$$

where $t_s\,[sec]$ is the time-step.

3 Model Parameters

All model parameters except inductor values which are missing are included in the following two tables:

Table 1 – Base values of the pressure–volume relationship parameters used in the CVS model					
Parameter (units)	E_{es} (kPa/l)	V_d (l)	V_o (l)	λ (1/l)	P_o (kPa)
Left ventricle free wall (lvf)	454	0.005	0.005	15	0.17
Right ventricle free wall (rvf)	87	0.005	0.005	15	0.16
Septum free wall (spt)	6500	0.002	0.002	435	0.148
Pericardium (pcd)	–	–	0.2	30	0.0667
Vena-cava (vc)	1.5	2.83	–	–	–
Pulmonary artery (pa)	45	0.16	–	–	–
Pulmonary vein (pu)	0.8	0.2	–	–	–
Aorta (ao)	94	0.8	–	–	–

Table 2 – Base values of the resistances and other parameters in the CVS model	
Parameter	Value
Mitral valve (R_{mt})	0.06 kPa s/l
Aortic valve (R_{av})	1.4 kPa s/l
Tricuspid valve (R_{tc})	0.18 kPa s/l
Pulmonary valve (R_{pv})	0.48 kPa s/l
Pulmonary circulation (R_{pul})	19 kPa s/l
Systemic circulation (R_{sys})	140 kPa s/l
Heart rate (HR)	80 bpm
Total blood volume (V_{tot})	5.5 l
Thoracic cavity pressure (P_{pl})	−4 mmHg

In order to run the model we should also set initial conditions - the different volumes of the chambers and the currents. I guess the initial currents can be set to zero.

3.1 Physical units

Since $R = \frac{8\mu l}{\pi {r_0}^4}$ and $\mu - \left[\frac{N}{m^2}s\right]$, $l - [m]$, $r_0 - [m]$ than $R - \left[\frac{\frac{N}{m^2}s}{m^3}\right]$. Pressure units are $[Pascal] = [Pa] = \left[\frac{N}{m^2}\right]$, liter units are $[liter] = [l] = [10^{-3}m^3]$, therefore $R - \left[10^3\frac{Pas}{l}\right]$.

$Q = \frac{P_2 - P_1}{R}$ so Q has units: $\left[\frac{10^{-3}l}{s}\right]$

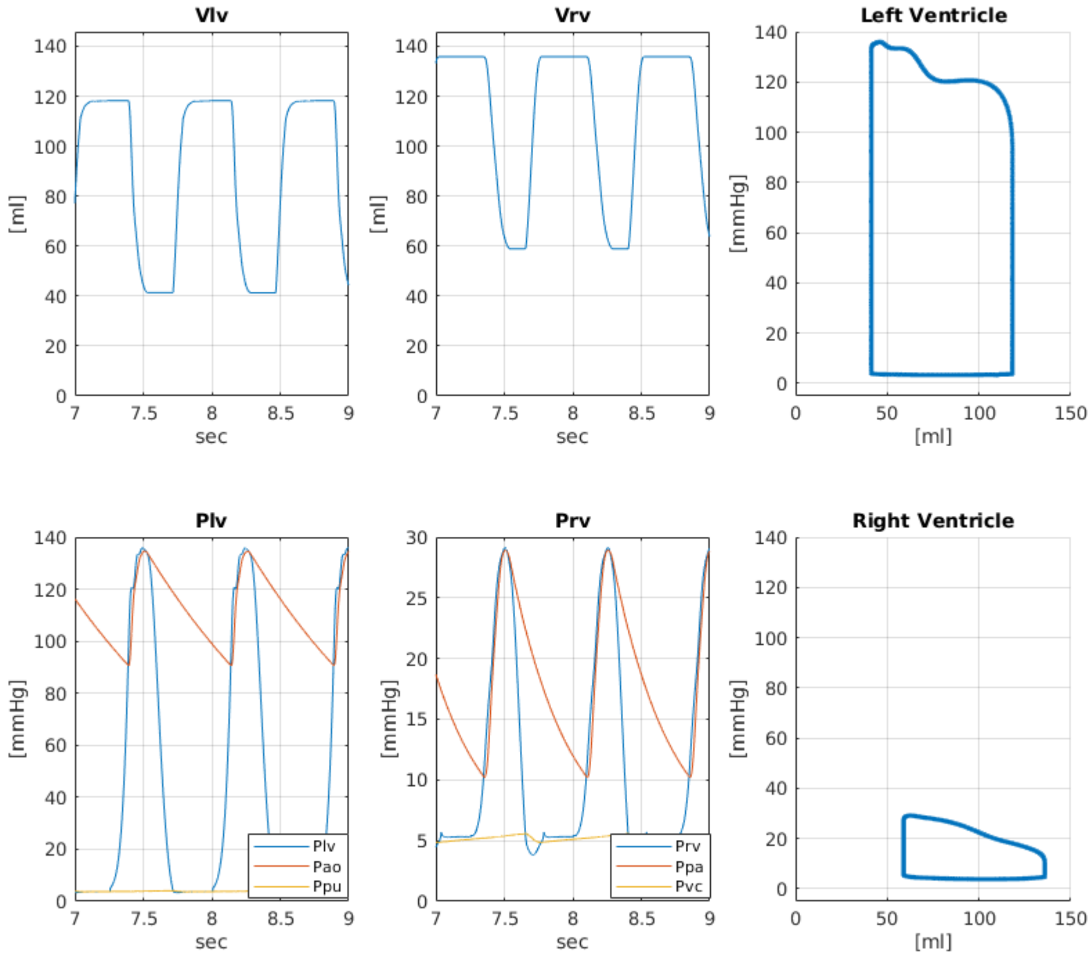
$P_0 - [kPa]$, $\lambda - \left[\frac{1}{l}\right]$, $V_0, V_d - [l]$, $E_{es} - \left[\frac{kPa}{l}\right]$

$[mmHg] = [133.322387415Pa]$

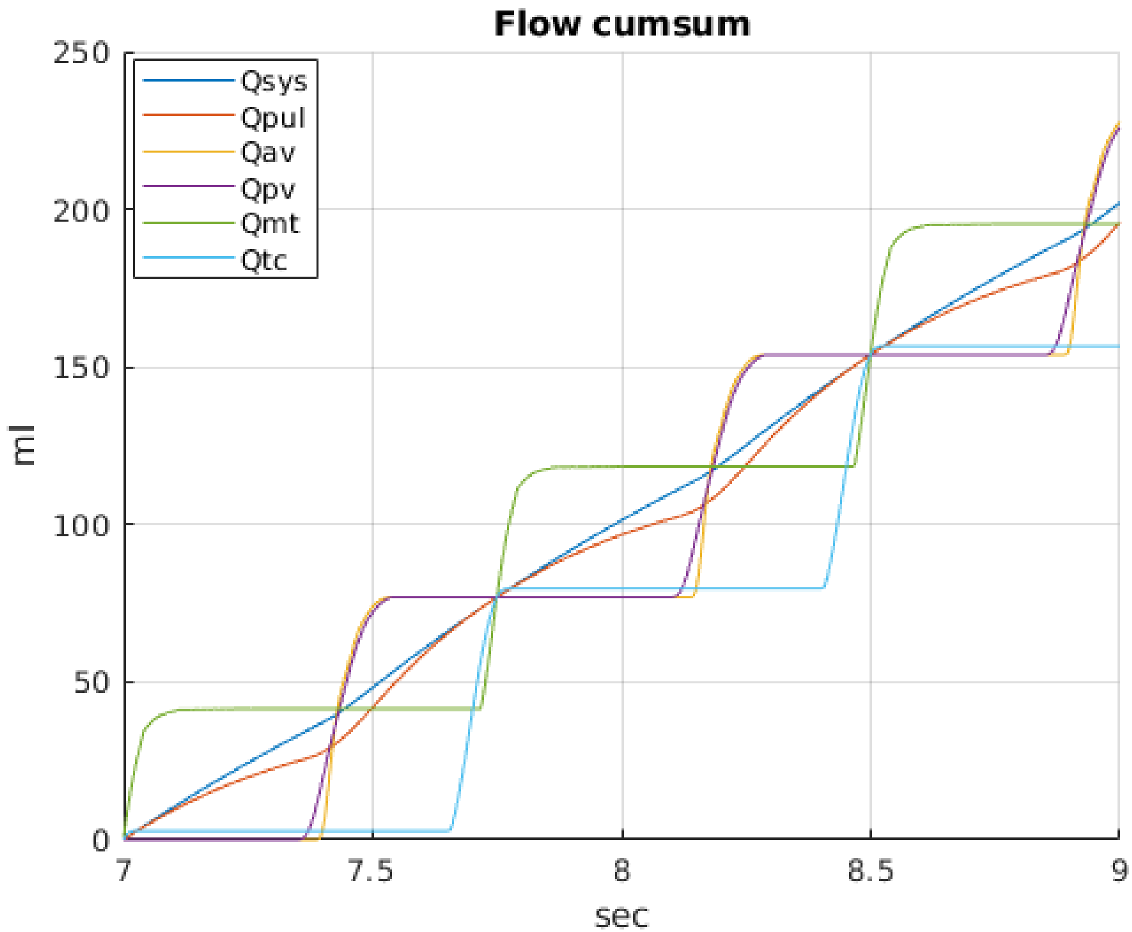
Our use-full functions:
$e(t) - nounits$
$f, g - [kPa]$
$h - \left[\frac{10^{-3}l}{s}\right]$ therefore the incuctance has units $L = \left[\frac{kPa}{l}s^2\right]$

3.2 Results

Some results from running the model which seems to be OK, i.e - values are in the vicinity of normal human measurements.// Right, Left ventricles pressures and volumes:



Vlv,Plv,Vrv,Prv - ventricles volumes and pressures.
Pao - aortic pressure; Ppu - pulmonary vein pressure; Ppa - pulmonary artery pressure; Pvc - vena-cava pressure.
Corresponding blood flow - it can be seen that a stroke volume is about 70 ml:



Qsys - systemic flow; Qpul - pulmonary flow; Qav - flow through aortic valve; Qpv - flow through pulmonary valve; Qmt - flow through mitral valve; Qtc - flow through tricuspid valve;