# MAD 2020-21, Assignment 3

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**General comments:** The assignments in MAD must be completed and written individually. You are allowed (and encouraged) to discuss the exercises in small groups. If you do so, you are required to list your group partners in the submission. The report must be written completely by yourself. In order to pass the assignment, you will need to get at least 40% of the available points. The data needed for the assignment can be found in the assignment folder that you download from Absalon.

**Submission instructions:** Submit your report as a PDF, not zipped up with the rest. Please add your source code to the submission, both as executable files and as part of your report in appendix. To include it in your report, you can use the `lstlisting` environment in LaTeX, or you can include a "print to pdf" output in your pdf report. In some exercises we will ask you to include a code snippet as part of your solution text - a code snippet is only the most essential lines of code needed for solving the problem, this does not include import statements, other forms of boiler plate code, as well as plotting code.

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## Principal Components Analysis

**Exercise 1** (**Implement PCA, 8 points**). See the Jupyter notebook file `pca_StudentVersion.ipynb` for the detailed questions and hints.

a) Implement PCA on the diatoms database. You are supposed to write your own implementation and thus is not allowed to use implementations found in various libraries. Please output the proportion of variance explained by each of the first 10 components (5 points)
b) Visualize fourth component of the PCA (3 points)

*Deliverables.* a) The jupyter notebook with your solution and write in the report the proportion of variance explained by the first 10 components, b) include the visualisations in the report and comment on the results.

## Statistics

**Exercise 2** (**2 points. (based on Blitzstein & Hwang Exercise 10.7.6) Inequalities).** Let X be a random variable with mean $\mu$ and variance $\sigma^2$. Show that

$$E\left[(X-\mu)^4\right] \geq \sigma^4 \ .$$

Hint: Consider if you can use Jensen's inequality.

*Deliverables.* Include the steps and argumentation of the proof of the expression.

**Exercise 3** (**4 points. Confidence Intervals).** Let $\gamma \in \mathbb{R}$ be fixed and let $X_1, \ldots, X_n$ be i.i.d. with Normal distribution $\mathcal{N}(\mu, \sigma^2)$. We estimate $\mu$ by the sample mean $\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i$. In the lecture, we have seen that

$$\sqrt{n}\,\frac{\hat{\mu}-\mu}{\sigma} \sim \mathcal{N}(0,1). \tag{1}$$

a) Pretend that (1) holds, even if we replace $\sigma$ by the estimator $\hat{\sigma} := \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\hat{\mu})^2}$ (i.e. the sample standard deviation). Construct a $\gamma$-confidence interval for $\mu$ by using the procedure explained in the lecture.
b) We can make a simulation of many experiments at a fixed $n$, and compute the probability of the correct $\mu$ value being outside the confidence interval – if our estimated confidence interval is a good fit then this probability should be $< (1-\gamma)$. Modify the code in `confidenceinterv.py` (here, $n = 9$) and report, how often (out of 10000 experiments) the correct parameter lies outside the confidence interval. (Hint: Numpy's `np.var` divides by $n$, not by $n-1$. To correct for this, set the parameter `ddof=1` of `np.var`.)

c) In fact, (1) does not hold if we replace $\sigma$ with $\hat{\sigma}$. Instead, we have

$$\sqrt{n}\,\frac{\hat{\mu}-\mu}{\hat{\sigma}} \sim t_{n-1}$$

where $t_{n-1}$ is a student-$t$ distribution with $n-1$ degrees of freedom. Again, report the corresponding confidence interval, modify the notebook, and report, how often the correct parameter is not covered. (Hint: Use `scipy.stats.t.ppf(q, n-1)` to compute the critical value by reverse look up in the CDF of the $t$ distribution of $n-1$ degrees of freedom.)

*Deliverables.* a) Write the correct expression for the $\gamma$-confidence interval under these assumptions, b) your modified version of the code and your answer to the question, c) write the correct expression for the $\gamma$-confidence interval under these assumptions, and include your modified code and your answer to the question.

**Exercise 4 (4 points. Hypothesis Testing).** A scientist claims that he has found a single gene that has an influence on the flowering time of a plant. In order to see whether his claim is true, he obtains five pairs $(X_1, Y_1), \ldots, (X_5, Y_5)$ of two genetically identical replicates. In each second replicate $(Y_1, \ldots, Y_5)$ he has knockedout the gene. The following table shows the flowering time (in days).

| Plant | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Replicate 1 without knockout | 4.1 | 4.8 | 4.0 | 4.5 | 4.0 |
| Replicate 2 with knockout | 3.1 | 4.3 | 4.5 | 3.0 | 3.5 |

Assume that the differences $X_i - Y_i$ (that is, flowering time replicate X minus flowering time replicate Y) are normally distributed with mean $\mu$ and variance $\sigma^2$.

a) Choose the null hypothesis and briefly justify your answer.
b) Perform the corresponding $t$-test to the level 0.05 (Hint: Use the "six steps" from the lecture).
c) Can the scientist change the test result by (illegally) copying the data set, that is, by writing down each data point $k$ times and pretending that he has investigated $5 \cdot k$ independent pairs of plants? Justify your answer.

*Deliverables.* a) Your justified answer, b) explain the six steps you go through in the t-test and whether or not you can reject the null hypothesis, c) your justified answer.