MAD Exam Answers

Exam no: 161

January 17, 2022

Exercise 1 (Maximum Likelihood Estimation)

$\underset{(\mu,\sigma)}{\arg\max} f(x) = \underset{(\mu,\sigma)}{\arg\max} \log f(x) =$

$\underset{(\mu,\sigma)}{\arg\max} \log(\prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} \cdot \frac{1}{x} \cdot \exp(-\frac{1}{2}(\frac{\log(x)-\mu}{\sigma})^2)) =$

$\underset{(\mu,\sigma)}{\arg\max} \sum_{i=1}^{n} \log(\sigma^{-1} \cdot (2\pi)^{-\frac{1}{2}} \cdot x^{-1} \cdot \exp(-\frac{1}{2}\sigma^{-2}(\log(x)-\mu)^2)) =$

$\underset{(\mu,\sigma)}{\arg\max} \sum_{i=1}^{n}(-\log(\sigma) - \frac{1}{2}\log(2\pi) - \log(x) - \frac{1}{2}\sigma^{-2}(\log(x)-\mu)^2) =$

$-n\log(\sigma) - \frac{n}{2}\log(2\pi) - n\log(x) - \frac{1}{2}\sigma^{-2}\sum_{i=1}^{n}(\log(x)-\mu)^2$

$\frac{\partial f}{\partial \mu} = 0$

$-\frac{1}{2}\sigma^{-2}\sum_{i=1}^{n} 2(\log(x)-\mu)(-1) = 0$

$\sigma^{-2}\sum_{i=1}^{n}(\log(x)-\mu) = 0$

$\log(X) - n\mu = 0$

$\mu = \frac{1}{n}\log(X)$

$\hat{\mu} = \log(\sqrt[n]{X})$

$\hat{\mu} = \log(\sqrt[n]{x_1 \cdot x_2 \cdot ... \cdot x_n})$

Exercise 2 (Statistics)

a)

$\arg\max_{\beta} f_\beta(x) = \arg\max_{\beta} \log f_\beta(x) = \arg\max_{\beta} \log(\prod_{i=1}^{n} \frac{2}{\beta^2} \cdot \beta - x) = \arg\max_{\beta} \log(\prod_{i=1}^{n} 2\beta^{-2} \cdot (\beta - x)) = \arg\max_{\beta} \sum_{i=1}^{n} [-4\log(\beta) + \log(\beta - x)]$

$\frac{\partial f}{\partial \beta} = 0$

$\sum_{i=1}^{n} [\frac{-4}{\beta} + \frac{1}{\beta - x}] = 0$

$\frac{-4n}{\beta} + \frac{n}{\beta - x} = 0$

$\frac{-4n(\beta - x)}{\beta} + \frac{n\beta}{\beta - x} = 0$

$\frac{-4n(\beta - x) + n\beta}{\beta(\beta - x)} = 0$

$\frac{-4n\beta + 4nx + n\beta}{\beta(\beta - x)} = 0$

$3n\beta = 4nx$

$\beta = \frac{4x}{3}$

When $x_1 = 3$, $\beta = 4$

When $x_2 = 4$, $\beta = \frac{16}{3}$

b)

Model: $X \sim Bin(n, \theta)$

Null hypothesis: $H_0 : \theta = 0.5$

Alternative: $H_1 : \theta \neq 0.5$

Trials: $n = 20$

Test statistic: $T = \frac{\sqrt{n}(\bar{X} - H_0)}{\sigma}$

Distribution of T under H_0 :T$\sim t_{n-1} = t_{19}$

Level: $\alpha = 0.05$

Rejection region: $R = \{0, ..., a\} \cup \{20 - a, ..., 20\} = \{0, ..., 2\} \cup \{18, ..., 20\}$

Computing T: $\sigma^2 = n\theta(1 - \theta) = 20 \cdot 0.5 \cdot 0.5 = 5$

$\sigma = \sqrt{5}$

$T = \frac{\sqrt{20}(13 - 20 \cdot 0.5)}{\sqrt{5}} = 6$

Test decision: $T \notin R \implies H_0$ not rejected

Exercise 3 (Principal Component Analysis)

1)

$x_{mean} = \frac{0.6+0.5+1.1-0.5+0.8+0.2-0.1+1}{8} = \frac{3.6}{8} = 0.45$

$y_{mean} = \frac{1.1+1+2+0.2-0.1-0.1-1.5+2.5}{8} = \frac{5.1}{8} = 0.6375$

Mean point of the data is $\begin{bmatrix} 0.45 \\ 0.6375 \end{bmatrix}$

2)

$$data_{center} = \begin{bmatrix} x - 0.45 \\ y - 0.6375 \end{bmatrix} = \begin{bmatrix} 0.15 & 0.05 & 0.65 & -0.95 & 0.35 & -0.25 & -0.55 & 0.55 \\ 0.4625 & 0.3625 & 1.3625 & -0.4375 & -0.7375 & -0.7375 & -2.1375 & 1.8625 \end{bmatrix}$$

$$covariancematrix = \frac{1}{N} \sum_{n=1}^{N} data_{center} \cdot data_{center}^{\intercal} = \frac{1}{N} \sum_{n=1}^{N}$$

$$\begin{bmatrix} 0.15 & 0.05 & 0.65 & -0.95 & 0.35 & -0.25 & -0.55 & 0.55 \\ 0.4625 & 0.3625 & 1.3625 & -0.4375 & -0.7375 & -0.7375 & -2.1375 & 1.8625 \end{bmatrix} \cdot \begin{bmatrix} 0.15 & 0.4625 \\ 0.05 & 0.3625 \\ 0.65 & 1.3625 \\ -0.95 & -0.4375 \\ 0.35 & -0.7375 \\ -0.25 & -0.7375 \\ -0.55 & -2.1375 \\ 0.55 & 1.8625 \end{bmatrix} =$$

$$\frac{1}{N} \sum_{n=1}^{N} \begin{bmatrix} \frac{107}{50} & \frac{703}{200} \\ \frac{703}{200} & \frac{1843}{160} \end{bmatrix} = \frac{1}{N} \sum_{n=1}^{N} \begin{bmatrix} 2.14 & 3.515 \\ 3.515 & 11.519 \end{bmatrix} = \begin{bmatrix} \frac{2.14}{8-1} & \frac{3.515}{8-1} \\ \frac{3.515}{8-1} & \frac{11.519}{8-1} \end{bmatrix} = \begin{bmatrix} \frac{107}{350} & \frac{703}{1400} \\ \frac{703}{1400} & \frac{1843}{1120} \end{bmatrix} =$$

$$\begin{bmatrix} 0.30571429 & 0.50214286 \\ 0.50214286 & 1.64557143 \end{bmatrix}$$

To find eigenvalues $\lambda :$

$$\det \begin{bmatrix} 0.30571429-\lambda & 0.50214286 \\ 0.50214286 & 1.64557143-\lambda \end{bmatrix} = 0$$

$$(\tfrac{107}{350}-\lambda)(\tfrac{1843}{1120}-\lambda)-\tfrac{703}{1400}\cdot\tfrac{703}{1400}=0$$

$$\lambda^2-(\tfrac{107}{350}+\tfrac{1843}{1120})\lambda+\tfrac{107}{350}\cdot\tfrac{1843}{1120}-(\tfrac{703}{1400})^2=0$$

$$\lambda^2-(\tfrac{1561}{800})\lambda+\tfrac{197201}{392000}-\tfrac{494209}{1960000}=0$$

$$\lambda^2-(\tfrac{1561}{800})\lambda+\tfrac{122949}{490000}=0$$

$$\lambda_1=\tfrac{10927+\sqrt{87924385}}{11200}=1.81283932$$

$$\lambda_2=\tfrac{10927-\sqrt{87924385}}{11200}=0.13841068$$

Eigenvalues are 1.81283932, 0.13841068

3)

When $\lambda_1=\frac{10927+\sqrt{87924385}}{11200}=1.81283932$

$$\begin{bmatrix} \frac{107}{350}-\frac{10927+\sqrt{87924385}}{11200} & \frac{703}{1400} \\ \frac{703}{1400} & \frac{1843}{1120}-\frac{10927+\sqrt{87924385}}{11200} \end{bmatrix}\cdot\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{-7{,}503-\sqrt{87924385}}{11200} & \frac{703}{1400} \\ \frac{703}{1400} & \frac{7{,}503-\sqrt{87924385}}{11200} \end{bmatrix}\cdot\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\tfrac{-7{,}503-\sqrt{87924385}}{11200}x+\tfrac{703}{1400}y=0$$

Let $x=k.$ then $y=\frac{7{,}503+\sqrt{87924385}}{11200}k\cdot\frac{1400}{703}=\frac{7{,}503+\sqrt{87924385}}{5{,}624}k$

$$\sqrt{1^2+(\frac{7{,}503+\sqrt{87924385}}{5{,}624})^2} = 3.16359349$$

$$Eigenvector = \begin{bmatrix} \frac{1}{3.16359349} \\ \\ \frac{7{,}503+\sqrt{87924385}}{5{,}624\cdot 3.16359349} \end{bmatrix} = \begin{bmatrix} 0.31609624 \\ \\ 0.94872713 \end{bmatrix}$$

When $\lambda_2 = \frac{10927-\sqrt{87924385}}{11200} = 0.13841068$

$$\begin{bmatrix} \frac{107}{350} - \frac{10927-\sqrt{87924385}}{11200} & \frac{703}{1400} \\ \\ \frac{703}{1400} & \frac{1843}{1120} - \frac{10927-\sqrt{87924385}}{11200} \end{bmatrix} \cdot \begin{bmatrix} x \\ \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{-7{,}503+\sqrt{87924385}}{11200} & \frac{703}{1400} \\ \\ \frac{703}{1400} & \frac{7{,}503+\sqrt{87924385}}{11200} \end{bmatrix} \cdot \begin{bmatrix} x \\ \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ \\ 0 \end{bmatrix}$$

$$\frac{-7{,}503+\sqrt{87924385}}{11200}x + \frac{703}{1400}y = 0$$

Let $x = k$. then $y = \frac{7{,}503-\sqrt{87924385}}{11200}k \cdot \frac{1400}{703} = \frac{7{,}503-\sqrt{87924385}}{5{,}624}k$

$$\sqrt{1^2+(\frac{7{,}503-\sqrt{87924385}}{5{,}624})^2} = 1.0540438526240894300912667001583$$

$$Eigenvector = \begin{bmatrix} \frac{1}{1.05404385} \\ \\ \frac{7{,}503-\sqrt{87924385}}{5{,}624\cdot 1.05404385} \end{bmatrix} = \begin{bmatrix} 0.94872713 \\ \\ -0.3160962 \end{bmatrix}$$

Eigenvectors are $\begin{bmatrix} 0.31609624 & -0.94872713 \\ \\ 0.94872713 & 0.31609624 \end{bmatrix}$

Exercise 4 (Regression)

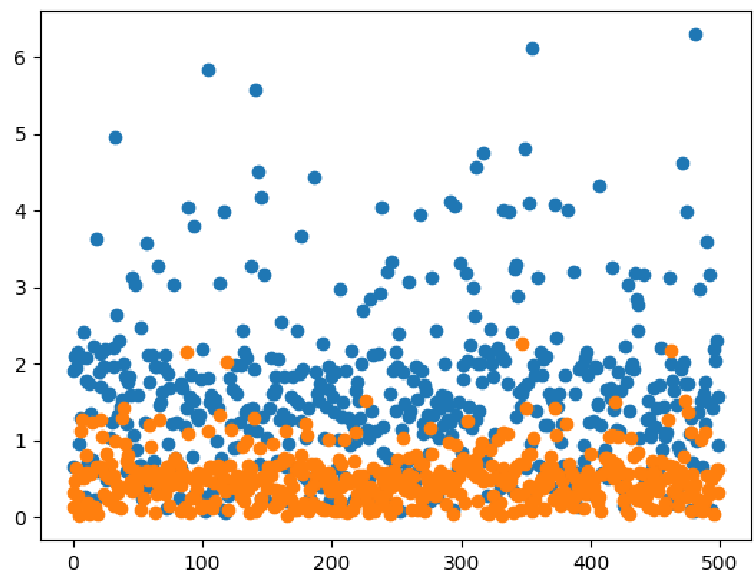
1)

a) RMSE:

1.4352299881906454

1.4083244892693603

1.5272255823198648



b) RMSE:

1.4352299881906454

1.4083244892693603

1.5272255823198648