MAD Assignment 4

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Exercise 1 (Maximum Likelihood)

$\arg\max_\theta p(\theta) = \arg\max_\theta \log P(\theta) = \arg\max_\theta \log(\prod_{i=1}^n (1-\theta)^{x_i-1}\theta) = \arg\max_\theta \sum_{i=1}^n (x_i - 1)\log(1-\theta) + \log\theta$

$\frac{\partial p}{\partial \theta} = 0$

$\sum_{i=1}^n -\frac{x_i-1}{1-\theta} + \frac{1}{\theta} = 0$

$\sum_{i=1}^n \frac{1}{\theta} = \sum_{i=1}^n \frac{x_i-1}{1-\theta}$

$\frac{1}{\theta}\sum_{i=1}^n 1 = \frac{1}{1-\theta}\sum_{i=1}^n (x_i - 1)$

$(1-\theta)\sum_{i=1}^n 1 = \theta\sum_{i=1}^n (x_i - 1)$

$\sum_{i=1}^n 1 - \theta\sum_{i=1}^n 1 = \theta\sum_{i=1}^n (x_i - 1)$

$n = \theta\sum_{i=1}^n (x_i - 1 + 1)$

$\theta = \frac{n}{\sum_{i=1}^n x_i}$

Since $\frac{\partial^2 p}{\partial\theta\partial\theta} = -\frac{x_i-1}{(1-\theta)^2} - \frac{1}{\theta^2} < 0$, the estimate $\hat{\theta} = \frac{n}{\sum_{i=1}^n x_i}$ is a global maximum.

(Since $0 < \sum_{i=1}^n x_i < n$ and $0 < \theta < 1$)

Exercise 2 (4-dimensional Maximum Likelihood)

a)

$Pr(x_{\text{min}} \leq x \leq x_{\text{max}}, y_{\text{min}} \leq y \leq y_{\text{max}}) = \int_{x_{\text{min}}}^{x_{\text{max}}} \int_{y_{\text{min}}}^{y_{\text{max}}} f(x,y)dxdy = c(x_{\text{max}} - x_{\text{min}})(y_{\text{max}} - y_{\text{min}}) = 1$

$c = \frac{1}{(x_{\text{max}} - x_{\text{min}})(y_{\text{max}} - y_{\text{min}}}$

b)

$\theta_1 = (-1, 4, -1, 3)$

$\theta_2 = (-2, 5, -3, 6)$

$c_{\theta_1} = \frac{1}{(4-(-1))(3-(-1)} = \frac{1}{20}$

$c_{\theta_2} = \frac{1}{(5-(-2))(6-(-3)} = \frac{1}{63}$

c)

$\theta = (0, 2, 0, 2)$

$c_{\theta} = \frac{1}{(2-0)(2-0)} = \frac{1}{4}$

Exercise 3 (Coin Game)

a)

$p(r) = 1 \ (0 \leq r \leq 1)$

$p(y_N|r) = \binom{N}{y_N} r^{y_N}(1-r)^{N-y_N}$

$p(r|y_N) = \frac{p(y_N|r)p(r)}{p(y_N)}$

$p(r|y_N) \propto p(y_N|r)p(r)$

$p(r|y_N) \propto \binom{N}{y_N} r^{y_N}(1-r)^{N-y_N} \cdot 1$

$p(r|y_N) \propto r^{y_N}(1-r)^{N-y_N}$

$p(r|y_N) \propto r^{\delta}(1-r)^{\gamma}$ with $\delta = y_N$ and $\gamma = N - y_N$

b)

$p(r) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} r^{\alpha-1}(1-r)^{\beta-1}$

$r^{\alpha-1}(1-r)^{\beta-1} = 2r$

$p(r) = 2r \ (0 \leq r \leq 1)$

$p(y_N|r) = \binom{N}{y_N} r^{y_N}(1-r)^{N-y_N}$

$p(r|y_N) = \frac{p(y_N|r)p(r)}{p(y_N)}$

$p(r|y_N) \propto p(y_N|r)p(r)$

$p(r|y_N) \propto \binom{N}{y_N} r^{y_N}(1-r)^{N-y_N} \cdot 2r$

$p(r|y_N) \propto r^{y_N+1}(1-r)^{N-y_N}$

$p(r|y_N) \propto r^{\delta}(1-r)^{\gamma}$ with $\delta = y_N + 1$ and $\gamma = N - y_N$

c)

$p(r) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} r^{\alpha-1}(1-r)^{\beta-1}$

$r^{\alpha-1}(1-r)^{\beta-1} = 3r^2$

$p(r) = 2r \ (0 \leq r \leq 1)$

$p(y_N|r) = \binom{N}{y_N} r^{y_N}(1-r)^{N-y_N}$

$p(r|y_N) = \frac{p(y_N|r)p(r)}{p(y_N)}$

$p(r|y_N) \propto p(y_N|r)p(r)$

$p(r|y_N) \propto \binom{N}{y_N} r^{y_N}(1-r)^{N-y_N} \cdot 3r^2$

$p(r|y_N) \propto r^{y_N+2}(1-r)^{N-y_N}$

$p(r|y_N) \propto r^{\delta}(1-r)^{\gamma}$ with $\delta = y_N + 2$ and $\gamma = N - y_N$

Exercise 4 (Probabilistic Regression)

a)

b)

c)

d)