Introduction to Quantum Computing

Assignment 1: Entangled states, mixed states and Bernstein-Vazirani algorithm
University of Copenhagen

This assignment is 2 pages long and consists of 3 problems. Students are requested to formulate their solutions individually. Cases of overlaps that indicate copying of other students' work will not be accepted and are handled by the general rules of the faculty. Please hand in your solutions via Absalon - both LaTeX and readable pictures of your hand-written solutions are accepted. Please include a front page with your name and date of birth as well as an indication of the total number of pages.

Exercise 1: (Maximally entangled state (40%))

Step 1 Alice prepares 2-qubit state

$$\left|\Psi^+\right\rangle := \frac{1}{\sqrt{2}}\left|00\right\rangle + \frac{1}{\sqrt{2}}\left|11\right\rangle.$$

and sends the second qubit to Bob.

Step 2 Alice chooses a random bit $b \in \{0,1\}$, applies H^b to her remaining qubit, and measures it in standard basis. She records her measurement outcome $a \in \{0,1\}$.

Step 3 Alice reveals her basis choice, b, from Step 1 to Bob.

Answer the following questions and **remember to show your work** rather than just giving the final answer.

(a) Compute Bob's reduced density matrix $\rho_B := \mathrm{Tr}_A(\left|\Psi^+\right\rangle\left\langle\Psi^+\right|)$ after Step 1.

(b) For each value of the random bits a, b, compute Bob's reduced density matrix $\rho_B^{(a,b)}$ after Step 2.

(c) Let $\rho_B^{(a=0)}$ and $\rho_B^{(a=1)}$ be the density matrices for Bob's reduced state after Step 2, given that Alice's bit is $a = 0$ and $a = 1$, respectively. Compute $\rho_B^{(a=0)}$ and $\rho_B^{(a=1)}$. Describe a strategy allowing Bob to guess, before Step 3, Alice's bit a with probability larger than $1/2$. Compute the success probability achieved by your strategy.

(d) Can Bob guess Alice's bit b before Step 3 with probability better than $1/2$? Justify your answer.

(e) What should Bob do after Step 3 to guess the value of Alice's bit a with certainty? Justify your answer.

Exercise 2: (Mixed states and the Bloch sphere (30%)**)** In the lecture and exercises so far, we saw how to represent a pure qubit state on the Bloch sphere. We are now going to discuss how to extend this representation to mixed states. First, recall the definition of the Pauli matrices:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and for $r = (r_1, r_2, r_3) \in \mathbb{R}^3$ define the matrix $\rho(r) \in \mathcal{M}_2$ as

$$\rho(r) = \frac{I}{2} + \frac{1}{2}(r_1 X + r_2 Y + r_3 Z).$$

(a) Show that $\rho(r)$ is a density matrix if and only if $\|r\| \leq 1$. Conclude that we may identify the set of qubit density matrices with the unit ball of $\mathbb{R}^3$.

(b) For a Pauli matrix $P \in X, Y, Z$, let $\rho(r') = P\rho(r)P$. Express r' in terms of r for each one of the Pauli matrices. Interpret your results geometrically on the Bloch sphere.

(c) For the Hadamard matrix H, let $\rho(r') = H\rho(r)H$. Express r' in terms of r. Interpret your results geometrically on the Bloch sphere.

Exercise 3: (Bernstein-Vazirani algorithm (30%).)

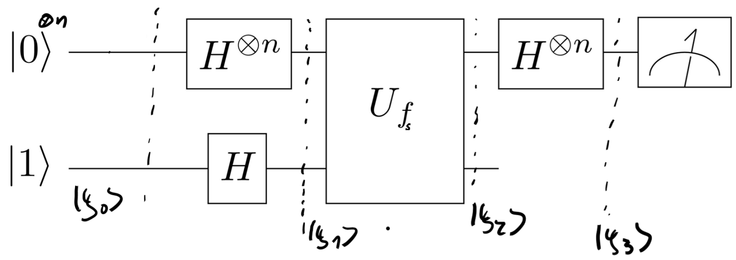

Figure 1: Quantum circuit to solve Deutsch-Jozsa and Bernstein-Vazirani algorithm.

We consider in this exercise boolean functions $f_s : \{0,1\}^n \mapsto \{0,1\}$ defined via a bit-string $s \in \{0,1\}^n$ such that $f_s(x) = x.s = \sum_{i=0}^{n-1} x_i \cdot s_i \mod 2$.

a) Show that for $n = 3$, the function $f_s(x) = x_2 \oplus x_3$, can be realized by $s = 011$. What s' implements $f_{s'}(x) = x_1 \oplus x_2$?

b) Denote by U_{f_s} the quantum oracle for a function f_s as described above. How does U_{f_s} act on $|x\rangle |a\rangle$ for $x \in \{0,1\}^n$ and $a \in \{0,1\}$. What is the inverse of U_{f_s}?

c) Analyze the quantum algorithm shown in Fig. 1. Determine $|\Psi_i\rangle$ for $i = 0, 1, 2, 3$. What are the possible outcomes of a computational basis measurement of $|\Psi_3\rangle$ and how does this help us to determine s?