Introduction to Quantum Computing

Exercise sheet 1: QM: states & tensor product
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Exercise 1: (Bra-Ket-Notation) In the lecture, we introduced the Bra-Ket, or Dirac-notation. To familiarize ourselves a bit more with this notion, let us consider $\mathbb{C}^3$ with the standard ONB $\{|1\rangle, |2\rangle, |3\rangle\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ and let us define the following vectors

$$|\alpha\rangle := 7i\,|1\rangle - |2\rangle + 3\,|3\rangle \qquad |\beta\rangle := 4\,i\,|1\rangle + 2\,|2\rangle$$

(a) Derive the bra-vectors $\langle\alpha|$ und $\langle\beta|$ as well as the vector representations of $|\alpha\rangle$ und $|\beta\rangle$ with respect to the standard basis of $\mathbb{C}^3$.

(b) Compute the scalar products $\langle\alpha|\beta\rangle$ and $\langle\beta|\alpha\rangle$ as well the normalized version of $|\alpha\rangle$.

(c) Compute the matrix representation of the operators $|\alpha\rangle\langle\beta|$ and $|\alpha\rangle\langle\alpha|$. Which one of them is Hermitian?

Exercise 2: (Qubit & Bloch Sphere) In quantum computing, the pure states of a qubit are described by a normalized vector in $\mathbb{C}^2$:

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle \tag{1}$$

with $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$. We will now introduce an alternative representation of these states, namely the Bloch sphere representation.

(a) Show that $|\psi\rangle$ from (1) is indeed normalized.

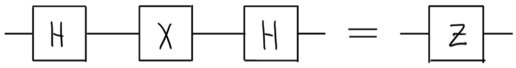
(b) Two vectors that differ only by a global phase factor are considered to describe the same quantum state, i.e. the state $|\phi\rangle$ is identified with $e^{i\gamma}\,|\phi\rangle$ for any $\gamma \in \mathbb{R}$. Use this identification in order to show that any state according to (1) can be re-parametrized according to

$$|\Psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\omega}\sin\left(\frac{\theta}{2}\right)|1\rangle \tag{2}$$

(c) Treating the parameters θ and ω as angles, we see that they parametrize a sphere in $\mathbb{R}^3$, the so called Bloch sphere. Visualise the Bloch sphere representation of the states $|0\rangle$, $|1\rangle$ $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$, $-|0\rangle$ and $e^{-42i}\,|-\rangle$. What is the relative angle of two orthogonal states on the Bloch sphere?

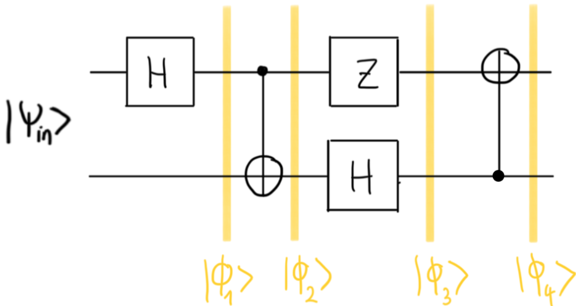
Exercise 3: (Transformations & Circuits) Transformations and Circuits are central concepts in quantum computing. In the following, we are going to work with some examples.

(a) Show that a Pauli X operation, preceded and followed by Hadamard transforms, equals a Pauli Z operation: $HXH = Z$, as illustrated in the circuit diagram below.



(b) Show that conjugating a CNOT gate with Hadamard gates on both qubits switches the role of the control- and the target-bit. In other words, if a CNOT gate where the first qubit controls the state of the second qubit is conjugated with Hadamard gates, it becomes a CNOT gate where the second qubit controls the first qubit. Draw the corresponding circuit diagram.

(c) Consider the following quantum circuit C:



(I) Track the state $|00\rangle$ throughout the circuit, i.e. compute $|\phi_1\rangle$, $|\phi_2\rangle$, $|\phi_3\rangle$ and $|\phi_4\rangle$ if the input state is $|\psi_{in}\rangle = |00\rangle$. What are the outcome probabilities of a computational basis measurement of $|\phi_4\rangle$?

(II) What is the unitary operation U corresponding to the circuit C (with respect to the computational basis)?

(III) Draw the circuit corresponding to the inverse operation U^{-1}.

Exercise 4: (Tensor products) Recall that for A a $d_1 \times d_2$ matrix and B a $d_3 \times d_4$ matrix, their *tensor product* $A \otimes B$ is defined as the $d_1 d_3 \times d_2 d_4$ matrix:

$$A \otimes B = \begin{pmatrix} A_{11}B & \cdots & A_{1d_2}B \\ A_{21}B & \cdots & A_{2d_2}B \\ & \ddots & \\ A_{d_1 1}B & \cdots & A_{d_1 d_2}B \end{pmatrix}.$$

This definition naturally extends to the tensor product $v \otimes w$ of vectors $v \in \mathbb{C}^{d_1}, w \in \mathbb{C}^{d_2}$ by interpreting them as $1 \times d_1$ and $1 \times d_2$ matrices, respectively.

(a) Compute

$$\begin{pmatrix} 1 \\ 1+i \\ 2e^{i\frac{\pi}{4}} \end{pmatrix} \otimes \begin{pmatrix} 2-2i \\ (1-i)^{-1} \\ -5 \end{pmatrix}$$

and

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} i & 1 \\ -2 & 0 \end{pmatrix}.$$

(b) Show that $(A \otimes B)(v \otimes w) = (Av) \otimes (Bw)$ for $A, B \in \mathbb{C}^{d \times d}$ and $v, w \in \mathbb{C}^d$.

(c) Show that $A \otimes (B + C) = A \otimes B + A \otimes C$.

(d) Given $|\phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle)$ and $|\psi\rangle = \frac{1}{\sqrt{2}}(i\,|0\rangle - |1\rangle)$ compute $|\phi\rangle \otimes |\psi\rangle$.