Recaps / Summaries

- ▷ Complex numbers
- ▷ LinAlg
- ▷ Quantum Mechanics Basics

Recap: Complex numbers

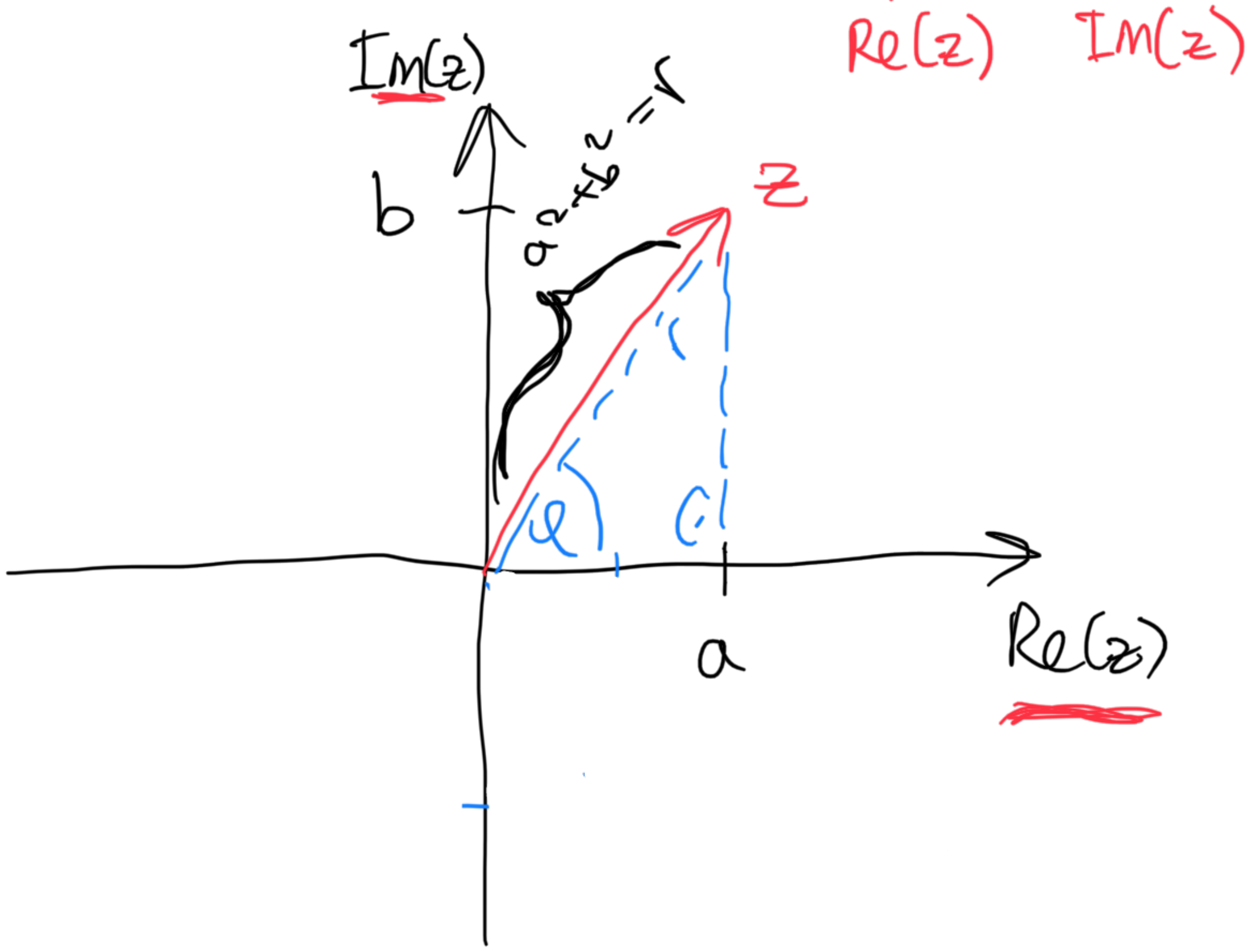
see also videos on Absalon

real numbers: $7, \frac{2}{5}, \sqrt{2}, \ldots$

$x^2 = 2$

$\underline{x^2 = -1}$ $\longrightarrow$ imaginary unit i

complex numbers: $z = \underbrace{a}_{Re(z)} + i\underbrace{b}_{Im(z)}$



Polar representation:

Polar representation:

$$z = r\cos\varphi + ir\sin\varphi = re^{i\varphi}$$

$$z_1 = a_1 + ib_1 \,, \quad z_2 = a_2 + ib_2$$

Addition: $z_1 + z_2 = a_1 + ib_1 + a_2 + ib_2$

$$= (a_1 + a_2) + i(b_1 + b_2)$$

Multiplication: $z_1 \cdot z_2 = (a_1 + ib_1)(a_2 + ib_2)$

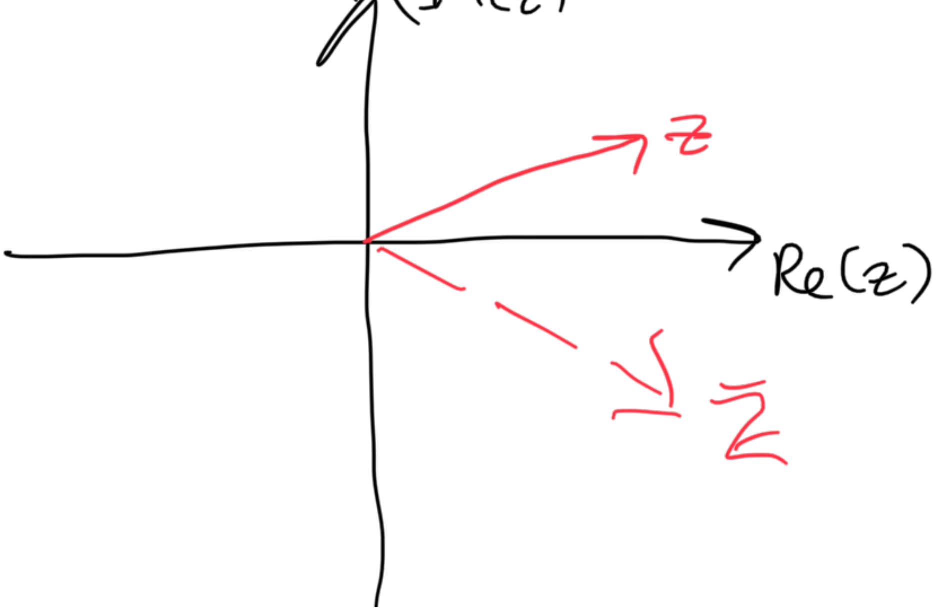
$$= a_1 \cdot a_2 + a_1 \cdot ib_2 + ib_1 \cdot a_2 + ib_1 \cdot ib_2$$

$$= a_1 \cdot a_2 + \underbrace{i^2}_{-1} b_1 b_2 + i(b_1 a_2 + a_1 b_2)$$

$$= a_1 a_2 - b_1 b_2 + i(b_1 a_2 + a_1 b_2)$$

Complex conjugate: $\bar{z} = a - ib$

$$= re^{-i\varphi}$$



$$\mathrm{Re}(z) = \frac{1}{2}(z+\bar{z}) = \frac{1}{2}(a+ib+a-ib)$$
$$= \frac{1}{2}\cdot 2a = a$$

$$\mathrm{Im}(z) = -\frac{i}{2}(z-\bar{z}) = -\frac{i}{2}(a+ib-a+ib)$$
$$= -\frac{i}{2}(2ib) = b$$

Absolute value:

$$|z|^2 = z\cdot\bar{z} = (a+ib)(a-ib)$$
$$= a^2 + iba - iba - i^2b^2$$
$$= a^2 + b^2$$

Division: $\displaystyle \frac{z_1}{z_2} = \frac{z_1}{z_2}\cdot\frac{\bar{z}_2}{\bar{z}_2} = \frac{z_1\cdot\bar{z}_2}{|z_2|}$

Ex. Addition: $z_1 = 5+i$, $z_2 = 2+3i$

$$z_1 + z_2 = 5+i+2+3i = 7+4i$$
$$z_1 - z_2 = 5+i-2-3i = (5-2) + i(1-3)$$
$$= 3-2i$$

What you should know now:

- ways to represent complex numbers
- how to compute with them

Recap: Linear Algebra

see Renes / Renner lecture notes, Mathematical

Homomorphism: $f: \mathcal{H} \to \mathcal{H}'$

Homomorphism: $S: \mathcal{H} \to \mathcal{H}'$ Background

Endomorphism: $S: \mathcal{H} \to \mathcal{H}$

adjoint: $S^*: \mathcal{H}' \to \mathcal{H}$

s.t. $\forall x \in \mathcal{H}, \forall y \in \mathcal{H}'$:

$$\langle y, Sx \rangle_{\mathcal{H}'} = \langle S^* y, x \rangle_{\mathcal{H}}$$

Unitary: $SS^* = S^*S = \mathbb{1}$

Hermitian: $S: \mathcal{H} \to \mathcal{H}$
(self adjoint) $S^* = S$

Positive:
For S Hermitian, we say $S \geq 0$
if $\forall x \in \mathcal{H}$: $\langle x, Sx \rangle \geq 0$.

Projector: $S^2 = S$

Spectral theorem: Watrous lecture notes. 1.1.3

$S: \mathcal{H} \to \mathcal{H}$ Hermitian

Then, you can find a unitary U

and diagonal matrix $D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$

s.t. $\quad S = U D U^*$

How to diagonalize:

1) Find eigenvalues through characteristic polynomial

$$\chi(\lambda) = \det(S - \lambda \cdot \mathbb{1})$$

λ is an eigenvalue when $\chi(\lambda) = 0$

$\lambda_1, \lambda_2, \lambda_3$

2) Find eigenvectors

$$\ker(S - \lambda \mathbb{1}) = \operatorname{span}\begin{pmatrix} \text{eigenvectors} \\ \text{corresponding to } \lambda \end{pmatrix}$$

$\lambda_1 \to u_1$, $\lambda_2 \to u_2$, $\lambda_3 \to u_3, u_4$

3) Find an ONB of $\ker(S - \lambda \mathbb{1})$ for each eigenvalue

Then, $U = \begin{pmatrix} u_1 & u_2 & u_3 & u_4 \end{pmatrix}$

$S = U D U^*$

$$S = U D U^*$$

$$D = U^* S U$$

$$D = \begin{pmatrix} \lambda_1 & \ddots & & & & 0 \\ & & \lambda_1 & \lambda_2 & & \\ & & & & \ddots \lambda_2 & \\ 0 & & & & & \ddots \end{pmatrix}$$

Recap: Quantum Mechanics Basics

* Braket / Dirac notation

ket: basis vector: $|e_i\rangle = \begin{pmatrix}0\\0\\\vdots\\1\\\vdots\\0\end{pmatrix} \leftarrow$ ith place

basis of space: $\{|e_i\rangle\}$

computational basis: $\{|i\rangle\} = \{|0\rangle, |1\rangle, \dots\}$

Bra: $\langle i| = (0\ 0\ 0 \cdots 1 \cdots 0) = (|i\rangle)^\dagger$

Braket: $\langle i|j\rangle = (0 \cdots \underset{\substack{\uparrow \\ \text{ith place}}}{1} \cdots 0)\begin{pmatrix}0\\\vdots\\1\\0\end{pmatrix} \leftarrow \text{jth place} = \begin{cases}1 & i=j\\0 & i\neq j\end{cases}$

ketbra: $|i\rangle\langle j| = \begin{pmatrix}0\\\vdots\\1\\0\end{pmatrix}(0\cdots1\cdots0) = \begin{array}{cl} \begin{pmatrix}0&0&0&0&0\\ \cdots&\boxed{1}&0&\cdots&\\0&\cdots&&0&\end{pmatrix} & \text{ith row} \\ \text{jth column} & \end{array}$

Example: identity matrix: $\mathbb{1} = \sum_i |i\rangle\langle i|$

vectors in a Hilbert space: pure states

→ Write any pure state as $|\psi\rangle = \sum_i \alpha_i |i\rangle$

$|\psi\rangle = \sum_i |i\rangle \underbrace{\langle i|\psi\rangle}_{\alpha_i}$

Task: Write this vector $|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ in computational basis

$|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\begin{matrix}\leftarrow |0\rangle\\ \leftarrow |1\rangle\end{matrix}$ $\quad |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

What is $\langle +|$ in computational basis? $\dagger$ $\dagger$

$\langle +| = (|+\rangle)^\dagger = \frac{1}{\sqrt{2}}(1, 1) = \frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)$

matrices in the Hilbert space:

$M = \sum_{ij} M_{ij}|i\rangle\langle j| = \begin{array}{cl}\begin{pmatrix} M_{11} \cdots M_{12} & & \\ \vdots & & \\ M_{21} \cdots M_{ij} & & \\ & & \end{pmatrix} & \leftarrow \text{ith row} \\ \uparrow \text{jth column} & \end{array}$

$M = \sum |i\rangle\underline{\langle i|M|j\rangle}\langle j|$

$ij \quad M_{ij}$

Task: Write down the matrix for $M = |+\rangle\langle+|$

$$M = |+\rangle\langle+| = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)$$

$$= \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 0| + |1\rangle\langle 0| + |1\rangle\langle 1|)$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

<u>Density operators</u>: more general quantum states

mixed states: $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| = \sum \rho_{ij} |i\rangle\langle j|$

↳ mix of pure states

$\mathrm{tr}\,\rho = 1$

$\rho \geq 0$

$\rho^\dagger = \rho$

pure states: $|\psi\rangle \longleftrightarrow |\psi\rangle\langle\psi|$

$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ with $p_i = 1$ for some i, $p_j = 0 \ \forall j \neq i$.

<u>Measurement/ Observables</u>: set of operators $\{A_i\}$,

which are positive: $A_i \geq 0$

and add to identity: $\sum_i A_i = \mathbb{1}$

Task: Is $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ a measurement?

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = |0\rangle\langle 0| \geq 0$ $\quad |0\rangle\langle 0| + |1\rangle\langle 1| = \sum_i |i\rangle\langle i| = \mathbb{1}$

$|1\rangle\langle 1| \geq 0$

<u>Trace</u>: $\mathrm{tr}(M) = \sum_i \langle i | M | i \rangle$

Show: This is equivalent to $\mathrm{tr}(M) = \sum_i M_{ii}$.

$$\mathrm{tr}(M) = \sum_{ikl} \langle i| \; M_{kl} |k\rangle\langle l | i\rangle$$

$\hookrightarrow 1 \quad l = i$

$= \sum M_{ii}$ $\quad \{ 0 \quad k \neq i$

Expectation values of measurements:

probability to measure j for a system in state ρ:

$$p(j) = \operatorname{tr}(A_j \rho)$$

$$p(j) \geq 0 \quad (\text{bc } A_j \geq 0,\ \rho \geq 0)$$

$$\sum_j p(j) = \sum_j \operatorname{tr} A_j \rho = \operatorname{tr}\Big(\underbrace{\big(\sum_j A_j\big)}_{\mathbb{1}}\rho\Big) = \operatorname{tr}\rho = 1.$$

Task: What if ρ is pure?

$$\rho = |\psi\rangle\langle\psi|,\ |\psi\rangle = \sum \alpha_i |i\rangle$$

$$p(j) = \operatorname{tr} A_j |\psi\rangle\langle\psi|$$

$$= \sum_i \langle i| A_j |\psi\rangle\langle\psi| i\rangle$$

$$= \sum_{ik} \langle i| A_j |\psi\rangle \bar{\alpha}_k \langle k| i\rangle \qquad 1 \Leftrightarrow k = i \quad \langle i|i\rangle = 1$$

$$= \sum_i \underbrace{\bar{\alpha}_i \langle i|}_{\langle\psi|} A_j |\psi\rangle = \langle\psi|A_j|\psi\rangle$$

Task: A system is in state $|+\rangle$.
What is the probability to measure
0 and 1 in the computational basis?

$$|+\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

meas. 0: $A_0 = |0\rangle\langle 0|$

meas. 1: $A_1 = |1\rangle\langle 1|$

$$p(0) = \langle + | 0\rangle\langle 0| +\rangle$$

$$= \langle + |0\rangle \cdot \langle 0| \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$= \langle + |0\rangle \cdot \tfrac{1}{\sqrt{2}}\big(\langle 0|0\rangle + \cancel{\langle 0|1\rangle}\big)$$

$$= \frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)|0\rangle \cdot \frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}(\underbrace{\langle 0|0\rangle}_{1} + \underbrace{\langle 1|0\rangle}_{0}) \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}$$

$$p(1) = \langle +| A_1 |+\rangle = \langle +|1\rangle\langle 1|+\rangle$$

$$= \langle +|1\rangle \cdot \overline{\langle +|1\rangle} = |\langle +|1\rangle|^2$$

$$= |\tfrac{1}{\sqrt{2}}(\langle 0| - \langle 1|)|1\rangle|^2$$

$$= \frac{1}{2}(|\underbrace{\langle 0|1\rangle}_{0} + \underbrace{\langle 1|1\rangle}_{1}|^2 = \frac{1}{2}$$

<u>Tensor</u> <u>product</u>

$$|ij\rangle = |i\rangle \otimes |j\rangle = \begin{array}{r} \\ \\ i\text{th place} \\ \\ \\ \end{array}\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \otimes \begin{array}{r} \\ \\ j\text{'th place} \\ \\ \\ \end{array}\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \\ \\ \vdots \\ \\ \\ 1 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}\begin{array}{l} 00 \\ 01 \\ 02 \\ \vdots \\ 0d \\ 10 \\ 11 \\ \vdots \\ ij \\ ij{+}1 \\ \vdots \\ dd \end{array}$$

$\rightarrow$ new basis element $|ij\rangle$

Task: $|v_1\rangle = |0\rangle$, $|v_2\rangle = |+\rangle$

What is $|v_1\rangle \otimes |v_2\rangle$?
What is $|v_2\rangle \otimes |v_1\rangle$?

$$|v_1\rangle \otimes |v_2\rangle = |0\rangle \otimes |+\rangle = |0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$= \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle) = \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)$$

$$|v_2\rangle \otimes |v_1\rangle = |+\rangle \otimes |0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle$$

$$= \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |0\rangle)$$

$$= \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$

For matrices: $M_1 \otimes M_2 = \begin{pmatrix} M_1^{11} \cdot M_2 & M^{12} \cdot M_2 & \\ & M_1^{22} \cdot M_2 & \\ & & \ddots \end{pmatrix}$

$$|i\rangle\langle j| \otimes |k\rangle\langle l| = \begin{pmatrix} 0 & \cdots & 0 \\ & 1 & \\ 0 & & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & \cdots & 0 \\ & 1 & \\ 0 & & 0 \end{pmatrix}$$

$$= \begin{array}{c} \\ \begin{pmatrix} 0 & 0 & 0 & \\ & & \underbrace{\boxed{1}}_{l} \} k & \end{pmatrix} \begin{array}{l} \} \text{ith block} \\ \\ \end{array} \\ \underbrace{\qquad\qquad\qquad}_{\text{jth block}} \end{array}$$

$$M_1 \otimes M_2 = \sum_{ijkl} |i\rangle\langle i| M_1 |j\rangle\langle j| \otimes |k\rangle\langle k| M_2 |l\rangle\langle l|$$
$$= \sum_{ijkl} M^1_{ij} \, M^2_{kl} \; |i\rangle\langle j| \otimes |k\rangle\langle l|$$

Task: What is $\mathbb{1} \otimes \sigma_z$?

$$\mathbb{1} \otimes \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 \cdot \sigma_z & 0 \\ 0 & 1 \sigma_z \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & 1 & \\ 0 & & & -1 \end{pmatrix}$$

<u>Partial trace</u> $\quad \mathrm{tr}_A = \underbrace{\mathrm{tr}}_{\text{on system A}} \otimes \overbrace{\mathbb{1}_B}^{\text{on system B}}$

reduced density matrix

$$\rho_B = \mathrm{tr}_A \, \rho_{AB} = \sum_i (\langle i| \otimes \mathbb{1}) \, \rho_{AB} \, (|i\rangle \otimes \mathbb{1})$$

Task: $\rho_{AB} = \frac{1}{2}(|00\rangle\langle 00| + |01\rangle\langle 00| + |00\rangle\langle 01| + |01\rangle\langle 01|)$

What is $\rho_B = \mathrm{tr}_A \, \rho_{AB}$?

$$\sum_i \Big((\langle i| \otimes \mathbb{1}) \big(|00\rangle\langle 00| + |01\rangle\langle 00| + |00\rangle\langle 10| + |01\rangle\langle 01| \big) (|i\rangle \otimes \mathbb{1}) \Big)$$

$i=0$

$$= \sum_i (\langle i| \otimes \mathbb{1}) |00\rangle\langle 00| (|i\rangle \otimes \mathbb{1})$$

$|00\rangle\langle 00| = |0\rangle \otimes |0\rangle \cdot \langle 0| \otimes \langle 0|$

$$+ \sum_i (\langle i| \otimes \mathbb{1}) |01\rangle\langle 00| (|i\rangle \otimes \mathbb{1})$$

$= |0\rangle\langle 0| \otimes |0\rangle\langle 0|$

$$+ \sum_i (\langle i| \otimes \mathbb{1}) |00\rangle\langle 10| (|i\rangle \otimes \mathbb{1})$$

$$+ \sum_i (\langle i| \otimes \mathbb{1}) |01\rangle\langle 01| (|i\rangle \otimes \mathbb{1})$$

$$= \sum_i \langle i|0\rangle\langle 0|i\rangle \; \boxed{|0\rangle\langle 0|}$$

$$+ \sum_i \langle i|0\rangle\langle 0|i\rangle \; \boxed{|1\rangle\langle 0|}$$

$i=0$ $\quad$ $i=1$

$$+ \sum_i \langle i|0\rangle\langle 1|i\rangle \; \boxed{|0\rangle\langle 0|} \to 0$$

$$+ \sum_i \langle i|0\rangle\langle 0|i\rangle \; \boxed{|1\rangle\langle 1|}$$

$1 \quad i=0$

$$= 1 \cdot |0\rangle\langle 0| + 1 \cdot |1\rangle\langle 0| + 0 \cdot \cancel{|0\rangle\langle 0|} + 1 \cdot |1\rangle\langle 1|$$

What is $\rho_A = tr_B \, \rho_{AB}$?

$$tr_B(\rho) = \sum_i (\mathbb{1} \otimes \langle i|) \, \rho \, (\mathbb{1} \otimes |i\rangle)$$

$$= \sum_i (\mathbb{1} \otimes \langle i|) \big(|00\rangle\langle 00| + |01\rangle\langle 00| + |00\rangle\langle 10| + |01\rangle\langle 01| \big) (\mathbb{1} \otimes |i\rangle)$$

$|0\rangle\langle 0| \otimes |0\rangle\langle 0|$

$$= \sum_i (\mathbb{1} \otimes \langle i|) |00\rangle\langle 00| (\mathbb{1} \otimes |i\rangle)$$

$\mathbb{1} |0\rangle\langle 0| \mathbb{1} \otimes \langle i|0\rangle\langle 0|i\rangle$

$$+ \sum_i (\mathbb{1} \otimes \langle i|) |01\rangle\langle 00| (\mathbb{1} \otimes |i\rangle)$$

$|0\rangle\langle 0| \otimes |1\rangle\langle 0|$

$$+ \sum_i (\mathbb{1} \otimes \langle i|) |00\rangle\langle 10| (\mathbb{1} \otimes |i\rangle)$$

$|0\rangle\langle 1| \otimes |0\rangle\langle 0|$

$$+ \sum_i (\mathbb{1} \otimes \langle i|) |01\rangle\langle 01| (\mathbb{1} \otimes |i\rangle)$$

$|0\rangle\langle 0| \otimes |1\rangle\langle 1|$

$$= |0\rangle\langle 0| \cdot \sum_i \langle i|0\rangle\langle 0|i\rangle \qquad i=0 \to 1,\quad i=1 \to 0$$

$$+ |0\rangle\langle 0| \cdot \sum_i \langle i|1\rangle\langle 0|i\rangle \qquad \left.\begin{matrix} i=0 \to 0 \\ i=1 \to 0 \end{matrix}\right\} \text{always } 0$$

$$+ |0\rangle\langle 1| \cdot \sum_i \langle i|0\rangle\langle 0|i\rangle$$

$$+ |0\rangle\langle 0| \cdot \sum_i \langle i|1\rangle\langle 1|i\rangle$$

$$= |0\rangle\langle 0| + |0\rangle\langle 1| + |0\rangle\langle 0|$$