Introduction to Quantum Computing

Problem Set 2: Quantum sates and measurements. Basic quantum protocols.
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Exercise 1: (Density Matrices)

(a) Let ρ be a $d \times d$ matrix. Show that the following two statements are equivalent

(i) We can write ρ as

$$\rho = \sum_{i=1}^{n} p_i \left|\psi_i\right\rangle \left\langle\psi_i\right|$$

for some ensemble of pure states $\{(p_1, |\psi_1\rangle), \ldots (p_n, |\psi_n\rangle)\}$, where $p_1 + \ldots + p_n = 1$ and $p_i \geq 0\ \forall i = 1, ..., n$.

(ii) $\mathrm{Tr}(\rho) = 1$ and $\rho \succeq 0$.

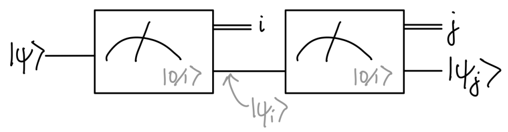
Hint: You can use the fact that $\rho \succeq 0$ *if and only if* $\langle\psi|\,\rho\,|\psi\rangle \geq 0$ *for any* $|\psi\rangle \in \mathbb{C}^d$.

(b) Show that a density matrix ρ describes a pure quantum state if and only if $\mathrm{Tr}\big(\rho^2\big) = 1$.
Recall that, by definition, a density matrix ρ *describes a pure state if* $\rho = |\psi\rangle\langle\psi|$ *for some pure quantum state* $|\psi\rangle$.

Exercise 2: (Consecutive measurements)

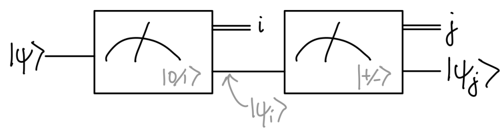
(a) Suppose we measure a pure qubit state $|\psi\rangle \in \mathbb{C}^2$ in $|0/1\rangle$-basis and obtain outcome $i \in \{0, 1\}$. Denote the corresponding post-measurement state by $|\psi_i\rangle$.

(i) Suppose we measure the post-measurement state $|\psi_i\rangle$ in $|0/1\rangle$-basis again:



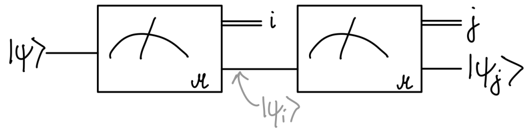
Compute the outcome probabilities and the associated post-measurement states.

(ii) Suppose we measure the post-measurement state $|\psi_i\rangle$ in $|+/-\rangle$-basis:



Compute the outcome probabilities and the associated post-measurement states.

(b) More generally, suppose we measure quantum state $|\psi\rangle \in \mathbb{C}^d$ with a *projective* measurement, $\mathcal{M}$, given by $M_0, \ldots, M_{d-1}$ twice in a row:



Show that the two measurement outcomes and post-measurement states coincide, *i.e.* $i = j$ and $|\psi_i\rangle = |\psi_j\rangle$.

Exercise 4: (Classical & quantum strategies) Q is visiting the Enterprise and proposes a simple game to Captain Picard in order to show his superiority.

1. A coin is put on the table with heads-up and covered so neither of them can see it during the game.
2. Without Picard looking, Q can now decide to leave the coin heads up or to change it to tails.
3. Without looking at the coin first, Picard can make the same move (turning over the coin or not) without revealing his choice to Q.
4. Q is allowed a final move - again without looking at the coin beforehand.
5. The coin is uncovered. If it shows heads, Q wins a bar of gold-pressed latinum.

(a) Picard and Q play many rounds of the game. What is Q's winning probability if Picard chooses to turn the coin with probability p for each of Q's four possible moves? What p should Picard choose in order to make the game fair (both have winning probability 1/2)?

(b) Identifying the states head and tail with the computational basis states $\{|0\rangle, |1\rangle\}$ of $\mathbb{C}^2$, show that we can describe the game by modeling the coin flip as a Pauli X matrix, not-flipping as the identity matrix and the final check as a measurement in the computational basis.

(c) Unbeknownst to Picard, Q exercises his superior powers by choosing a quantum strategy: Instead of applying X or $\mathbb{1}$, he can apply an arbitrary pair U, V of unitary 2×2-matrices. Find U, V such that Q wins the game with certainty.

(d) Finally Picard becomes tired of Q's cheating and demands to change their roles in the game: Show that if Picard makes the first and last move Q cannot win with certainty - even if Picard still can only play classically.

Exercise (Bonus): (Impossibility of deletion) Show that unitaries cannot "delete" information: there is no 1-qubit unitary U that maps $|\psi\rangle \mapsto |0\rangle$ for every 1-qubit state $|\psi\rangle$.