Exercise Sheet 1 — solutions

1. $|\alpha\rangle = 7i|1\rangle - |2\rangle + 3|3\rangle$

   $|\beta\rangle = 4i|1\rangle + 2|2\rangle$

a) $\langle\alpha| = (|\alpha\rangle)^\dagger = -7i\langle 1| - \langle 2| + 3\langle 3|$

$\langle\beta| = -4i\langle 1| + 2\langle 2|$

b) $\langle\alpha|\beta\rangle = (-7i\langle 1| - \langle 2| + 3\langle 3|)(4i|1\rangle + 2|2\rangle)$

$= -7i\cdot 4i\langle 1|1\rangle - 2\cdot 7i\langle 1|2\rangle - 2\langle 2|2\rangle$

$- 4i\langle 1|2\rangle + 3\cdot 4i\langle 3|1\rangle + 3\cdot 2\langle 3|2\rangle$

$= -28i^2 - 2 = 28 - 2 = 26$

$\langle\beta|\alpha\rangle = (-4i\langle 1| + 2\langle 2|)(7i|1\rangle - |2\rangle + 3|3\rangle)$

$$= -28i^2 - 2 = 26$$

$$\|\alpha\|^2 = \langle\alpha|\alpha\rangle = \left(-7i\langle 1| - \langle 2| + 3\langle 3|\right)\left(7i|1\rangle - |2\rangle + 3|3\rangle\right)$$

$$= -49i^2\langle 1|1\rangle + \langle 2|2\rangle + 3^2\langle 3|3\rangle$$

$$= 49 + 1 + 9 = 59$$

normalized version of $|\alpha\rangle$: $|\tilde{\alpha}\rangle = \frac{1}{\sqrt{59}}|\alpha\rangle$

c) $|\alpha\rangle\langle\beta| = \left(7i|1\rangle - |2\rangle + 3|3\rangle\right)\left(-4i\langle 1| + 2\langle 2|\right)$

$$= -28i^2|1\rangle\langle 1| + 2\cdot 7i\,|1\rangle\langle 2| - 4i\,|2\rangle\langle 1|$$

$$-2|2\rangle\langle 2| - 4\cdot 3i\,|3\rangle\langle 1| + 3\cdot 2|3\rangle\langle 2|$$

$$= \begin{pmatrix} 28 & 14i & 0 \\ -4i & -2 & 0 \\ -12i & 6 & 0 \end{pmatrix}$$

$$(|\alpha\rangle\langle\beta|)^\dagger = \begin{pmatrix} 28 & -4i & -12i \\ 14i & -2 & 6 \\ 0 & 0 & 0 \end{pmatrix} \neq |\alpha\rangle\langle\beta|$$

$\rightsquigarrow$ not Hermitian

$$|\alpha\rangle\langle\alpha| = \begin{pmatrix} 49 & -7i & 21i \\ 7i & 1 & -3 \\ -21i & -3 & 9 \end{pmatrix}$$

$(|\alpha\rangle\langle\alpha|)^\dagger = |\alpha\rangle\langle\alpha|$ is Hermitian.

② $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

a) $\||\psi\rangle\|^2 = \langle\psi|\psi\rangle = (\bar{\alpha}\langle 0| + \bar{\beta}\langle 1|)(\alpha|0\rangle + \beta|1\rangle)$

$$= \bar{\alpha}\alpha\langle 0|0\rangle + \bar{\beta}\beta\langle 1|1\rangle$$

$$= |\alpha|^2 + |\beta|^2 = 1$$

b) $|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

$\alpha \in \mathbb{C} \rightsquigarrow$ can be written as $\alpha = \underbrace{\phi_0}_{\in \mathbb{R}} e^{i \underbrace{\theta_0}_{\in \mathbb{R}}}$

in terms of parameters $\phi_0, \theta_0 \in \mathbb{R}$

$\beta \in \mathbb{C} \rightsquigarrow \beta = \phi_1 e^{i\theta_1}$

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \phi_0 e^{i\theta_0} \\ \phi_1 e^{i\theta_1} \end{pmatrix} = \begin{pmatrix} \phi_0 e^{i\theta_0} \\ \sqrt{1-\phi_0^2}\, e^{i\theta_1} \end{pmatrix} = e^{i\theta_0} \begin{pmatrix} \phi_0 \\ \sqrt{1-\phi_0^2}\, e^{i(\theta_1-\theta_0)} \end{pmatrix}$$

$$|\alpha|^2 + |\beta|^2 = \phi_0^2 + \phi_1^2$$

$$\overset{\text{global phase}}{=} \begin{pmatrix} \phi_0 \\ \sqrt{1-\phi_0^2}\; e^{i(\theta_1-\theta_2)} \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \cdot e^{i\omega} \end{pmatrix}$$

can reparametrize: $\phi_0 = \cos\frac{\theta}{2}$

$\theta_1 - \theta_2 = \omega$

c) spherical coordinates: $\vec{r}_{\theta,\omega} = \begin{pmatrix} \sin\theta\cos\omega \\ \sin\theta\sin\omega \\ \cos\theta \end{pmatrix}$

$$|0\rangle \stackrel{!}{=} \cos\frac{\theta}{2}|0\rangle + e^{i\omega}\sin\frac{\theta}{2}|1\rangle$$

$\rightsquigarrow \cos\frac{\theta}{2} = 1 \rightsquigarrow \theta = 0$

$\sin\frac{\theta}{2} = 0$ $\quad$ ω can be anything

$\omega = 0$

$$\vec{r}_{|0\rangle} = \vec{r}_{0,0} = \begin{pmatrix} 0 \\ 0 \\ \cos(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$|1\rangle \stackrel{!}{=} \cos\frac{\theta}{2}|0\rangle + e^{i\omega}\sin\frac{\theta}{2}|1\rangle$$

$\rightsquigarrow e^{i\omega}\sin\frac{\theta}{2} = 1 \Big\} \; \theta = \pi$

$$\cos\frac{\theta}{2} = 1 \quad \Big\} \quad \omega = 0$$

$$\vec{r}_{|1\rangle} = \vec{r}_{\pi,0} = \begin{pmatrix} \sin(\pi)\cos(0) \\ \sin(\pi)\sin(0) \\ \cos(\pi) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$\rightsquigarrow \left.\begin{aligned} \cos\frac{\theta}{2} &\overset{!}{=} \frac{1}{\sqrt{2}} \\ \sin\frac{\theta}{2}e^{i\omega} &\overset{!}{=} \frac{1}{\sqrt{2}} \end{aligned}\right\} \begin{aligned} \theta &= \pi/2 \\ \omega &= 0 \end{aligned}$$

$$\vec{r}_{|+\rangle} = \vec{r}_{\pi/2,0} = \begin{pmatrix} \sin\frac{\pi}{2}\cos 0 \\ \sin\pi/2\,\sin 0 \\ \cos\frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

$$\left.\begin{aligned} \cos\frac{\theta}{2} &= \frac{1}{\sqrt{2}} \end{aligned}\right\} \begin{aligned} \theta &= 3\pi/2 \\ \omega &= 0 \end{aligned}$$

$\sin\frac{\theta}{2} e^{i\omega} = -\frac{1}{\sqrt{2}} \Big\}$

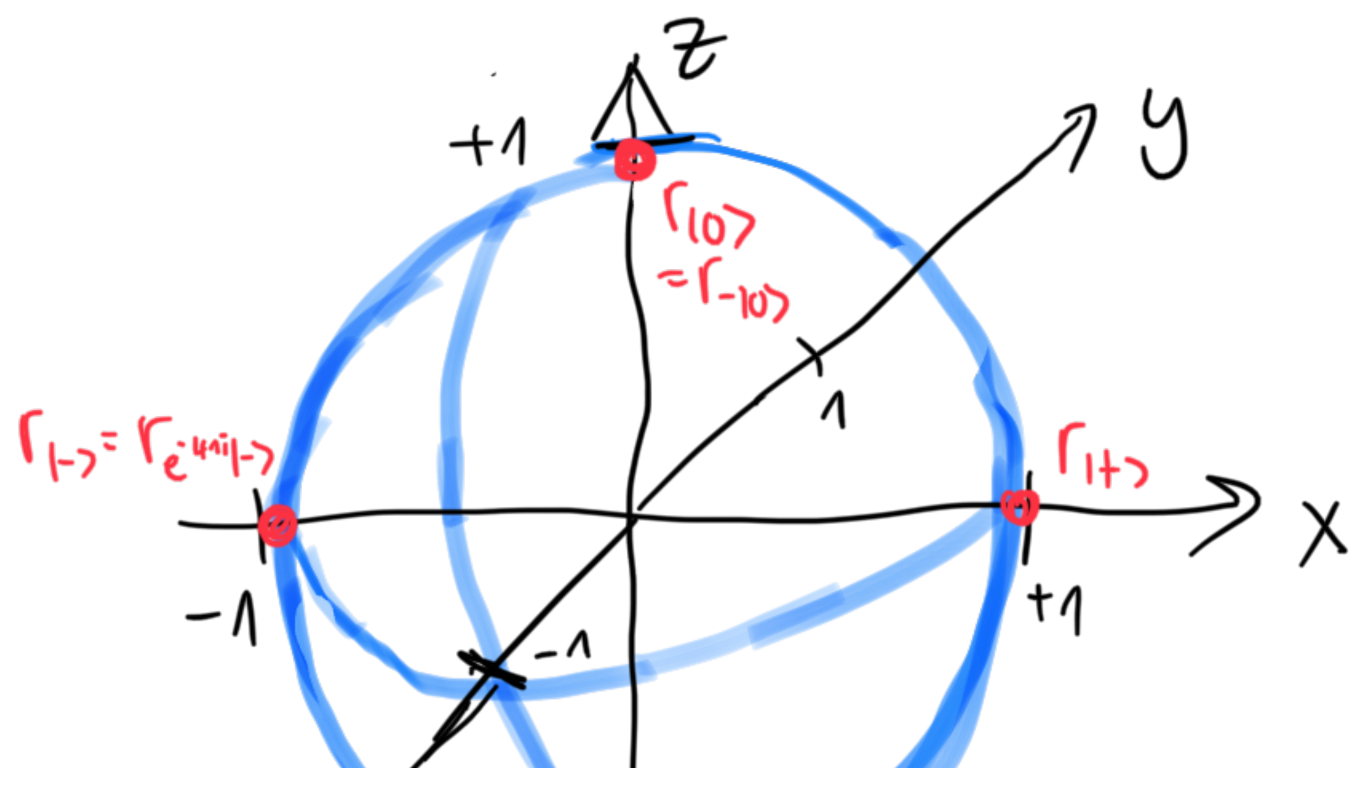
$$r_{|-\rangle} = \vec{r}_{3\pi/2,\,0} = \begin{pmatrix}-1\\0\\0\end{pmatrix}$$

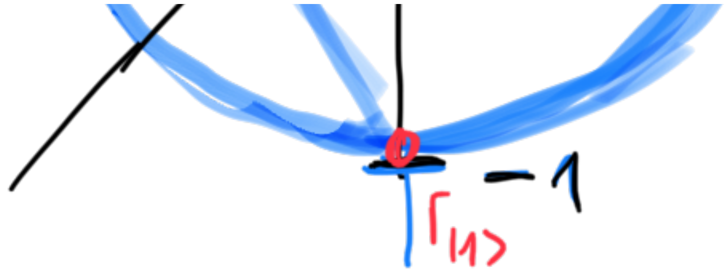
$-|0\rangle$: $\cos\frac{\theta}{2} \overset{!}{=} -1 \rightarrow \theta = 2\pi$

$\omega = 0$

$$r_{-|0\rangle} = \vec{r}_{2\pi,0} = r_{0,0}$$

$e^{-41i}|-\rangle = |-\rangle \rightsquigarrow$ same vector $\vec{r}_{e^{-41i}|-\rangle} = \vec{r}_{|-\rangle}$





relative angle:

$$|\psi_i\rangle = \cos\frac{\theta_i}{2}\,|0\rangle + e^{i\omega_i}\sin\frac{\theta_i}{2}\,|1\rangle \qquad i=1,2$$

$$0 = \langle\psi_1|\psi_2\rangle = \left(\cos\frac{\theta_1}{2}\,\langle 0| + e^{-i\omega_1}\sin\frac{\theta_1}{2}\,\langle 1|\right)$$

$$\cdot\left(\cos\frac{\theta_2}{2}\,|0\rangle + e^{+i\omega_2}\sin\frac{\theta_2}{2}\,|1\rangle\right)$$

$$= \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + e^{i(\omega_2-\omega_1)}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}$$

$\cos(x+y) = \cos x\cos y + \sin x \sin y$

$\cos(x-y) = \cos x\cos y - \sin x \sin y$

$\Rightarrow 2\cos x\cos y = \cos(x+y) + \cos(x-y)$

$$= \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + e^{i(\omega_2-\omega_1)}\left(\cos\frac{\theta_1+\theta_2}{2} - \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\right)$$

$$= e^{i(\omega_2-\omega_1)}\cos\frac{\theta_1+\theta_2}{2} + \left(1-e^{i(\omega_2-\omega_1)}\right)\underbrace{\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}}$$

$$\frac{1}{2}\left(\cos\frac{\theta_1+\theta_2}{2}+\cos\frac{\theta_1-\theta_2}{2}\right)$$

$$=\left(\frac{1-e^{i(\omega_2-\omega_1)}}{2}+e^{i(\omega_2-\omega_1)}\right)\cos\frac{\theta_1+\theta_2}{2}+\left(1-e^{i(\omega_2-\omega_1)}\right)\cos\frac{\theta_1-\theta_2}{2}$$

$$=\frac{1}{2}\left(\left(1+e^{i(\omega_2-\omega_1)}\right)\cos\frac{\theta_1+\theta_2}{2}+\left(1-e^{i(\omega_2-\omega_1)}\right)\cos\frac{\theta_1-\theta_2}{2}\right)$$

$$\left(1+e^{i(\omega_2-\omega_1)}\right)\cos\frac{\theta_1-\theta_2}{2}=0=\left(1-e^{i(\omega_2-\omega_1)}\right)\cos\frac{\theta_1-\theta_2}{2}$$

a) $\omega_2-\omega_1=\pi \to e^{i\pi}=-1 \longrightarrow \quad \omega_2=\pi+\omega_1$

$$\frac{\theta_1+\theta_2}{2}=\frac{\pi}{2} \quad\longrightarrow\quad \theta_2=\pi-\theta_1$$

$$\begin{pmatrix}\sin\theta_2\cos\omega_2\\ \sin\theta_2\sin\omega_2\\ \cos\theta_2\end{pmatrix}=\begin{pmatrix}\sin(\pi-\theta_1)\cos(\omega_1+\pi)\\ \sin(\pi-\theta_1)\sin(\omega_1+\pi)\\ \cos(\pi-\theta_1)\end{pmatrix}$$

$$=\begin{pmatrix}\sin(\theta_1)(-\cos\omega_1)\\ \sin(\theta_1)(-\sin\omega_1)\\ -\cos(\theta_1)\end{pmatrix}=-\begin{pmatrix}\sin\theta_1\cos\omega_1\\ \sin\theta_1\sin\omega_1\\ \cos\theta_1\end{pmatrix}$$

b) $\omega_2 - \omega_1 = 0 \rightarrow e^{i0} = 1 \rightarrow \omega_1 = \omega_2$

$$\frac{\theta_1 - \theta_2}{2} = \frac{\pi}{2} \rightsquigarrow \theta_1 = \pi + \theta_1$$

$$\begin{pmatrix} \sin\theta_2 \cos\omega_2 \\ \sin\theta_2 \sin\omega_2 \\ \cos\theta_2 \end{pmatrix} = \begin{pmatrix} \sin(\pi + \theta_1)\cos(\omega_1) \\ \sin(\pi + \theta_1)\sin(\omega_1) \\ \cos(\pi + \theta_1) \end{pmatrix} = - \begin{pmatrix} \sin\theta_1 \cos\omega_1 \\ \sin\theta_1 \sin\omega_1 \\ \cos\theta_1 \end{pmatrix}$$

$\Rightarrow$ Orthogonal states are at antipodes on the sphere.

3 a) how Pauli Z acts:

$$Z|0\rangle = |0\rangle$$

$$Z|1\rangle = -|1\rangle$$

Action of HXH:

$$HXH|0\rangle = HX|+\rangle = HX \cdot \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$= \frac{1}{\sqrt{2}} H(X|0\rangle + X|1\rangle)$$

$$= \frac{1}{\sqrt{2}} H(|1\rangle + |0\rangle)$$

$$= \frac{1}{\sqrt{2}}(|-\rangle + |+\rangle)$$

$$= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle + |0\rangle + |1\rangle)$$

$$= \frac{1}{2} \cdot 2|0\rangle = |0\rangle = Z|0\rangle$$

$$HXH|1\rangle = HX|-\rangle$$

$$= \frac{1}{\sqrt{2}} H\left(X|0\rangle - X|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}} H\left(|1\rangle - |0\rangle\right)$$

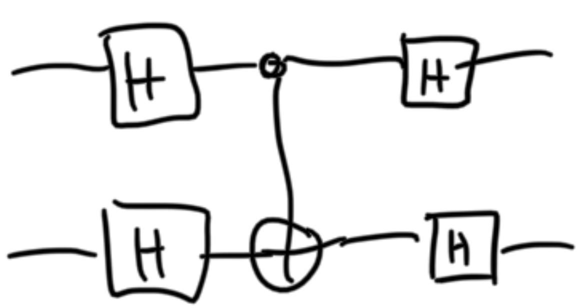
$$= \frac{1}{\sqrt{2}} \left(|-\rangle - |+\rangle\right)$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \left(|0\rangle - |1\rangle - |0\rangle - |1\rangle\right)$$

$$= \frac{1}{2} \cdot \left(-2|1\rangle\right) = -|1\rangle = Z|1\rangle$$

$$\rightarrow HXH = Z$$

b)



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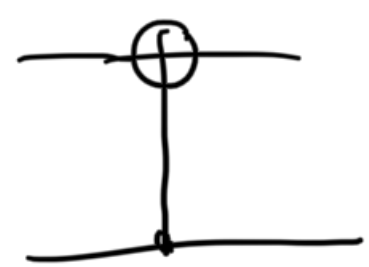


$$(H\otimes H)\,CNOT_1\,(H\otimes H) = CNOT_2$$

Note:

$$(H\otimes H)|00\rangle = |{+}{+}\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

$$(H\otimes H)|01\rangle = |{+}{-}\rangle = \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle)$$

$$(H\otimes H)|10\rangle = |{-}{+}\rangle = \frac{1}{2}(|00\rangle + |01\rangle - |10\rangle - |11\rangle)$$

$$(H\otimes H)|11\rangle = |{-}{-}\rangle = \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle)$$

& $H\otimes H|{+}{+}\rangle = |00\rangle$ etc

$|00\rangle$:

$$H\otimes H\,|00\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

$$CNOT_1(H\otimes H)\,|00\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |11\rangle + |01\rangle)$$

$$= |{+}{+}\rangle$$

$$(H\otimes H)\,CNOT_1\,(H\otimes H)|00\rangle = (H\otimes H)|{+}{+}\rangle$$

$$= |00\rangle = CNOT_2|00\rangle$$

$|01\rangle$:

$$CNOT_1(H\otimes H)|01\rangle = \frac{1}{2}CNOT\,(|00\rangle - |01\rangle + |10\rangle - |11\rangle)$$

$$= \frac{1}{2}(|00\rangle - |01\rangle + |11\rangle - |10\rangle)$$

$$(H\otimes H)\,CNOT_1\,(H\otimes H)|01\rangle = |11\rangle = CNOT_2|01\rangle$$

$|10\rangle$:

$$CNOT_1\,(H\otimes H)|10\rangle = \frac{1}{2}CNOT\,(|00\rangle + |01\rangle - |10\rangle - |11\rangle)$$

$$= \frac{1}{2}(|00\rangle + |01\rangle - |10\rangle - |11\rangle)$$

$$(H\otimes H)(CNOT_1)(H\otimes H)|10\rangle = |10\rangle = CNOT_2|10\rangle$$

$|11\rangle$:

$$CNOT_1(H\otimes H)|11\rangle = CNOT_1 \cdot \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle)$$

$$= \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle)$$

$$(H\otimes H)\,CNOT_1\,(H\otimes H)|11\rangle = |01\rangle = CNOT_2|11\rangle$$

$$\Rightarrow (H\otimes H)\,CNOT_1\,(H\otimes H) = CNOT_2$$

a) $|\psi_{in}\rangle = |00\rangle$

$\downarrow$

$$|\phi_1\rangle = H\otimes \mathbb{1}\,|00\rangle = |+0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$

$\downarrow$

$$|\phi_2\rangle = CNOT_1 |\phi_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$\downarrow$$

$$|\phi_3\rangle = (Z \otimes H)|\phi_2\rangle = \frac{1}{\sqrt{2}}(|0+\rangle - |1-\rangle)$$

$$= \frac{1}{\sqrt{2}\cdot\sqrt{2}}(|00\rangle + |01\rangle - |10\rangle + |11\rangle)$$

$$\downarrow$$

$$|\phi_4\rangle = CNOT_2 |\phi_3\rangle$$

$$= \frac{1}{2}(|00\rangle + |11\rangle - |10\rangle + |01\rangle)$$

b) $|01\rangle \xrightarrow{H\otimes 1} |+1\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |11\rangle)$

$\xrightarrow{CNOT_1}$

$\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \xrightarrow{Z\otimes H}$

$$\longrightarrow \frac{1}{\sqrt{2}}\left(|0-\rangle - |1+\rangle\right)$$

$$= \frac{1}{\sqrt{2}\sqrt{2}}\left(|00\rangle - |01\rangle - |10\rangle - |11\rangle\right)$$

$$\xrightarrow{CNOT_2} \frac{1}{2}\left(|00\rangle - |11\rangle - |10\rangle - |01\rangle\right)$$

analogously:

$$|10\rangle \rightarrow \frac{1}{2}\left(|00\rangle - |01\rangle + |10\rangle + |11\rangle\right)$$

$$|11\rangle \rightarrow \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle - |11\rangle\right)$$

$$\Rightarrow \quad U = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

in basis: $(|00\rangle, |01\rangle, |10\rangle, |11\rangle)$

c) Since U is unitary, $UU = 1 \Rightarrow U^{-1} = U^*$

$$U = CNOT_2 \cdot (Z \otimes H) \cdot CNOT_1 \cdot (H \otimes 1)$$

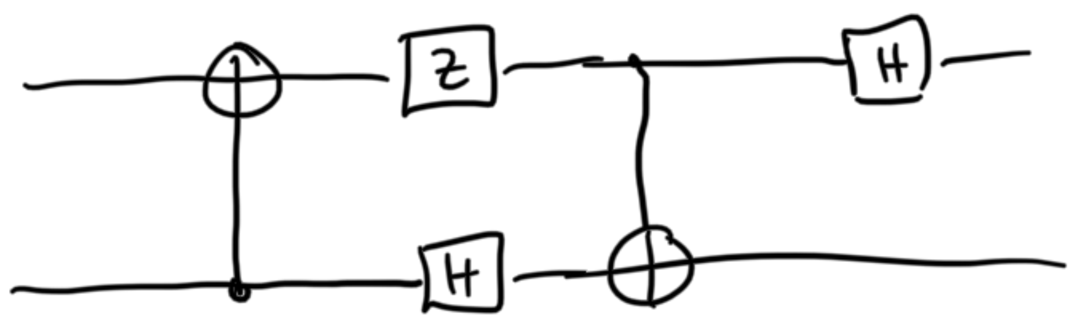
$$U^\dagger = (CNOT_2 \cdot (Z \otimes H) \cdot CNOT_1 \cdot (H \otimes 1))^\dagger$$

$$= (H \otimes 1)^\dagger \cdot CNOT_1^\dagger \cdot (Z \otimes H)^\dagger \cdot CNOT_2^\dagger$$

H, Z, CNOT are unitary.

$$= (H \otimes 1) \cdot CNOT_1 \cdot (Z \otimes H) \cdot CNOT_2$$

U^{-1}:



(4) a)

$$\begin{pmatrix} 1 \\ 1+i \\ 2e^{i\pi/4} \end{pmatrix} \otimes \begin{pmatrix} 2-2i \\ (1-i)^{-1} \\ -5 \end{pmatrix} = \begin{pmatrix} 2-2i \\ (1-i)^{-1} \\ -5 \\ (1+i)(2-2i) \\ (1+i)(1-i)^{-1} \\ -(1+i)\cdot 5 \\ 2e^{i\pi/4}(2-2i) \\ 2e^{i\pi/4}(1-i)^{-1} \\ -10e^{i\pi/4} \end{pmatrix} = \begin{pmatrix} 2-2i \\ (1-i)^{-1} \\ -5 \\ 4 \\ i \\ -(5+5i) \\ 4\sqrt{2} \\ \sqrt{2}i \\ -5\sqrt{2}(1+i) \end{pmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} i & 1 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} i & 1 & i & 1 \\ -2 & 0 & -2 & 0 \\ i & 1 & -i & -1 \\ -2 & 0 & 2 & 0 \end{pmatrix}$$

b) $$A \otimes B = \begin{pmatrix} A_{11}B & A_{12}B & \cdots & A_{1d}B \\ A_{21}\cdot B & & & \\ \vdots & & & \\ A_{d1}\cdot B & & & \end{pmatrix}$$

$$v \otimes w = \begin{pmatrix} v_1 \cdot w \\ v_2 \cdot w \\ \vdots \\ v_d \cdot w \end{pmatrix}$$

$$(A \otimes B)(v \otimes w) = \begin{pmatrix} A_{11}B & A_{12}B & \cdots & A_{1d}B \\ A_{21}B & & & \\ \vdots & & & \\ A_{d1}B & & & \end{pmatrix} \begin{pmatrix} v_1 \cdot w \\ v_2 \cdot w \\ \vdots \\ v_d \cdot w \end{pmatrix}$$

$$= \begin{pmatrix} A_{11} \cdot B \cdot v_1 w + A_{12} \cdot B \cdot v_2 \cdot w + \dots A_{1d} B \cdot v_d \cdot w \\ A_{21} B \cdot v_1 \cdot w + A_{22} \cdot B \cdot v_2 \cdot w + \cdots \\ \end{pmatrix}$$

$$\begin{pmatrix} \vdots \\ A_{d1} B v_1 w + \dots \end{pmatrix}$$

$$= \begin{pmatrix} (A_{11} v_1 + A_{12} v_2 + \dots + A_{1d} v_d)\, Bw \\ (A_{21} v_1 + A_{22} v_2 + \dots + A_{2d} v_d)\, Bw \\ \vdots \\ (A_{d1} v_1 + \dots + A_{dd} v_d)\, Bw \end{pmatrix}$$

$$= \begin{pmatrix} (A \cdot v)_1\, B \cdot w \\ (A \cdot v)_2\, B \cdot w \\ \vdots \\ (Av)_d\, Bw \end{pmatrix} = Av \otimes Bw$$

c) $A \otimes (B + C)$

$$\begin{pmatrix} A_{11}(B+C) & A_{12}(B+C) & \cdots & A_{1d}(B+C) \\ \vdots \end{pmatrix}$$

$$= \begin{pmatrix} A_{21}(B+C) & \cdots \\ \vdots & \\ A_{d1}(B+C) & \end{pmatrix}$$

$>(\ell)$

$$= \begin{pmatrix} A_{11}B + A_{11}C & \cdots \\ \vdots & \\ & \end{pmatrix}$$

$$= \begin{pmatrix} A_{11}B & \cdots \\ \vdots & \\ & \end{pmatrix} + \begin{pmatrix} A_{11}C & \cdots \\ \vdots & \\ & \end{pmatrix}$$

$$= A \otimes B + A \otimes C$$

d) $|\phi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right)$

$|\psi\rangle = \frac{1}{\sqrt{2}}\left(i|0\rangle - |1\rangle\right)$

$$|\phi\rangle \otimes |\psi\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right) \otimes \left(i|0\rangle - |1\rangle\right)$$

$$= \frac{1}{2}\left(i|0\rangle \otimes |0\rangle - |0\rangle \otimes |1\rangle + \underbrace{i^2}_{-1}|1\rangle \otimes |0\rangle - i|1\rangle \otimes |1\rangle\right)$$

$$= \frac{1}{2}\left(i|00\rangle - |01\rangle - |10\rangle - i|11\rangle\right)$$