Introduction to Quantum Computing

Problem Set 4: Quantum algorithms II
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Exercise 1: (Simon's algorithm) We will have a look at Simon's algorithm in this exercise. In particular, we want to discuss a concrete example on $n = 3$ qubits, which allows us to consider $N = 2^n = 8$.

a) Give a description of Simon's problem in your own words including the relevant input and output quantities.

b) We now consider the concrete case of $x = (x_0, \dots, x_7)$ given by

$$x_{000} = x_{101} = 01, \quad x_{001} = x_{100} = 10, \quad x_{010} = x_{111} = 11, \quad x_{011} = x_{110} = 00,$$

where we again identify $(1, \dots, 8)$ with $\{0,1\}^3$. Argue that $s = 101$.

c) Draw the quantum circuit diagram of Simon's algorithm for $n = 3$.

d) Follow the steps of the algorithm and give the intermediate states (i) after the initial Hadamards, (ii) after the oracle query is executed (iii) after the measurement of the second register in computational basis (Assume the result was 01) and (iv) after the final Hadamards.

e) Discuss why a measurement of the first 3 qubits in the computational basis of the final state gives information about s?

f) Give a sequence of possible outcomes of the algorithm that will lead to full information about s and compute s from it.

Exercise 2: (Quantum Fourier transform) We are going to discuss some properties of the quantum fourier transform and its implementation as a quantum circuit. Given $N = 2^n$ and $\omega_N = e^{2\pi i/N}$ the discrete Fourier transformation F_N is defined by the matrix elements $(F_N)_{i,j} = \frac{\omega_N^{i \cdot j}}{\sqrt{N}}$ for $i, j \in \{0, \dots, 2^n - 1\}$ in the computational basis on n qubits.

a) Check that the columns of F_N form an orthonormal basis.

b) Show that computational basis state indexed by $k \in \{0,1\}^n$ is transformed under the Fourier transformation according to

$$|k\rangle \mapsto \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \omega_N^{k \cdot j} |j\rangle \tag{1}$$

c) Verify that the mapping in b) can be equivalently described as

$$|k\rangle \mapsto \sum_{j=0}^{N-1} \prod_{l=1}^{n} e^{2\pi i k j_l / 2^l} |j_1 \dots j_n\rangle = \bigotimes_{l=1}^{n} \frac{1}{\sqrt{2}} \Big(|0\rangle + e^{2\pi i k/2^l} |1\rangle \Big) \tag{2}$$

d) Verify that the matrix representation of F_4 is given by

$$F_4 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \tag{3}$$

c) Show that by flipping the middle two columns of F_4 you arrive at the matrix $\widetilde{F}_4 = \begin{pmatrix} H & S \cdot H \\ H & -S \cdot H \end{pmatrix}$, where H is the Hadamard matrix. Argue that we can implement this flip of the columns by physically exchanging the two qubits.

d) Verify that $\widetilde{F}_4$ decomposes as $(H \otimes \mathbb{1})(|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes S)(\mathbb{1} \otimes H)$. Draw the corresponding quantum circuit.