Exercise 3.

a) $f_s(x) = x \cdot s = \sum_{i=0}^{n-1} x_i \cdot s_i \mod 2$, $n = 3$

$\Rightarrow \sum_{i=0}^{2} x_i \cdot s_i = x_0 \cdot s_0 \oplus x_1 \cdot s_1 \oplus x_2 \cdot s_2$, $s = 011$

~~s=011~~, $x=$	$x_2 \oplus x_3$	$x_0 \cdot s_0 \oplus x_1 \cdot s_1 \oplus x_2 \cdot s_2$
000	$0 \oplus 0 = 0$	$0 \oplus 0 \oplus 0 = 0$
001	$0 \oplus 1 = 1$	$0 \oplus 0 \oplus 1 = 1$
010	$1 \oplus 0 = 1$	$0 \oplus 1 \oplus 0 = 1$
100	$0 \oplus 0 = 0$	$0 \oplus 0 \oplus 0 = 0$
101	$0 \oplus 1 = 1$	$0 \oplus 0 \oplus 1 = 1$
110	$1 \oplus 0 = 1$	$0 \oplus 1 \oplus 0 = 1$
111	$1 \oplus 1 = 0$	$0 \oplus 1 \oplus 1 = 0$

$x=$	$x_1 \oplus x_2$	$x_0 \cdot s'_0 \oplus x_1 \cdot s'_1 \oplus x_2 \cdot s'_2$	
000	$0 \oplus 0 = 0$	$0 \oplus 0 \oplus 0 = 0$	
001	$0 \oplus 0 = 0$	$0 \oplus 0 \oplus s'_2 = 0$	$\Rightarrow s'_2 = 0$
010	$0 \oplus 1 = 1$	$0 \oplus s'_1 \oplus 0 = 1$	$\Rightarrow s'_1 = 1$
100	$1 \oplus 0 = 1$	$s'_0 \oplus 0 \oplus 0 = 1$	$\Rightarrow s'_0 = 1$
101	$1 \oplus 0 = 1$	$s'_0 \oplus 0 \oplus s'_2 = 1$	$\Rightarrow 1 \oplus 0 \oplus 0 = 1$ ✓
110	$1 \oplus 1 = 0$	$s'_0 \oplus s'_1 \oplus 0 = 0$	$\Rightarrow 1 \oplus 1 \oplus 0 = 0$ ✓
111	$1 \oplus 1 = 0$	$s'_0 \oplus s'_1 \oplus s'_2 = 0$	$\Rightarrow 1 \oplus 1 \oplus 0 = 0$ ✓

$\Rightarrow s' = 110$.

b) $U_{f_s}|x\rangle|a\rangle = |x\rangle \otimes |a \oplus f_s\rangle$

if $f_s = 0$, $\Rightarrow |a \oplus 0\rangle$.

$\Rightarrow \begin{cases} \text{if } a = 0, & |0 \oplus 0\rangle = |0\rangle \Rightarrow |x\rangle|0\rangle. \\ \text{if } a = 1, & |1 \oplus 0\rangle = |1\rangle \Rightarrow |x\rangle|1\rangle, \end{cases}$

if $f_s = 1$, $\Rightarrow |a \oplus 1\rangle$.

$\Rightarrow \begin{cases} \text{if } a = 0, & |0 \oplus 1\rangle = |1\rangle \Rightarrow |x\rangle|1\rangle \\ \text{if } a = 1, & |1 \oplus 1\rangle = |0\rangle \Rightarrow |x\rangle|0\rangle \end{cases}$

$\therefore U_{f_s}|x\rangle|a\rangle = |x\rangle \otimes |a \oplus f_s\rangle =$ ~~$|x\rangle \otimes |1\rangle$~~ $|x\rangle|0\rangle$ or $|x\rangle|1\rangle$.

with $x \in \{0,1\}^n$.

c) $|\psi_0\rangle = |0\rangle^{\otimes n}|1\rangle$.

$|\psi_1\rangle = H^{\otimes n}|0\rangle^{\otimes n} \cdot H|1\rangle = |+\rangle^{\otimes n} \cdot |-\rangle$.

$|\psi_2\rangle = U_{f_s}|+\rangle^{\otimes n} \cdot |-\rangle = U_{f_s}|+\rangle^{\otimes n} \cdot \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}}(|+\rangle^{\otimes n}|0 \oplus f_s\rangle - |+\rangle^{\otimes n}|1 \oplus f_s\rangle)$

$\left.\begin{array}{l} \text{if } f_s = 0, \text{ then } \frac{1}{\sqrt{2}}(|+\rangle^{\otimes n}|0\rangle - |+\rangle^{\otimes n}|1\rangle) = |+\rangle^{\otimes n}|-\rangle \\ \text{if } f_s = 1, \text{ then } \frac{1}{\sqrt{2}}(|+\rangle^{\otimes n}|1\rangle - |+\rangle^{\otimes n}|0\rangle) = |+\rangle^{\otimes n}(-1)|-\rangle \end{array}\right\} \Rightarrow |+\rangle^{\otimes n}(-1)^{f_s}|-\rangle$.

$|\psi_3\rangle = H^{\otimes n}|+\rangle^{\otimes n}(-1)^{f_s}|-\rangle = |0\rangle^{\otimes n}(-1)^{f_s}|-\rangle$.

Exercise 2.

a) $\rho(r) = \frac{I}{2} + \frac{1}{2}(r_1 X + r_2 Y + r_3 Z) = \frac{1}{2}(I + r_1 X + r_2 Y + r_3 Z)$

$= \frac{1}{2}\left(\begin{bmatrix}1&0\\0&1\end{bmatrix} + \begin{bmatrix}0&r_1\\r_1&0\end{bmatrix} + \begin{bmatrix}0&-ir_2\\ir_2&0\end{bmatrix} + \begin{bmatrix}r_3&0\\0&-r_3\end{bmatrix}\right)$

$= \frac{1}{2}\begin{bmatrix}1+r_3 & r_1-r_2 i\\ r_1+r_2 i & 1-r_3\end{bmatrix}$.

~~$Tr(\rho(r)) = \frac{1}{2}(1+r_3+1-r_3) = 1$~~

$\rho(r)^2 = \frac{1}{2}\begin{bmatrix}1+r_3 & r_1-r_2 i\\ r_1+r_2 i & 1-r_3\end{bmatrix}\cdot\frac{1}{2}\begin{bmatrix}1+r_3 & r_1-r_2 i\\ r_1+r_2 i & 1-r_3\end{bmatrix} = \frac{1}{4}\begin{bmatrix}(1+r_3)^2+r_1^2+r_2^2 & 2(r_1-r_2 i)\\ 2(r_1+r_2 i) & (1+r_3)^2+r_1^2+r_2^2\end{bmatrix}$

$\because \rho(r)^2 \neq \rho \quad \therefore \|r\| < 1$

mix state.

b) $\rho(r') = P\rho(r)P = (r_1 X + r_2 Y + r_3 Z)\cdot\frac{1}{2}(I + r_1 X + r_2 Y + r_3 Z)\cdot(r_1 X + r_2 Y + r_3 Z)$

$= \frac{1}{2}(r_1 X + r_1^2 X^2 + r_1 r_2 XY + r_1 r_3 XZ + r_2 Y + r_1 r_2 YX + r_2^2 Y^2 + r_2 r_3 YZ + r_3 Z$
$+ r_3 r_1 ZX + r_3 r_2 ZY + r_3^2 Z^2)(r_1 X + r_2 Y + r_3 Z)$

$= \frac{1}{2}(r_1 X + r_1^2 + r_1 r_2 XY + r_1 r_3 XZ + r_2 Y + r_1 YX + r_2^2 + r_2 r_3 YZ + r_3 Z$
$+ r_3 r_1 ZX + r_3 r_2 ZY + r_3^2)(r_1 X + r_2 Y + r_3 Z)$

$= \frac{1}{2}(r_1^2 X^2 + r_1 r_2 XY + r_1 r_3 XZ + r_1^3 X + r_1^2 r_2 Y + r_1^2 r_3 Z + r_1^2 r_2 XYX + r_1 r_2^2 XY^2$
$+ r_1 r_2 XYZ + r_1^2 r_3 XZX + r_1 r_2 r_3 XZY + r_1 r_3^2 XZ^2 + r_2 r_1 YX + r_2^2 Y^2 + r_2 r_3 YZ$
$+ r_1^2 YX^2 + r_1 r_2 YXY + r_1 r_3 YXZ + r_2^2 r_1 X + r_2^3 Y + r_2^2 r_3 Z + r_2 r_3 r_1 YZX$
$+ r_2^2 r_3 YZY + r_2 r_3^2 YZ^2 + r_3 r_1 ZX + r_3 r_2 ZY + r_3^2 Z^2 + r_3 r_1^2 ZX^2 +$
$r_3 r_1 r_2 ZXY + r_3^2 r_1 ZXZ + r_3 r_2 r_1 ZYX + r_3 r_2^2 ZY^2 + r_3^2 r_2 ZYZ + r_3^2 r_1 X$
$+ r_3^2 r_2 Y + r_3^3 Z)$

$= \frac{1}{2}(r_1^2 + r_1 r_2 XY + r_1 r_3 XZ + r_1^3 X + r_1^2 r_2 Y + r_1^2 r_3 + r_1^2 r_2(-Y) + r_1 r_2^2 X$
$+ r_1 r_2 r_3 i + r_1^2 r_3(-1) + r_1 r_2 r_3(-XY) + r_1 r_3^2 X + r_2 r_1 YX + r_2^2 + r_2 r_3 YZ$
$+ r_1^2 Y + r_1 r_2(-X) + r_1 r_3(-i) + r_2^2 r_1 X + r_2^3 Y + r_2^2 r_3 Z + r_2 r_3 r_1 i$
$+ r_2^2 r_3(-Z) + r_2 r_3^2 Y + r_3 r_1 ZX + r_3 r_2 ZY + r_3^2 + r_3 r_1^2 Z + r_3 r_1 r_2 XY$
$+ r_3^2 r_1(-X) + r_3 r_2 r_1 (-i) + r_3 r_2^2 Z + r_3^2 r_2 (YZ) + r_3^2 r_1 X + r_3^2 r_2 Y + r_3^3 Z)$

$= \frac{1}{2}[r_1^2(1+Y) + r_2^2 + r_3^2 + r_1 r_2(XY + YX - X) + r_1 r_3(XZ - i + ZX) + r_1^3 X$
$+ r_1^2 r_2(Y - Y) + r_1^2 r_3(-1+Z) + r_1 r_2^2(X+X) + r_1 r_2 r_3(i - XY + XY + i - i)$
$+ r_1 r_3^2(X + X - X) + r_2 r_3(YZ + ZY) + r_2^3 Y + r_2^2 r_3(Z - Z + Z) + r_2 r_3^2(Y + YZ + Y) + r_3^3 Z]$

$$= \frac{1}{2}\left[r_1^2(1+Y) + r_2^2 + r_3^2 + r_1 r_2(-X) + r_1 r_3(-i) + r_1^3 X + r_1^2 r_3 Z + r_1 r_2^2(2X) \right.$$

$$\left. + r_1 r_2 r_3 i + r_1 r_3^2 X + r_2^3 Y + r_2^2 r_3 Z + r_2 r_3^2(2Y + YZ) + r_3^3 Z \right]$$

$$= \frac{1}{2}\left[r_1^2 + r_2^2 + r_3^2 + r_1^3 X + r_2^3 Y + r_3^3 Z + r_1^2 Y - r_1 r_2 X - r_1 r_3 i + r_1 r_2^2 \cdot 2X \right.$$

$$\left. + r_1 r_3^2 X + r_1^2 r_3 Z + r_2 r_3^2 \cdot 2Y + r_2 r_3^2 YZ + r_2^2 r_3 Z + r_1 r_2 r_3 i \right).$$

c) $\rho(r') = H \rho(r) H$

$$= \frac{1}{\sqrt{2}}(X+Z) \cdot \frac{1}{2}(I + r_1 X + r_2 Y + r_3 Z) \cdot \frac{1}{\sqrt{2}}(X+Z)$$

$$= \frac{1}{4}(X + r_1 X^2 + r_2 XY + r_3 XZ + Z + r_1 ZX + r_2 ZY + r_3 Z^2)(X+Z)$$

$$= \frac{1}{4}(X^2 + XZ + r_1 X^2 X + r_1 X^2 Z + r_2 XYX + r_2 XYZ + r_3 XZX + r_3 XZ^2 + ZX + Z^2$$

$$+ r_1 ZX^2 + r_1 ZXZ + r_2 ZYX + r_2 ZYZ + r_3 Z^2 X + r_3 Z^2 Z).$$

$$= \frac{1}{4}(1 + XZ + r_1 X + r_1 Z + r_2(-Y) + r_2 i + r_3(-1) + r_3 X + ZX + 1 + r_1 Z$$

$$+ r_1(-X) + r_2(-i) + r_2 YZ + r_3 X + r_3 Z)$$

$$= \frac{1}{4}(2 + XZ + ZX + r_1 X + r_1 Z - r_1 X + r_1 Z - r_2 Y + r_2 i - r_2 i + r_2 YZ - r_3$$

$$+ r_3 X + r_3 X + r_3 Z)$$

$$= \frac{1}{4}(2 + 2 r_1 Z - r_2 Y + r_2 Y Z - r_3 + 2 r_3 X + r_3 Z).$$

Exercise 1

1.a) $\rho_B = Tr_A(|\psi^+\rangle\langle\psi^+|)$; $|\psi^+\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$= \sum_{i,j,k=0}^{1} (\langle i| \otimes 1) \frac{1}{2}[(|j\rangle\langle k|) \otimes (|j\rangle\langle k|)](|i\rangle \otimes 1)$; $= \frac{1}{\sqrt{2}} \sum_{j=0}^{1} |j,j\rangle$

$= \frac{1}{2} \sum_{i,j,k=0}^{1} [(\langle i|j\rangle\langle k|) \otimes (1 \otimes |j\rangle\langle k|)](|i\rangle \otimes 1)$; $|\psi^+\rangle\langle\psi^+| = \frac{1}{2} \sum_{j,k=0}^{1} |j,j\rangle\langle k,k|$

$= \frac{1}{2} \sum_{i,j,k=0}^{1} (\langle i|j\rangle\langle k|i\rangle)(1 \otimes |j\rangle\langle k| \otimes 1)$; $= \frac{1}{2} \sum_{j,k=0}^{1} (|j\rangle\langle k|) \otimes (|j\rangle\langle k|)$

$= \frac{1}{2} \sum_{i,j,k=0}^{1} \delta_{ij}\delta_{ik} |i\rangle\langle i|$

$= \frac{1}{2} \sum_{i=0}^{1} |i\rangle\langle i|$

$= \frac{1}{2} \cdot \mathbb{1}$

b) $b \in \{0,1\}$

$H^b \in \{+,-\} \Rightarrow \{\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\}$

$|\psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$|\psi'\rangle = H|\psi^+\rangle = \frac{1}{\sqrt{2}}(H|00\rangle + H|11\rangle)$

$= \frac{1}{\sqrt{2}}[\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) + \frac{1}{\sqrt{2}}(|01\rangle - |11\rangle)]$

$= \frac{1}{2}(|00\rangle + |10\rangle + |01\rangle - |11\rangle)$

$|\psi'\rangle\langle\psi'| = \frac{1}{2}(|00\rangle + |10\rangle + |01\rangle - |11\rangle) \cdot \frac{1}{2}(\langle 00| + \langle 10| + \langle 01| - \langle 11|)$

$= \frac{1}{4}(|00\rangle\langle 00| + |00\rangle\langle 10| + |00\rangle\langle 01| + |00\rangle\langle 11| + |10\rangle\langle 00| + |10\rangle\langle 10| + |10\rangle\langle 01|$
$+ |10\rangle\langle 11| + |01\rangle\langle 00| + |01\rangle\langle 10| + |01\rangle\langle 01| + |01\rangle\langle 11| - |11\rangle\langle 00|$
$- |11\rangle\langle 10| - |11\rangle\langle 01| + |11\rangle\langle 11|)$

$= \frac{1}{4}(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 1| \otimes |0\rangle\langle 0| + |0\rangle\langle 0| \otimes |0\rangle\langle 1| + |0\rangle\langle 1| \otimes |0\rangle\langle 1| + |1\rangle\langle 0| \otimes |0\rangle\langle 0|$
$+ |1\rangle\langle 1| \otimes |0\rangle\langle 0| + |1\rangle\langle 0| \otimes |0\rangle\langle 1| + |1\rangle\langle 1| \otimes |0\rangle\langle 1| + |0\rangle\langle 0| \otimes |1\rangle\langle 0| + |0\rangle\langle 1| \otimes |1\rangle\langle 0|$
$+ |0\rangle\langle 0| \otimes |1\rangle\langle 1| + |0\rangle\langle 1| \otimes |1\rangle\langle 1| - |1\rangle\langle 0| \otimes |1\rangle\langle 0| - |1\rangle\langle 1| \otimes |1\rangle\langle 0| - |1\rangle\langle 0| \otimes |1\rangle\langle 1|$
$+ |1\rangle\langle 1| \otimes |1\rangle\langle 1|)$

$= \frac{1}{4}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1| + |0\rangle\langle 0| + |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|)$

$= \frac{1}{2}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 1|)$

$Tr_A(|\psi'\rangle\langle\psi'|) = \frac{1}{2} \cdot 2 = 1$