Introduction to Quantum Computing

Exercise sheet 1: linear algebra primer
University of Copenhagen

The purpose of this exercise sheet is to cover some of the background material in linear algebra we will be using frequently during the course and setting it into the context of finite dimensional complex vector spaces.

Exercise 1: (Vectors with complex entries & the scalar product) The main mathematical players, we will encounter in this course are finite dimensional vector spaces over the complex numbers. To be more precise, we denote by $\mathbb{C}^d$ the complex vector space of dimension d meaning, that all its elements are given by d-tuples of complex numbers. In the following, we are going to work with some of the most important concepts. To this end let us consider the case $d = 3$ and define the vectors

$$v = \begin{pmatrix} 1 \\ 1+i \\ 2e^{i\frac{\pi}{4}} \end{pmatrix}, \quad w = \begin{pmatrix} 2-2i \\ (1-i)^{-1} \\ -5 \end{pmatrix}.$$

(a) There are two natural operations on vectors, we are going to use frequently: multiplication of a vector by a scalar and addition of two vectors (called forming a superposition in quantum mechanics). Both operations are defined element-wise. Compute the vectors $2 \cdot v + w$ and $\frac{i-1}{2}v + \frac{1}{i}w$.

(b) Analogous to Euclidean vector spaces over the real numbers such as $\mathbb{R}^3$, we can define a scalar product on $\mathbb{C}^d$ via

$$\langle v|w\rangle = \sum_{i=1}^{d} \bar{v}_i w_i \,,$$

where $\bar{a}$ is the complex conjugate of the complex number a. Compute $\langle v|w\rangle$ for v and w as defined above.

(c) Based on the scalar product introduced in (b), we can define the norm of a vector via scalar product on $\mathbb{C}^d$ via

$$\|v\|^2 = \langle v|v\rangle = \sum_{i=1}^{d} \bar{v}_i v_i \,.$$

Compute the norms of the vectors v and w.

(d) We call two vectors $x, y \in C^d$ orthogonal if $\langle x|y\rangle = 0$. Find a vector that is orthogonal to v. Do the same for w.

Exercise 2: (Linear transformations & unitarity) In the course, quantum computations will be written in terms of linear transformations or more precisely unitary transformations of complex vectors. We will now consider the vector space $\mathbb{C}^2$ and define the vectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad v_3 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

(a) Compute the norms $\|v_i\|$ of the v_i and define the vectors $\widetilde{v}_i = \frac{1}{\|v_i\|} v_i$. What is the norm of these vectors?

(b) Compute the matrix vector products Av_i for the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

(c) Find a matrix T that satisfies $T \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix}$ for all $a, b \in \mathbb{C}$.

(d) For a matrix M, we define its adjoint $M^\dagger$ entry-wise via $\left(M^\dagger\right)_{i,j} = \bar{M}_{i,j}$. Show for A defined in (b) the relation $\langle v_2 | A v_3 \rangle = \langle A^\dagger v_2 | v_3 \rangle$. Note, this relation holds independent of the matrix A and the dimension d.

(e) We call a matrix U unitary if $U^\dagger U = U U^\dagger = \mathbb{1}$, where we denote by $\mathbb{1}$ the identity matrix. Show that

$$U = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, \quad |a|^2 + |b|^2 = 1 \, .$$

(f) Show that $\|v\| = \|Uv\|$ and $\langle U^\dagger v | U w \rangle = \langle v | w \rangle$ for U unitary.is a unitary 2×2-matrix with determinant 1.

Exercise 3: (Eigenvalues and diagonalization) *Warning: the following exercise might be a bit more challenging - do not feel bad if you cannot solve them directly, part of them we will discuss in the first weeks's lecture.* Recall that $\lambda \in \mathbb{C}$ is called an *eigenvalue* of a matrix $A \in \mathbb{C}^{d\times d}$ if there exists $v \in \mathbb{C}^d \setminus \{0\}$ such that:

$$Av = \lambda v.$$

The vector v is called an *eigenvector*. The matrix A is called *diagonalizable* if there exist a diagonal matrix D and an invertible matrix X such that:

$$A = XDX^{-1}.$$

Moreover, the diagonal of D consists of the eigenvalues of A. Also note that if A is *normal* ($AA^* = A^*A$), then we can pick X unitary.

(a) Diagonalize the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

i.e. find a matrix X and a diagonal matrix D such that $A = XDX^{-1}$. Can you pick X unitary? Compute A^{42}.

(b) A is called *Hermitian* if $A^* = A$. Show that if A is hermitian and λ one of its eigenvalues, then $\lambda \in \mathbb{R}$.

(c) Show that if a matrix A has an eigenvalue λ, then so does $B = XAX^{-1}$ for X invertible. How can you construct the corresponding eigenvector of B to λ from the eigenvector of A to λ?

(d) Show that unitary matrices are normal.

(e) Show that eigenvalues of unitary matrices are complex numbers of modulus 1.