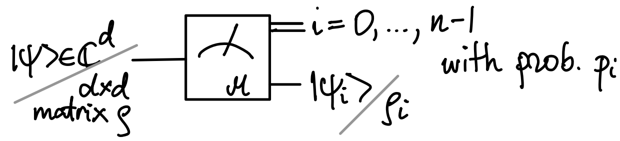
Lecture 4.1

Recap: measurements

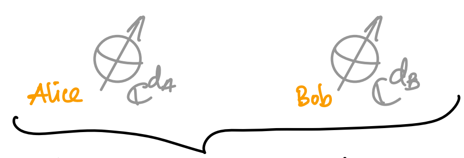
Consider n-outcome measurement $\mathcal{M} = (M_0, \dots, M_{n-1})$ that can be applied to a d-dimensional quantum system (We have $\sum_{i=0}^{n-1} M_i^\dagger M_i = \mathbb{1}_d$)



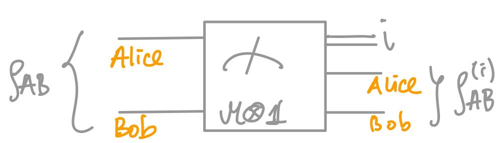
input	p_i	post-measurement state, given outcome i
pure state $\lvert\psi\rangle$	$p_i = \lVert M_i \lvert\psi\rangle \rVert^2 = \langle\psi\rvert M_i^\dagger M_i \lvert\psi\rangle$	$\lvert\psi_i\rangle = M_i \lvert\psi\rangle / \sqrt{p_i}$
density matrix ρ	$p_i = \mathrm{Tr}(M_i \rho M_i^\dagger)$	$\rho_i = M_i \rho M_i^\dagger / p_i$

"Entanglement cannot be used to communicate"

Suppose Alice and Bob share a bipartite state ρ_{AB} and Alice measures her system with $\mathcal{M} = (M_0, \dots, M_{n-1})$



$(d_A d_B) \times (d_A d_B)$ density matrix ρ_{AB}



Examples:

- $\rho_{AB} = |\psi^+\rangle\langle\psi^+|$ where $|\psi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$
- $\rho_{AB} = \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11|$

Claim 1: The outcome probabilities p_i and Alice's post-measurement state $\mathrm{Tr}_B(\rho_{AB}^{(i)}) =: \rho_A^{(i)}$ only depend on Alice's reduced input state $\rho_A := \mathrm{Tr}_B(\rho_{AB})$

Recall: $\mathrm{Tr}_B(\rho) = \sum_{j=0}^{d_B - 1} (\mathbb{1}_{d_A} \otimes \langle j|)\, \rho\, (\mathbb{1}_{d_A} \otimes |j\rangle)$

Proof. Recall that $\mathcal{M}$ corresponds to measurement $\mathcal{M}'$ with operators $M_i' = M_i \otimes \mathbb{1}_B$.

$$p_i = \mathrm{Tr}\left((M_i \otimes \mathbb{1})\, \rho_{AB}\, (M_i^\dagger \otimes \mathbb{1})\right)$$

think why? $\rightarrow$ $= \mathrm{Tr}_A \mathrm{Tr}_B (\quad — \,//\, — \quad)$

Recall: $(A_1 \otimes B_1)(A_2 \otimes B_2) = (A_1 A_2) \cdot (B_1 B_2)$

$$= \mathrm{Tr}_A\left[\sum_{j=0}^{d_B - 1} \underbrace{(\overset{A_1}{\mathbb{1}_A} \otimes \overset{B_1}{\langle j|})(\overset{A_2}{M_i} \otimes \overset{B_2}{\mathbb{1}})}_{(\mathbb{1}\cdot M_i)\otimes(\langle j|\mathbb{1}) = M_i \otimes \langle j| = (M_i \cdot \mathbb{1})\otimes(1\cdot\langle j|) = (M_i \otimes 1)\cdot(\mathbb{1}\otimes\langle j|) = M_i \cdot (\mathbb{1}\otimes\langle j|)} \rho_{AB}\, (M_i^\dagger \otimes \mathbb{1})(\mathbb{1}_A \otimes |j\rangle) \right]$$

$$= \mathrm{Tr}_A\left[\sum_{j=0}^{d_B-1} M_i\, (\mathbb{1}_A \otimes \langle j|)\, \rho_{AB}\, (\mathbb{1}_A \otimes |j\rangle)\, M_i^\dagger \right]$$

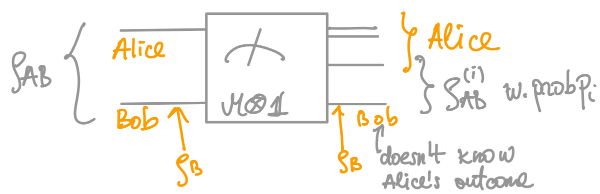
Q: what should be the next step?

$$= \mathrm{Tr}\left(M_i\, \mathrm{Tr}_B(\rho_{AB})\, M_i^\dagger\right) = \mathrm{Tr}\left(M_i\, \rho_A\, M_i^\dagger\right)$$

Post-measurement state:

$$\rho_A^{(i)} = \mathrm{Tr}_B\left[(M_i \otimes \mathbb{1})\, \rho_{AB}\, (M_i^\dagger \otimes \mathbb{1})\right] / p_i = M_i\, \rho_A\, M_i^\dagger / p_i \qquad \square$$

Claim 2. Bob's reduced state $\rho_B = \mathrm{Tr}_A(\rho_{AB})$ is not affected by Alice's measurement.



<u>Proof.</u>

Bob's state after Alice's measurement is

$$\sum_{i=0}^{n-1} p_i\, \mathrm{Tr}_A(\rho_{AB}^i) = \sum_{i=0}^{n-1} \mathrm{Tr}_A\left[(M_i \otimes \mathbb{1})\, \rho_{AB}\, (M_i^\dagger \otimes \mathbb{1})\right]$$

$$= \sum_{i=0}^{n-1} \mathrm{Tr}_A\left[(M_i^\dagger M_i \otimes \mathbb{1})\, \rho_{AB}\right]$$

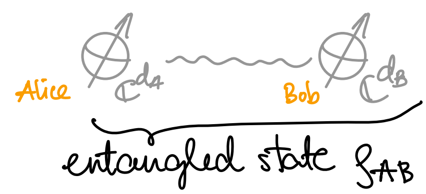
$$= \mathrm{Tr}_A(\rho_{AB}) = \rho_B \qquad \square$$

Properties of trace
- $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$
- $\mathrm{Tr}_A(M(A \otimes \mathbb{1})) = \mathrm{Tr}_A((A \otimes \mathbb{1})M)$

To summarize (claims 1&2): Local measurement outcomes only depend on the (local) reduced state. Moreover, the actions of another party have no effect on this state.

- In particular, Alice cannot transmit information to Bob by choosing which measurement to perform on her part of the shared entangled state (hoping to alter Bob's state in this way)

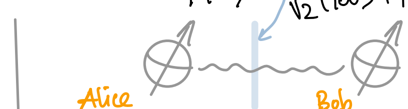
Alice's actions have no effect on Bob's state ρ_B and are thus undetectable by him



SPOOKY ACTION AT A DISTANCE

Alice & Bob are "space-like separated"

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$



0 — $\rho_A^{t=0} = \frac{1}{2}\mathbb{1}$ $\qquad$ $\rho_B^{t=0} = \frac{1}{2}\mathbb{1}$

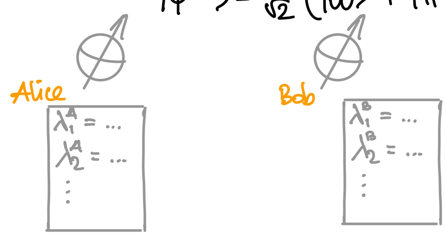
Alice measures in $|0/1\rangle$-basis and gets outcome $i=0/1$ ⟶ 1 — $\rho_A^{t=1} = |i\rangle\langle i|$ $\qquad$ $\rho_B^{t=1} = |i\rangle\langle i|$ with prob. $\frac{1}{2}$

time

What if Alice measured in $|+/-\rangle$-basis at $t=1$?

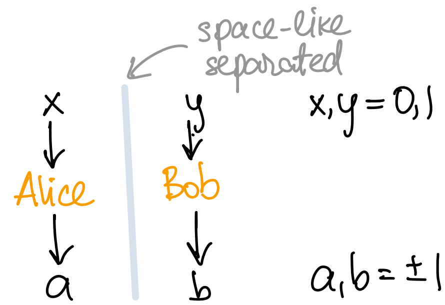
LOCAL HIDDEN VARIABLES (LHVs)

$|\psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$



is QM incomplete? In particular, could there exist local hidden variables whose value determines the outcomes of local measurements?

CHSH Bell scenario



Let's aim to maximize the

CHSH-expression (★)

$$\mathbb{E}(a_0 b_0) + \mathbb{E}(a_0 b_1) + \mathbb{E}(a_1 b_0) - \mathbb{E}(a_1 b_1)$$

a_i / b_i is the random variable representing Alice's/Bob's output on input $i = 0,1$.

To maximize (★):

on $xy = 00, 01, 10$ A&B should output $a = b$

On $xy = 11$ A&B should output $a \neq b$

How big of a value can we obtain in an LHV world?

Since LHVs $\vec{\lambda}^A$ and $\vec{\lambda}^B$ determine Alice's and Bob's measurement outcomes, $\begin{matrix} a_i = \tilde{a}_i \\ b_i = \tilde{b}_i \end{matrix}$ with prob. one for some $\tilde{a}_i, \tilde{b}_i \in \{\pm 1\}$ and $i = 0,1$

So in LHV-world (☆) becomes: $\tilde{a}_0\tilde{b}_0 + \tilde{a}_0\tilde{b}_1 + \tilde{a}_1\tilde{b}_0 - \tilde{a}_1\tilde{b}_1 =: T$

Q: what is the largest value T can take?

Ans: $T = \tilde{a}_0 \underbrace{(\tilde{b}_0 + \tilde{b}_1)} + \tilde{a}_1 \underbrace{(\tilde{b}_0 - \tilde{b}_1)}$

One of these is 0
and the other is +2 or −2.

So $T \leq 2$. We can get $T=2$ by letting $\tilde{a}_i = \tilde{b}_i = 1$ for $i=0,1$.

How big a value can we get according to QM?

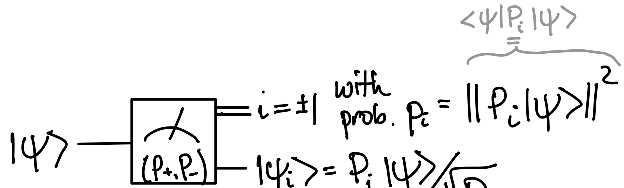
Need to talk about observables first!

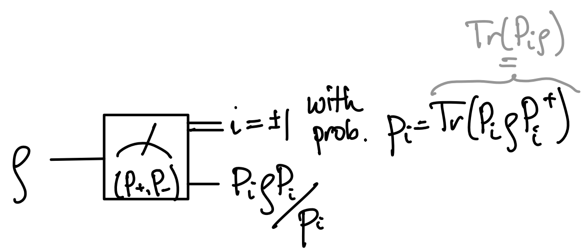
OBSERVABLES

2-outcome projective measurements $\mathcal{P} = \{P_+, P_-\}$ can conveniently be encoded in a single matrix, O, that we refer to as observable.

Recall: $P_+ + P_- = \mathbb{1}$, $P_i^2 = P_i = P_i^+$
This implies $P_+ \perp P_-$ i.e. $P_+P_- = 0 = P_-P_+$

matrix full of zeroes





Correspondence

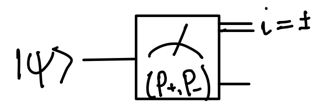
2-outcome projective measurement $\mathcal{P}=(P_+,P_-)$ $\Longleftrightarrow$ Observable O, which is a Hermitian unitary matrix (i.e. $O^\dagger = O$ and $O^2=\mathbb{1}$)

$\Rightarrow$ Given $\mathcal{P}=(P_+,P_-)$, take $O=P_+-P_-$

Check: $O^\dagger = P_+^\dagger - P_-^\dagger = P_+ - P_-$ ✓

$O^2 = (P_+-P_-)^2 = P_+^2 - P_+P_- - P_-P_+ + P_-^2 = P_+ + P_- = \mathbb{1}$

$\Leftarrow$ Given O where $O^\dagger = O$ and $O^2=\mathbb{1}$, take its spectral decomposition $O=\sum_i \lambda_i P_i$. Note that P_i's are mutually orthogonal projectors and $\lambda_i \in \{\pm 1\}$ (since $O^2=\mathbb{1}$)
So $O=(+1)\cdot P_1 + (-1)P_2$ and we can take $\mathcal{P}=(P_1,P_2)$.



Expected value, $\mathbb{E}_{|\psi\rangle}(O)$, of the classical outcome $i=\pm1$

$$\mathbb{E}_{|\psi\rangle}(O) = p_+\cdot(+1) + p_-\cdot(-1) = p_+ - p_- = \langle\psi|P_+|\psi\rangle - \langle\psi|P_-|\psi\rangle = \langle\psi|(P_+-P_-)|\psi\rangle$$
$$= \langle\psi|O|\psi\rangle$$

Q: Find observable, O, for $|+/-\rangle$-basis measurement $(P_+=|+\rangle\langle+|,\ P_-=|-\rangle\langle-|)$
Compute $\mathbb{E}_\psi(O)$ for $|\psi\rangle=|0\rangle$ and $|\psi\rangle=|-\rangle$.

Ans: $O = |+\rangle\langle+| - |-\rangle\langle-| = \frac{1}{2}\left(\begin{pmatrix}1&1\\1&1\end{pmatrix} - \begin{pmatrix}1&-1\\-1&1\end{pmatrix}\right) = \begin{pmatrix}0&1\\1&0\end{pmatrix} = X$

$\mathbb{E}_{|0\rangle}(X) = \langle 0|X|0\rangle = 0 \qquad \mathbb{E}_{|-\rangle}(X) = \langle -|X|-\rangle = -1.$

Now that we've seen observables, let's return to our question.

How big a value can we get according to QM?

$x, y = 0,1$

$x \to$ Alice $\to a$ | $y \to$ Bob $\to b$

$a, b = \pm 1$

$$\boxed{\text{CHSH-expression} \quad (\star) \quad \mathbb{E}(a_0 b_0) + \mathbb{E}(a_0 b_1) + \mathbb{E}(a_1 b_0) - \mathbb{E}(a_1 b_1)}$$

Strategy:

Alice & Bob share state $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

On input	Alice measures	Bob measures
0	Z	$\frac{1}{\sqrt{2}}(Z+X)$
1	X	$\frac{1}{\sqrt{2}}(Z-X)$

Q: Are Bob's measurements valid 2-outcome observables?

Ans: Hermitian? ✓

involution? $\left(\frac{1}{\sqrt{2}}(Z+X)\right)^2 = \frac{1}{2}\left(Z^2 + \overbrace{ZX + XZ}^{0} + X^2\right) = \frac{1}{2}(2\mathbb{1}) = \mathbb{1}$ ✓

$\left(\frac{1}{\sqrt{2}}(Z-X)\right)^2 = \frac{1}{2}\left(Z^2 \underbrace{-ZX - XZ}_{0} + X^2\right) = \mathbb{1}$ ✓

Q: Consider observables A, B. 1) Is $A \otimes B$ an observable?

2) What expectation value does $\langle\psi|(A\otimes B)|\psi\rangle$ describe?

Ans: 1) Yes since $(A\otimes B)^\dagger = A^\dagger \otimes B^\dagger = A\otimes B$ and $(A\otimes B)^2 = (A^2 \otimes B^2) = \mathbb{1}$

2) We have spectral decompositions $A = A_+ - A_-$ and $B = B_+ + B_-$.

Then $A\otimes B = \underbrace{(A_+\otimes B_+ + A_-\otimes B_-)}_{\substack{(+1)\text{-eigenspace} \\ \text{A\&B answer the same } (a=b \Leftrightarrow ab=1)}} - \underbrace{(A_+\otimes B_- + A_-\otimes B_+)}_{\substack{(-1)\text{-eigenspace} \\ \text{A\&B answer different } (a\neq b \Leftrightarrow ab=-1)}}$

Let a/b be the random variable corresponding to Alice's/Bob's outcome when they use observables A/B to measure their part of $|\psi\rangle$.
So $\langle\psi|(A\otimes B)|\psi\rangle = \Pr(ab=1) - \Pr(ab=-1) = \mathbb{E}(ab)$

The above strategy, gives the following value in $(\star)$

$$\mathbb{E}(a_0b_0) + \mathbb{E}(a_0b_1) + \mathbb{E}(a_1b_0) - \mathbb{E}(a_1b_1) =$$

$$= \langle\psi^+| \Big(\underbrace{Z\otimes \tfrac{1}{\sqrt{2}}(Z+X)}_{a_0b_0} + \underbrace{Z\otimes\tfrac{1}{\sqrt{2}}(Z-X)}_{a_0b_1} + \underbrace{X\otimes\tfrac{1}{\sqrt{2}}(Z+X)}_{a_1b_0} + \underbrace{X\otimes\tfrac{1}{\sqrt{2}}(Z-X)}_{a_1b_1}\Big)|\psi\rangle$$

$$= \frac{2}{\sqrt{2}}\langle\psi^+|(Z\otimes Z) + (X\otimes X)|\psi^+\rangle = 2\sqrt{2}$$

Q: Compute
- $(X\otimes X)|\psi^+\rangle = \frac{1}{\sqrt{2}}(X\otimes X)(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}|11\rangle + |00\rangle = |\psi^+\rangle$
- $(Z\otimes Z)|\psi^+\rangle = \frac{1}{\sqrt{2}}(Z\otimes Z)(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}|00\rangle + (-|1\rangle)\otimes(-|1\rangle)$
  $= |\psi^+\rangle$

Recall that in LHV-world we could get value at most $2 < 2\sqrt{2}$.
So QM & LHV models lead to different predictions, which can be tested experimentally.

(this has been done proving QM to be the theory preferred by nature)

Is the above quantum strategy optimal?

Consider an arbitrary* quantum strategy for CHSH
- A_x / B_y are observables for Alice/Bob for question x/y.
- Shared state: $|\psi\rangle$

We have

$$\langle\psi|\left[(A_0\otimes B_0) + (A_0\otimes B_1) + (A_1\otimes B_0) - (A_1\otimes B_1)\right]|\psi\rangle$$

$$= \langle\psi|\left[\underbrace{A_0\otimes(B_0+B_1) + A_1\otimes(B_0-B_1)}_{B}\right]|\psi\rangle \leq \lambda_{max}(B)$$

Commutator
$[A,B] := AB - BA$

Anticommutator
$\{A,B\} := AB + BA$

$$B^2 = \overset{\mathbb{1}}{A_0^2}\otimes(B_0^2 + \cancel{\{B_0,B_1\}} + B_1^2) + \overset{\mathbb{1}}{A_1^2}\otimes(B_0^2 - \cancel{\{B_0,B_1\}} + B_1^2) +$$

$$+ (A_0A_1)\otimes(\cancel{B_0^2} + [B_1,B_0] - \cancel{B_1^2}) + (A_1A_0)\otimes(\cancel{B_0^2} + [B_0,B_1] - \cancel{B_1^2})$$

$$= \mathbb{1}\otimes(4\mathbb{1}) + (A_0A_1 - A_1A_0)\otimes[B_1,B_0]$$

$$= 4\mathbb{1} + [A_0,A_1]\otimes[B_1,B_0]$$

Q: Upper bound $\lambda_{max}[A_0,A_1]$.

Ans: $\lambda_{max}[A_0,A_1] \leq 2$

$$\lambda_{max}(B) \leq \sqrt{\lambda_{max}(B^2)} \leq \sqrt{4+2\cdot 2} = 2\sqrt{2}$$

So the quantum strategy we saw is optimal!

SUPPLEMENTARY MATERIAL: PROVING PROPERTIES OF PARTIAL TRACE.

1) $\boxed{Tr(M) = Tr_A Tr_B(M)}$ M is $(d_A d_B)\times(d_A d_B)$ matrix

proof:

after tracing out, Bob's system becomes 1-dimensional

$$Tr_A Tr_B(M) = \sum_{i=0}^{d_A-1}\sum_{j=0}^{d_B-1} (\langle i|\otimes 1)(\mathbb{1}_{d_A}\otimes\langle j|)\, M\, (\mathbb{1}_A\otimes|j\rangle)(|i\rangle\otimes 1)$$

$$= \sum_{i=0}^{d_A-1}\sum_{j=0}^{d_B-1} \langle i,j|\, M\, \underbrace{|i,j\rangle}_{\text{standard basis of } \mathbb{C}^{d_A}\otimes\mathbb{C}^{d_B}} = Tr(M)$$

2) $\boxed{Tr_A(M(A\otimes\mathbb{1})) = Tr_A((A\otimes\mathbb{1})M)}$

proof:

$$Tr_A(M(A\otimes\mathbb{1})) = \sum_{i=0}^{d_A-1} (\langle i|\otimes\mathbb{1})\, M\,(A\otimes\mathbb{1})(|i\rangle\otimes\mathbb{1})$$

$$= \sum_{i=0}^{d_A-1} (\langle i|\otimes\mathbb{1})\, M\,\overbrace{\left(\sum_{k=0}^{d_A-1}|k\rangle\langle k|\otimes\mathbb{1}\right)}^{\mathbb{1}_{AB}}(A\otimes\mathbb{1})(|i\rangle\otimes\mathbb{1})$$

$$= \sum_{i,k=0}^{d_A-1} \underline{(\langle i|\otimes\mathbb{1})\, M\,(|k\rangle\otimes\mathbb{1})}\;\underline{(\langle k|A|i\rangle\otimes\mathbb{1})}$$

can swap these, since const$\cdot\mathbb{1}$ commutes with any matrix

$$= \sum_{i,k=0}^{d_A-1} (\langle k|A|i\rangle\otimes\mathbb{1})(\langle i|\otimes\mathbb{1})\, M\,(|k\rangle\otimes\mathbb{1})$$

$$= \sum_{i,k=0}^{d_A-1} (\langle k|\otimes\mathbb{1})(A\otimes\mathbb{1})(|i\rangle\langle i|\otimes\mathbb{1})\, M\,(|k\rangle\otimes\mathbb{1})$$

$$= \sum_{k=0}^{d_A-1} (\langle k|\otimes\mathbb{1})(A\otimes\mathbb{1})\overbrace{\left(\sum_{i=0}^{d_A-1}|i\rangle\langle i|\otimes\mathbb{1}\right)}^{\mathbb{1}}\, M\,(|k\rangle\otimes\mathbb{1})$$

$$= Tr_A((A\otimes\mathbb{1})M)$$