# Exercise Sheet 1 – solutions

**1**

$$v = \begin{pmatrix} 1 \\ 1+i \\ 2e^{i\pi/4} \end{pmatrix}, \quad w = \begin{pmatrix} 2-2i \\ (1-i)^{-1} \\ -5 \end{pmatrix}$$

a) $$2v + w = \begin{pmatrix} 2 \\ 1+i \\ 2e^{i\pi/4} \end{pmatrix} + \begin{pmatrix} 2-2i \\ (1-i)^{-1} \\ -5 \end{pmatrix} = \begin{pmatrix} 4-2i \\ 1+i+(1-i)^{-1} \\ 2e^{i\pi/4}-5 \end{pmatrix}$$

$$\frac{1}{i} = \frac{i}{i}\frac{1}{i} = -i$$

$$(1+i)(1-i) = +2$$

$$\frac{1}{(1-i)i} = \frac{1}{i+1}$$

$$\frac{1}{i+1} = \frac{i-1}{i-1}\,\frac{1}{i+1} = \frac{i-1}{-2} = \frac{1}{2} - \frac{i}{2}$$

$$\frac{i-1}{2}v + \frac{1}{i}w = \begin{pmatrix} \frac{i-1}{2} + \frac{1}{i}(2-2i) \\ (1+i)\cdot\frac{(i-1)}{2} + \frac{1}{(1-i)i} \\ \frac{i-1}{\cancel{2}}\cdot\cancel{2}e^{i\pi/4} - \frac{5}{i} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{i}{2} - \frac{1}{2} + \frac{2}{i} - 2 \\ -1 + \frac{1}{i+1} \\ -\sqrt{2} + 5i \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{3}{2}i - \frac{5}{2} \\ -\frac{1}{2} - \frac{i}{2} \\ -\sqrt{2} + 5i \end{pmatrix}$$

$$e^{i\pi/4} = \frac{1}{\sqrt{2}}(i+1)$$

$$(i-1)\frac{1}{\sqrt{2}}\cdot(i+1)$$

$$= \frac{-2}{\sqrt{2}} = -\sqrt{2}$$

b) $$\langle v | w \rangle = \sum_{i=1}^{d} \overline{v_i}\, w_i$$

$$= 1\cdot(2-2i) + (1-i)\cdot(1-i)^{-1} + 2e^{-i\pi/4}\cdot(-5)$$

$$= 2(1-i) + 1 - 5\cdot 2\cdot\Big(\underbrace{\cos(-\pi/4)}_{=\cos\pi/4 = 1/\sqrt{2}} + i\underbrace{\sin(-\pi/4)}_{=-\sin(\pi/4)}\Big)$$

$$-\cos \pi/4 = 1/\sqrt{2} \qquad = -\frac{1}{\sqrt{2}}$$

$$= 2(1-i) + 1 - 5\sqrt{2}\,(1-i)$$

$$= 2 - 2i + 1 - 5\sqrt{2} + 5\sqrt{2}\, i$$

$$= (3 + 5\sqrt{2}) + (5\sqrt{2} - 2)\, i$$

c) $\|v\|^2 = \langle v | v \rangle$

$$\|v\|^2 = (1,\ 1-i,\ 2e^{-i\pi/4}) \begin{pmatrix} 1 \\ 1+i \\ 2e^{i\pi/4} \end{pmatrix}$$

$$= 1 \cdot 1 + (1-i)(1+i) + 2e^{-i\pi/4} \cdot 2e^{i\pi/4}$$

$$= 1 + 2 + 4 = 6$$

$$\|w\|^2 = (2+2i)(2-2i) + (1+i)^{-1}(1-i)^{-1} + (-5)^2$$

$$= 4 - 4i^2 + \frac{1}{(1+i)(1-i)} + 25$$

$$= 4 + 4 + \frac{1}{2} + 25 = \frac{67}{2}$$

d) $\langle x | v \rangle = 0$?

$$\langle x | v \rangle = \bar{x}_1 + (1+i)\,\bar{x}_2 + 2e^{i\pi/4}\,\bar{x}_3 \overset{!}{=} 0$$

$$x_3 = \frac{1}{2} e^{i\pi/4} \quad \rightarrow \quad \bar{x}_3 \cdot 2e^{i\pi/4} = 1$$

$$x_2 = (1+i) \quad \rightarrow \quad \bar{x}_2 \cdot (1+i) = (1-i)(1+i) = 2$$

$$\Rightarrow \quad x_1 = -3 \quad \rightarrow \quad \langle x | v \rangle = -3 + 2 + 1 = 0.$$

$$\langle x|w\rangle = \bar{x}_1 \cdot (2-2i) + \frac{1}{1-i} \cdot \bar{x}_2 - 5 \cdot \bar{x}_3 \overset{!}{=} 0$$

$$\bar{x}_1 = 0 \Rightarrow x_1 = 0$$

$$\bar{x}_2 = 1-i \Rightarrow x_2 = 1+i \Rightarrow \frac{1}{1-i} \cdot \bar{x}_2 = 1$$

$$\bar{x}_3 = \frac{1}{5}, \quad x_3 = \frac{1}{5}$$

$$\Rightarrow \langle x|w\rangle = 0 + 1 - 5 \cdot \frac{1}{5} = 0$$

(2) $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 \\ i \end{pmatrix}$

a) $\|v_1\| = \sqrt{\|v_1\|^2} = 1$

$$\|v_2\| = \sqrt{1+1} = \sqrt{2}$$

$$\|v_3\| = \sqrt{1-i^2} = \sqrt{2}$$

$$\tilde{v}_1 = v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\tilde{v}_2 = \frac{1}{\sqrt{2}} v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\tilde{v}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

b) $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$$Av_1 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$Av_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

$$Av_3 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 2+i \\ 1+2i \end{pmatrix}$$

c) $T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix} \; \forall a,b \in \mathbb{C}$. $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} T_{11}a + T_{12}b \\ T_{21}a + T_{22}b \end{pmatrix} \overset{!}{=} \begin{pmatrix} b \\ a \end{pmatrix}$$

$$T_{11} = 0 = T_{22}$$

$$T_{12} = 1 = T_{21}$$

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

d) $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$$A^\dagger = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = A$$

$$\langle v_2 | A v_3 \rangle = (1, 1) \begin{pmatrix} 2+i \\ 1+2i \end{pmatrix} = (2+i) + (1+2i)$$
$$= 3 + 3i$$

$$\langle A^\dagger v_2 | v_3 \rangle = (|Av_2\rangle)^\dagger |v_3\rangle = (3,3) \begin{pmatrix} 1 \\ i \end{pmatrix}$$
$$= 3 + 3i = \langle v_2 | A v_3 \rangle$$

Note: holds independent of dimension. Proof

Step 1: What is the i-th component of $A \cdot w$?

$$(A w)_i = \sum_{j=1}^d A_{ij} w_j$$

$$\langle v | A w \rangle = \sum_{i=1}^d \bar{v}_i \cdot (Aw)_i$$

$$= \sum_{i,j=1}^d \bar{v}_i \cdot A_{ij} \cdot w_j$$

det. from sheet

$$A_{ij} = \overline{\overline{A_{ij}}} = \overline{(A^{+})_{ji}}$$

complex conj. twice

$$= \sum_{i,j} \underbrace{\overline{A^{+}_{ji}}\ \overline{v_i}}_{\overline{(A^{+}v)_j}}\ w_j$$

$$= \sum_j \overline{(A^{+}v)_j}\ w_j = \langle A^{+}v, w \rangle$$

e) $$U = \begin{pmatrix} a & b \\ -e^{i\varphi}\bar{b} & e^{i\varphi}\bar{a} \end{pmatrix} \quad \text{for } \begin{array}{l} e^{i\varphi} = 1 \\ e^{-i\varphi} = 1 \end{array}$$

$$U^{+} = \begin{pmatrix} \bar{a} & -e^{-i\varphi} b \\ \bar{b} & e^{-i\varphi} a \end{pmatrix}$$

$$U^{+} \cdot U = \begin{pmatrix} \bar{a} & -e^{-i\varphi} b \\ \bar{b} & e^{-i\varphi} a \end{pmatrix} \begin{pmatrix} a & b \\ -e^{i\varphi}\bar{b} & e^{i\varphi}\bar{a} \end{pmatrix}$$

$$= \begin{pmatrix} a\bar{a} + b\bar{b}e^{i\varphi}e^{-i\varphi} & b\bar{a} - \bar{a}b \cdot e^{i\varphi}e^{-i\varphi} \\ a\bar{b} - a\bar{b}e^{-i\varphi}e^{i\varphi} & b\bar{b} + a\bar{a}\underbrace{e^{i\varphi}e^{-i\varphi}}_{= e^{0} = 1} \end{pmatrix}$$

$$= \begin{pmatrix} |a|^2 + |b|^2 & 0 \\ \ldots & \ldots \end{pmatrix}$$

$$\begin{pmatrix} \ \\ 0 & & |a|^2+|b|^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}$$

$$U = U \cdot \mathbb{1} = U \cdot U^\dagger \cdot U \stackrel{!}{=} U$$

$$\rightarrow U \cdot U^\dagger = \mathbb{1}$$

f) $U$ is unitary $\Rightarrow U \cdot U^\dagger = \mathbb{1} = UU^\dagger$

$$\langle U v | U w \rangle = \langle U^\dagger U v | w \rangle = \langle v | w \rangle$$

$$\| U v \| = \sqrt{\langle U v, U v \rangle} = \sqrt{\langle v, v \rangle} = \| v \|$$

3) a) 1. Characteristic Polynomial → eigenvalues

$$\det(A-\lambda \mathbb{1}) = \det\begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix}$$

$$= (2-\lambda)^2 - 1 \overset{!}{=} 0$$

$$(2-\lambda)^2 = 1 \longrightarrow \lambda = 2 \pm 1 \begin{cases} \rightarrow \lambda_1 = 3 \\ \rightarrow \lambda_2 = 1 \end{cases}$$

2. eigenvectors

$$\lambda_1 = 3: \quad \ker\begin{pmatrix} 2-3 & 1 \\ 1 & 2-3 \end{pmatrix} = \ker\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \operatorname{span}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$$

$$\longrightarrow |v_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 1: \quad \ker\begin{pmatrix} 2-1 & 1 \\ 1 & 2-1 \end{pmatrix} = \ker\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= \operatorname{span}\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$$

$$\longrightarrow |v_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

3. Find matrices X & D

$$X = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

unitary: columns normalized

$$X^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$D = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A^{42} = X D X^{-1} X D X^{-1} \cdots X D X^{-1}$$

$$= X \underbrace{D^{42}} X^{-1}$$

$$D^{42} = \begin{pmatrix} 3^{42} & 0 \\ 0 & 1^{42} \end{pmatrix} = \begin{pmatrix} 3^{42} & 0 \\ 0 & 1 \end{pmatrix}$$

$$X D^{42} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3^{42} & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 3^{42} & 1 \\ 3^{42} & -1 \end{pmatrix}$$

$$X D^{42} X^{-1} = \frac{1}{2} \begin{pmatrix} 3^{42} & 1 \\ 3^{42} & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{\phantom{2}} \begin{pmatrix} 3^{42}+1 & 3^{42}-1 \end{pmatrix}$$

$$2 \mid 3^{42}-1 \qquad 3^{42}+1 \,/$$

$$3^{42} = 109\,418\,989\,131\,512\,359\,209$$

b)

Let $\lambda$ be an eigenvalue of $A$

$\rightarrow A|v\rangle = \lambda|v\rangle.$

$\Rightarrow \langle v|A|v\rangle = \lambda\langle v|v\rangle.$

$\circledast$ $\langle v|A^{+} \underset{\uparrow}{=} (A|v\rangle)^{+} \underset{\uparrow}{=} (\lambda|v\rangle)^{+} = \bar{\lambda}\langle v|$

Lecture $\Rightarrow$ $(AB)^{+} = B^{+}A^{+}$

$\langle v|^{+} = |v\rangle$

$|v\rangle$ is eigenvector of $A$

$\langle v|A^{+}|v\rangle = \bar{\lambda}\langle v|v\rangle$

A Hermitian:

$A^{+} = A$

$\bar{\lambda}\langle v|v\rangle = \langle v|A^{+}|v\rangle = \langle v|A|v\rangle = \lambda\langle v|v\rangle$

$\rightarrow \bar{\lambda} = \lambda \rightarrow \lambda$ is real

c) Let $\lambda$ be eigenvalue of $A$:

$A|v\rangle = \lambda|v\rangle$

$B = XAX^{-1} \rightarrow BX = XA$

$BX|v\rangle = XA|v\rangle = X(\lambda|v\rangle) = \lambda \cdot X|v\rangle$

$\Rightarrow B(X|v\rangle) = \lambda \cdot (X|v\rangle)$

$\rightsquigarrow$ any eigenvalue of $A$ is an eigenvalue of $B$ with eigenvector $X|v\rangle$.

d) $U$ unitary $\Rightarrow$ $U$ normal

unitary:

$U^\dagger U = \mathbb{1}$

$U^\dagger \underline{UU^\dagger} = \mathbb{1} U^\dagger = U^\dagger = U^\dagger \underline{\mathbb{1}} \rightsquigarrow UU^\dagger = \mathbb{1}$

(follows from $U^\dagger U = \mathbb{1}$)

$\Rightarrow U^\dagger U = \mathbb{1} = UU^\dagger$

normal

e) $\langle v| A^\dagger A |v\rangle \underset{\uparrow}{=} \langle v|\mathbb{1}|v\rangle = \langle v|v\rangle$

$A^\dagger A = \mathbb{1}$ unitary

$\langle v|A^\dagger A|v\rangle = \langle v|A^\dagger|v\rangle \cdot \lambda$

$\overset{*}{=} \langle v|v\rangle \cdot \bar{\lambda} \cdot \lambda$

$= \langle v|v\rangle \cdot |\lambda|^2$

$\Rightarrow \langle v|v\rangle = |\lambda|^2 \cdot \langle v|v\rangle$

$\Rightarrow |\lambda|^2 = 1$