Exercise Sheet 2 - solutions

1 a) $\underline{i \Rightarrow ii}$

$$\mathrm{tr}(\rho) = \mathrm{tr}\left(\sum_{i=1}^{n} p_i |\psi_i\rangle\langle\psi_i|\right) = \sum_{i=1}^{n} p_i \underbrace{\mathrm{tr}\, |\psi_i\rangle\langle\psi_i|}_{=1}$$

$$= \sum_i p_i = 1.$$

$$\rho \geq 0 \iff \langle\psi|\rho|\psi\rangle \geq 0$$

$$\langle\psi|\rho|\psi\rangle = \langle\psi| \sum_{i=1}^{n} p_i |\psi_i\rangle\langle\psi_i|\psi\rangle$$

$$= \sum_i p_i \langle\psi|\psi_i\rangle\langle\psi_i|\psi\rangle$$

$$= \sum_i \underbrace{p_i}_{\geq 0} \underbrace{|\langle\psi|\psi_i\rangle|}_{\text{absolute value}, \geq 0} \geq 0$$

$\underline{ii \Rightarrow i}$ $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$

using that ρ is diagonalizable

can always write this, but conditions on p must not necessarily be true.
Now we show that conditions on p follow from ii.

$$1 = \mathrm{tr}(\rho) = \mathrm{tr}\left(\sum_i p_i |\psi_i\rangle\langle\psi_i|\right) = \sum_i p_i$$

$$0 \leq \langle\psi|\rho|\psi\rangle = \sum_i p_i |\langle\psi_i|\psi\rangle| \quad \forall |\psi\rangle$$

this includes e.g. $|\psi\rangle = |\psi_i\rangle \;\forall i$
$\rightsquigarrow p_i \geq 0$.

b) $$\mathrm{Tr}(\rho^2) \underset{\uparrow}{=} \mathrm{Tr}(|\psi\rangle\langle\psi|)^2 = \mathrm{Tr}(|\psi\rangle\langle\psi|) = 1$$

ρ pure

$$\mathrm{Tr}(\rho^2) = 1 \Rightarrow \mathrm{Tr}\left(\sum_{ij} p_i p_j |\psi_i\rangle \overbrace{\langle\psi_i|\psi_j\rangle}^{\substack{1 \text{ if } i=j \\ 0 \text{ if } i\neq j}} \langle\psi_j|\right)$$

$$= \mathrm{Tr}\left(\sum p_i^2 |\psi_i\rangle\langle\psi_i|\right)$$

$= \mathrm{Tr}\left(\sum_i p_i |\psi_i\rangle\langle\psi_i|\right)$

$= \sum_i p_i^2 \underbrace{\mathrm{Tr}\, |\psi_i\rangle\langle\psi_i|}_{\wedge}$

$= \sum_i p_i^2 \overset{!}{=} 1$

$0 \leq p_i \leq 1 \rightsquigarrow$ One $p_i = 1$, the rest 0.

$p_i^2 \leq p_i \rightsquigarrow \sum p_i^2 \leq \sum p_i$

equals only if $p_i^2 = p_i$

2 a) $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

Measurement: set $\{M_0, M_1, \dots\}$

projective measurement:

e.g. Measuring in $|0/1\rangle$ basis: $M_0 = |0\rangle\langle 0|$, $M_1 = |1\rangle\langle 1|$

$p(\text{outcome } j) = \mathrm{tr}(M_j |\psi\rangle\langle\psi|) = |\langle j|\psi\rangle|^2$

state after meas: $|\psi_j\rangle = \frac{M_j |\psi\rangle}{\sqrt{p(j)}} = |j\rangle$

e.g. Measuring in $|+/-\rangle$ basis: $M_0 = |+\rangle\langle +|$, $M_1 = |-\rangle\langle -|$

First measurement:

If $i=0$: $|\psi_0\rangle = |0\rangle$ state after 1st measurement

$i=1$ $|\psi_1\rangle = |1\rangle$

(i) If $i=0$, measure:

states after 2nd measurement

$j=0$ w/ prob. $|\langle 0|\psi_0\rangle|^2 = |\langle 0|0\rangle|^2 = 1 \rightarrow |\psi_0\rangle = |0\rangle$

$j=1$ w/ prob $|\langle 1|\psi_0\rangle|^2 = |\langle 1|0\rangle|^2 = 0 \rightsquigarrow |\psi_1\rangle = |1\rangle$

If $i=1$, measure

$j=0$ w/ prob. $|\langle 0|\psi_1\rangle|^2 = 0 \rightsquigarrow |\psi_0\rangle = |0\rangle$

$j=1$ w\ prob $|\langle 1|\psi_1\rangle|^2=1 \rightsquigarrow |\psi_1\rangle=|1\rangle$

$$\left|\tfrac{1}{\sqrt{2}}(\langle 0|+\langle 1|)\,|0\rangle\right|^2=\left|\tfrac{1}{\sqrt{2}}\langle 0|0\rangle\right|^2=\tfrac{1}{2}$$

(ii) If $i=0$, measure

$j=0$ w\ prob $|\langle +|\psi_0\rangle|^2=\frac{1}{2} \rightsquigarrow |\psi_0\rangle=|+\rangle$

$j=1$ w\ prob $|\langle -|\psi_0\rangle|^2=\frac{1}{2} \rightsquigarrow |\psi_1\rangle=|-\rangle$

If $i=1$, measure

$j=0$ w\ prob $|\langle +|\psi_1\rangle|^2=\frac{1}{2} \rightarrow |\psi_0\rangle=|+\rangle$

$j=1$ w\ prob $|\langle -|\psi_1\rangle|^2=\frac{1}{2} \rightsquigarrow |\psi_1\rangle=|-\rangle$

b) $|\psi\rangle\in\mathbb{C}^d$

$$\mathcal{M}=\{M_0,\dots M_{d-1}\}$$

$$|\psi_i\rangle=\frac{M_i\,|\psi\rangle}{\sqrt{p(i)}}$$

$$|\psi_j\rangle=\frac{M_j\,|\psi_i\rangle}{\sqrt{p(j)}}$$

$$=\frac{M_j\,M_i\,|\psi\rangle}{\sqrt{p(j)}}$$

projective measurements: orthogonal

$$M_jM_i=\begin{cases}M_i & \text{if } i=j\\ 0 & \text{else}\end{cases}$$

$\rightarrow$ result $i=j$,

and state after 2nd measurement,

bc $M_i^2=M_i \rightsquigarrow |\psi_j\rangle=|\psi_i\rangle$

③ Q can flip & flip
do nothing & flip
flip & do nothing
do nothing & do nothing

If Q does flip & flip, Q only wins if Picard does nothing. If Picard flips with prob. p, he would win with prob 1-p here.

Q flips & flips → Q wins if P doesn't flip → prob 1-p

Q flips & does nothing → Q wins if P flips → prob p

Q does nothing & flips → Q wins if P flips → prob p

Q does nothing & does nothing → Q wins if P does nothing → prob 1-p.

To make it fair, the winning probability should be $\frac{1}{2}$, no matter what Q chooses

50% chance of Q winning, 50% chance of P winning

This is the case for $p = 1/2$.

b) heads up → system in state $|0\rangle$

tails up → system in state $|1\rangle$

flipping the coin: applying X : $X|0\rangle = |1\rangle$ ✓
$X|1\rangle = |0\rangle$

doing nothing: applying $\mathbb{1}$: $\mathbb{1}|0\rangle = |0\rangle$
$\mathbb{1}|1\rangle = |1\rangle$ ✓

measurement in computational basis:

measurement in computational basis:

$\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$

measures which basis state it is in

$\rightarrow$ reveals if the "coin" shows heads or tails

$\Rightarrow$ The game can be modelled by a measurement in computational basis on the state

$$Q_2 \circ P \circ Q_1 \, |0\rangle$$

$$\text{with } Q_2, P, Q_1 \in \{X, \mathbb{1}\}.$$

Rechecking the cases from before:

If Q flips & flips:

If Picard flips: $X X X |0\rangle = X |0\rangle = |1\rangle$

$\rightarrow$ the measurement result will be tails, Q looses

If Picard doesn't flip:

$$X \mathbb{1} X |0\rangle = |0\rangle$$

$\rightarrow$ the measurement result is heads, Q wins

c) Situation: $U P V |0\rangle$

Q wants to win in any scenario

$\rightarrow$ he wants to find U & V s.t.

$$U X \boxed{V |0\rangle} = |0\rangle$$

<u>and</u> $$U \mathbb{1} \boxed{V |0\rangle} = |0\rangle$$

Note: What could be a state $|\psi\rangle = V|0\rangle$

s.t. $X|\psi\rangle = \mathbb{1}|\psi\rangle = |\psi\rangle$?

$\Rightarrow$ eigenstate of X with eigenvalue 1.

$$|+\rangle = \frac{1}{\sqrt{2}} |0\rangle + |1\rangle$$

$$X|+\rangle = |+\rangle \checkmark$$

$$\mathbb{1}|+\rangle = |+\rangle \checkmark$$

$\Rightarrow$ want V to map $|0\rangle$ to $|+\rangle$.

This is achieved by Hadamard matrix

$$H = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(|+\rangle\langle 0| + |-\rangle\langle 1|\right)$$

$\Rightarrow$ Select $V = H$.

$$XH|0\rangle = |+\rangle$$

$$\mathbb{1}\, H|0\rangle = |+\rangle$$

$\Rightarrow$ To get measurement result "heads" every time, must map $|+\rangle$ back to $|0\rangle$ with unitary U.

This is done by the inverse of H, which is $H^{-1} = H$ (because $H^2 = 0$, see 2d)

$\Rightarrow$ select $U = H$.

$\Rightarrow$ Q wins with certainty if he performs H at every step.

d) Q wants a unitary U, s.t. $XUX|0\rangle = |0\rangle$

$$XU\mathbb{1}\,|0\rangle = |0\rangle$$

$$\mathbb{1}\,UX|0\rangle = |0\rangle$$

$$\mathbb{1}\, U\, \mathbb{1}\, |0\rangle = |0\rangle$$

can we find a U such that this always gives $|0\rangle$?

No. For any U, Picard can select to flip or not flip the coin in the final step.

He would need a state $|\psi\rangle$ s.t.

$\mathbb{1}|\psi\rangle = |0\rangle \rightarrow$ only true for $|\psi\rangle = |0\rangle$

$X|\psi\rangle = |0\rangle \rightarrow$ only true for $|\psi\rangle = |1\rangle$ $\lightning$

$\rightarrow$ such a $|\psi\rangle$ does not exist.

(4) $U|\psi\rangle = |0\rangle \quad \forall |\psi\rangle$

$|\psi\rangle = |0\rangle \rightsquigarrow U = \mathbb{1}$

$|\psi\rangle = |1\rangle \rightsquigarrow U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$U(\alpha|0\rangle + \beta|1\rangle) = \begin{pmatrix} U_{00} & U_{01} \\ U_{10} & U_{11} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$= \begin{pmatrix} U_{00}\alpha + U_{01}\beta \\ U_{10}\alpha + U_{11}\beta \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\rightarrow U_{10} = U_{11} = 0.$