Exercise 3.

a) $f_s(x) = x \cdot s = \sum_{i=0}^{n-1} x_i \cdot s_i \mod 2$

$\Rightarrow \sum_{i=0}^{2} x_i \cdot s_i = x_0 \cdot s_0 \oplus x_1 \cdot s_1 \oplus x_2 \cdot s_2$

$x =$	$x_2 \oplus x_3$	$x_0 \cdot s_0 \oplus x_1 \cdot s_1 \oplus x_2 \cdot s_2$
0 0 0	$0 \oplus 0 = 0$	$0 \oplus 0 \oplus 0 = 0$
0 0 1	$0 \oplus 1 = 1$	$0 \oplus 0 \oplus 1 = 1$
0 1 0	$1 \oplus 0 = 1$	$0 \oplus 1 \oplus 0 = 1$
1 0 0	$0 \oplus 0 = 0$	$0 \oplus 0 \oplus 0 = 0$
1 0 1	$0 \oplus 1 = 1$	$0 \oplus 0 \oplus 1 = 1$
1 1 0	$1 \oplus 0 = 1$	$0 \oplus 1 \oplus 0 = 1$
1 1 1	$1 \oplus 1 = 0$	$0 \oplus 1 \oplus 1 = 0$

$x =$	$x_1 \oplus x_2$	$x_0 \cdot s'_0 \oplus x_1 \cdot s'_1 \oplus x_2 \cdot s'_2$	
0 0 0	$0 \oplus 0 = 0$	$0 \oplus 0 \oplus 0 = 0$	$\Rightarrow s'_2 = 0.$
0 0 1	$0 \oplus 0 = 0$	$0 \oplus 0 \oplus s'_2 = 0$	$\Rightarrow s'_1 = 1$
0 1 0	$0 \oplus 1 = 1$	$0 \oplus s'_1 \oplus 0 = 1$	$\Rightarrow s'_0 = 1$
1 0 0	$1 \oplus 0 = 1$	$s'_0 \oplus 0 \oplus 0 = 1$	$\Rightarrow 0 \oplus 0 \oplus 0 = 1$ ✓
1 0 1	$1 \oplus 0 = 1$	$s'_0 \oplus 0 \oplus s'_2 = 1$	$\Rightarrow 1 \oplus 1 \oplus 0 = 0$ ✓
1 1 0	$1 \oplus 1 = 0$	$s'_0 \oplus s'_1 \oplus 0 = 0$	$\Rightarrow 1 \oplus 1 \oplus 0 = 0$ ✓
1 1 1	$1 \oplus 1 = 0$	$s'_0 \oplus s'_1 \oplus s'_2 = 0$	

$\Rightarrow s' = 110.$

✓

b) $U_{f_s}|x\rangle|a\rangle = |x\rangle \otimes |a \oplus f_s\rangle$

if $f_s = 0, \Rightarrow |a \oplus 0\rangle.$ if $f_s = 1, \Rightarrow |a \oplus 1\rangle.$

$\Rightarrow \begin{cases} \text{if } a = 0, & |0 \oplus 0\rangle = |0\rangle \Rightarrow |x\rangle \otimes |0\rangle. \\ \text{if } a = 1, & |1 \oplus 0\rangle = |1\rangle \Rightarrow |x\rangle|1\rangle, \end{cases}$ $\Rightarrow \begin{cases} \text{if } a = 0, & |0 \oplus 1\rangle = |1\rangle \Rightarrow |x\rangle|1\rangle \\ \text{if } a = 1, & |1 \oplus 1\rangle = |0\rangle \Rightarrow |x\rangle|0\rangle \end{cases}$

$\therefore U_{f_s}|x\rangle|a\rangle = |x\rangle \otimes |a \oplus f_s\rangle = |x\rangle|0\rangle$ or $|x\rangle|1\rangle.$

with $x \in \{0,1\}^n.$

c) $|\psi_0\rangle = |0\rangle^{\otimes n}|1\rangle.$

$|\psi_1\rangle = H^{\otimes n}|0\rangle^{\otimes n} \cdot H|1\rangle = |+\rangle^{\otimes n} \cdot |-\rangle.$

$|\psi_2\rangle = U_{f_s}|+\rangle^{\otimes n} \cdot |-\rangle, = U_{f_s}|+\rangle^{\otimes n} \cdot \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}}(|+\rangle^{\otimes n}|0 \oplus f_s\rangle - |+\rangle^{\otimes n}|1 \oplus f_s\rangle)$

if $f_s = 0$, then $\frac{1}{\sqrt{2}}(|+\rangle^{\otimes n}|0\rangle - |+\rangle^{\otimes n}|1\rangle) = |+\rangle^{\otimes n}|-\rangle.$ $\Big\} \Rightarrow |+\rangle^{\otimes n}(-1)^{f_s}|-\rangle.$

if $f_s = 1$, then $\frac{1}{\sqrt{2}}(|+\rangle^{\otimes n}|1\rangle - |+\rangle^{\otimes n}|0\rangle) = |+\rangle^{\otimes n}(-1)|-\rangle$

$|\psi_3\rangle = H^{\otimes n}|+\rangle^{\otimes n}(-1)^{f_s}|-\rangle = |0\rangle^{\otimes n}(-1)^{f_s}|-\rangle.$

Exercise 2.

a) $\rho(r) = \frac{I}{2} + \frac{1}{2}(r_1 X + r_2 Y + r_3 Z) = \frac{1}{2}(I + r_1 X + r_2 Y + r_3 Z)$.

$$= \frac{1}{2}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & r_1 \\ r_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -ir_2 \\ ir_2 & 0 \end{bmatrix} + \begin{bmatrix} r_3 & 0 \\ 0 & -r_3 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 1 + r_3 & r_1 - r_2 i \\ r_1 + r_2 i & 1 - r_3 \end{bmatrix}.$$

~~$Tr(\rho(r)) = \frac{1}{2}(1 + r_3 + 1 - r_3) = 1$~~

$$\rho(r)^2 = \frac{1}{2}\begin{bmatrix} 1 + r_3 & r_1 - r_2 i \\ r_1 + r_2 i & 1 - r_2 \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 1 + r_3 & r_1 - r_2 i \\ r_1 + r_2 i & 1 - r_3 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} (1+r_3)^2 + r_1^2 + r_2^2 & 2(r_1 - r_2 i) \\ 2(r_1 + r_2 i) & (1 - r_3)^2 + r_1^2 + r_2^2 \end{bmatrix}$$

$\therefore \rho(r)^2 \neq \rho$ $\therefore \|r\| < 1$

mix state.

↳ not related to question posed in exercise

b) $\rho(r)' = P\rho(r)P = (r_1 X + r_2 Y + r_3 Z) \cdot \frac{1}{2}(I + r_1 X + r_2 Y + r_3 Z) \cdot (r_1 X + r_2 Y + r_3 Z)$

$= \frac{1}{2}(r_1 X + r_1^2 X^2 + r_1 r_2 XY + r_1 r_3 XZ + r_2 Y + r_1 r_2 XYX + r_2^2 Y^2 + r_2 r_3 YZ + r_3 Z$

$\quad + r_3 r_1 ZX + r_3 r_2 ZY + r_3^2 Z^2)(r_1 X + r_2 Y + r_3 Z)$

$= \frac{1}{2}(r_1 X + r_1^2 + r_1 r_2 XY + r_1 r_3 XZ + r_2 Y + r_1 Y X + r_2^2 + r_2 r_3 YZ + r_3 Z$

$\quad + r_3 r_1 ZX + r_3 r_2 ZY + r_3^2)(r_1 X + r_2 Y + r_3 Z)$

$= \frac{1}{2}(r_1^2 X^2 + r_1 r_2 XY + r_1 r_3 XZ + r_1^3 X + r_1^2 r_2 Y + r_1^2 r_3 Z + r_1^2 r_2 X^* YX + r_1 r_2^2 XY^2$

$\quad + r_1 r_2 r_3 XYZ + r_1^2 r_3 XZX + r_1 r_2 r_3 XZY + r_1 r_3^2 XZ^2 + r_2 r_1 YX + r_2^2 Y^2 + r_2 r_3 YZ$

$\quad + r_1^2 YX^2 + r_1 r_2 YXY + r_1 r_3 YXZ + r_2^2 r_1 X + r_2^3 Y + r_2^2 r_3 Z + r_2 r_3 r_1 YZX$

$\quad + r_2^2 r_3 YZY + r_2 r_3^2 YZ^2 + r_3 r_1 ZX + r_3 r_2 ZY + r_3^2 Z^2 + r_3 r_1^2 ZX^2 + \cancel{r_3}$

$\quad r_3 r_1 r_2 ZXY + r_3^2 r_1 ZXZ + r_3 r_2 r_1 ZYX + r_3 r_2^2 ZY^2 + r_3^2 r_2 ZYZ + r_3^2 r_1 X$

$\quad + r_3^2 r_2 Y + r_3^3 Z)$

$= \frac{1}{2}(r_1^2 + r_1 r_2 XY + r_1 r_3 XZ + r_1^3 X + r_1^2 r_2 Y + r_1^2 r_3 + r_1^2 r_2(-Y) + r_1 r_2^2 X$

$\quad + r_1 r_2 r_3 i + r_1^2 r_3(-1) + r_1 r_3 r_2(-XY) + r_1 r_3^2 X + r_2 r_1 YX + r_2^2 + r_2 r_3 YZ$

$\quad + r_1^2 Y + r_1 r_2(-X) + r_1 r_3(-i) + r_2^2 r_1 X + r_2^3 Y + r_2^2 r_3 Z + r_2 r_3 r_1 i$

$\quad + r_2^2 r_3(-Z) + r_2 r_3^2 Y + r_3 r_1 ZX + r_3 r_2 ZY + r_3^2 + r_3 r_1^2 Z + r_3 r_1 r_2 XY$

$\quad + r_3^2 r_1(-X) + r_3 r_2 r_1(-i) + r_3 r_2^2 Z + r_3^2 r_2(Y) + r_3^2 r_1 X + r_3^2 r_2 Y + r_3^3 Z)$

$= \frac{1}{2}[r_1^2(1 + Y) + r_2^2 + r_3^2 + r_1 r_2(XY + YX - X) + r_1 r_3(XZ - i + ZX) + r_1^3 X$

$\quad + r_1^2 r_2(Y - Y) + r_1^2 r_3(-1 + Z) + r_1 r_2^2(X + X) + r_1 r_2 r_3(i - XY + XY + i - i)$

$\quad + r_1 r_3^2(X + X - X) + r_2 r_3(YZ + ZY) + r_2^3 Y + r_2^2 r_3(Z - Z + Z) + r_3 r_2^2(Y + YZ + Y) + r_3^3 Z]$

$$= \frac{1}{2}\left[r_1^2(1+Y) + r_2^2 + r_3^2 + r_1 r_2(-X) + r_1 r_3(-i) + r_1^3 x + r_1^2 r_3 z + r_1 r_2^2(2X) \right.$$

$$\left. + r_1 r_2 r_3 i + r_1 r_3^2 x + r_2^3 Y + r_2^2 r_3 z + r_2 r_3^2(2Y+Yz) + r_3^3 z \right]$$

$$= \frac{1}{2}\left[r_1^2 + r_2^2 + r_3^2 + r_1^3 x + r_2^3 Y + r_3^3 z + r_1^2 Y - r_1 r_2 x - r_1 r_3 i + r_1 r_2^2 \cdot 2x \right.$$

$$\left. + r_1 r_3^2 x + r_1^2 r_3 z + r_2 r_3^2 \cdot 2Y + r_2 r_3^2 Yz + r_2^2 r_3 z + r_1 r_2 r_3 i \right).$$

$\textcolor{red}{\hookrightarrow \text{it is impossible to follow and read off result}}$

c) $\rho(r') = H \rho(r) H$

$$= \frac{1}{\sqrt{2}}(x+z) \cdot \frac{1}{2}(I + r_1 x + r_2 Y + r_3 z) \cdot \frac{1}{\sqrt{2}}(x+z)$$

$$= \frac{1}{4}(x + r_1 x^2 + r_2 xY + r_3 xz + z + r_1 zx + r_2 zY + r_3 z^2)(x+z)$$

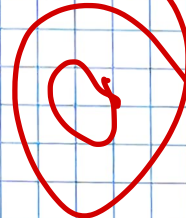$$= \frac{1}{4}(x^2 + \cancel{xz} + xz + r_1 x^2 \cdot x + r_1 x^2 z + r_2 xYx + r_2 xYz + r_3 xzx + r_3 xz^2 + zx + z^2$$

$$+ r_1 zx^2 + r_1 zxz + r_2 zYx + r_2 zYz + r_3 z^2 x + r_3 z^2 z)$$

$$= \frac{1}{4}(1 + xz + \cancel{r_1 X} + r_1 z + r_2(-Y) + r_2 i + r_3(-1) + r_3 x + zx + 1 + r_1 z$$

$$+ r_1(-x) + r_2(-i) + r_2 z + r_3 x + r_3 z)$$

$$= \frac{1}{4}(2 + xz + zx + r_1 x + r_1 z - r_1 x + r_1 z - r_2 Y + r_2 i - r_2 i + r_2 Yz - r_3$$

$$+ r_3 x + r_3 x + r_3 z)$$

$$= \frac{1}{4}(2 + \cancel{xz + zx} + 2 r_1 z - r_2 Y + r_2 Yz - r_3 + 2 r_3 x + r_3 z).$$

$\textcolor{red}{\hookrightarrow \text{Same}}$

$\textcolor{red}{\bigcirc}$

Exercise 1

1.a)$\rho_B = \text{Tr}_A(|\psi^+\rangle\langle\psi^+|)$ $\qquad$ $|\psi^+\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$\qquad = \sum_{i,j,k=0}^{1} (\langle i|\otimes 1) \frac{1}{2}(|j\rangle\langle k|)(|j\rangle\langle k|)(|i\rangle\otimes 1)$ $\qquad = \frac{1}{\sqrt{2}} \sum_{j=0}^{1} |j,j\rangle$

$\qquad = \frac{1}{2} \sum_{i,j,k=0}^{1} [(\langle i|j\rangle\langle k|)\otimes(1\otimes|j\rangle\langle k|)](|i\rangle\otimes 1)$ $\qquad |\psi^+\rangle\langle\psi^+| = \frac{1}{2}\sum_{j,k=0}^{1} |j,j\rangle\langle k,k|$

$\qquad = \frac{1}{2} \sum_{i,j,k=0}^{1} (\langle i|j\rangle\langle k|1\rangle)(1\otimes|j\rangle\langle k|\otimes 1)$ $\qquad = \frac{1}{2}\sum_{j,k=0}^{1} (|j\rangle\langle k|)\otimes(|j\rangle\langle k|)$

$\qquad = \frac{1}{2} \sum_{i,j,k=0}^{1} \delta_{ij}\delta_{ik} |i\rangle\langle i|$

$\qquad = \frac{1}{2} \sum_{i=0}^{1} |i\rangle\langle i|$

$\qquad = \frac{1}{2}\cdot 1$

✓ (0.5)

b) $b \in \{0,1\}$.
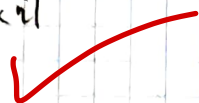
$\qquad H^b \in \{+,-\}. \Rightarrow \{\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\}$

$\qquad$ ~~contr.~~ $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$|\psi'\rangle = H|\psi^+\rangle = \frac{1}{\sqrt{2}}(H|00\rangle + H|11\rangle)$

$\qquad = \frac{1}{\sqrt{2}}[\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) + \frac{1}{\sqrt{2}}(|01\rangle - |11\rangle)]$

$\qquad = \frac{1}{2}(|00\rangle + |10\rangle + |01\rangle - |11\rangle)$

$|\psi'\rangle\langle\psi'| = \frac{1}{2}(|00\rangle + |10\rangle + |01\rangle - |11\rangle)\cdot\frac{1}{2}(\langle 00| + \langle 10| + \langle 01| - \langle 11|)$

$\qquad = \frac{1}{4}(|00\rangle\langle00| + |00\rangle\langle10| + |00\rangle\langle01| + |00\rangle\langle11| + |10\rangle\langle00| + |10\rangle\langle10| + |10\rangle\langle01|$

$\qquad\qquad + |10\rangle\langle11| + |01\rangle\langle00| + |01\rangle\langle10| + |01\rangle\langle01| + |01\rangle\langle11| - |11\rangle\langle00|$

$\qquad\qquad - |11\rangle\langle10| - |11\rangle\langle01| + |11\rangle\langle11|)$

$\qquad = \frac{1}{4}(|0\rangle\langle0|\otimes|0\rangle\langle0| + |0\rangle\langle1|\otimes|0\rangle\langle0| + |0\rangle\langle0|\otimes|0\rangle\langle1| + |0\rangle\langle1|\otimes|0\rangle\langle1| + |1\rangle\langle0|\otimes|0\rangle\langle0|$

$\qquad\qquad + |1\rangle\langle1|\otimes|0\rangle\langle0| + |1\rangle\langle0|\otimes|0\rangle\langle1| + |1\rangle\langle1|\otimes|0\rangle\langle1| + |0\rangle\langle0|\otimes|1\rangle\langle0| + |0\rangle\langle1|\otimes|1\rangle\langle0|$

$\qquad\qquad + |0\rangle\langle0|\otimes|1\rangle\langle1| + |0\rangle\langle1|\otimes|1\rangle\langle1| - |1\rangle\langle0|\otimes|1\rangle\langle0| - |1\rangle\langle1|\otimes|1\rangle\langle0| - |1\rangle\langle0|\otimes|1\rangle\langle1|$

$\qquad\qquad + |1\rangle\langle1|\otimes|1\rangle\langle1|)$

$\qquad = \frac{1}{4}(|0\rangle\langle0| + |0\rangle\langle1| + |1\rangle\langle0| + |1\rangle\langle1| + |0\rangle\langle0| + |0\rangle\langle1| - |1\rangle\langle0| + |1\rangle\langle1|)$

$\qquad = \frac{1}{2}(|0\rangle\langle0| + |0\rangle\langle1| + |1\rangle\langle1|)$

$\text{Tr}_A(|\psi'\rangle\langle\psi'|) = \frac{1}{2}\cdot 2 = 1$ ✗

↳ not correct