Lecture 6.2: Stabilizers & Steane Code

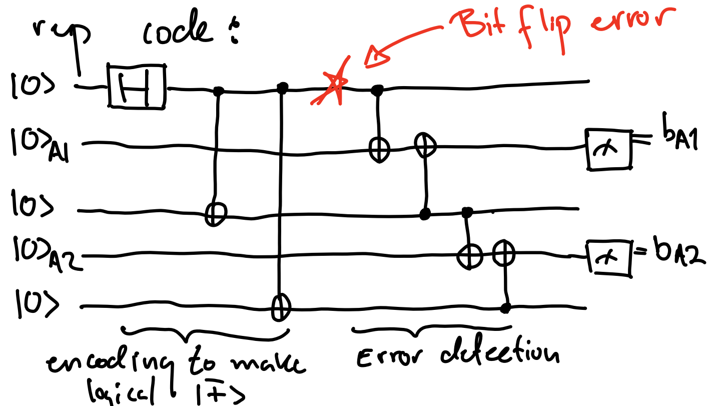
Schedule

- $9^{15} - 9^{30}$: Recap
- $9^{30} - 9^{50}$: Diagnosing general errors
- $9^{50} - 10^{10}$: General view of Stabilizers
- $10^{20} - 10^{40}$: Steane code!
- $10^{40} - 11^{00}$: Logical ops Steane
- *$11^{15} - 11^{30}$: Fault tolerance
- *$11^{30} - 11^{45}$: Large scale QEC

*/ Not in reading

Reading: Mermin S. 4, S. 6, S. 7

Recap from monday: 3-qubit rep code:



- Logical states:

$$|\bar{0}\rangle = |000\rangle, \quad |\bar{1}\rangle = |111\rangle$$

- Stabilizers: $\{Z_2 Z_1, Z_1 Z_0\}$
- Corrects one bit flip error

	no error	Q_0	Q_1	Q_2
b_{A1}	0	1	1	0
b_{A2}	0	0	1	1

- We saw: $Z_2 Z_1 |\bar{\Psi}\rangle = Z_1 Z_0 |\bar{\Psi}\rangle = +1 |\bar{\Psi}\rangle$ if $|\bar{\Psi}\rangle$ error free
- Lets add error: $X_0 |\bar{\Psi}\rangle = |\bar{\Psi}_0\rangle$. Recall: $X_i Z_j = Z_j X_i$ if $i \neq j$; $X_i Z_j = -Z_j X_i$ if $i = j$

$$S_1: Z_1 Z_0 |\bar{\Psi}_0\rangle = Z_1 Z_0 X_0 |\bar{\Psi}\rangle = -Z_1 X_0 Z_0 |\bar{\Psi}\rangle = -X_0 Z_1 Z_0 |\bar{\Psi}\rangle = -X_0 |\bar{\Psi}\rangle = -1 |\bar{\Psi}_0\rangle$$

$$S_2: Z_2 Z_1 |\bar{\Psi}_0\rangle = Z_2 Z_1 X_0 |\bar{\Psi}\rangle = \underline{\qquad\qquad} = \underline{\qquad\qquad} = \underline{\qquad} = +1 |\bar{\Psi}_0\rangle$$

	$\mathbb{1}$	X_0	X_1	X_2
$S_1 = Z_1 Z_0$	+1	−1	−1	+1
$S_2 = Z_2 Z_1$	+1	+1	−1	−1

- Lets try a Z_0 error: $Z_0 |\bar{\Psi}\rangle = |\bar{\Psi}_z\rangle$:

$$S_1: Z_1 Z_0 |\bar{\Psi}_z\rangle = Z_1 Z_0 Z_0 |\bar{\Psi}\rangle = +Z_1 Z_0 Z_0 |\bar{\Psi}\rangle = +Z_0 Z_1 Z_0 |\bar{\Psi}\rangle = +Z_0 |\bar{\Psi}\rangle = +|\bar{\Psi}_z\rangle$$

$$S_2: Z_2 Z_1 |\bar{\Psi}_z\rangle = Z_2 Z_1 Z_0 |\bar{\Psi}\rangle = \underline{\qquad\qquad} = \underline{\qquad\qquad} = \underline{\qquad} = +|\bar{\Psi}_z\rangle$$

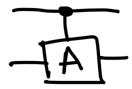
- CANNOT detect Z errors!
- General state with errors:

$$\boxed{|e\rangle |\bar{\Psi}\rangle \longmapsto \left[|e_a\rangle \mathbb{1} + \sum_{i=0}^{n-1} |e_{bi}\rangle Z_i + |e_{ci}\rangle X_i + |e_{di}\rangle Y_i \right] |\bar{\Psi}\rangle}$$

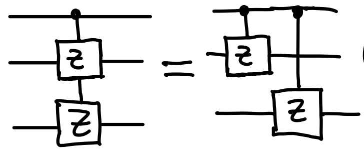
Diagnosing errors (finding „error syndromes")

- Wish to determine general structure of circuits to find an error
- Consider operators of form: $A^2 = \mathbb{1} \Rightarrow$ Eigenvalues of A must be ± 1
- All pauli matrices satisfy these properties
- If A satisfies above criterion, we can form projection operators:

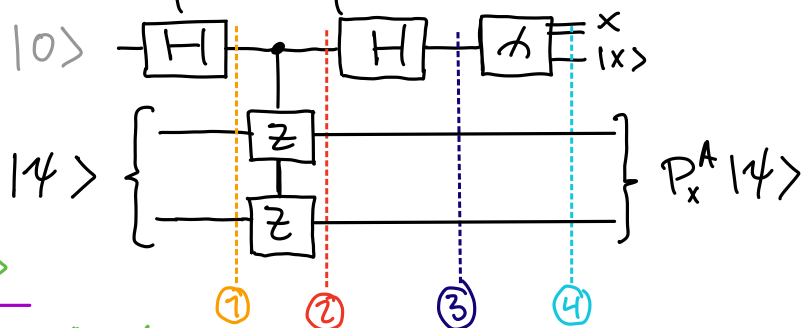
$$P_0^A = \tfrac{1}{2}(\mathbb{1} + A) \quad , \quad P_1^A = \tfrac{1}{2}(\mathbb{1} - A)$$

- Define now „controlled-A", with some ancilla qubit: 

$A: X \Rightarrow CA = CX$ = CNOT = 

$A: Z_1 Z_0 \Rightarrow CZ_1 CZ_0 =$  (Here A was a stabilizer!)

- Consider now the following circuit for $A = Z_1 Z_0$



① $(|0\rangle + |1\rangle) \otimes |\psi\rangle$

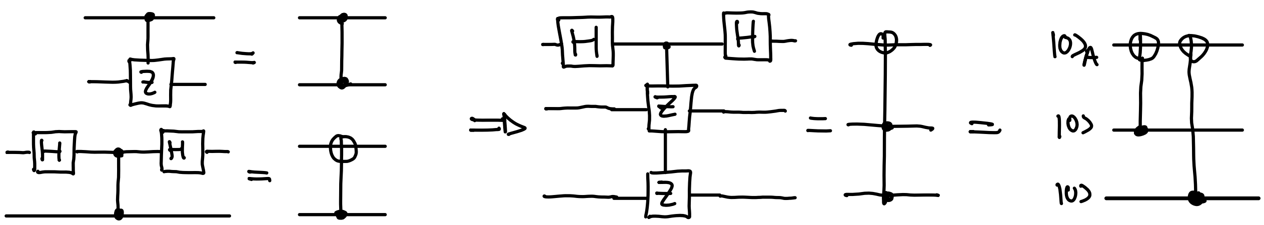
② $|0\rangle \otimes |\psi\rangle + |1\rangle \otimes A|\psi\rangle$

③ $(|0\rangle + |1\rangle)\otimes|\psi\rangle + (|0\rangle - |1\rangle)\otimes A|\psi\rangle = |0\rangle \otimes (|\psi\rangle + A|\psi\rangle) + |1\rangle \otimes (|\psi\rangle - A|\psi\rangle)$

$= |0\rangle \otimes P_0^A|\psi\rangle + |1\rangle \otimes P_1^A|\psi\rangle$

④ Now measure ancilla: if $b=0 \Rightarrow P_0^A|\psi\rangle$: $+1$ eigenstate of A !!
if $b=1 \Rightarrow P_1^A|\psi\rangle$: -1 eigenstate of A !!

- Finally, recall that:



- So the circuit we saw in 3 qubit repetition code was in fact using this whole formalism „behind the scenes"

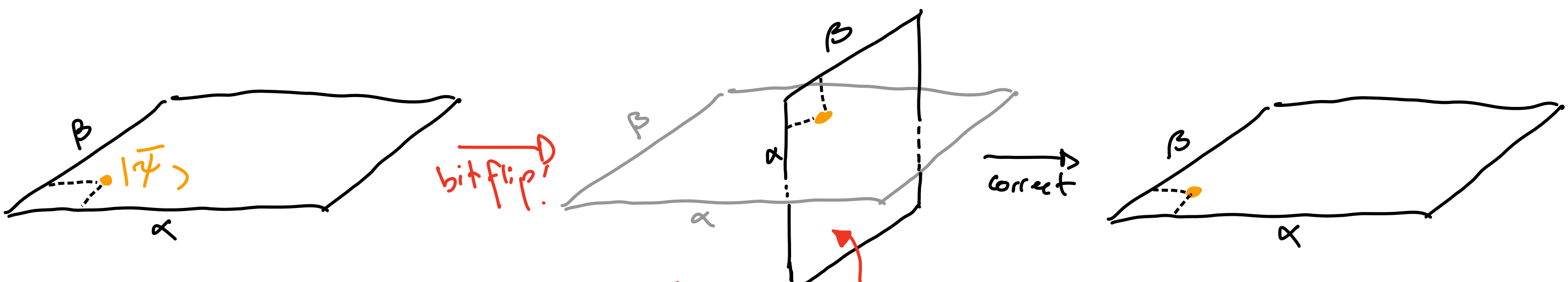
High level of (stabilizer) QEC

- Given A's they define a subspace (via P_x^A).
- We saw that 3 types of errors exist: X, Y, Z
- We need one subspace to be „the logical qubit"
  $\Rightarrow$ We need $1+3n$ orthogonal subspaces
- n qubits: 2^n states, but one should be logical: 2^{n-1}
- We must now choose enough qubits that there is enough subspaces:

$$\boxed{2^{n-1} \geq 1+3n}$$

- Lowest n: 5 : Laflamme–Miguel–Paz '96
  7 : Steane code '96
  9 : Shor code '95
- Consider a general logical qubit: Assuming 3 qubit repetition

$$|\bar{\psi}\rangle = \alpha|\bar{0}\rangle + \beta|\bar{1}\rangle = \alpha|000\rangle + \beta|111\rangle$$



$$S_1|\bar{\psi}\rangle = Z_1 Z_0|\bar{\psi}\rangle = -|\bar{\psi}\rangle$$

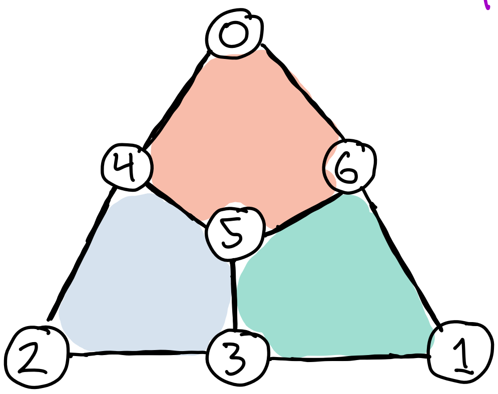
- Stabilizer told you subspace changed, but without revealing where in the subspace the qubit is!
- Some notation: $[[n, k, d]]$-code:
  n: physical qubits, k: logical qubits d: distance
- Number of errors code can correct: $\left\lfloor \frac{d-1}{2} \right\rfloor$ errors

> Repetition code
> $[[3, 1, 3^*]]$ -code with stabilizers $\{Z_1 Z_0, Z_2 Z_1\}$

- With $d=3$ how many errors can rep. code correct? 1^*
- Note: Rep code corrects only bit-flip or phaseflip (not both)

The Steane code: A [[7, 1, 3]] code

- 7 physical 1 logical qubits how many stabilizers? 6



- What are the stabilizers?

$M_0 = X_0 X_4 X_5 X_6$ $N_0 = Z_0 Z_4 Z_5 Z_6$

$M_1 = X_1 X_3 X_5 X_6$ $N_1 = Z_1 Z_3 Z_5 Z_6$

$M_2 = X_2 X_3 X_4 X_6$ $N_2 = Z_2 Z_3 Z_4 Z_6$

- Do the 6 stabilizers satisfy $A^2 = \mathbb{1}$? Yes: $P_i^2 = \mathbb{1}$ and $P_i P_j = P_j P_i$ if $i \neq j$
- Logical states defined via:

$$|\bar{0}\rangle = P_+^{M_0} P_+^{M_1} P_+^{M_2} |0000000\rangle = \frac{1}{\sqrt{2}^3}(\mathbb{1} + M_0)(\mathbb{1} + M_1)(\mathbb{1} + M_2)|0\rangle^{\otimes 7}$$

$$|\bar{1}\rangle = P_+^{M_0} P_+^{M_1} P_+^{M_2} \bar{X} |0000000\rangle,$$

- where $\bar{X}$ is logical bitflip:

$$\bar{X}|\bar{0}\rangle = |\bar{1}\rangle \Longleftrightarrow XXXXXXX|0000000\rangle = |1111111\rangle$$

- Observe: $\langle \bar{0} | \bar{1} \rangle = 0$: $|\bar{0}\rangle$ has even number of $|1\rangle$'s; $|\bar{1}\rangle$ has odd number of $|1\rangle$'s
- A general error in the steane code:

$$|e\rangle|\bar{\Psi}_s\rangle \longmapsto \left(|e_a\rangle \mathbb{1} + \sum_{i=1}^{7} \left[|e_b\rangle X_i + |e_c\rangle Y_i + |e_d\rangle Z_i \right] \right) |\bar{\Psi}_s\rangle$$

- How many possible single-qubit errors can exist? $21 = 7 \times 3$
- How to determine if state $|\bar{\Psi}'\rangle_{\text{Steane}}$ has error? Measure all 6 stabilizers

	$\mathbb{1}$	X_0	X_1	X_2	X_3	X_4	X_5	X_6
N_0	+1	___						
N_1	+1	___						
N_2	+1	___						

	$\mathbb{1}$	Z_0	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6
M_0	+1	___						
M_1	+1	___						
M_2	+1	___						

- An example: Measure all stabilizers and get:

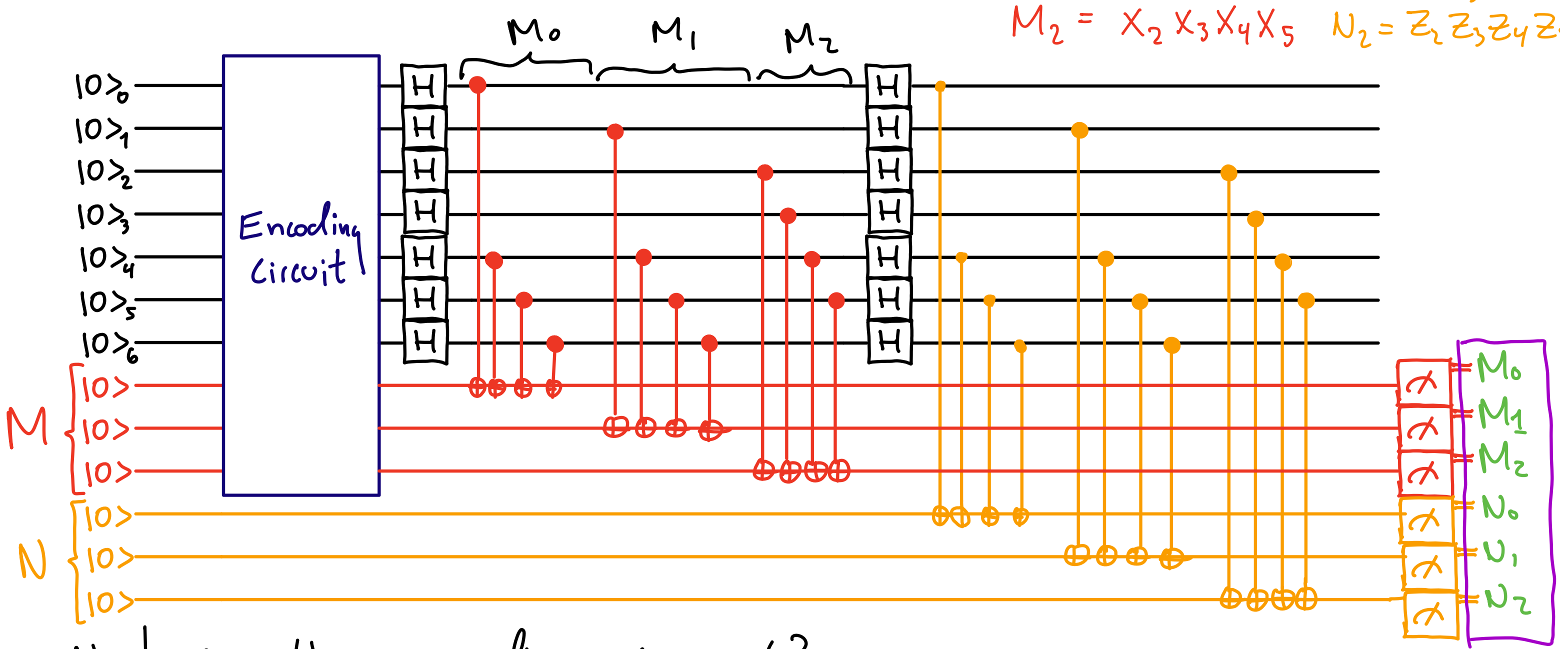
$N_0 = -1$, $N_1 = +1$, $N_2 = -1$
$M_0 = +1$, $M_1 = +1$, $M_2 = +1$ } Which error occurred? X_4

$N_0 = +1$, $N_1 = -1$, $N_2 = +1$
$M_0 = +1$, $M_1 = -1$, $M_2 = +1$ } Which error occurred? $X_1 Z_1 = Y_1$

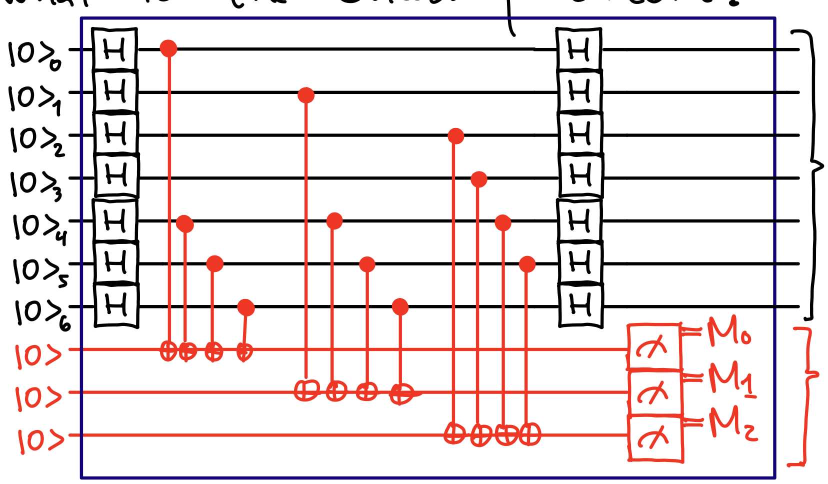
- Stop for a second and think: We are finding errors in a $2^7 = 128$ dimensional Hilbert space.
- Process of determining errors from pattern of ancillas: DECODING
- # of physical qubits needed to implement logical qubit in Steane code?
  7 for logical qubit, 6 to measure every stabilizer

$M_0 = X_0 X_4 X_5 X_6$ $\quad N_0 = Z_0 Z_4 Z_5 Z_6$
$M_1 = X_1 X_3 X_5 X_6$ $\quad N_1 = Z_1 Z_3 Z_5 Z_6$
$M_2 = X_2 X_3 X_4 X_5$ $\quad N_2 = Z_2 Z_3 Z_4 Z_5$

- Full Steane code circuit



- What is the encoding circuit?



$$P_+^{M_2} P_+^{M_1} P_+^{M_0} |0\rangle^{\otimes 7} = \frac{1}{\sqrt{2}^3}(\mathbb{1}+M_0)(\mathbb{1}+M_1)(\mathbb{1}+M_2)|0\rangle^{\otimes 7}$$

If all $M_i = +1$: Have logical state
If some $M_i \neq +1$: Apply Z_i to correct

$\Downarrow$

Left with logical $|0\rangle = |\bar{0}\rangle$

Logical operations in Steane code

- Why did I choose the $n=7$ code? Logical operations easier than $n=5$

Universal gate sets

To do universal QC (meaning „go anywhere on Bloch Sphere") you need a universal gate set:

$G_1 = \{e^{i\varphi}, R_x(\varphi), R_y(\varphi), R_z(\varphi), \text{CNOT}\}$

$G_2 = \{X, Z, H, \text{CNOT}, T\}$ $\quad T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/8} \end{bmatrix}$

$G_3 = \{H, \text{TOFFOLI}\}$ $\quad$ TOFFOLI =

- G_1 is hard to prove stuff with, G_2 is „workhorse", G_3 interesting
- Which Logical operations can be done in Steane code?

Logical X:

$\bar{X} = X^{\otimes 7} = XXXXXXX$: $\bar{X}$ commute with M_i and flibs all bits:

$\bar{X}|\bar{0}\rangle = |\bar{1}\rangle \quad , \quad \bar{X}|\bar{1}\rangle = |\bar{0}\rangle$

Logical Z:

$\bar{Z} = Z^{\otimes 7} = ZZZZZZZ$: $\bar{Z}$ commute with M_i, anticommute with $\bar{X}$ and leaves $|0\rangle_7$ invariant:

$\bar{Z}|\bar{0}\rangle = |\bar{0}\rangle \quad , \quad \bar{Z}|\bar{1}\rangle = -|\bar{1}\rangle$

Logical Hadamard:

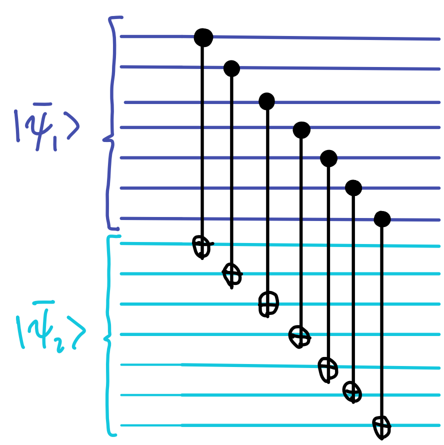
$\bar{H} = H^{\otimes 7} = HHHHHHH$: Recall $HXH = Z$: Due to definitions above, straight forward to see that:

$\bar{H}\bar{X}\bar{H} = \bar{Z} \Rightarrow \bar{H}|\bar{0}\rangle = |\bar{+}\rangle \quad , \quad \bar{H}|\bar{1}\rangle = |\bar{-}\rangle$

Logical CNOT:

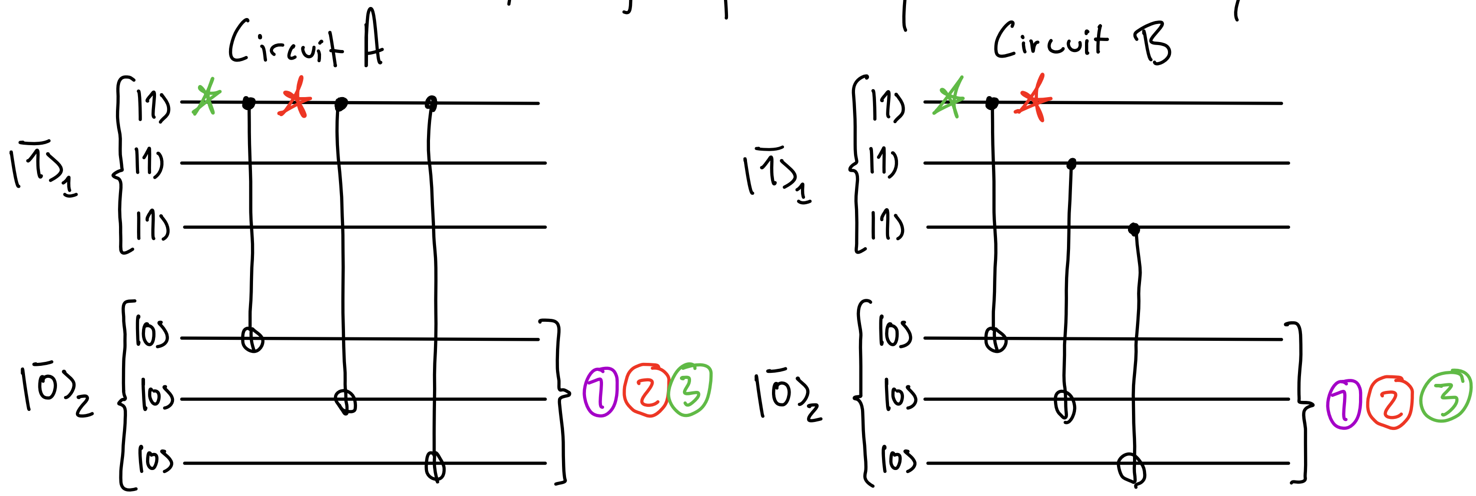
Since we have $\bar{X}$: $\text{CNOT} = |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes X$:

$\overline{\text{CNOT}} = |\bar{0}\rangle\langle\bar{0}| \otimes \bar{\mathbb{1}} + |\bar{1}\rangle\langle\bar{1}| \otimes \bar{X}$:



Fault tolerance and the limits of QC

- Nearly there, just need T. Turns out, cannot be done fault tolerantly.
- Consider now 2 ways of implementing same (nominally) identical:



What is state of logical qubit?

① Circuit A: $|111\rangle = |\bar{1}\rangle$ ② $|100\rangle$ ③ $|000\rangle$

Circuit B: $|111\rangle = |\bar{1}\rangle$ ② $|111\rangle = |\bar{1}\rangle$ ③ $|011\rangle$

correctable!

- Obviously big difference! Circuit B is fault tolerant
- Steane code cannot implement T-gate fault tolerantly!

> **Eastin-Knill theorem** (informally) '08
>
> No code capable of detecting single-qubit errors has a universal fault tolerant gate set

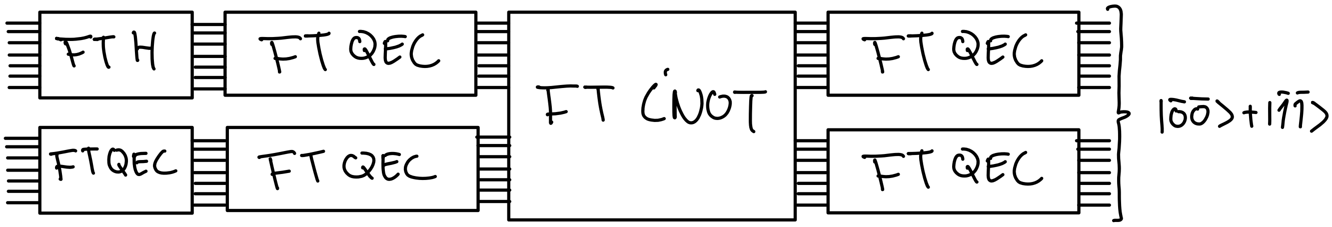
- ok, but maybe $\{X, Z, H, CNOT\}$ is enough to do quantum speedup? NO

> **Gottesman-Knill theorem** (informally) '97
>
> Any quantum computation consisting solely of $\{X, Z, H, CNOT\}$ gates can be efficiently simulated classically

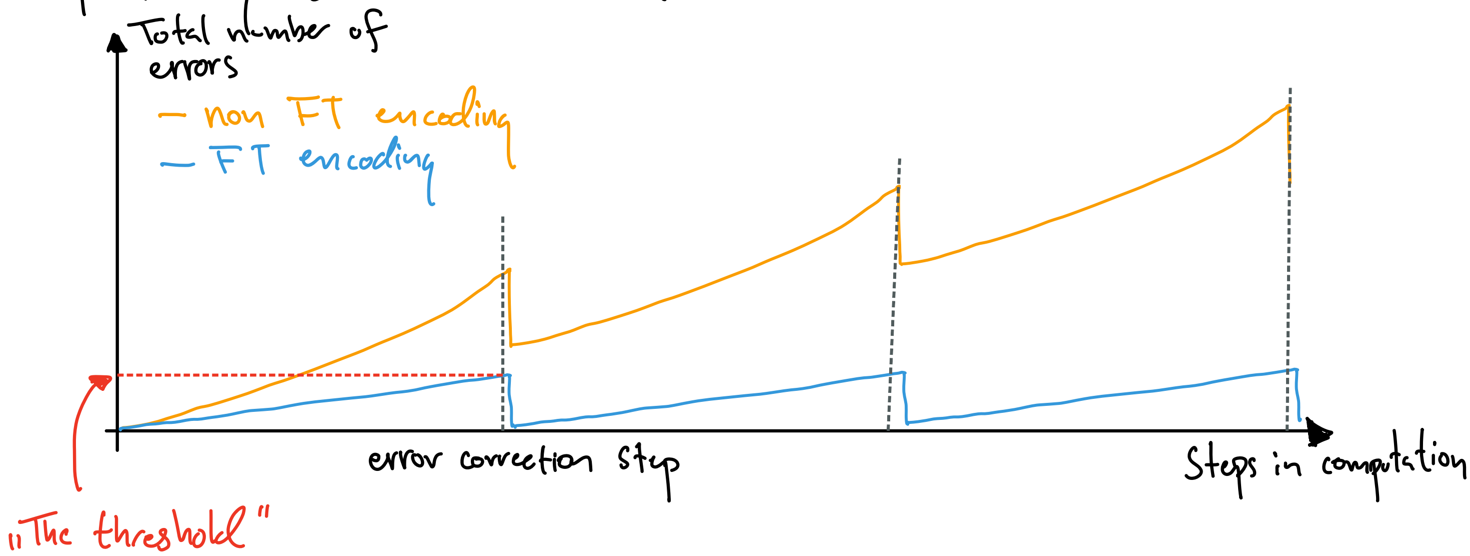
- Thus in order to get universal quantum computing we supplement the fault tolerant gate set. Typically done by adding T-gate via something called „magic state distillation". Work on this in assignment 2

Big picture view of FT-QC

- So, a large scale error corrected quantum computer with Steane code:



- A final way to understand FT



- Threshold for Steane code: 0.08% → 99.92%
- Threshold for "Surface code": 1% → 99%
- Current world best: 99.8% (Song, PRX, 11, 021058 (2021))