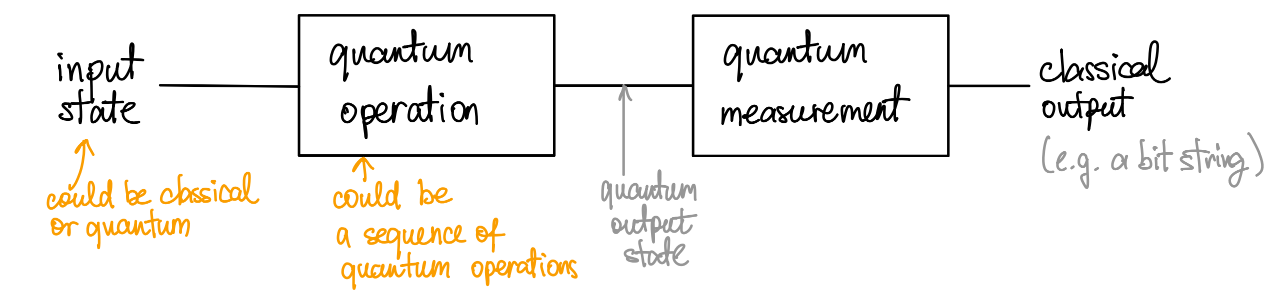
LECTURE 2.1

Plan
- General quantum states
- General quantum measurements (+ special classes)

* • (more) general quantum operations

A quantum protocol/algorithm



QUANTUM STATES

Recall from Week 1:

- n-qubit quantum state

$$|\psi\rangle = \sum_{x\in\{0,1\}^n} \alpha_x |x\rangle = \begin{pmatrix} \alpha_{0\ldots0} \\ \alpha_{0\ldots1} \\ \vdots \\ \alpha_{1\ldots} \end{pmatrix} \in \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2 = \mathbb{C}^{2^n}$$

where $\underbrace{\||\psi\|| = 1}_{\text{unit vector}}$ $\left(\Leftrightarrow \langle\psi|\psi\rangle = 1 \quad \Leftrightarrow \sum_{x\in\{0,1\}^n} |\alpha_x|^2 = 1 \right)$

- We can also consider d-dimensional quantum states for any $d \in \mathbb{Z}^+$:

$$|\psi\rangle = \sum_{i=0}^{d-1} \alpha_i |i\rangle = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{d-1} \end{pmatrix} \in \mathbb{C}^d \quad , \quad \text{where } \||\psi\|| = 1$$

$$|i\rangle := e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i^{th} \text{ position}$$

$|i\rangle \in \mathbb{C}^d$

We call such states $|\psi\rangle$ **pure quantum states**.

Def. A pure quantum state is a unit vector in $\mathbb{C}^d$
$$|\psi\rangle = \sum_{i=0}^{d-1} \alpha_i |i\rangle$$
for some $d \in \mathbb{Z}^+$

More general quantum states

with probability $\xrightarrow{1/3}$ prepare $|0\rangle$

with probability $\xrightarrow{2/3}$ prepare $|+\rangle$

How do we describe such a state?

- Ensemble: $\{ (\frac{1}{3}, |0\rangle), (\frac{2}{3}, |+\rangle) \}$

  Density matrix: $\frac{1}{3} |0\rangle\langle 0| + \frac{2}{3} |+\rangle\langle +| =$

- If pure state $|\psi_i\rangle$ is given with prob p_i, the density matrix is
$$\boxed{\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|}$$

Q: Find the density matrices for the following:

1) state $|0\rangle$
2) $\{(\frac{1}{2}, |0\rangle), (\frac{1}{2}, |1\rangle)\}$ "state $|i\rangle$ given with prob. $\frac{1}{2}$ for $i=0,1$"
3) $\{(\frac{1}{2}, |+\rangle), (\frac{1}{2}, |-\rangle)\}$

A:

1) $|0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
2) $\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}\mathbb{1}$ ← completely mixed state
3) $\frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -| = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\mathbb{1}$

Observe: Different ensembles can have the same density matrix.

$\Rightarrow$ Since the density matrix determines measurement statistics, it is impossible to distinguish (2) from (3) (without access to extra information)

Def. A mixed quantum state is given by a density matrix $\rho \in \mathbb{M}_d$ (for some $d \in \mathbb{Z}^+$) satisfying

(1) $\mathrm{Tr}(\rho) = 1$

(2) $\rho \geq 0$ (ρ is positive semidefinite)

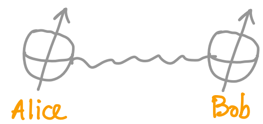
- Any $\rho \in \mathbb{M}_d$ satisfying (1) & (2) is a valid quantum state.
- in p-set 2, you will show that

$\rho = \sum_{i=1}^{n} p_i \|\psi_i\rangle\langle\psi_i\|$ for some ensemble $\{(p_i, \|\psi_i\rangle)\}_{i=1}^{n}$	$\Longleftrightarrow$	$\mathrm{Tr}(\rho) = 1$ and $\rho \geq 0$

QUANTUM STATE OF A SUBSYSTEM

Recall the Bell state

$$|\psi^+\rangle := \big(|00\rangle + |11\rangle\big)/\sqrt{2} \in \mathbb{C}^2\otimes\mathbb{C}^2$$



How do we describe the state of Alice's electron alone?

DETOUR: PARTIAL TRACE

Recall: For matrix $M\in \mathbb{M}_d$: $\mathrm{Tr}(M) = \sum_{i=0}^{d-1} M_{ii} = \sum_{i=0}^{d-1} \underbrace{\langle i|\, M\, |i\rangle}_{(0\cdots010\cdots0)\; M \begin{pmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{pmatrix}}$

For matrix $M\in \mathbb{M}_{d_1 d_2}$ (which acts on $\mathbb{C}^{d_1}\otimes\mathbb{C}^{d_2}$)

system 1 system 2

we can "trace out" one of the two systems:

- trace out system 1 : $\boxed{\mathrm{Tr}_1(M) = \sum_{i=0}^{d_1-1} (\langle i|\otimes \mathbb{1})\, M\, (|i\rangle\otimes\mathbb{1})}$
- —— " —— 2 : $\boxed{\mathrm{Tr}_2(M) = \sum_{i=0}^{d_2-1} (\mathbb{1}\otimes\langle i|)\, M\, (\mathbb{1}\otimes|i\rangle)}$

Example:

$$\mathrm{Tr}_1\big(|\psi^+\rangle\langle\psi^+|\big)$$

Note: $|\psi^+\rangle = \frac{1}{\sqrt{2}}\sum_{j=0}^{1} |j,j\rangle$

$|\psi^+\rangle\langle\psi^+| = \frac{1}{2}\sum_{j,k=0}^{1} |j,j\rangle\langle k,k|$

$= \frac{1}{2}\sum_{j,k=0}^{1} (|j\rangle\langle k|)\otimes(|j\rangle\langle k|)$

$$= \sum_i \sum_{j,k} \big(\overset{A_1}{\langle i|}\otimes \overset{B_1}{\mathbb{1}}\big)\, \frac{1}{2}\big(\overset{A_2}{|j\rangle\langle k|}\otimes \overset{B_2}{|j\rangle\langle k|}\big)\big(\overset{A_3}{|i\rangle}\otimes \overset{C_3}{\mathbb{1}}\big) =$$

use $(A_1\otimes B_1)(A_2\otimes B_2)(A_3\otimes B_3) = (A_1A_2A_3)\otimes(B_1B_2B_3)$

$$= \frac{1}{2}\sum_{i,j,k=0}^{1} \big(\langle i|j\rangle\langle k|i\rangle\big)\otimes\big(\mathbb{1}\,|j\rangle\langle k|\,\mathbb{1}\big)$$

$$= \frac{1}{2} \sum_{i,j,k=0}^{1} \delta_{ij} \delta_{ik} \; |i\rangle\langle i| = \frac{1}{2} \sum_{i=0}^{1} |i\rangle\langle i| = \frac{1}{2} \mathbb{1}$$

in fact, also $\mathrm{Tr}_2 (|\psi^+\rangle\langle\psi^+|) = \frac{1}{2}\mathbb{1}$

Q: • Compute $\mathrm{Tr}_1(|\psi\rangle\langle\psi|)$ and $\mathrm{Tr}_2(|\psi\rangle\langle\psi|)$ for $|\psi\rangle = |0\rangle \otimes |+\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$

• Try to guess $\mathrm{Tr}_1(\rho)$ and $\mathrm{Tr}_2(\rho)$ for $\rho = \rho_1 \otimes \rho_2$?

Ans:

$$\mathrm{Tr}_1(|\psi\rangle\langle\psi|) = \mathrm{Tr}_1(|0\rangle\langle 0| \otimes |+\rangle\langle +|) =$$

$$= \sum_{i=0}^{1} (\underset{A_1}{\langle i|} \otimes \underset{B_1}{\mathbb{1}}) (\underset{A_2}{|0\rangle\langle 0|} \otimes \underset{B_2}{|+\rangle\langle +|}) (\underset{A_3}{|i\rangle} \otimes \underset{C_3}{\mathbb{1}})$$

$$(A_1 \otimes B_1)(A_2 \otimes B_2)(A_3 \otimes B_3) = (A_1 A_2 A_3) \otimes (B_1 B_2 B_3)$$

$$= \sum_{i=0}^{1} \underbrace{(\langle i|0\rangle\langle 0|i\rangle)}_{\delta_{i,0}} \otimes (\mathbb{1}\, |+\rangle\langle +|\, \mathbb{1}) = |+\rangle\langle +|$$

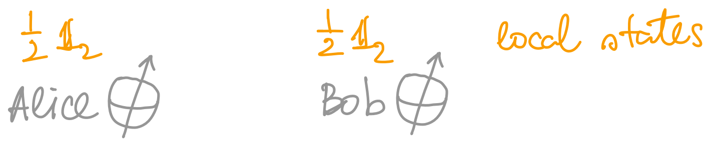
$$\mathrm{Tr}_2(|\psi\rangle\langle\psi|) = \sum_{i=0}^{1} (\underset{A_1}{\mathbb{1}} \otimes \underset{B_1}{\langle i|}) (\underset{A_2}{|0\rangle\langle 0|} \otimes \underset{B_2}{|+\rangle\langle +|}) (\underset{A_3}{|i\rangle} \otimes \underset{B_3}{\mathbb{1}}) =$$

$$= \sum_{i=0}^{1} (\mathbb{1}\, |0\rangle\langle 0|\, \mathbb{1}) \otimes (\underbrace{\langle i|+\rangle}_{\frac{1}{\sqrt{2}}}\langle +|i\rangle) = |0\rangle\langle 0| \otimes \left(\sum_{i=0}^{1} \frac{1}{2}\right) = |0\rangle\langle 0|$$

In general,

$$\boxed{\mathrm{Tr}_1(\rho_1 \otimes \rho_2) = \rho_2 \qquad \mathrm{Tr}_2(\rho_1 \otimes \rho_2) = \rho_1}$$

Suppose Alice & Bob each prepare a completely mixed state $\frac{1}{2}\mathbb{1}$



joint state $\frac{1}{2}\mathbb{1}\otimes\frac{1}{2}\mathbb{1}=\frac{1}{4}\mathbb{1}_4$

Recall that for Bell state $|\psi^+\rangle$ the reduced states were also $\frac{1}{2}\mathbb{1}$!

$\Rightarrow$ Alice & Bob locally cannot tell apart the completely mixed state $\frac{1}{4}\mathbb{1}_4$ from $|\psi^+\rangle\langle\psi^+|$

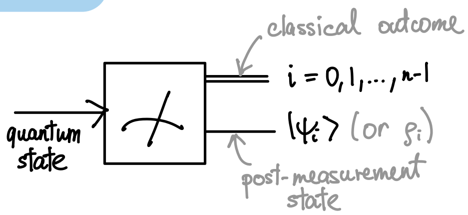
mixed, not entangled — pure, entangled

> A pure state $|\psi\rangle\in\mathbb{C}^{d_1}\otimes\mathbb{C}^{d_2}$ is entangled, if we cannot write it as $|\alpha\rangle\otimes|\beta\rangle$ for $|\alpha\rangle\in\mathbb{C}^{d_1}, |\beta\rangle\in\mathbb{C}^{d_2}$

$\Rightarrow$ mixed states $\rho=\rho_1\otimes\rho_2$ are not entangled

There are, however, states $\rho\neq\rho_1\otimes\rho_2$ which are nevertheless entangled.

MEASUREMENTS

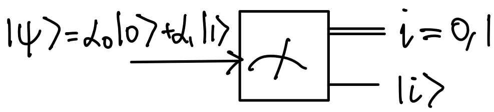


with probability p_i

n-outcome measurement

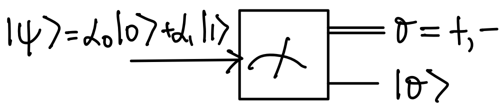
In Week 1 we saw

- Standard basis $|0/1\rangle$ measurement



with prob. $p_i = |\langle i|\psi\rangle|^2 = |\alpha_i|^2$

- $|+/-\rangle$-basis measurement



with prob. $p_\sigma = |\langle\sigma|\psi\rangle|^2$

Def. An **n-outcome measurement** is given by a list of operators $M_0, \dots, M_{n-1} \in \mathbb{M}_{d \times d}$ such that

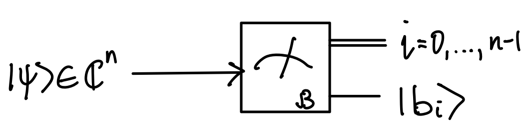
$$\sum_{i=0}^{n-1} M_i^\dagger M_i = \mathbb{1} \quad (\star)$$

if M_i's are projectors (i.e. $M_i^\dagger = M_i = M_i^2$), then

$$(\star) \iff \sum_{i=0}^{n-1} M_i \qquad (\text{since } M_i^\dagger M_i = M_i^2 = M_i)$$

Def. We say that measurement, $\mathcal{M}$, given by $M_0, \dots, M_{n-1}$ is **projective** if all M_i's are projectors.

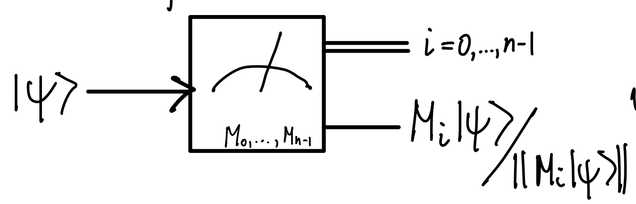
For an orthonormal basis $\mathcal{B}$: $|b_0\rangle, \dots, |b_{n-1}\rangle \in \mathbb{C}^n$, the corresponding projective measurement is given by $M_i = |b_i\rangle\langle b_i|$. We refer to $\mathcal{M}$ as "**measurement in basis $\mathcal{B}$**".



with prob. $p_i = |\langle b_i|\psi\rangle|^2$

Probabilities & post-measurement states

- For pure states:



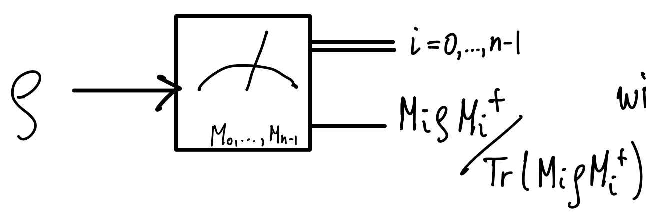
with prob. $p_i = \| M_i |\psi\rangle \|^2$

$= \langle\psi| \underbrace{M_i^\dagger}_{} \underbrace{M_i |\psi\rangle}_{}$

$= Tr(M_i |\psi\rangle\langle\psi| M_i^\dagger)$

use $Tr(XY) = Tr(YX)$
"cyclicity of trace"

- For mixed states



with prob. $p_i = Tr(M_i \rho M_i^\dagger)$

Recall from Week 1

1) Operators for $|0/1\rangle$-basis measurement: $M_0 = |0\rangle\langle 0|$ $\quad M_1 = |1\rangle\langle 1|$
2) — " — $|+/-\rangle$-basis — " — : $M_+ = |+\rangle\langle +|$ $\quad M_- = |-\rangle\langle -|$

Let's check the def.: 1) $M_0 + M_1 = \begin{pmatrix}1&0\\0&0\end{pmatrix} + \begin{pmatrix}0&0\\0&1\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix}$ ✓

2) $M_+ + M_- = \frac{1}{2}\begin{pmatrix}1&1\\1&1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1&-1\\-1&1\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix}$ ✓

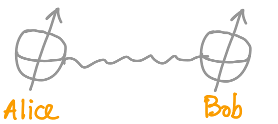
So (★) is satisfied!

Reality check: Do both ways of calculating p_+, p_- match?

Let $\sigma = +, -$

$$p_\sigma = \| M_\sigma |\psi\rangle \|^2 = \langle\psi| M_\sigma^\dagger M_\sigma |\psi\rangle = \langle\psi| \cdot \overbrace{|\sigma\rangle \underbrace{\langle\sigma|}_{1}}^{P_\sigma^\dagger} \cdot \overbrace{\underbrace{|\sigma\rangle}_{} \langle\sigma|}^{P_\sigma} |\psi\rangle = |\langle\sigma|\psi\rangle|^2 \checkmark$$

PARTIAL MEASUREMENTS



$$|\psi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Suppose Alice measures her qubit in $|0/1\rangle$-basis.

Q: What is her probability to get outcome $i = 0, 1$?
What is the post-measurement state? (Discuss with a peer)

How do we describe Alice's measurement $\mathcal{M} = \left(M_0 = |0\rangle\langle 0|, M_1 = |1\rangle\langle 1| \right)$ as a measurement of the joint system of 2 qubits?

Ans: $\mathcal{M}$ corresponds to global measurement $\mathcal{M}'$ given by

$$M_0' = |0\rangle\langle 0| \otimes \mathbb{1}_2, \quad M_1' = |1\rangle\langle 1| \otimes \mathbb{1}_2$$

"do nothing to Bob's qubit"

> Measurement, $\mathcal{M}$, given by $M_0, \dots, M_{n-1}$ on the first system of $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ corresponds to the global measurement $\mathcal{M}'$:
> $$M_0' = M_0 \otimes \mathbb{1}_{d_2}, \quad \dots, \quad M_{n-1}' = M_{n-1} \otimes \mathbb{1}_{d_2}$$

$$p_i = \| M_i' |\psi^+\rangle \|^2$$

$\boxed{i=0}$

$$= \| (|0\rangle\langle 0| \otimes \mathbb{1}) |\psi^+\rangle \|^2$$

$$(|0\rangle\langle 0| \otimes \mathbb{1}) \tfrac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$$

$$= \tfrac{1}{\sqrt{2}} \left[(|0\rangle\langle 0| \otimes \mathbb{1}) |00\rangle + (|0\rangle\langle 0| \otimes \mathbb{1}) |11\rangle \right]$$

$A_1 \quad B_1 \quad A_2 \quad B_2$

$$= \tfrac{1}{\sqrt{2}} \left[(|0\rangle\underbrace{\langle 0|0\rangle}_{1}) \otimes (\mathbb{1}|0\rangle) + (|0\rangle\underbrace{\langle 0|1\rangle}_{0}) \otimes (\mathbb{1}|1\rangle) \right]$$

$$= \tfrac{1}{\sqrt{2}} |00\rangle$$

$$= \| \tfrac{1}{\sqrt{2}} |00\rangle \|^2$$

$$= \tfrac{1}{2}$$

So $p_0 = \frac{1}{2}$ and $p_1 = \frac{1}{2}$. Post-measurement states: $|\psi_0\rangle = |00\rangle$, $|\psi_1\rangle = |11\rangle$

Q: If Bob also measures his qubit in $|0/1\rangle$-basis, how will his outcome, j, compare to the outcome, i, obtained by Alice?

Ans: Alice and Bob always obtain the same outcome (i.e. $i=j$).

Another approach to computing p_i.

Recall: Alice's reduced state is $\rho_A = \mathrm{Tr}_2(|\Psi^+\rangle\langle\Psi^+|) = \frac{1}{2}\mathbb{1}$

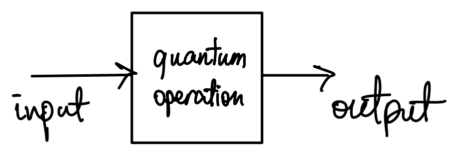
So $p_i = \mathrm{Tr}(M_i \rho_A M_i^\dagger)$

$= \mathrm{Tr}(\frac{1}{2}|i\rangle\langle i|)$

$= \frac{1}{2}$

$$M_i \rho_A M_i^\dagger = |i\rangle \underbrace{\langle i| \cdot \tfrac{1}{2}\mathbb{1} \cdot |i\rangle}_{1/2} \langle i| = \tfrac{1}{2}|i\rangle\langle i|$$

QUANTUM OPERATIONS



Recap: we can act on qubits with unitary gates (ex: H, CNOT, Z etc).

In general, we can apply a $d\times d$ unitary U ($U^\dagger U = UU^\dagger = \mathbb{1}_d$) on pure state $|\psi_{in}\rangle \in \mathbb{C}^d$, to get

$$|\psi_{out}\rangle = U|\psi_{in}\rangle$$

Q: Check that $|\psi_{out}\rangle$ is a valid quantum state.

Ans:

$$\| |\psi_{out}\rangle \| = \langle \psi_{out} | \psi_{out}\rangle = \langle \psi_{in} | \underbrace{U^\dagger U}_{\mathbb{1}} | \psi_{in}\rangle = \| |\psi_{in}\rangle \| = 1 \ \checkmark$$

- If input state is $\rho_{in} \in M_d$, then output is
$$\rho_{out} = U \rho_{in} U^{\dagger}.$$

Mixed unitary channel: apply unitary U_i with probability p_i.

$$\rho_{in} \longrightarrow \boxed{(p_i, U_i)} \longrightarrow \rho_{out} = \sum_{i=1}^{n} p_i U_i \rho_{in} U_i^{\dagger}$$