Lecture 1.2: Qubits, gates, measurements

Schedule:

- $9^{15} - 9^{25}$: Recap
- $9^{25} - 9^{30}$: Operators on qbits (From Monday)
- $9^{30} - 9^{45}$: Bras, kets, projectors
- $9^{45} - 10^{00}$: Hadamard operation
- $10^{15} - 10^{30}$: CNOT and CZ
- $10^{30} - 10^{45}$: Quantum circuit diagrams
- $10^{45} - 11^{00}$: Measurements (I)
- $11^{15} - 11^{35}$: Measurements (II)
- $11^{35} - 12^{00}$: Bloch Sphere

BRACKET

- Recall: $|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle = \begin{pmatrix}\alpha_0\\ \alpha_1\end{pmatrix}$, $(|\alpha_0|^2 + |\alpha_1|^2 = 1)$

  "a ket"

- Define: $\langle\psi| = \alpha_0^*\langle 0| + \alpha_1^*\langle 1| = (\alpha_0^* \;\; \alpha_1^*)$

  "a bra"

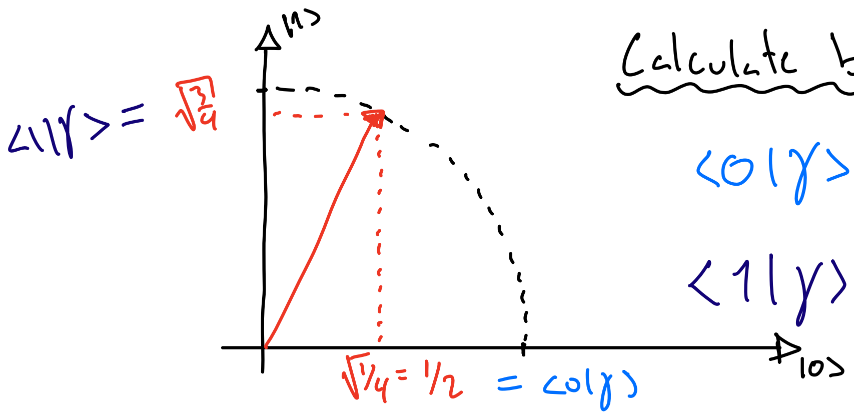
> Now define the BRACKET:
>
> $|\psi\rangle = \sum_{i=0}^{n} \alpha_i |i\rangle$ and $|\Phi\rangle = \sum_{i=0}^{n} \varphi_i |i\rangle$ then
>
> $\langle\psi|\Phi\rangle = \sum_{i=0}^{n} \alpha_i^* \varphi_i$: "Braket"

Q: What is $\langle 0|1\rangle$? What is $\langle 0|\psi\rangle$? What is $\langle\psi|\psi\rangle$?

A: 0, α_0, 1

- Sometimes called "inner product". We will use it a lot.
- Has a nice geometric interpretation:

$$|\gamma\rangle = \sqrt{\tfrac{1}{4}}\,|0\rangle + \sqrt{\tfrac{3}{4}}\,|1\rangle$$



Calculate brackets:

$$\langle 0|\gamma\rangle = \langle 0|\left(\sqrt{\tfrac{1}{4}}|0\rangle + \sqrt{\tfrac{3}{4}}|1\rangle\right) = \sqrt{\tfrac{1}{4}}$$

$$\langle 1|\gamma\rangle = \langle 1|\left(\sqrt{\tfrac{1}{4}}|0\rangle + \sqrt{\tfrac{3}{4}}|1\rangle\right) = \sqrt{\tfrac{3}{4}}$$

KETBRAs

- Consider the operators:

$$\tilde{n} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{matrix} \tilde{n}|0\rangle = |0\rangle \\ \tilde{n}|1\rangle = 0 \end{matrix} \quad \& \quad n = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} : \begin{cases} n|0\rangle = 0 \\ n|1\rangle = |1\rangle \end{cases} \qquad (*)$$

- $\tilde{n}$ has an appealing intuitive interpretation in terms of „ketbra"
- The ketbra of a state $|\psi\rangle$: $|\psi\rangle\langle\psi|$

> What does $|a\rangle\langle b|$ mean? (outer product)
>
> The outer product: $|a\rangle\langle b| = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} [b_0^* \; b_1^* \; \cdots \; b_n^*] = \begin{bmatrix} a_0 b_0^* & a_0 b_1^* & \cdots & a_0 b_n^* \\ a_1 b_0^* & a_1 b_1^* & \cdots & \vdots \\ \vdots & & \ddots & \vdots \\ a_n b_0^* & \cdots & \cdots & a_n b_n^* \end{bmatrix}$

Example

- For state $|0\rangle$:

  ketbra is: $|0\rangle\langle 0| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1 \;\; 0) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \tilde{n}$

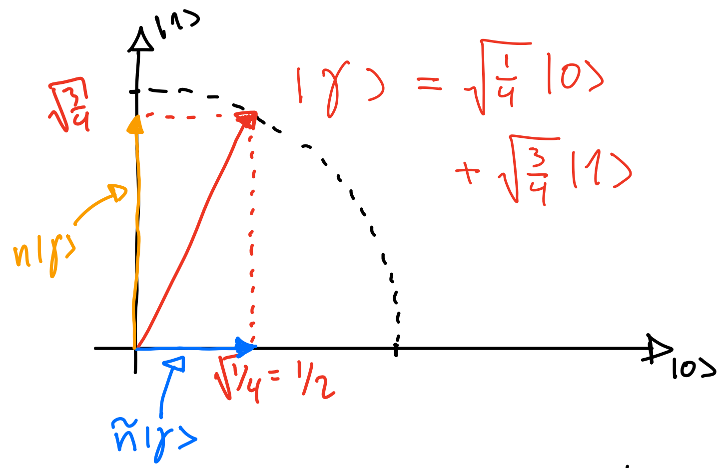
- Writing $\tilde{n}$ in this way is very helpful:

$$\tilde{n}|0\rangle = |0\rangle\underbrace{\langle 0|0\rangle}_{\text{1 Bracket!}} = |0\rangle$$

Q: Matrix form of $|1\rangle\langle 1|$? Matrix form of $|0\rangle\langle 0| + |1\rangle\langle 1|$?

A: $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = n$ and $\mathbb{1}$

- Also a nice geometric interpretation



$$\tilde{n}|\gamma\rangle = |0\rangle\langle 0|\left(\sqrt{\tfrac{1}{4}}|0\rangle + \sqrt{\tfrac{3}{4}}|1\rangle\right) = \sqrt{\tfrac{1}{4}}|0\rangle$$

$$n|\gamma\rangle = |1\rangle\langle 1|\left(\sqrt{\tfrac{1}{4}}|0\rangle + \sqrt{\tfrac{3}{4}}|1\rangle\right) = \sqrt{\tfrac{3}{4}}|1\rangle$$

- Ketbras are „projection operators"
- Projection operators are a powerful tool (much more on them later):

$$\text{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \boxed{|0\rangle\langle 0|} \otimes \mathbb{1} + \boxed{|1\rangle\langle 1|} \otimes X$$

IMPORTANT QBIT OPERATIONS

- Consider the operator:

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$$

Q: Anyone recognize this matrix?
A: Pauli σ_z matrix

$$Z|0\rangle = 1|0\rangle \quad , \quad Z|1\rangle = -1|1\rangle$$

computational states are eigenstates of Z-operator

- Z anticommutes with X:

$$ZX = -XZ$$

- Note: $n = \frac{1}{2}(\mathbb{1} - Z)$, $\tilde{n} = \frac{1}{2}(\mathbb{1} + Z)$

- We will very often need the following 2 states

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \quad \text{and} \quad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

- Observe the nice symmetry:

$Z|0\rangle = +1|0\rangle$ $\qquad$ $Z|1\rangle = -1|1\rangle$

$Z|+\rangle = |-\rangle$ $\qquad$ $Z|-\rangle = |+\rangle$

$X|+\rangle = |+\rangle$ $\qquad$ $X|-\rangle = -|-\rangle$

$X|0\rangle = |1\rangle$ $\qquad$ $X|1\rangle = |0\rangle$

eigenstates

— flips

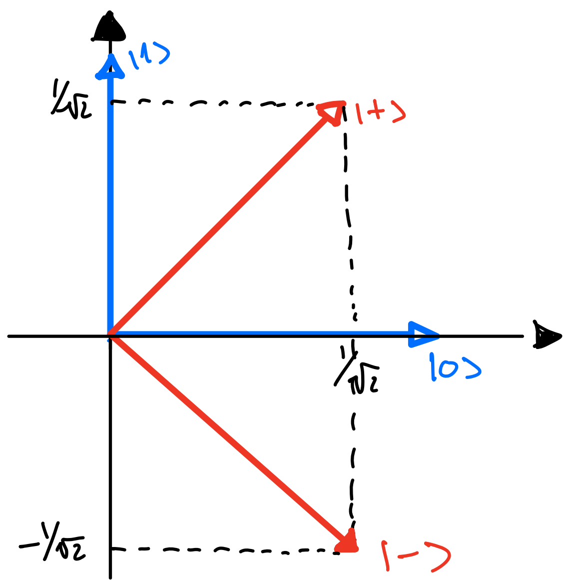
- X and Z act similarly, but in two different <u>basis</u>

"comp basis"
"Z basis" $= \{|0\rangle, |1\rangle\}$

- Eigenvectors of Z
- Flip operator is X

"± basis"
"X basis" $= \{|+\rangle, |-\rangle\}$

- Eigenvectors of X
- Flip operator is Z



Hadamard Operation:

- The operation that rotates between the Z- and X-basis:

$$H = \frac{1}{\sqrt{2}}(X+Z) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

- EXCEEDINGLY important operator!
- Not to be confused with the Hamiltonian operator!

Q: What is $H|0\rangle$? $H|-\rangle$?

A: $|+\rangle$ and $|1\rangle$

- Some important properties of H:

$$H^2 = \mathbb{1} \qquad \text{and} \qquad HXH = Z \qquad (*)$$

Q: What is HZH? A: X

Controlled-Z

- In analogy with CNOT controllably applying X, we can imagine controllably applying Z:

$$\boxed{CZ = |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes Z}$$

"applies Z to qubit 2 if qubit 1 is in state $|1\rangle$"

Q: fill in

$\lvert b_1\rangle$	$\lvert b_2\rangle$	$CZ_{01}\lvert b_1 b_2\rangle$
$\lvert 0\rangle$	$\lvert 0\rangle$	$\lvert 00\rangle$
$\lvert 0\rangle$	$\lvert 1\rangle$	$\lvert 01\rangle$
$\lvert 1\rangle$	$\lvert 0\rangle$	$\lvert 10\rangle$
$\lvert 1\rangle$	$\lvert 1\rangle$	$-\lvert 11\rangle$

- Notice: Completely symmetric in target & control!
- Because of $(*)$ we see:

$$(\mathbb{1} \otimes H)\, CNOT\, (\mathbb{1} \otimes H) = CZ \qquad (\dagger)$$

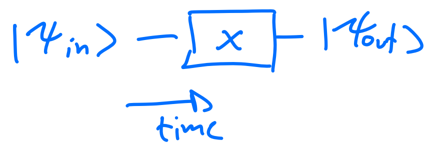
General controlled operation:

- Any controlled operation can be expressed as:

$$CU = |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes U$$

QUANTUM CIRCUIT DIAGRAMS

- A tool that we will use frequently throughout class:

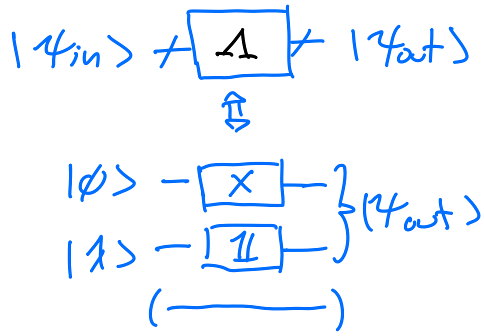
$|\psi_{out}\rangle = X|\psi_{in}\rangle \iff$ 

- A couple of examples:

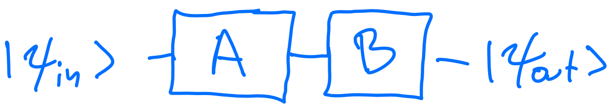
$|\psi_{out}\rangle = \Lambda|\psi_{in}\rangle$

$|\psi_{in}\rangle = |\phi\rangle \otimes |\lambda\rangle \iff$

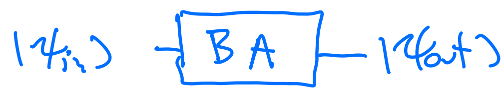
$\Lambda = X \otimes \mathbb{1}$



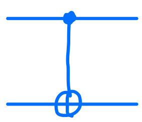
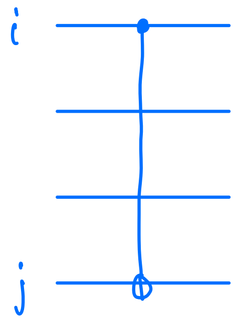
- Ordering is important:

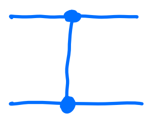


$\iff |\psi_{out}\rangle = B\,A|\psi_{in}\rangle$

$\iff$ 

- Multiqubit gates:

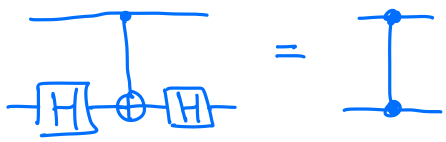
CNOT = , $CNOT_{ij}$ = 

, CZ = 

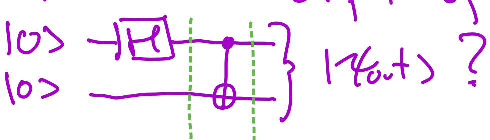
Q: Why is this symmetric?

A: CZ symmetric in

- The identity we saw in (+) can be expressed neatly:

$(\mathbb{1}\otimes H)\,CNOT\,(\mathbb{1}\otimes H) = CZ \iff$ 

Q: What is the output of



A: $\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)|0\rangle$ $\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)$ = Bell state! Entanglement!

Q: Can you 

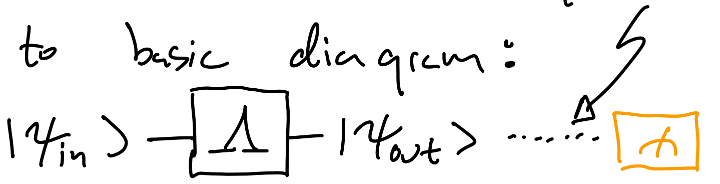
to an equivalent circuit, with less gates?

A: 

MEASUREMENTS

How do you know the state?

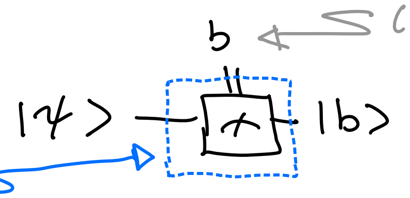
- Return to basic diagram: ⇒ YOU MEASURE



- In classical computing: NO problem just look at bits whenever you feel like it! | From any single bit, you learn 1 bit ("1" or "0")
- Not possible in QC! Recall state: $|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$:

> There exists no single operation that will reveal all bits of e.g. $\alpha_0 = \frac{1}{\sqrt{2}} = 0.7071067\ldots$

- To build intuition for quantum measurements, consider: $|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$
- Heres how measurement operation works (in computational basis)



> The probability that $b = 0$ is $|\alpha_0|^2$
>
> The probability that $b = 1$ is $|\alpha_1|^2$

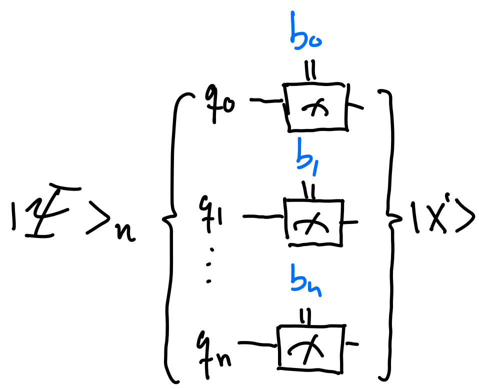
"The Born Rule"

- A link between (complex) amplitudes α_i and measurable probabilities p_i
- Take a more general state:

$$|\Psi\rangle_n = \sum_{0 \le x < 2^n} \alpha_x |X\rangle_n$$

$$X = \underbrace{b_0 b_1 b_2 b_3 b_4 b_5 \ldots}_{n}$$

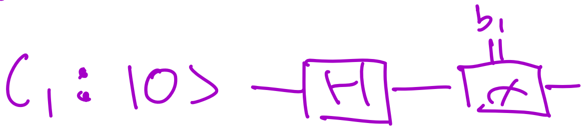
Assumed here it was a qubit



> Measurement result $X' = b_0 b_1 \cdots b_n$ with probability $|\alpha_x|^2$

- The fact that the resulting quantum state is just the $|x\rangle$'s component (as opposed to 'any' of the other $2^n - 1$ possibilities, is called wavefunction collapse.

Q: Consider the circuits



For both circuits answer:

1: What are the possible values of b?

2: What is the probability of obtaining each b?

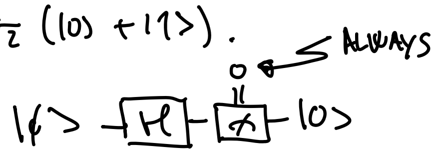
A: 1: $b = 0, 1$ 2: $1/2, 1/2$ (for both C_1 and C_2)

Important: The phase $|0\rangle - |1\rangle = |0\rangle + e^{i\pi}|1\rangle$ Played no role in the outcome probability! Because $|e^{i\varphi}| = 1$ and Born rule is $|\cdot|^2$

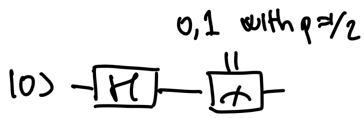
State of qubit before measurement?

Suppose you have $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

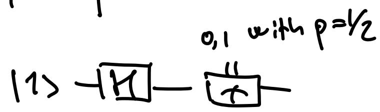
Now apply $H|+\rangle = |0\rangle$:



But imagine now $|+\rangle$ was either $|0\rangle$ with $p = 1/2$ or $|1\rangle$ with $p = 1/2$, and apply again H:



OR



The outcome would be different! SO Superposition is not "its either $|0\rangle$ or $|1\rangle$ with $p = 1/2$", its simply just something very different.

- There is a lot of philosophy here. We will essentially NOT worry about. We will simply take it as a prescription for interacting with qubits.

- Note: If all n qubits are restricted to single-component computational basis states, its clear that q. comp $\longleftrightarrow$ reversible classical comp.

MEASUREMENTS (II) [DEVIATE FROM MERMIN]

- Measurements can be thought of as a set of *projections*
- Recall the comp. basis: $\{|0\rangle, |1\rangle\}$. Associated projection operators:

$$P_0 = |0\rangle\langle 0| = \begin{pmatrix}1\\0\end{pmatrix}(1 \quad 0) = \begin{bmatrix}1&0\\0&0\end{bmatrix}, \quad P_1 = |1\rangle\langle 1| = \begin{bmatrix}0&0\\0&1\end{bmatrix}$$

- Now define probability $p(i)$:

$$p(i) = \| P_i|\psi\rangle\|^2 = \langle\psi|P_i^\dagger P_i|\psi\rangle = \langle\psi|i\rangle\overbrace{\langle i|i\rangle}^{1}\langle i|\psi\rangle$$

$$= \langle\psi|i\rangle\langle i|\psi\rangle = |\langle\psi|i\rangle|^2 = |\alpha_i|^2$$

Small p = probability
Large P = projector

- A measurement in the computational basis:

$$|\psi\rangle \longrightarrow \begin{cases} \text{With probability } p(0): |\psi\rangle \to \mathcal{N} P_0|\psi\rangle \\ \text{With probability } p(1): |\psi\rangle \to \mathcal{N} P_1|\psi\rangle \end{cases}$$

(where $\mathcal{N}$ is some appropriate normalization factor)

Example of projection operators in action

- Consider circuit: $|0\rangle$ —[H]—[measure]—

$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

- Possible output states

$$P_0|+\rangle = |0\rangle\langle 0|\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$1|0\rangle$

$0\cdot|0\rangle$

$$= |0\rangle\frac{1}{\sqrt{2}}\cdot\mathcal{N}$$

$$P_1|+\rangle = ?$$

A: $|1\rangle\frac{1}{\sqrt{2}}\mathcal{N}$

- Probabilities:

$$p(0) = |\langle +|0\rangle|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

$$p(1) = |\langle +|1\rangle|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

- Ask yourself: whats special about $\{|0\rangle, |1\rangle\}$-basis? Nothing.
- Need a slightly technical result to generalize to other basis

Spectral decomposition

Any normal ($AA^\dagger = A^\dagger A$) operator M on a vector space V is diagonal wrt some orthonormal basis for V: $M = \sum_i \lambda_i P_i$

$M = Z = \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

- Orthonormal basis:

$$P_+ = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } P_- = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow Z = 1 P_+ - 1 P_- = \sum_i \lambda_i P_i$$

Notice:

① $\sum_{i=0}^{n} P_i = \mathbb{1}$ (completeness), ② $P_i^\dagger = P_i$ (hermitian)

③ $P_i^2 = P_i$ (idempotent operators), ④ $P_i P_j = \delta_{ij} P_i$ (mutually orthogonal)

- When ①-④ satisfied we say that the collection $\{P_0, \dots, P_n\}$ is an n-outcome projective measurement.
- This is why the computational basis is sometimes called z-basis
- Operators that can be written as a projective measurement are called observable. Correspond to a physical quantity (loosely speaking)

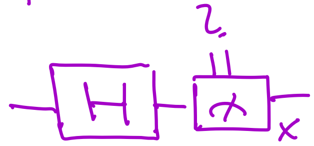
$M = X = \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

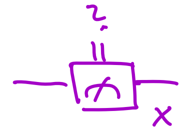
- Have seen that $X|+\rangle = |+\rangle$ and $X|-\rangle = -|-\rangle$ and $\langle + | - \rangle = 0$

$$\Rightarrow P_+^x = |+\rangle\langle +| = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } P_-^x = |-\rangle\langle -| = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

- Satisfies (1) - (4): A projective measurement in x-basis

Q: What are possible outcomes and probabilities of the following:

C_1: $|0\rangle$ 

C_2: $|0\rangle$ 

Possible outcomes	Probabilities	Possible outcomes	Probabilities
$P_+ \lvert +\rangle = \lvert +\rangle \cdot ✓$	$p(+) = 1$	$P_+ \lvert 0\rangle = \lvert +\rangle \cdot ✓$	$P_+ = 1/2$
$P_- \lvert +\rangle = \lvert 0\rangle \cdot ✓$	$p(-) = 0$	$P_- \lvert 0\rangle = \lvert -\rangle \cdot ✓$	$P_- = 1/2$

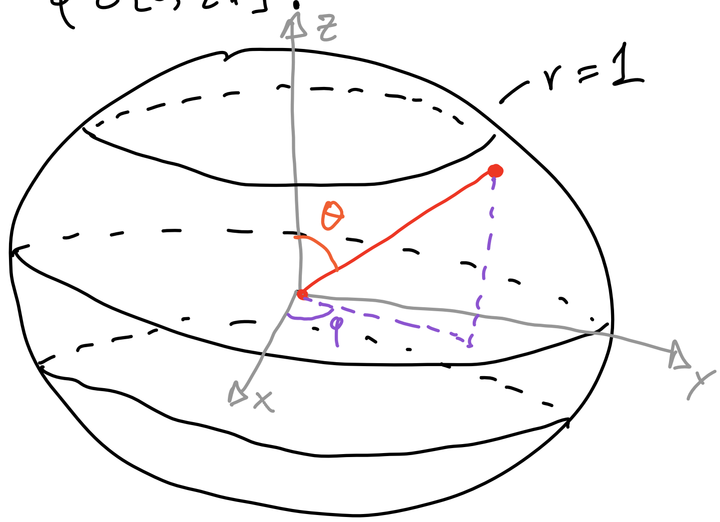
- Unless explicitly noted, measurements typically in comp basis.

BLOCH SPHERE [DEVIATE FROM MERMIN]

- A general (pure) single-qubit state can be written as:

$$|\psi\rangle = e^{i\gamma}\left(\cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle\right)$$

- γ an overall phase: No observable influence on state
- $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$.



Q: Most general state is $|\psi\rangle = a|0\rangle + b|1\rangle$, $a, b \in \mathbb{C}$:
$a = r_a e^{i\varphi_a}$, $b = r_b e^{i\varphi_b}$: 4 parameter
why is only 3 enough? $(\gamma, \varphi, \theta)$.

A: One is fixed by normalization

- Note: Orthogonal states ($|0\rangle$, $|1\rangle$) are antipodal on BS.
- Suppose I give you $|\lambda\rangle$. How to determine points on BS?

$$\begin{pmatrix} x_{|\lambda\rangle} \\ y_{|\lambda\rangle} \\ z_{|\lambda\rangle} \end{pmatrix} = \begin{pmatrix} \mathbb{E}_{|\lambda\rangle}(X) \\ \mathbb{E}_{|\lambda\rangle}(Y) \\ \mathbb{E}_{|\lambda\rangle}(Z) \end{pmatrix}$$

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = iXZ$$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- where: $\mathbb{E}_{|\lambda\rangle}(O) = \langle\lambda|O|\lambda\rangle$ is called expectation value

- Example

$$|0\rangle : \begin{pmatrix} \mathbb{E}_{|0\rangle}(X) \\ \mathbb{E}_{|0\rangle}(Y) \\ \mathbb{E}_{|0\rangle}(Z) \end{pmatrix} = \begin{pmatrix} \langle 0|X|0\rangle \\ \langle 0|Y|0\rangle \\ \langle 0|Z|0\rangle \end{pmatrix} = \begin{pmatrix} \langle 0|1\rangle \\ \langle 0|i|1\rangle \\ \langle 0|0\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Q : Calculate each expectation value in the Bloch vector:

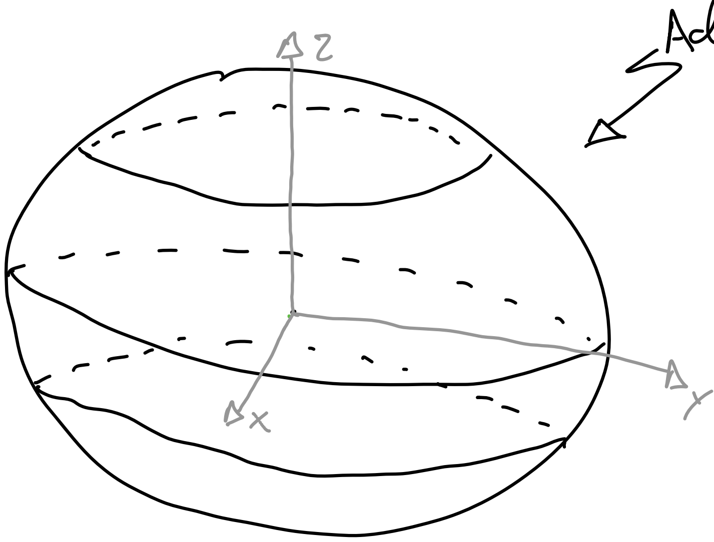
$\langle 0|X|0\rangle =$

$\langle 0|Y|0\rangle =$

$\langle 0|Z|0\rangle =$

Q: What is the Bloch vector for the $|+\rangle$-state?

A:
$$\begin{pmatrix} \mathbb{E}_{|+\rangle}(X) \\ \mathbb{E}_{|+\rangle}(Y) \\ \mathbb{E}_{|+\rangle}(Z) \end{pmatrix} = \begin{pmatrix} \langle +|X|+\rangle \\ \langle +|Y|+\rangle \\ \langle +|Z|+\rangle \end{pmatrix} = \begin{pmatrix} 1 \\ i\langle +|X|-\rangle \\ \langle +|-\rangle \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$



Add your states to this Bloch sphere!