Lecture 6.1: Introduction to QEC

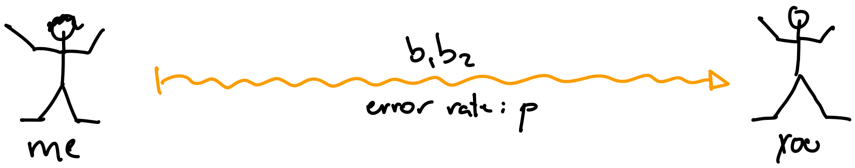
Schedule

$15^{15} - 15^{35}$: Classical error correction
$15^{35} - 15^{50}$: Issues for Quantum error correction
$15^{50} - 16^{30}$: 3 qubit repetition code
16^{30} - 17ish : General quantum errors

Reading:
Mermin 5.1
5.2
5.3

Classical Error correction

Imagine wanting to send bitstring "$b_1 b_2$" over crappy wifi:



With some probability p the bit flips: $b_1 \mapsto \sim b_1$, $b_2 \mapsto \sim b_2$

Can I somehow protect against the bit flip? YES! $\Rightarrow$ Error correction!

Repetition code

Instead of b, I send "bbb" and you just take a majority vote:

$$10 \mapsto \underbrace{(111)}_{\text{Logical bit 1}}(000) \xrightarrow[\text{error rate: } p]{} (101)(001) \rightsquigarrow (111)(000)$$

Use 3-bit repetition code   Take majority vote in each block

Without repetition code: Prob. of uncorrectable error: ___ p

With repetition code:

How many errors can 3-bit repetition code correct? ___ 1

Prob. of uncorrectable error: ______ $P_{3\text{flips}} + P_{2\text{flips}}$

Probabilities of errors:

$$P_{3\text{flips}} = p^3 \quad , \quad P_{2\text{flips}} = 3p^2(1-p)$$

When is repetition code better?

$$P_{\text{error, rep}} < P_{\text{error, bare}} : \quad p^3 + 3p^2(1-p) < p \; ?$$

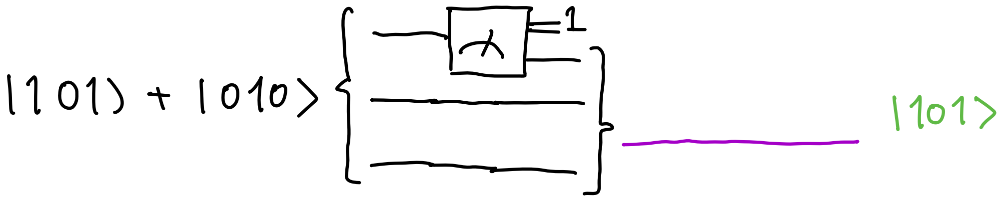
Some algebra yields: $p < 1/2 \Rightarrow$

3 bit repetition code reduces error probability if $p < 1/2$.

General considerations for quantum error correction

- Naively copying repetition code approach has several problems:
  1. No-cloning theorem:

     CANNOT do $|\psi\rangle \mapsto |\psi\rangle|\psi\rangle|\psi\rangle$
  2. Measurement collapses wavefunction:



     $\Rightarrow$ Measurement destroys superposition/entanglement :-(
  3. Quantum bits can have more than just bit-flip errors:

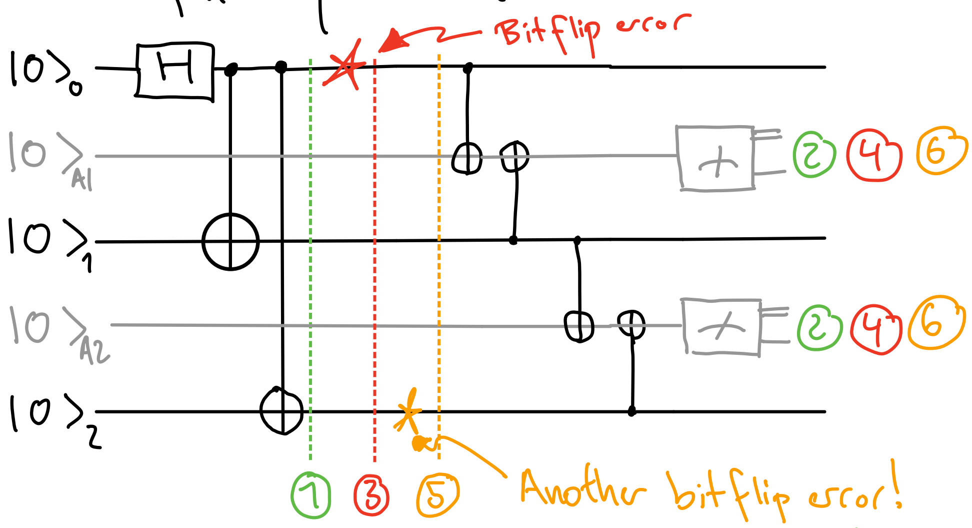
     $$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \rightsquigarrow \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \text{ : Phase flip}$$
  4. Quantum errors are continuous:

     $$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \rightsquigarrow \frac{1}{\sqrt{2}}(|0\rangle + e^{i\varphi}|1\rangle)$$

- This list made some physicists in the late 80ies / early 90ies to say QC could never work!
- Enter Peter Shor (again!) in 1994 (and many others right after)

Quantum 3 qubit repetition code

- Consider the following circuit



1. What is state of black qubits? ______ $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$
2. What is measurement outcome for A1 and A2? ______ $A_1 = |0\rangle$, $A_2 = |0\rangle$

- In the case of bitflip error at red ✱ on qubit 0:

3. What is state of black qubits? ______ $\frac{1}{\sqrt{2}}(|100\rangle + |011\rangle)$
4. What is measurement outcome for A1 and A2? ______ $A_1 = |1\rangle$, $A_2 = |0\rangle$

- In the case of bitflip error at orange ✱ (ignore red star):

5. What is state of black qubits? (ignore red error) ______ $\frac{1}{\sqrt{2}}(|001\rangle + |110\rangle)$
6. What is measurement outcome for A1 and A2? ______ $A_1 = |0\rangle$, $A_2 = |1\rangle$

- So, lets make a table:

	no error	Qubit 0	Qubit 1	Qubit 2
A1	0	1	1	0
A2	0	0	1	1

- Note: Measurement outcome is <u>distinct</u> for all 4 cases!
- Can determine procedure from pattern of 0's and 1's!
- If measurement outcome is 11, answer:

  1: Which qubit had an error? ______ qubit 1

  2: Which operation should you apply to correct error? ______ X_2

- Observe the following:

$$Z_1 Z_0 (|000\rangle + |111\rangle) = +1(|000\rangle + |111\rangle)$$

$$Z_2 Z_1 (|000\rangle + |111\rangle) = +1(|000\rangle + |111\rangle)$$

- Why would we care? Take a state with an error:

$$Z_1 Z_0 (|100\rangle + |011\rangle) = \boxed{-1}(|100\rangle + |011\rangle)$$

$$Z_2 Z_1 (|100\rangle + |011\rangle) = \boxed{+1}(|100\rangle + |011\rangle)$$

or

$$Z_1 Z_0 (|001\rangle + |110\rangle) = \boxed{+1}(|001\rangle + |110\rangle)$$

$$Z_2 Z_1 (|001\rangle + |110\rangle) = \boxed{-1}(|001\rangle + |110\rangle)$$

- Eigenvalues of $Z_1 Z_0$, $Z_2 Z_1$ are same as measurement outcomes!
- We call $S = \{Z_1 Z_0, Z_2 Z_1\}$ the stabilizers of the 3 qubit repetition

$$\mathcal{O}|\psi\rangle = +1|\psi\rangle \quad \text{where } \mathcal{O} \in S$$

- We say „$|\psi\rangle$ is stabilized by $\mathcal{O}$".
- Observe now that corrupted states are of the form

$$|\Psi_j\rangle = X_j |\Psi\rangle$$

- Now recall: $Z_i X_j = X_j Z_i$ if $i \neq j$ and $Z_i X_j = -X_j Z_i$ if $i = j$
- What happens to stabilizers if there's an error on Qubit 0?

$$S_1 : Z_1 Z_0 |\Psi_0\rangle = Z_1 Z_0 X_0 |\Psi\rangle = -Z_1 X_0 Z_0 |\Psi\rangle = -X_0 Z_1 Z_0 |\Psi\rangle = -X_0 |\Psi\rangle = -1|\Psi_0\rangle$$

$$S_2 : Z_2 Z_1 |\Psi_0\rangle = Z_2 Z_1 X_0 |\Psi\rangle = \_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_ = \_\_\_\_\_ = +1|\Psi_0\rangle$$

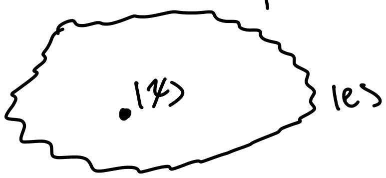
- Make a table of commutation signs:

	$\mathbb{1}$	X_0	X_1	X_2
$S_1 = Z_1 Z_0$	+1	−1	−1	+1
$S_2 = Z_2 Z_1$	+1	+1	−1	−1

- Wow! This the <u>same</u> table as our ancilla based table!

More general quantum errors

- We assumed A LOT in the preceeding section.
- Only discrete bit flips: What about e.g. the partial phases? How can we ever hope to correct those?
- Consider our qubit(s) interacting with an environment:



$$|e\rangle|0\rangle \mapsto |e_0\rangle|0\rangle + \underbrace{|e_1\rangle|1\rangle}_{\text{Bit flips}}$$

$$|e\rangle|1\rangle \mapsto |e_2\rangle|0\rangle + |e_3\rangle|1\rangle$$

- Lets rewrite into a more suggestive form:
- Recall: $P_0 = |0\rangle\langle 0| = \frac{1}{2}(\mathbb{1} + Z)$, $P_1 = |1\rangle\langle 1| = \frac{1}{2}(\mathbb{1} - Z)$

$$|e\rangle|x\rangle \mapsto \left(|e_0\rangle\otimes\mathbb{1} + |e_1\rangle\otimes X\right)\left(\mathbb{1}\otimes P_0|x\rangle\right) + \left(|e_2\rangle\otimes\mathbb{1} + |e_3\rangle\otimes X\right)\left(\mathbb{1}\otimes P_1|x\rangle\right)$$

- Recall: $Y = iZX$, Write out and do a lot of algebra:

$$|e\rangle|x\rangle \mapsto \frac{1}{2}\Big[\underbrace{(|e_0\rangle+|e_3\rangle)}_{|e_a\rangle}\otimes\mathbb{1} + \underbrace{(|e_0\rangle-|e_3\rangle)}_{|e_b\rangle}\otimes Z + \underbrace{(|e_2\rangle+|e_1\rangle)}_{|e_c\rangle}\otimes X + \underbrace{i(|e_2\rangle-|e_1\rangle)}_{|e_d\rangle}\otimes Y\Big](\mathbb{1}\otimes|x\rangle)$$

- This result generalizes to any state $|\psi\rangle$:

$$|e\rangle|\psi\rangle \mapsto \left[|e_a\rangle\mathbb{1} + |e_b\rangle Z + |e_c\rangle X + |e_d\rangle Y\right]|\psi\rangle$$

Nothing happened

There was a phaseflip

There was a bitflip

There was a combined bit/phase flip

- Generalize to n-qubit system:

$$|e\rangle|\Psi\rangle \mapsto \sum_{\mu_1=0}^{3}\cdots\sum_{\mu_n=0}^{3} |e_{\mu_1\dots\mu_n}\rangle P^{(\mu_1)}\otimes\cdots\otimes P^{(\mu_n)}|\Psi\rangle$$

- Where $P^{(0)} = \mathbb{1}$, $P^{(1)} = X$, $P^{(2)} = Y$, $P^{(3)} = Z$
- Leads to general expression for errors on n qubits:

$$\boxed{|e\rangle|\Psi\rangle \mapsto \left[|e_a\rangle\mathbb{1} + \sum_{i=0}^{n-1}|e_{bi}\rangle Z_i + |e_{ci}\rangle X_i + |e_{di}\rangle Y_i\right]|\Psi_i\rangle}$$

- „Just" need to determine which stabilizers to measure to diagnose errors!