Assignment 2.

1: Magic states & T-gates.

a) $CNOT|\phi\rangle = CNOT(\alpha|0\rangle + \beta|1\rangle)$.

$= (|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes X)(\alpha|0\rangle + \beta|1\rangle)$.

$= \alpha|0\rangle\langle 0|0\rangle + \beta|0\rangle\langle 0|1\rangle + \alpha|1\rangle\langle 1|0\rangle \otimes X + \beta|1\rangle\langle 1|1\rangle \otimes X$.

$= \alpha|0\rangle + \beta|1\rangle \otimes X$.

$CNOT|\psi\rangle = CNOT\frac{1}{\sqrt{2}}(|0\rangle + e^{i\frac{\pi}{4}}|1\rangle)$ $\qquad e^{i\frac{\pi}{4}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$

$= \frac{1}{\sqrt{2}} \cdot (|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes X)(|0\rangle + \frac{1}{\sqrt{2}}(1+i)|1\rangle)$ $\qquad = \frac{1}{\sqrt{2}}(1+i)$

$= \frac{1}{\sqrt{2}}[|0\rangle\langle 0|0\rangle + |1\rangle\langle 1|1\rangle \cdot \frac{1}{\sqrt{2}}(1+i) \cdot X]$.

$= \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}(1+i)|1\rangle \otimes X$.

b) $P_{|0\rangle} = |\frac{1}{\sqrt{2}}|^2 = \frac{1}{2}$ $\qquad P_{|1\rangle} = |\frac{1}{2}(1+i)|^2 = \frac{1}{4}(1+2i+i^2) = \frac{1}{2}i$.

c) $T|\phi\rangle = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{8}} \end{bmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ e^{i\frac{\pi}{8}}\beta \end{pmatrix} = \alpha|0\rangle + e^{i\frac{\pi}{8}}\beta|1\rangle$.

$\cancel{e^{i\frac{\pi}{8}} = \cos\frac{\pi}{8} + i\sin\frac{\pi}{8}}$

$\beta = 0, \Rightarrow \alpha|0\rangle$.

d) $T|\phi\rangle = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{8}} \end{bmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ e^{i\frac{\pi}{8}}\beta \end{pmatrix} = \alpha|0\rangle + e^{i\frac{\pi}{8}}\beta|1\rangle$.

2: Period Finding.

$f(x) = 7^x \mod 10.$

a). periodicity is 10.

b). i) $|00\rangle$

ii) $QFT_d |00\rangle = QFT_{128} |00\rangle$

$$= \frac{1}{\sqrt{128}} \sum_{k \in \{0, p, 2p, \ldots, d p\}} e^{2\pi \cdot i \cdot k \cdot l / d} |0\rangle |0\rangle$$

$\Rightarrow \sum_{k \in \{0, p, 2p \ldots, d \cdot p\}} e^{2\pi \cdot i \cdot k \cdot l / d}$, k = integer multiples of p., d = multiple of p.

· if l = integer multiple of $\frac{d}{p}$.

$\Rightarrow$ ~~integer~~ $\frac{k \cdot l}{d} = \frac{\text{integer multiples of } p \cdot \text{integer multiple of } \frac{d}{p}}{\text{multiple of } p.} = \frac{d}{p}.$

$$= \frac{1}{\sqrt{128}} \frac{d}{p} . |0\rangle |0\rangle$$

$$= \frac{1}{\sqrt{128}} \cdot \frac{128}{p} |0\rangle |0\rangle.$$

$$= \frac{\sqrt{128}}{p} |0\rangle |0\rangle.$$

iii). $O_{f(x)} QFT_d |00\rangle = O_{f(x)} \cdot \frac{\sqrt{128}}{p} |00\rangle.$

$$= \frac{\sqrt{128}}{p} |f(x)\rangle |0\rangle.$$

iv). $QFT_d O_{f(x)} QFT_d |00\rangle = QFT_d \cdot \frac{\sqrt{128}}{p} |f(x)\rangle |0\rangle$

$$= \frac{1}{\sqrt{128}} \cdot \frac{d}{p} \cdot \frac{\sqrt{128}}{p} |f(x)\rangle |0\rangle$$

$$= \frac{d}{p^2} |f(x)\rangle |0\rangle.$$

3. Error-Correcting codes.

a). $CNOT_1 |\bar{0}\rangle = CNOT_1 \frac{1}{2}(|00\rangle + |11\rangle) \otimes (|00\rangle + |11\rangle) = \frac{1}{2}(|00\rangle + |10\rangle) \otimes (|00\rangle + |10\rangle)$

$CNOT_2 |\bar{1}\rangle = CNOT_2 \frac{1}{2}(|00\rangle - |11\rangle) \otimes (|00\rangle - |11\rangle) = \frac{1}{2}(|00\rangle \cancel{+} - |01\rangle) \otimes (|00\rangle - |01\rangle)$

b). ~~CNOT~~ $H |\bar{0}\rangle = H \frac{1}{2}(|00\rangle + |11\rangle) \otimes (|00\rangle + |11\rangle) = \frac{1}{2}(|+0\rangle + |-1\rangle \otimes (|00\rangle + |11\rangle)$

$H |\bar{1}\rangle = H \frac{1}{2}(|00\rangle - |11\rangle) \otimes (|00\rangle - |11\rangle) = \frac{1}{2}(|+0\rangle - |-1\rangle) \otimes (|00\rangle - |11\rangle)$.