Exercise Sheet 3 – solutions

1 a) $\|\mathbb{1} - Z\|_{op} = \left\| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\|_{op}$

$$= \left\| \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \right\|_{op} = \max_{\vec{v} \text{ s.t. } \|\vec{v}\|=1} \left\| \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \vec{v} \right\|$$

$$= \left\| \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\| = \sqrt{\left\| \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\|^2} = \sqrt{4} = 2$$

the best $\vec{v}$ will be $\vec{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\|\mathbb{1} - X\|_{op} = \left\| \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right\|_{op}$$

$$= \max_{\vec{v}} \left\| \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \vec{v} \right\| =$$

$$= \max \sqrt{|v_1 - v_2|^2 + |v_2 - v_1|^2}$$

$\vec{v}$

$$= \max_{v} \sqrt{2|v_1 - v_2|^2}$$

$$= \sqrt{2} \max_{\vec{v}} \sqrt{|v_1 - v_2|^2}$$

the best $\vec{v}$ will be $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$

$$\leq \sqrt{2} \cdot \sqrt{\left(\frac{2}{\sqrt{2}}\right)^2} = 2$$

b) $\|U - V\|_{op} = \max_{\vec{v}} \|(U-V)\vec{v}\|$

U unitary
→ $U^\dagger$ is unitary
→ preserves norm (sheet 0)

$$= \max_{\vec{v}} \|U^\dagger (U - V)\vec{v}\|$$

$$= \|U^\dagger (U-V)\|_{op}$$

$$= \|\mathbb{1} - U^\dagger V\|_{op}$$

i) Operator norm satisfies triangle inequality:

$$\| A + B \|_{op} = \max_{\vec{v}} \|(A+B)v\|$$

$$= \max_{\vec{v}} \| Av + Bv \|$$

triangle inequality for the norm from sheet 1. maximizing over different vectors can only make it even bigger.

$$\leq \max_{\vec{v}} \|Av\| + \max_{\vec{\omega}} \|B\vec{\omega}\|$$

$$= \|A\|_{op} + \|B\|_{op}$$

$$\|U - V\|_{op} = \max_{\vec{v}} \|(U_T U_{T-1} \cdots U_1 - V_T V_{T-1} \cdots V_1)\vec{v}\|$$

start: $U = U_T U_{T-1} \cdots U_{j+1} U_j U_{j-1} \cdots U_1$

replace a

single unitary $U^{(j)} = U_T U_{T-1} \cdots U_{j+1} V_j U_{j-1} \cdots U_1$

$$\| U - U^{(j)} \|_{op} = \| (U_T \cdots U_{j+1})(U_j - V_j)(U_{j-1} \cdots U_1) \|_{op}$$

unitaries conserve norm

$$\overset{\downarrow}{=} \| U_j - V_j \|_{op} \quad \forall j$$

replacing two

Unitaries: $U^{(j,k)}$

$$\| U - U^{(j,k)} \|_{op} = \| U_T \cdots U_j \cdots U_k \cdots U_1 - U_T \cdots V_j \cdots V_k \cdots U_1 \|_{op}$$

adding zero

$$\overset{\downarrow}{=} \| U_T \cdots U_j \cdots U_k \cdots U_1 + U_T \cdots V_j \cdots U_k \cdots U_1$$
$$- U_T \cdots V_j \cdots U_k \cdots U_1 - U_T \cdots V_j \cdots V_k \cdots U_1 \|_{op}$$

triangle inequality

$$\overset{\downarrow}{\leq} \| U_T \cdots (U_j - V_j) \cdots U_k \cdots U_1 \|_{op} + \| U_T \cdots V_j \cdots (U_k - V_k) \cdots U_1 \|_{op}$$

$$= \|U_j - V_j\|_{op} + \|U_k - V_k\|_{op}$$

U, V:

adding a lot of zeros...

$$\|U - V\|_{op} = \|U - U^{(1)} + U^{(1)} - U^{(1,2)} + U^{(1,2)}$$

$$\dots - U^{(1,2,\dots T-1)} + U^{(1,2\dots T-1)} - V\|_{op}$$

$$\leq \underbrace{\|U - U^{(1)}\|_{op}}_{\|U_1 - V_1\|_{op}} + \underbrace{\|U^{(1)} - U^{(1,2)}\|_{op}}_{\|U_2 - V_2\|_{op}} + \dots + \underbrace{\|U^{(1,2,\dots T-1)} - V\|_{op}}_{\|U_T - V_T\|_{op}}$$

$$= \sum_j \|U_j - V_j\|_{op}$$

d)

$$p_j(i) = |\langle\psi_j|\phi_i\rangle|^2 = |\langle\psi_j|U|\phi_i\rangle|^2 = \langle\phi|U^\dagger|\psi_j\rangle\langle\psi_j|U|\phi\rangle$$

$$|p_j(1) - p_j(2)| = \left| |\langle\phi|U_1|\psi_j\rangle|^2 - |\langle\phi|U_2|\psi_j\rangle|^2 \right|$$

reformulation of the prob.

$$= \left|\langle\phi|U_1^\dagger|\psi_j\rangle\langle\psi_j|U_1|\phi\rangle - \langle\phi|U_2^\dagger|\psi_j\rangle\langle\psi_j|U_2|\phi\rangle\right|$$

adding zero

$$= \left|\langle\phi|U_1^\dagger|\psi_j\rangle\langle\psi_j|(U_1-U_2)|\phi\rangle - \langle\phi|U_1^\dagger|\psi_j\rangle\langle\psi_j|U_2|\phi\rangle - \langle\phi|U_2^\dagger|\psi_j\rangle\langle\psi_j|U_2|\phi\rangle\right|$$

$$= \left|\langle\phi|U_1^\dagger|\psi_j\rangle\langle\psi_j|(U_1-U_2)|\phi\rangle + \langle\phi|(U_1^\dagger-U_2^\dagger)|\psi_j\rangle\langle\psi_j|U_2|\phi\rangle\right|$$

triangle inequality

$$\leq \underbrace{|\langle\phi|U_1^\dagger|\psi_j\rangle\langle\psi_j|(U_1-U_2)|\phi\rangle|} + \underbrace{|\langle\phi|(U_1^\dagger-U_2^\dagger)|\psi_j\rangle\langle\psi_j|U_2|\phi\rangle|}_{=|\langle w|v\rangle|}$$

scalar product $\langle v|w\rangle$

w) $|v\rangle = |\psi_j\rangle\langle\psi_j|U_1|\phi\rangle$

$|w\rangle = (U_1-U_2)|\phi\rangle$

use: $|\langle v|w\rangle| \leq \|v\|\|w\|$

$$= \| |\psi_j\rangle\langle\psi_j|U_1|\phi\rangle \| \cdot \| (U_1 - U_2)|\phi\rangle \|$$

$$\leq \| |\psi_j\rangle\langle\psi_j|U_1|\phi\rangle \| \cdot \| U_1 - U_2 \|_{op}$$

$$\leq 2 \| |\psi_j\rangle\langle\psi_j|U_1|\phi\rangle \| \cdot \| U_1 - U_2 \|_{op}$$

$$\leq 2 \cdot \| U_1 - U_2 \|_{op}$$

$$\| |\psi_j\rangle\langle\psi_j| U_j |\phi\rangle \| = \underbrace{\| |\psi_j\rangle\langle\psi_j| \|_{op}}_{\substack{\text{has eigenvalues} \\ 1 \,\&\, \text{zeroes}}} \underbrace{\| U|\phi\rangle \|}_{= \| |\phi\rangle \| = 1} \leq 1$$

2) a) $|\psi_1\rangle = H^{\otimes n}|0\rangle \otimes H|1\rangle$

$$= \frac{1}{\sqrt{2^n}} \sum_{x \in S_n} |x\rangle \otimes |-\rangle$$

$$= \frac{1}{\sqrt{2^{n+1}}} \left[\sum_{x \in S_n} |x\rangle \otimes |0\rangle - |x\rangle \otimes |1\rangle \right]$$

b) $|\psi_2\rangle = \frac{1}{\sqrt{2^{n+1}}}\left[\sum_{x\in S_n} U_f(|x\rangle\otimes|0\rangle) - U_f(|x\rangle\otimes|1\rangle)\right]$

$$= \frac{1}{\sqrt{2^{n+1}}}\left[\sum_{x\in S_n} |x\rangle\otimes|0\oplus f(x)\rangle - |x\rangle\otimes|1\oplus f(x)\rangle\right]$$

$$= \frac{1}{\sqrt{2^{n+1}}}\left[\sum_{x\in S_n} |x\rangle\otimes|f(x)\rangle - |x\rangle\otimes|1\oplus f(x)\rangle\right]$$

$$= \frac{1}{\sqrt{2^{n+1}}}\left(\sum_{x\in S_n} |x,f(x)\rangle - |x,1\oplus f(x)\rangle\right)$$

c) $|\psi_2\rangle = \frac{1}{\sqrt{2^{n+1}}}\left(\sum_{x\in S_n} |x,f(x)\rangle - |x,1\oplus f(x)\rangle\right)$

If $f(x)=0$, then $1\oplus f(x)=1 \Rightarrow$ have $|0\rangle-|1\rangle$

If $f(x)=1$, $1\oplus f(x)=0. \Rightarrow$ have $|1\rangle-|0\rangle$

$$= \frac{1}{\sqrt{2^n}} \Big(\sum_{\substack{x \in S_n \\ f(x)=0}} |x\rangle \otimes |-\rangle + \sum_{\substack{x \in S_n \\ f(x)=1}} |x\rangle \otimes \underbrace{(-|-\rangle)}_{\sum_x |x\rangle \otimes |-\rangle} \Big)$$

$$= \frac{1}{\sqrt{2^n}} \sum_{x \in S_n} (-1)^{f(x)} |x\rangle \otimes |-\rangle$$

→ there is a − sign whenever $f(x)=1$, and no sign change when $f(x)=0$.

d) $$H^{\otimes 1} |x\rangle = \frac{1}{\sqrt{2}} \sum_{x \in \{0,1\}} (-1)^{x \cdot y} |y\rangle$$

$$= \frac{1}{\sqrt{2}} \left(|0\rangle + (-1)^x |1\rangle \right)$$

for $x=0$: $H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ (✓)

$H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ (✓)

$$H^{\otimes n}|x\rangle = H^{\otimes n}|x_1 \dots x_n\rangle = H|x_1\rangle \otimes \dots \otimes H|x_n\rangle$$

$$= \frac{1}{\sqrt{2}^n} \sum_{y_1 \in \{0,1\}} (-1)^{y_1 x_1} |y_1\rangle \otimes \sum_{y_2 \in \{0,1\}} (-1)^{y_2 x_2} |y_2\rangle \otimes \dots \sum_{y_n \in \{0,1\}} (-1)^{y_n x_n} |y_n\rangle$$

$$= \frac{1}{\sqrt{2}^n} \sum_{\substack{y_1 \in \{0,1\} \\ y_2 \in \{0,1\} \\ \vdots \\ y_n \in \{0,1\}}} (-1)^{x_1 y_1 + x_2 y_2 + \dots + x_n y_n} |y_1\rangle \otimes |y_2\rangle \otimes \dots \otimes |y_n\rangle$$

$$= \frac{1}{\sqrt{2}^n} \sum_{y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle$$

e) $|\psi_3\rangle = (H^{\otimes n})|\psi_2\rangle$

$$= H^{\otimes n}\left(\frac{1}{\sqrt{2^n}} \sum_x (-1)^{f(x)} |x\rangle\right)$$

$$= \frac{1}{\sqrt{2^n}} \sum_x (-1)^{f(x)} \left(H^{\otimes n} |x\rangle\right)$$

$$= \frac{1}{\sqrt{2^n}} \sum_{x,y} (-1)^{f(x)} (-1)^{x \cdot y} |y\rangle$$

f) $|\langle 0 \cdots 0 | \psi_3 \rangle|^2$

only terms w/ $y = 0 \cdots 0$ survive

$$= \left| \frac{1}{\sqrt{2^n}} \sum_{x,y} (-1)^{f(x)} (-1)^{x \cdot y} \langle 0 \cdots 0 | y \rangle \right|^2$$

$$= \left| \frac{1}{2^n} \sum_x (-1)^{f(x)} \right|^2 = \left(\frac{1}{2^n} \left(\sum_{x \in S^n} (-1)^{f(x)} \right) \right)^2$$

g) Let f be balanced: There are equally many cases of $f(x) = 0$ and $f(x) = 1$

$$p(0\cdots0) = \left(\frac{1}{2^n}\right)^2 \Big(\sum_x (-1)^{f(x)} \Big)^2$$

$$= \left(\frac{1}{2^n}\right)^2 \big(\underbrace{1 - 1 + 1 - 1 + \cdots + 1 - 1}_{\substack{2^n \text{ nrs : exactly half are } +1, \\ \text{exactly half are } -1.}} \big)^2$$

$$= 0.$$

for $f(x)$ constant: $f(x) = f(y) \ \forall x, y$

$$p(0\cdots0) = \Big(\frac{1}{2^n} (-1)^{f(x_0)} \sum_{x \in S_n} 1 \Big)$$

$$= \Big(\frac{1}{2^n} (-1)^{f(x_0)} \cdot 2^n \Big)^2 = 1.$$

3 See Qiskit / Quantum Lab tutorials

In the link from Monday's tutorial:

$\Rightarrow$ how to make an oracle for any 1-bit function

Qiskit textbook:

Worked example for a 2-bit function

& example for a 3-bit function
(& general way)

From Monday's tutorial: ibm.biz/quantum-hands-on

Qiskit textbook: https://qiskit.org/textbook/ch-algorithms/deutsch-jozsa.html