Problem set 4

2) a) To show: let $y_j \in \mathbb{C}^N$ be the j-th column of the discrete Fourier transform. We need to show that

$\langle y_j, y_k \rangle = \delta_{k,l}$.

Proof:

$$\langle y_j, y_k \rangle = \sum_{l=0}^{2^n-1} \frac{1}{N} \overline{\left(w_N^{lj}\right)} \, w_N^{kl}$$

$$= \sum_{l=0}^{2^n-1} \frac{1}{N} e^{\frac{2\pi i l}{N}(k-j)}.$$

If $k = j$:

$$\langle y_j, y_k \rangle = \sum_{\ell=0}^{2^n-1} \frac{1}{N} = 1.$$

If $k \neq j$:

$\boxed{\text{Fact:}}$ if x is a N-th root of unity ($x^N = 1$), then $\sum_{\ell=0}^{2^n-1} x^\ell = 0$. To see this, note that:

$$x^N - 1 = (x-1)\left(\sum_{\ell=0}^{N-1} x^\ell\right).$$

Let $r = k-\ell \neq 0$. Then w_N^r is a N-th root of unity. Thus, we conclude that

$$\sum_{l=0}^{N-1} \frac{1}{N} \omega_N^{l(k-j)} = \frac{1}{N} \sum_{l=0}^{N-1} \left(\omega_N^{\tilde{j}}\right)^l$$

$$= 0 .$$

$\square$

b) Note that when a matrix acts on the k-th computational basis state, the output is just the corresponding k-th column.

c) Expand j as

$$j = \sum_{l=1}^{n} 2^{n-l} j_l .$$

Then $e^{2\pi i k \frac{j}{2^n}} =$

$$e^{2\pi i k \sum_{l=1}^{n} \frac{2^{n-l}}{2^n} j_l} = e^{2\pi i \sum_{l=1}^{n} j_l 2^{-l}} .$$

$$= \prod_{l=1}^{n} e^{2\pi i \frac{j_l}{2^l}} .$$

This shows that

$$|k\rangle \longmapsto \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \prod_{l=1}^{n} e^{2\pi k \frac{j_l}{2^l}} |j_1, \cdots, j_n\rangle$$

To show that the exp. above is equal to

$$\bigotimes_{l=1}^{n} \frac{1}{\sqrt{2}} \left(|0\rangle + e^{2\pi i k/2^l} |1\rangle \right),$$

let us proceed by induction.

For $n=1$: clear.

Now assume the equality holds for all $n \leq m$. We then have:

$$\bigotimes_{l=1}^{n+1} \frac{1}{\sqrt{2}} \left(|0\rangle + e^{2\pi i k / 2^l} |1\rangle \right) =$$

$$\bigotimes_{l=1}^{n} \frac{1}{\sqrt{2}} \left(|0\rangle + e^{2\pi i k / 2^l} |1\rangle \right)$$

$$\otimes \frac{1}{\sqrt{2}} \left(|0\rangle + e^{2\pi i k / 2^{n+1}} |1\rangle \right)$$

$$= \left(\sum_{j=0}^{N-1} \prod_{l=1}^{n} e^{\frac{2\pi i k j_l}{2^l}} |j_1 \cdots j_n\rangle \right) \otimes |\psi_{n+1}\rangle,$$

$\hookrightarrow$ induction hypothesis

where $|\psi_{n+1}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{\frac{2\pi i k}{2^{n+1}}}|1\rangle\right)$

We see

$$\frac{1}{\sqrt{N}}\left(\sum_{j=0}^{N-1} \prod_{l=1}^{n} e^{2\pi i k \frac{j_l}{2^l}} |j_1 \cdots j_n\rangle\right) \otimes |\psi_{n+1}\rangle =$$

$$\frac{1}{\sqrt{2N}} \sum_{j=0}^{N-1} \prod_{l=1}^{n} e^{2\pi i \frac{j_l}{2^l}} |j_1, \cdots, j_n\, 0\rangle \quad +$$

$$\frac{1}{\sqrt{2N}} \sum_{j=0}^{N-1} \prod_{l=1}^{n} e^{2\pi i \frac{j_l k}{2^l}} \cdot e^{\frac{2\pi i k}{2^{n+1}}} |j_1, \cdots, j_n, 1\rangle =$$

$$\frac{1}{\sqrt{2N}} \sum_{j=0}^{2N-1} \prod_{l=1}^{n+1} e^{2\pi i \frac{j_l k}{2^l}} |j_1, \cdots, j_n, j_{n+1}\rangle$$

which shows the claim for

$m = n+1$.

$\square$

d) clear.

e) We have $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$,

$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ and

$SH = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$. The fact

that $F_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} H & S \cdot H \\ H & -SH \end{pmatrix}$ follows

from direct inspection.

The unitary corresponding to exchanging qubits is given by

$Sw \, |i\rangle|j\rangle = |j\rangle|i\rangle$.

Thus $Sw \left[|i\rangle\langle j| \otimes |k\rangle\langle l| \right] =$

$|k\rangle\langle j| \otimes |i\rangle\langle j|$.

The claim then follows from direct inspection:

$$2F_4 = \ |0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 0| \otimes |0\rangle\langle 1| + |0\rangle\langle 0| \otimes |1\rangle\langle 0| + i\, |0\rangle\langle 0| \otimes |1\rangle\langle 1| \ldots$$

$$\Rightarrow 2\, S_W F_4 = F_4$$

Disclaimer: note that the normalization is wrong in the exercise sheet, it should be $\frac{1}{\sqrt{2}}\begin{pmatrix} H & SH \\ H & -SH \end{pmatrix}$

d)

$$H \otimes \mathbb{1} \left(|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes S \right)(\mathbb{1} \otimes H) =$$

$$|+\rangle\langle 0| \otimes \mathbb{1} + |-\rangle\langle 1| \otimes S)(\mathbb{1} \otimes H) =$$

$$|+\rangle\langle 0| \otimes H + |-\rangle\langle 1| \otimes SH =$$

$$\frac{1}{\sqrt{2}} \Big(|0\rangle\langle 0| \otimes H + |1\rangle\langle 0| \otimes H + |0\rangle\langle 1| \otimes SH - |1\rangle\langle 1| \otimes SH \Big)$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} H & SH \\ H & -SH \end{pmatrix}.$$

Circuit: note that

$|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes S$ corresponds to a controlled S-gate. We get

$\hat{F}_4 =$