Introduction to Quantum Computing

Problem Set 3: quantum circuits & first quantum algorithms.
University of Copenhagen

Exercise 1: (Approximating quantum gates) In this exercise we are going to follow up on the idea of approximating quantum circuits by each other. In the lecture, we defined the *operator norm* of a matrix $A \in \mathbb{C}^{d\times d}$ as

$$\|A\|_{\text{op}} = \max_{v\in\mathbb{C}^d \text{ s.t. } \|v\|=1} \|Av\|. \tag{1}$$

In turn, the *distance* between two matrices A, B can be defined as:

$$\|A - B\|_{\text{op}}.$$

(a) What is the distance between the 2×2 identity matrix and Z as well as between 2×2 the identity matrix and X?

(b) Show the relation $\|U - V\|_{\text{op}} = \|\mathbb{1} - U^\dagger V\|_{\text{op}}$ for U, V unitary.

(c) Let U and V be products of n-qubit unitaries $U = U_T U_{T-1} \cdots U_1$, $V = V_T V_{T-1} \cdots V_1$. Show that $\|U - V\|_{\text{op}} \leq \sum_j \|U_j - U_j\|_{\text{op}}$.

Hint: Argue first that the operator norm as defined in (1) *satisfies the triangle inequality, i.e.* $\|A + B\|_{op} \leq \|A\|_{op} + \|B\|_{op}$. *Try to compare* $U = U_T \cdots U_j \cdots U_1$ *and* $U' = U_T \cdots V_j \cdots U_1$, *where only the j-th gate has been changed.*

(d) Consider a pure quantum state $|\phi\rangle \in \mathbb{C}^d$ to which we apply unitary circuits U_1 or U_2 resulting in the states $|\phi_i\rangle = U_i |\phi\rangle$. Performing a projective measurement with respect to a basis $\{|\psi_j\rangle\}_{j=0}^{d-1}$, will lead to the outcome probabilities $p_j(i) = |\langle\psi_j|\phi_i\rangle|^2$. Show that the difference between $p_j(1)$ and $p_j(2)$ is bounded by twice the operator distance of U_1 and U_2.

Hint: Try to rewrite the expression for the difference of the probabilities so that $(U_1 - U_2)$ *appears. Recall that the scalar product of two vectors obeys the Cauchy-Schwarz-inequality:* $\langle v|w\rangle \leq \|v\|\|w\|$.

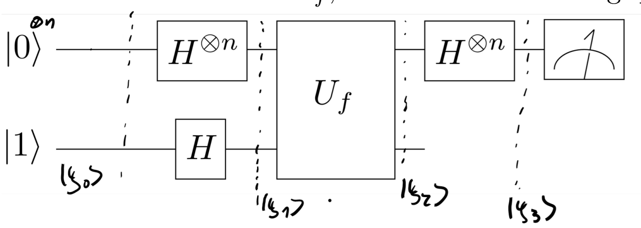
Exercise 2: (Deutsch-Jozsa algorithm) Consider a boolean function $f : \{0,1\}^n \mapsto \{0,1\}$ which satisfies one of the following two assumptions:

- f is constant, i.e. $f(x) = f(y)$ for all $x, y \in \{0,1\}^n$
- f is balanced, i.e. $|\{f(x) = 0\}| = |\{f(x) = 1\}| = 2^{n-1}$

Hence, we are promised that f is either constant or balanced, but we do not know which of the two it is. We want to understand why the following quantum algorithm allows us to resolve this question about f with a single query. As in the lecture, we define a quantum oracle for f via the unitary operation

$$U_f \, |x, b\rangle = |x, b \oplus f(x)\rangle \quad \text{with: } x \in \{0,1\}^n \,, b \in \{0,1\} \,. \tag{2}$$

With this definition of U_f, consider the following quantum circuit



where the measurement is done in the computational basis and $|\Psi_i\rangle$ are the intermediate states during the computation.

a) Determine the state $|\Psi_1\rangle$.

b) Show that $|\Psi_2\rangle = \frac{1}{\sqrt{2^{n+1}}} \sum_{i \in \{0,1\}^n} |i, f(i)\rangle - |i, 1 \oplus f(i)\rangle$

c) Deduce that $|\Psi_2\rangle$ can equivalently be written as

$$|\Psi_2\rangle = \left(\frac{1}{\sqrt{2^n}} \sum_{i \in \{0,1\}^n} (-1)^{f(i)} \, |i\rangle \right) |-\rangle \tag{3}$$

d) Derive the relation $H^{\otimes n} \, |i\rangle = \frac{1}{\sqrt{2^n}} \sum_{j \in \{0,1\}^n} (-1)^{i.j} \, |j\rangle$ for $i.j = \sum_{l=0}^{n-1} i_l \cdot j_l$, we discussed in the lecture.

e) Use the relation from d) to determine $|\Psi_3\rangle$.

f) Show that the probability to obtain the all zeros string as a measurement result in $|\Psi_3\rangle$ is given by $\left(\frac{1}{2^n} \sum_{j \in \{0,1\}^n} (-1)^{f(j)} \right)^2$.

g) Evaluate the expression for the outcome probability from f) depending on whether f is constant or balanced.

Exercise 3: (Getting started with qiskit) Next Monday, we will have an introductory course on IBM's quantum computing framework qiskit. Here, we put this knowledge to use for the Deutsch-Jozsa algorithm. Hence, following up on exercise 2 from this sheet, have a look at the presentation of the Deutsch-Jozsa algorithm in the qiskit textbook (`https://qiskit.org/textbook/ch-algorithms/deutsch-jozsa.html`).
Follow the discussion there on how to implement the oracle for balanced/constant functions for $n = 3$ and use this proposal to implement your own version of the Deutsch-Jozsa algorithm for an f of your choosing. Check that it generates the expected output behaviour. Let it run on a real quantum processor!