Tutorial 3

22 January 2022 10:59

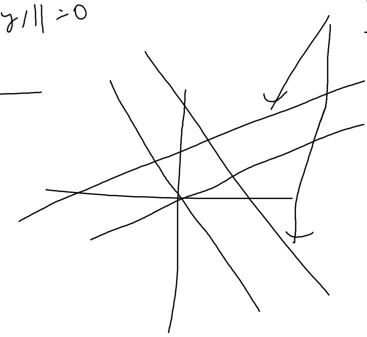
$f: S \longrightarrow \mathbb{R}$, $S \subseteq \mathbb{R}^n$, f is continuous, differentiable at x

$\forall y \in S$

$$f(y) = \underbrace{f(x) + \nabla f(x)^T(y-x)}_{\text{affine approximation}} + \underbrace{r_1(y)^T(y-x)}_{\text{small remainder}}$$

$\lim_{y \to x} \|r_1(y)\| = 0$

affine but not linear



$$\boxed{f(y) = f(x) + \nabla f(x)^T (y-x)}$$

For functions of single variables

$$f(y) = f(x) + f'(x)(y-x)$$

affine in y.

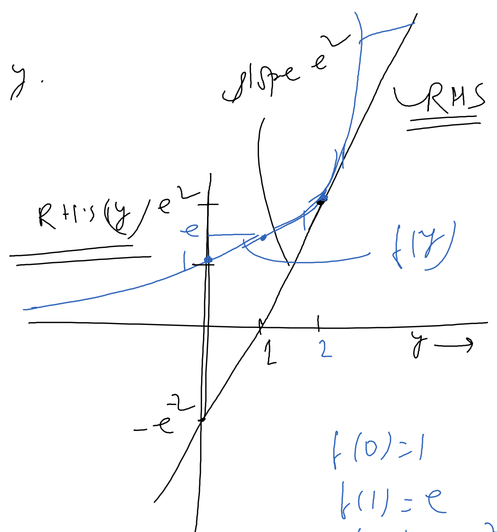
Example $f: \mathbb{R} \to \mathbb{R}$

$\boxed{f(x) = e^x}$

$f'(x) = e^x$

Fix $x = 2$

$$f(y) = e^2 + e^2(y-2)$$



$f(0) = 1$

$f(1) = e$

$f(2) = e^2$

$f'(x) = e^x$, $f'(2) = e^2$

approximation around $x = 5$

$= e^5 + e^5(y-5)$

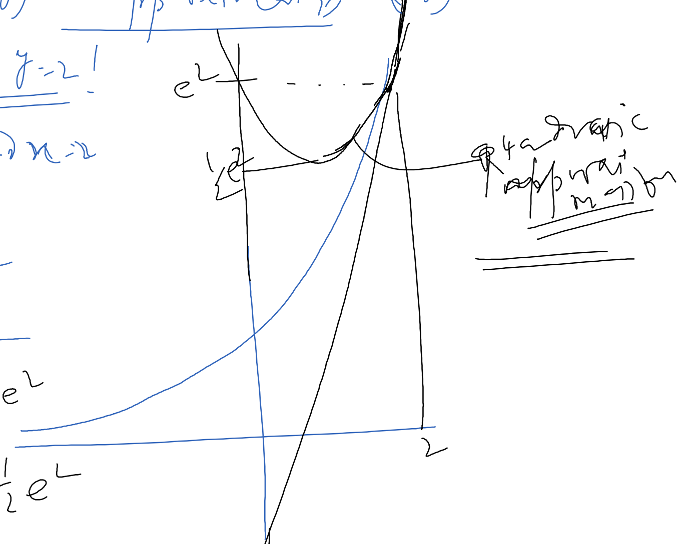
RHS(y) approximates f(y) around $y = 2$!

quadratic approximation around $x = 2$

$$e^2 + e^2(y-2) + \frac{1}{2}e^2(y-2)^2$$

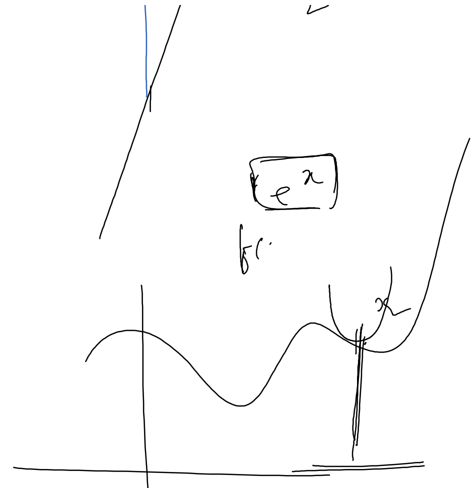
$\text{RHS}(0) = e^2 - \cancel{2e^2} + \cancel{2e^2} = e^2$

$\text{RHS}(1) = e^2 - e^2 + \frac{1}{2}e^2 = \frac{1}{2}e^2$



$\underline{RHS(1)} = e^2 - e^2 + \frac{1}{2}e^2 = \frac{1}{2}e^2$

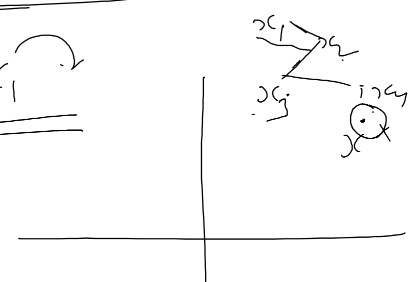
$RHS(2) = e^2$



$f(x_3) < f(x_2)$

$\min_{x \in S} f(x)$ — let the soln be x^*

x_1



$\tilde{f}(x)$ – quadratic approximation of $f(x)$ around x_2

$$\min_{x \in B_\epsilon(x_2)} \tilde{f}(x) \implies x_3 \text{ is soln to this problem}$$

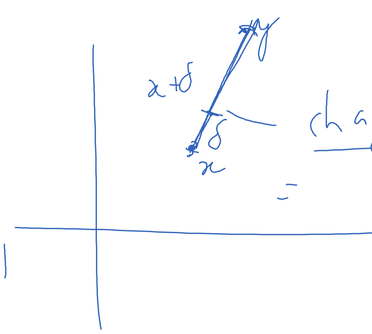
$\|x_3 - x_2\| \le \epsilon$

Affine approximation

$f(y) \approx f(x) + \nabla f(x)^T (y-x)$

$f(y) - f(x) \approx \nabla f(x)^T (y-x)$

$= \nabla f(x)^T \underbrace{\frac{(y-x)}{\|y-x\|}}_{u}$ $\|y-x\|$



change in fn

u. direction

= (directional derivative of f at x in the ... of $(y-x)$) × distance b/w y and x

$= d_{xy}$

$= d_{xy} \times \|y-x\|$

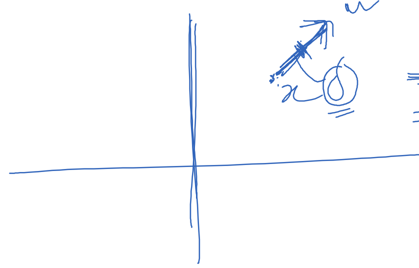
$\frac{f(y)-f(x)}{\|y-x\|} \approx d_{xy}$

rate of change of the fn

directional derivative of f at x in direction u is rate of change of the function at x in direction u

$f(x)$

$d_{x,u}$



$f(x+\delta)-f(x)$

$\delta\, d_{x,u}$

$\nabla f(x)^T \frac{u}{\|u\|}$

gradient

Exercise 1

Q.12 $G_n = [\frac{1}{n}, 1]$

Find $\bigcup_{i \in \mathbb{N}} G_i =$

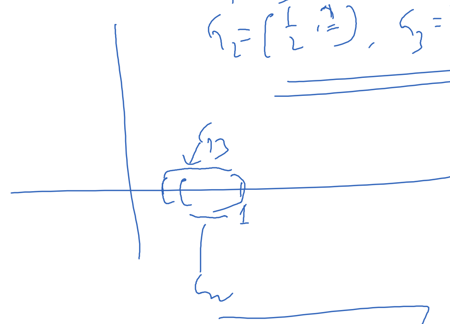
$G_1 = \{1\}$

$G_2 = [\frac{1}{2}, 1]$, $G_3 = [\frac{1}{3}, 1]$

$G_1 \cup G_2 = [\frac{1}{2}, 1]$

$\bigcup_{i=1}^{3} G_i = [\frac{1}{3}, 1]$

$\bigcup_{i=1}^{n} G_i = [\frac{1}{n}, 1]$



$1 \geq \epsilon > 0$

$\epsilon \in \bigcup_{i \in \mathbb{N}} G_i$

$\epsilon \in \bigcup G_i$?

$$\bigcup_{i=1}^{n} G_i = \underbrace{\left[\tfrac{1}{n}, 1\right]}_{G_n}$$

$(1 \geq \epsilon > 0)$

$10^{-6} \in \bigcup_{i\in\mathbb{N}} G_i$? ✓

$$10^{-6} \in \bigcup_{i=1}^{10^{10}} G_i = [10^{-10}, 1]$$

$$0 \in \bigcup_{i=1}^{10^{10}} G_i = [10^{-10}, 1] \text{ ? } \times$$

$$\bigcup_{i\in\mathbb{N}} G_i = (0, 1]$$

neither open nor closed!

- General approach to show $A = B$

Two steps

Step 1 $A \subseteq B$

take $x \in A$

show that $x \in B$

Step 2 $B \subseteq A$

take arbitrary $x \in B$

show that $x \in A$

$\bigcup_{i\in\mathbb{N}} G_i \subseteq (0,1]$

take $x \in \bigcup_{i\in\mathbb{N}} G_i$

$\Rightarrow \exists\, j$ s.t.

$x \in G_j$

$x \in [\frac{1}{j}, 1]$

$\Rightarrow x \in (0,1]$

take $x \in (0,1]$

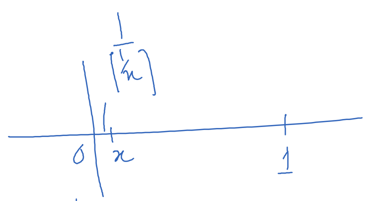
$\Rightarrow x \in \left[\frac{1}{\lceil \frac{1}{x} \rceil}, 1\right]$

$\Rightarrow x \in G_{\lceil \frac{1}{x} \rceil}$

$\Rightarrow x \in \bigcup_{i\in\mathbb{N}} G_i$

If $x \in \bigcup_{i\in\ell} A_i$

then there exists an $i \in \ell$ s.t. $x \in A_i$



$\frac{1}{\frac{1}{x}} = x$

$\frac{1}{\lceil \frac{1}{x} \rceil} \leq x$

$x \in \mathbb{R}$

floor(x) $\lfloor x \rfloor$ = largest integer $\leq x$

$\lfloor 2.5 \rfloor = 2$

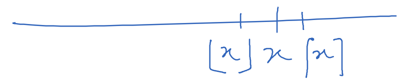
$\lfloor 3 \rfloor = 3$

$\lfloor 3.99999 \rfloor = 3$

ceil(x) $\lceil x \rceil$ = smallest integer $\geq x$

ceil(x) $\lceil x \rceil$ = smallest integer $\geq x$

$\lceil 2.0003 \rceil = 3$



— $\bigcup_{i=1}^{k} G_i = \left[\frac{1}{k}, 1\right]$

closed

— only finite unions of closed sets need to be closed.

$G_n^c = \underbrace{\left(-\infty, \frac{1}{n}\right)}_{L_n} \cup \underbrace{(1, \infty)}_{R_n}$

$\forall n$, G_n is closed

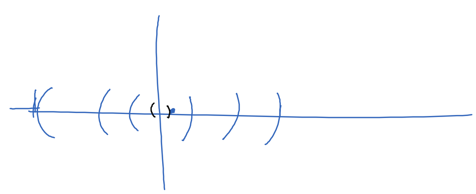
$\Rightarrow G_n^c$ is open.

$\Rightarrow \bigcup_{n \in \mathbb{N}} G_n^c$ is open //

$\Rightarrow \left(\bigcap_{n \in \mathbb{N}} G_n\right)^c$ is open

$\Rightarrow \bigcap_{n \in \mathbb{N}} G_n$ is closed //

Q.13 $G_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$

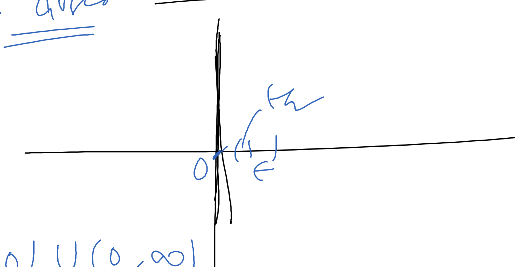


$\boxed{\bigcap_{n \in \mathbb{N}} G_n = \{0\}}$

closed

$\{0\}$ open? X

$\{0\}$ closed



$(-\infty, 0) \cup (0, \infty)$

... need to be open!!

– only finite intersection of open sets need to be open!!

lim inf and lim sup

$\{x_n\}$

$\lim_{n\to\infty} x_n$ may or may not exist.

$y_1 = \inf\{x_n\}$

$y_2 = \inf_{n\geq 2} x_n$

$\vdots$

$y_k = \inf_{n\geq k} x_n = \inf\{x_k, x_{k+1}, x_{k+2}, \ldots\}$

$$\underbrace{x_1, x_2, x_3, \ldots\ldots}_{y_1}$$

$\{y_k\}$ is monotonically increasing.

$$\liminf_{n\to\infty} x_n = \lim_{k\to\infty} y_k$$

– similarly defined z_n

$$y_n \leq x_n \leq z_n$$

$\Rightarrow y_n \leq z_n \quad \forall n$

$\Rightarrow \lim_{n\to\infty} y_n \leq \lim_{n\to\infty} z_n$

$$\boxed{\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n}$$

suppose $\lim_{n\to\infty} x_n$ exists

$$y_n \leq x_n \leq z_n \quad \forall n$$

$\Rightarrow \liminf_{n\to\infty} x_n \leq \lim_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$

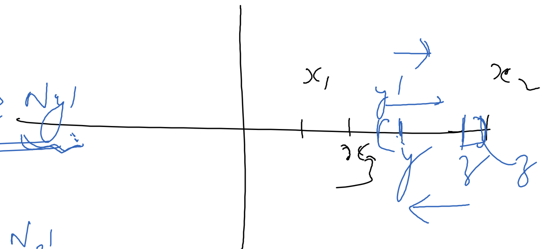
* if $\lim_{n\to\infty} x_n$ exists $\Leftarrow$ $\boxed{\lim_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n}$

* if $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$

$\exists \lim_{n\to\infty} x_n$ exists

* suppose, $y = \liminf_{n\to\infty} x_n$

, $z = \limsup_{n\to\infty} x_n$ $(z > y)$



. $\forall y' < y$, $\exists N_{y'}$ s.t. $x_n > y'$ $\forall n \geq N_{y'}$

. $\forall z' > z$ $\exists N_{z'}$ s.t. $x_n < z'$ $\forall n \geq N_{z'}$

$x_1, x_2, x_3, \dots, x_{N_{y'}}, x_{N_{y'}+1}, \dots$

iterative algorithm

$\to \lim_{n\to\infty} \|x_n - x^*\| = 0$

but we do not know whether $\lim_{n\to\infty} \|x_n - x^*\|$ exists

suppose we show $\limsup_{n\to\infty} \|x_n - x^*\| = 0$

$\liminf_{n\to\infty} \|x_n - x^*\| = 0$

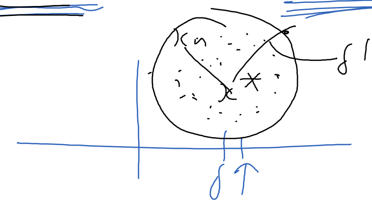
imply $\lim_{n\to\infty} \|x_n - x^*\| = 0$

suppose we only show

// $\limsup_{n\to\infty} \|x_n - x^*\| = \delta$ -10^{-3}

$\forall \delta' > \delta$, $\exists N_\delta$ s.t. $\|x_n - x^*\| < \delta'$ $\forall n \geq N_\delta$

$\|x_n - x^*\| < \delta' \quad \forall n \geq N$



$\liminf_{n \to \infty} x_n \qquad \limsup_{n \to \infty} x_n$

$\liminf_{x \to a} f(x) \qquad \limsup_{x \to a} f(x)$

$f : S \to \mathbb{R} \qquad S \subseteq \mathbb{R}^2$

$$\liminf_{n \to \infty} x_n = \lim_{n \to \infty} \left(\inf_{k \geq N} x_k \right)$$

$$\liminf_{x \to a} f(x) = \lim_{\delta_n \to 0} \left(\inf_{x \in B_a(\delta_n) \cap S} f(x) \right)$$

$\inf_{x \in B_a(\delta_n) \cap S} f(x)$

δ_n, a

$\inf_{x \in B_a(\delta_n) \cap S} f(x)$

$$\limsup_{x \to a} f(x) = \lim_{\delta_n \to 0} \left(\sup_{x \in B_a(\delta_n) \cap S} f(x) \right)$$

$\delta_n = \frac{1}{n}$

$= \inf\{ f(x) ; x \in B_a(\delta_n) \cap S \}$

Example (1) $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$

$f(x) = \sin\left(\frac{1}{x}\right)$

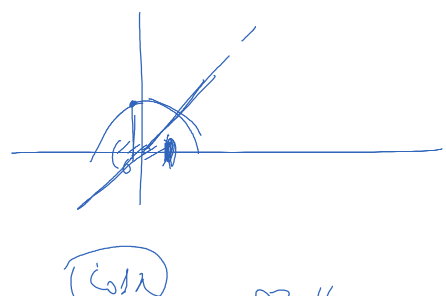
$\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exist

$$\liminf_{x \to 0} f(x) = \lim_{n \to \infty} \underbrace{\inf_{x \in B_0(\frac{1}{n}) \setminus \{0\}} \sin\left(\frac{1}{x}\right)}_{= -1}$$

$B_0(\frac{1}{n}) \cap (\mathbb{R} \setminus \{0\})$

$= -1$

$\limsup_{x \to 0} f(x) = 1$



(2) $f(x) = \frac{\cos x}{x}, \quad x \in \mathbb{R} \setminus \{0\}$

$$\liminf_{x \to 0} \frac{\cos x}{x} = \lim_{n \to \infty} \inf_{x \in B_0(\frac{1}{n}) \setminus \{0\}} \frac{\cos x}{x} = -\infty$$

$\limsup \frac{\cos x}{x}$ $\lim$ $\sup$ $\frac{\cos x}{x}$ ...

$$\liminf_{x\to 0} \frac{\cos x}{x} = \lim_{n\to\infty} \sup_{x\in B_0(\frac{1}{n})\setminus\{0\}} \frac{\cos x}{x} = \infty$$

$\mathbb{R}_{+}$

$f : S \longrightarrow \mathbb{R}$

suppose $\liminf_{x\to a} f(x) = \underline{f}$

and $\limsup_{x\to a} f(x) = \overline{f}$

$(\overline{f} > \underline{f})$

if $\overline{f} = \underline{f} \Rightarrow \lim_{x\to a} f(x)$ exists

$\forall f' < \underline{f}, \ \exists \delta > 0$ s.t. $f(x) > f' \ \forall x \in B_\delta(a) \cap S$

$\forall f' > \overline{f}, \ \exists \delta > 0$ s.t. $f(x) < f' \ \forall x \in B_\delta(a) \cap S$