1(a)

$$\min_{x} \frac{1}{2} x^T Q x + p^T x$$

$$\Rightarrow f(x) = \frac{1}{2} x^T Q x + p^T x$$

$$f'(x) = Qx + p$$

$$d_K = -f'(x) = -(Qx+p)$$

$$x_{K+1} = x_K + \alpha_K d_K$$

$$x_1 = x_0 + \alpha_0 d_0$$

Exact line search for α_K (α_0)

$$g(x) = \arg\min_{\alpha} f(x_1)$$

$$\frac{\partial f(x_1)}{\partial \alpha} = 0$$

$$f'(x_1)^T \frac{\partial x_1}{\partial \alpha} = 0$$

$$(Qx_1 + p)^T (d_0) = 0$$

$$[Q(x_0 + \alpha_0 d_0) + p]^T (d_0) = 0$$

$$[Qx_0 + p + Q\alpha_0 d_0]^T (d_0) = 0$$

$$[-d_0 + Q\alpha_0 d_0]^T (d_0) = 0$$

$$\alpha_0 d_0^T Q^T d_0 = d_0^T d_0$$

$$\alpha_0 = \frac{d_0^T d_0}{d_0^T Q d_0}$$

i.e. $\boxed{\alpha_K = \frac{d_K^T d_K}{d_K^T Q d_K}}$

This is exact line search α_K

$\alpha_{K+1} = x_K + \alpha_K d_K$.

Newton's method.

$$\nabla f(x) = Qx + P$$

$$\nabla^2 f(x) = Q$$

$$x_{k+1} = x_k - Q^{-1}(Qx + P)$$

$$\cancel{\begin{bmatrix} 100 & 0 \\ 0 & 800 \end{bmatrix} \begin{bmatrix} 1000 \\ 1 \end{bmatrix} = \begin{bmatrix} 100000 \\ 800 \end{bmatrix}}$$

2)
$$\min_x \quad x^2 + 1$$

$$\text{sub to} \quad (x-2)(x-4) \le 0$$

$$L(x;\mu) = x^2+1 + \mu(x^2 - 6x + 8)$$

$$\nabla_x L(x;\mu) = 2x + (2x-6)\mu = 0$$

$$2x + 2x\mu - 6\mu = 0$$

$$2x(1+\mu) = 6\mu$$

$$x = \frac{3\mu}{1+\mu}$$

If constraint is inactive, then $\mu = 0$ which will lead to $x = 0$

but $x^2 - 6x + 8 \le 0$

$8 \le 0$ - It is impossible.

Thus, by complementary slackness,

$$\mu g_i(x) = 0$$

$g_1(x)$ is active constraint, then.

$$x^2 - 6x + 8 = 0$$
$$\left(\frac{3\mu}{1+\mu}\right)^2 - 6\left(\frac{3\mu}{1+\mu}\right) + 8 = 0$$
$$\frac{9\mu^2}{(1+\mu)^2} - \frac{18\mu}{1+4} + 8 = 0$$

$$9\mu^2 - 18\mu(1+\mu) + 8(1+\mu)^2 = 0$$
$$9\mu^2 - 18\mu - 18\mu^2 + 8 + 16\mu + 8\mu^2 = 0$$

$$-\mu^2 - 2\mu + 8 = 0$$
$$\mu = -4, 2$$

According to KKT condition,

x^* is a minima if

$h_i(x^*) = 0$

$g_1(x^*) = 0$ $\quad$ $\mu g_1(x^*) = 0$

$\mu \geq 0$

Thus $\mu = -4$ is rejected, $\mu = 2$ is valid.

For $\mu = 2$, $g_i(x) = 0$.

$$x = \frac{3(2)}{1+2} = \underline{\underline{2}}$$

$\boxed{x^* = 2}$ at $\boxed{\mu = 2}$

(b) $\quad$ $g(\mu) = \min L(x, \mu)$

$= \min x^2 + 1 + \mu(x^2 - 6x + 8)$

$= \min (1+\mu)x^2 - 6x\mu + 1 + 8\mu$

$$\nabla_x L(x, \mu) = 2(1+\mu)x - 6\mu = 0$$
$$x = \frac{3\mu}{1+\mu}$$

i.e $\cancel{(1+\mu)}\ \cancel{3\mu}$

$$2(1+\mu)x = 6\mu$$
$$(1+\mu)x = 3\mu$$

i.e

$$\text{rem min} \quad (1+\mu)x^2 - 6x\mu + 1 + 8\mu$$
$$\text{i.e min} \quad 3x\mu - 6x\mu + 1 + 8\mu$$
$$\text{i.e min} \quad -3x\mu + 1 + 8\mu$$
$$\text{i.e} \quad -3\left(\frac{3\mu}{1+\mu}\right)\mu + 1 + 8\mu$$

$$g(\mu) = \frac{-9\mu^2}{1+\mu} + 1 + 8\mu$$

$$\text{Dual} = g(\mu) = \max. \frac{9\mu + 1 - \mu^2}{(1+\mu)}$$
$$\text{s.t} \quad \cancel{(1+\mu)x = 3\mu}$$
$$\mu \geq 0$$

primal problem $\Rightarrow$ $\min\limits_{x} \quad x^2+1$

sub to $(x-2)(x-4) \leq 0$

$= \max \; -(x^2+1)$

sub to $(x-2)(x-4) \leq 0$.

Dual problem $\Rightarrow$ $\max \dfrac{9\mu+1-\mu^2}{1+\mu}$

sub to $(1+\mu)x = 3\mu$

$= \min \; -\left(\dfrac{9\mu+1-\mu^2}{1+\mu}\right)$

sub to $(1+\mu)x = 3\mu$.

Page No.: Date: YOUVA

For $x=2$, $\mu=2$.

$c^Tx \Rightarrow$ primal solution
$= (-(x^2+1))$
$= -5$

Dual solution $\Rightarrow$ b^Ty
$= -\left(\frac{9\mu+1-\mu^2}{1+\mu}\right)$
$= -\left[\frac{+9(2)+1-4}{3}\right]$
$= -5.$

$\therefore c^Tx \leq b^Ty.$

Thus this verify weak duality.

(c) $g(\mu) = \frac{9\mu+1-\mu^2}{1+\mu}$

$\nabla g(\mu) = \frac{(1+\mu)(9-2\mu) - (9\mu+1-\mu^2)}{(1+\mu)^2} = 0.$

$(1+\mu)(9-2\mu) = 9\mu+1-\mu^2$
$9+7\mu-2\mu^2 = 9\mu+1-\mu^2$
$\mu^2+2\mu-8=0$
$(\mu+4)(\mu-2)=0$
$\mu=-4, \mu=2$

but since $\mu \geq 0$,
$\boxed{\mu=2}$

Dual value $= \frac{9\mu+1+\mu^2}{1+\mu} = 5$

Since $c^Tx = b^Ty \Rightarrow$ Primal sol^n = Dual sol^n.
Strong Duality holds.

g(u) is a maximization problem.

$$\nabla^2 g(u) = \frac{d}{du}\left[\frac{9-2u}{1+u} - \frac{(9u+1-u^2)}{(1+u)^2}\right]$$

$$= \frac{(1+u)(-2) - (9-2u)}{(1+u)^2} - \frac{(1+u)^2(9-2u) - (9u+1-u^2)2(1+u)}{(1+u)^4}$$

$$\nabla^2(g(u))\Big|_{u=2} = \frac{3(-2)-(9-4)}{9} - \frac{3^2(3) - (15)(2)(3)}{3^4}$$

$$= \frac{-6-5}{9} - \cancel{\frac{27-90}{3^4}}\ \frac{3-10}{3^2}$$

$$= -\frac{11}{9} + \frac{7}{9} = \underline{\underline{-\frac{4}{9}}}$$

Since it is negative, g(u) is concave function

g(u) is a concave maximization problem

Q.4.

(a) $$\min \ (x_1-4)^2 + (x_2-4)^2$$
$$\text{Sub to} \quad 5 - x_1 - x_2 \geq 0$$

Here $f(x) = (x_1-4)^2 + (x_2-4)^2$

$$\phi(x) = -\log(-5+x_1+x_2)$$

$$L(x) = \min_{x_1,x_2} \ (x_1-4)^2 + (x_2-4)^2 - \epsilon_K \log(-5+x_1+x_2)$$

$$\nabla L(x) = \begin{bmatrix} 2(x_1-4) - \dfrac{\epsilon_K}{x_1+x_2-5} \\ \\ 2(x_2-4) - \dfrac{\epsilon_K}{x_1+x_2-5} \end{bmatrix}$$

$$\nabla^2 L(x) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$$A = 2 + \frac{\epsilon_K}{(x_1+x_2-5)^2}$$

$$B = \frac{\epsilon_K}{(x_1+x_2-5)^2} = C$$

$$D = 2 + \frac{\epsilon_K}{(x_1+x_2-5)^2}$$

Penalty funct^n:-

$$L(x) = \min_{x_1 x_2} (x_1-4)^2 + (x_2-4)^2 + c_K(5-x_1-x_2)^2$$

$$\nabla L(x) = \begin{bmatrix} 2(x_1-4) - 2c_K(5-x_1-x_2) \\ 2(x_2-4) - 2c_K(5-x_1-x_2) \end{bmatrix}$$

$$\nabla^2 L = \begin{bmatrix} 2+2c_K & 2c_K \\ 2c_K & 2+2c_K \end{bmatrix}$$

5(b)

$$\min_x \; 5x_1^2 + 5x_2^2 + 5x_3^2 + 5x_4^2 - 4x_1x_3 - 4x_2x_4 + x_1 - x_2 + 2x_3 - 3x_4$$

$$[x_1 \; x_2 \; x_3 \; x_4] \begin{bmatrix} 5 & 0 & -2 & 0 \\ 0 & 5 & 0 & -2 \\ -2 & 0 & 5 & 0 \\ 0 & -2 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$+ [1 \; -1 \; 2 \; -3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$Q = \begin{bmatrix} 2.5 & 0 & -1 & 0 \\ 0 & 2.5 & 0 & -1 \\ -1 & 0 & 2.5 & 0 \\ 0 & -1 & 0 & 2.5 \end{bmatrix} \qquad p^T = [1 \; -1 \; 2 \; -3]$$

$$Q = \begin{bmatrix} 10 & 0 & -4 & 0 \\ 0 & 10 & 0 & -4 \\ -4 & 0 & 10 & 0 \\ 0 & -4 & 0 & 10 \end{bmatrix}$$