E1 215-O: Tutorial Questions

Linear and Non-linear Optimization

January 15, 2022

1. Let $x \in \mathbb{R}^n$ and $||x||_p = (\sum_{i=1}^{n} |x_i|)^{\frac{1}{p}}$ show that
$$||x||_2 \leq ||x||_1$$
2. Verify the parallelogram law
$$||x+y||_2^2 + ||x-y||_2^2 = 2(||x||_2^2 + ||y||_2^2)$$
3. Consider a matrix
$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$
under what conditions A is positive definite?
4. Prove that the matrix Q is orthogonal
$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$
5. Show that
$$A = \{(x,y) \in \mathcal{X} | x^2 + y^2 \neq 1\}$$
is open {Hint: find union of open sets }
6. Let for $x, y \in \mathbb{R}$, let $d(x,y)$ defined below be a notion of distance.
$$d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$
and let the ball be defined as $B_x(r) := \{y \in \mathbb{R}| \quad d(x,y) < r\}$. Show that under the above assumptions the singleton $\{1\}$ is an open set. {Hint: find an r such that $B_1(r) = \{1\}$}
7. State whether the following sets are compact in the space of $\mathbb{R}^2$
   (a) $A = \{[0,1] \times [0,1]\}$
   (b) $B = \{x \in \mathbb{R}^2 : \quad ||x||_2 \leq 2\}$
   (c) $C_1 = \{x \in \mathbb{R}^2 : \quad ||x||_2 \leq 2\}$
   (d) $C_2 = \{x \in \mathbb{R}^2 : \quad ||x||_2 \geq 1\}$
   (e) $C_1 \cap C_2$

8. Consider the sequence $x_n = 5 + \frac{1}{n^3}$. Does the sequence converge as n $\to \infty$. If $\epsilon = 10^{-6}$, find the value of N_ϵ such that the criterion in the definition of limit holds.

9. Does the sequence $x_n = n^2 \sin(\frac{1}{n}) \ \log(1 + \frac{1}{n})$ converge as n $\to \infty$? As in question 8, if $\epsilon = 10^{-2}$, find N_ϵ (you may want to use a calculator for evaluating logarithmic expressions).

10. Show that x^2 is continuous at $x = 1$ using the ϵ-δ definition.

11. if f(x) $= x^3$, using the definition of continuity if we assign $\delta = 10^{-1/3}$, find the corresponding ϵ. (Optional: If we consider the function f(x) $= x^{1/(2k+1)}$ for some large k, observe how the delta values for some small ϵ. keep on increasing k, Comment.)

12. Let $G_n = \left[\frac{1}{n}, 1\right]$. Find $\bigcup_{i \in \mathbb{N}} G_i$. Is it open or closed? What do you infer from it? {Hint: The set of natural numbers is an infinite set.}

13. Let $G_n = \left(\frac{-1}{n}, \frac{1}{n}\right)$. Find $\bigcap_{i \in \mathbb{N}} G_i$. Is it an open set? What do you infer from it?