Tutorial 13

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Example (Continuing from the last lecture)

Consider

$$2x_1 + x_2 + 2x_3 = 4$$
$$3x_1 + 3x_2 + x_3 = 3$$
$$x_1, x_2, x_3 \geq 0$$

Objective: obtain a basic feasible solution

Towards this, we solve

$$\min \sum_{i=1}^{2} u_i$$

s.t. $2x_1 + x_2 + 2x_3 + u_1 = 4$

$3x_1 + 3x_2 + x_3 + u_2 = 3$

$x_i \geq 0,\ u_i \geq 0.$

$$A = \begin{bmatrix} 2 & 1 & 2 & 1 & 0 \\ 3 & 3 & 1 & 0 & 1 \end{bmatrix}$$

- It is in canonical form, u_i's are basic variables
- $(0\ 0\ 0\ 4\ 3)^T$ is a basic feasible solution

We ~~can~~ employ simplex method

1st tableau

u_1 u_2 ← basic variables

x_1						
2	1	2	1	0	4	4/2
3	3	1	0	1	3	3/3
+5 $(-r_1)$	4 $(-r_2)$	3 $(-r_3)$	0	0	7 (objective value)	

r_j: $-5, -4, -3$

reduced cost vector r

$$r_j = c_j - \sum_{i=1}^{m} a_{ij} c_i$$

$\forall j = m+1, \ldots n$

$r_1 = 0 - 2 \times 1 - 3 \times 1 = -5$

current $r_3 = 0 - 2 \times 1 - 1 \times 1 = -3$

(In general,

$$\begin{matrix} A & b \\ -r^T & c^T x \end{matrix}$$

(x is the basic feasible soln)

we will follow this.

or,

$$\begin{matrix} A & b \\ r^T & -c^T x \end{matrix}$$

2nd tableau

1st row $-$ 2nd row $\times \frac{2}{3}$

x_1	x_2	x_3	u_1	u_2		
0	-1	$4/3$	1	$-2/3$	2	$\frac{2}{4/3} = \frac{3}{2}$

	x_1	x_2	x_3	y_1	y_2	
1st row $-$ 2 ... /3	0	-1	$4/3$	1	$-2/3$	2
2nd row/3	1	1	$1/3$	0	$1/3$	1
3rd row $-$ 2nd $\times$ 5/3	0	-1	$4/3$	0	$-5/3$	2

$\frac{2}{4/3} = 3/2$

$\frac{1}{1/3} = 3$

3rd tableau

	x_1	x_2	x_3	y_1	y_2	
	0	$-3/4$	1	$3/4$	$-1/2$	$3/2$
	1	$5/4$	0	$-1/4$	$1/2$	$1/2$
	0	0	0	-1	-1	0

$(-1) - 1 \times (-1)$
$= (-1) - (-1) = 0$

Hence, a optimal basic feasible solution is $\begin{bmatrix} 1/2 \\ 0 \\ 3/2 \\ 0 \\ 0 \end{bmatrix}$

Hence a basic feasible solution to the original problem is $\begin{bmatrix} 1/2 \\ 0 \\ 3/2 \end{bmatrix}$

A canonical form of the original set of equations

$$\begin{bmatrix} 0 & -3/4 & 1 \\ 1 & 5/4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}$$

Example

$$\max \quad 3x_1 + x_2 + 3x_3$$

s.t.

$$2x_1 + x_2 + x_3 \leq 2$$
$$x_1 + 2x_2 + 3x_3 \leq 5$$
$$2x_1 + 2x_2 + x_3 \leq 6$$
$$x_1, x_2, x_3 \geq 0$$

$3 \times \frac{1}{5} + 1 \times 0 + 3 \times \frac{8}{5} = \frac{27}{5}$

Equivalent problem is

$$\min \quad -3x_1 - x_2 - 3x_3$$

st.

$$2x_1 + x_2 + x_3 + y_1 = 2$$
$$x_1 + 2x_2 + 3x_3 + y_2 = 5$$

$$A = \begin{array}{cccccc} x_1 & x_2 & x_3 & y_1 & y_2 & y_3 \end{array}$$
$$A = \begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ & & 1 & 0 & 0 & 1 \end{bmatrix}$$

s.t. $2x_1 + x_2 + x_3 + u_1 = 2$

$x_1 + 2x_2 + 3x_3 + u_2 = 5$

$2x_1 + 2x_2 + x_3 + u_3 = 6$

$x_i \geq 0,\ u_i \geq 0$

$$A = \begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{bmatrix}$$

- It is in canonical form
- u_1, u_2, u_3 are basic variables
- $[0 \quad 0 \quad 0 \quad 2 \quad 5 \quad 6]^T$ is a basic feasible sol^n

1st tableau

x_1	x_2	x_3	u_1	u_2	u_3		
2	1	1	1	0	0	2	2/2
1	2	3	0	1	0	5	5/1
2	2	1	0	0	1	6	6/2
3	1	3	0	0	0	0	=

$\bar{c}_1 = -3 - (2\times 0 + 1 \times 0 + 2 \times 0) = -3$

2nd tableau

x_1	x_2	x_3	u_1	u_2	u_3	
1	1/2	1/2	1/2	0	0	1
0	3/2	5/2	-1/2	1	0	4
0	1	0	-1	0	1	4
0	-1/2	3/2	-3/2	0	0	-3

$\frac{1}{1/2} = 2$

$\frac{4}{5/2} = 8/5$

3rd tableau

x_1	x_2	x_3	u_1	u_2	u_3	
1	1/5	0	3/5	-1/5	0	1/5
0	3/5	1	-1/5	2/5	0	8/5
0	1	0	-1	0	1	4
0	-7/5	0	-6/5	-3/5	0	-27/5

$\frac{1}{2} - \frac{3}{2} \times \frac{1}{5}$

$\frac{1}{2} + \frac{1}{2} \times \frac{1}{5}$

$4 - \frac{1}{5} \times 4$

$-\frac{1}{2} - \frac{3}{5} \times \frac{3}{2}$

$-\frac{1}{2} - \frac{9}{10}$

$-\frac{3}{2} + \frac{3}{5} \times \frac{1}{2} =$

Hence

$[1/5 \quad 0 \quad 8/5 \quad 0 \quad 0 \quad 4]^T$ is an optimal basic feasible sol^n.

and ... $= \frac{27}{5}$.

$-\frac{3}{2}+\frac{3}{5}\times\frac{5}{2}=$

$-3\ \cancel{-}\ \frac{3}{5}\times 4$

$=-\frac{15}{5}-\frac{12}{5}$

Finally

$(\frac{1}{5}\ \ 0\ \ \frac{8}{5})^T$ is an optimal basic feasible soln to the original problem

The optimal value is $\frac{27}{5}$ - ...

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax=b \\ & x \geq 0 \end{array}$$

dual

$$\begin{array}{ll} \max & b^T y \\ \text{s.t.} & y^T A \leq C^T \end{array}$$

y' are free variables