Tutorial-6

Saturday, 19 February 2022 11:29 AM

$$f(x) = \frac{1}{2}x^T A x - b^T x \; ; \quad A = \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix} \; ; \; b = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$\underset{=}{\text{Pd}}$$

$$f(x) = 2x_1^2 - 2x_1x_2 + x_2^2 + 2x_1 - 2x_2 \quad (\text{mini}) = (0,1) = x^*$$

$$x^0 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

gradient descent (Fixed Step size):-

$$\alpha = 0.1$$

$$x^{k+1} = x^k - \alpha \nabla f(x^k)$$

$$x^1 = x^0 - \alpha \nabla f(x^0)$$

$$\nabla f(x) = \begin{bmatrix} 4x_1 - 2x_2 + 2 \\ -2x_1 + 2x_2 - 2 \end{bmatrix}$$

$Ax - b$

$Ax^0 - b$

$$\nabla f(x^0) = \begin{bmatrix} 14 \\ -10 \end{bmatrix}$$

$$x^1 = \begin{bmatrix} 2 \\ -2 \end{bmatrix} - 0.1 \begin{bmatrix} 14 \\ -10 \end{bmatrix} = \begin{bmatrix} 0.6 \\ -1 \end{bmatrix}$$

$$x^2 = x^1 - \alpha \nabla f(x^1) \qquad \nabla f(x^1) = \begin{bmatrix} 6.4 \\ -1.6 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 \\ -1 \end{bmatrix} - 0.1 \begin{bmatrix} 6.4 \\ -1.6 \end{bmatrix}$$

$$= \begin{bmatrix} -0.04 \\ -0.1920 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

optimal line search :-

$$f(x) = 2x_1^2 - 2x_1 x_2 + x_2^2 + 2x_1 - 2x_2$$

$$\nabla f(x) = \begin{bmatrix} 4x_1 - 2x_2 + 2 \\ -2x_1 + 2x_2 - 2 \end{bmatrix} \qquad x_0 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$x^1 = x^0 - \alpha \nabla f(x^0)$$

$$x^1 = \begin{bmatrix} 2 \\ -2 \end{bmatrix} - \alpha \begin{bmatrix} 14 \\ -10 \end{bmatrix}$$

$$x^1 = \begin{bmatrix} 2 - 14\alpha \\ -2 + 10\alpha \end{bmatrix}$$

$$f(x^1) = 2(2-14\alpha)^2 - 2(2-14\alpha)(-2+10\alpha) + (-2+10\alpha)^2 + 2(2-14\alpha) - 2(-2+10\alpha)$$

$$f(x^{k+1}) = g_{x^{k+1}}(\alpha)$$

function of α.

$\mathbb{R}^2 \to \mathbb{R}$

$x^{k+1} = h(\alpha)$ =

$\boxed{\arg\min g_{x^{k+1}}(\alpha)}$

$f(x^1) = g_{x^1}(\alpha)$ $\quad \mathbb{R} \to \mathbb{R}$

$g'_{x^1}(\alpha) = 0$

$\arg\min g_{x^1}(\alpha) = 0.191707$

$|g''_{x^1}(\alpha)| > 0$

$$x^1 = \begin{bmatrix} -0.6839 \\ -0.0829 \end{bmatrix} \to \qquad \nabla f(x^1) = \begin{bmatrix} \ \\ \ \end{bmatrix}$$

$$x^2 = x^1 - \alpha \nabla f(x^1)$$

$g \ (\alpha)$

$$x^2 = x^1 - \alpha \nabla f(x^1)$$

$$f(x^2) = g_{x^2}(\alpha) \Longrightarrow \text{argmin } g_{x^2}(\alpha)$$

$$\alpha = \qquad g'_{x^2}(\alpha) = 0$$

$$g''_{x^2}(\alpha) > 0$$

Armijo's Rule :-

$f: \mathbb{R} \to \mathbb{R}$; $f(x) = 3x^4 + 5x^3 - 4x^2 + 2$

$\nabla f(x) = 12x^3 + 15x^2 - 8x$

$$\left.\begin{array}{l} x^0 = 0.5 \\ \alpha = 1.5 \\ \beta = 0.98 \\ \sigma = 10^{-5} \end{array}\right\}$$

$$x^1 = x^0 - \alpha \nabla f(x^0)$$

$$= 0.5 - (1.5)(1.5 + 3.75 - 4)$$

$$= 0.5 - (1.5)(1.25)$$

$$= -1.375 \checkmark$$ (Is it your 1st iterate ??)

Verify

$$f(x^1) \leq f(x^0) - \alpha . \sigma . \|\nabla f(x^0)\|^2$$

$x^1 = -1.375$

$x^0 = 0.5$

$$-7.8292 \leq (\qquad) = \checkmark$$

$\times$

$$x^1 = -1.375$$

$x^2 = (\qquad)$ $\boxed{x^2 =}$

$\alpha^1 = (0.95)\,(1.5)$

$x^2 = x^1 - \alpha^1 \nabla f(x^1)$

$f(x^2) < f(x$

$\alpha^2 = \beta^2 \alpha$

TUTORIAL 6:

→ Discuss the Newton's Method for minimization

→ Demos → steepest descent (fixed step size)

(optimal line-search)

(back-tracking)

→ Newton's method

Rosen-broock valley-function

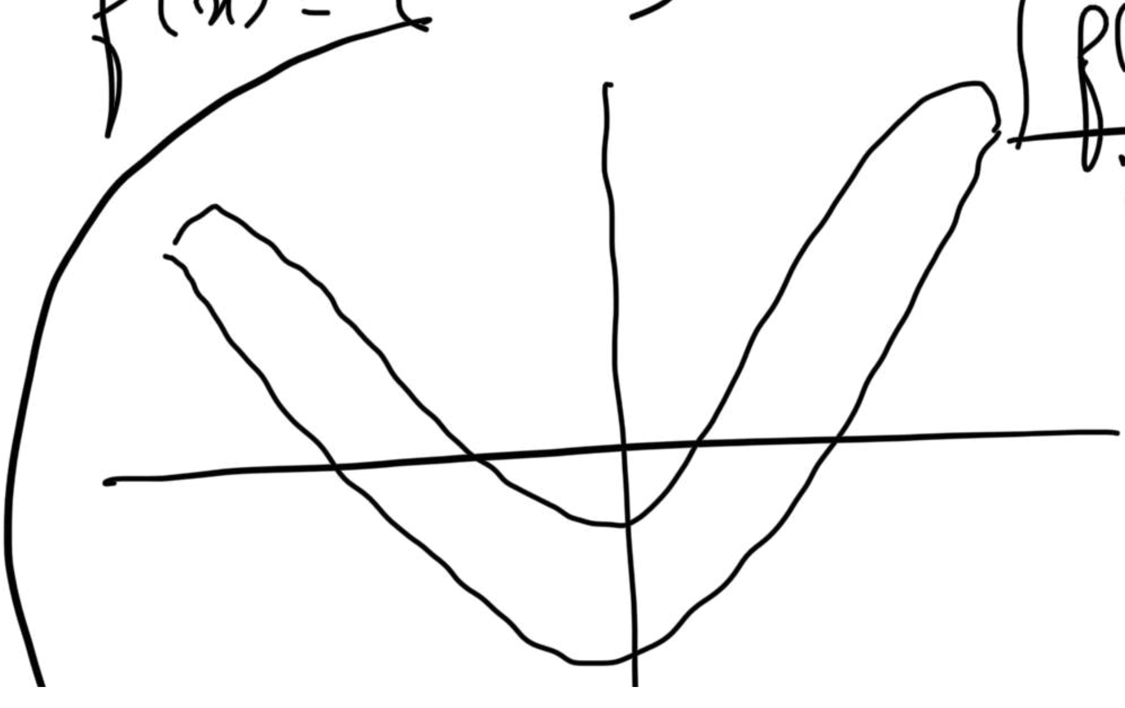
or Rosen-broock banana.

$f : \mathbb{R}^2 \to \mathbb{R}$

$f(x) = (a - x_1)^2 + b\,(x_2 - x_1^2)^2$

$\boxed{f(x^*) = 0}$

$x^* = \begin{pmatrix} a \\ a^2 \end{pmatrix}$



$$+ \qquad (a, b > 0)$$

$$\geq 0$$

$$\nabla f = \begin{Bmatrix} \dfrac{\partial f}{\partial x_1} \\[2ex] \dfrac{\partial f}{\partial x_2} \end{Bmatrix}$$

$$= \begin{Bmatrix} 2(a - x_1)(-1) + b\,2(x_2 - x_1^2)(-2x_1) \\ 2b(x_2 - x_1^2) \end{Bmatrix}$$

$$= \begin{Bmatrix} 2(x_1 - a) - 4b x_1 (x_2 - x_1^2) \\[2ex] 2b(x_2 - x_1^2) \end{Bmatrix}$$

$\nabla f = 0$ (first-order Necessary condition)

$$2(x_1 - a) = 4b x_1 (x_2 - x_1^2)$$

$$2b(x_2 - x_1^2) = 0$$

$$\boxed{x_2 = x_1^2}$$

$$\boxed{x_2 = x_1}$$

$$\boxed{x_1 = a} \qquad \boxed{x_2 = a^2}$$

$$\nabla^2 f = \begin{bmatrix} 2 - 4b(x_2 - 3x_1^2) & -4b\,x_1 \\ -4b\,x_1 & 2b \end{bmatrix}$$

$$\boxed{a = 1 \quad , \quad b = 100} \qquad x^* = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\nabla^2 f \left(x^* = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \begin{bmatrix} 2 - 400(1-3) & -400 \\ -400 & 200 \end{bmatrix}$$

$$= \begin{bmatrix} 802 & -400 \\ -400 & 200 \end{bmatrix}$$

$$\underline{(P\text{-}D.)}$$

$$x^0 = \begin{bmatrix} -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 \end{bmatrix}$$

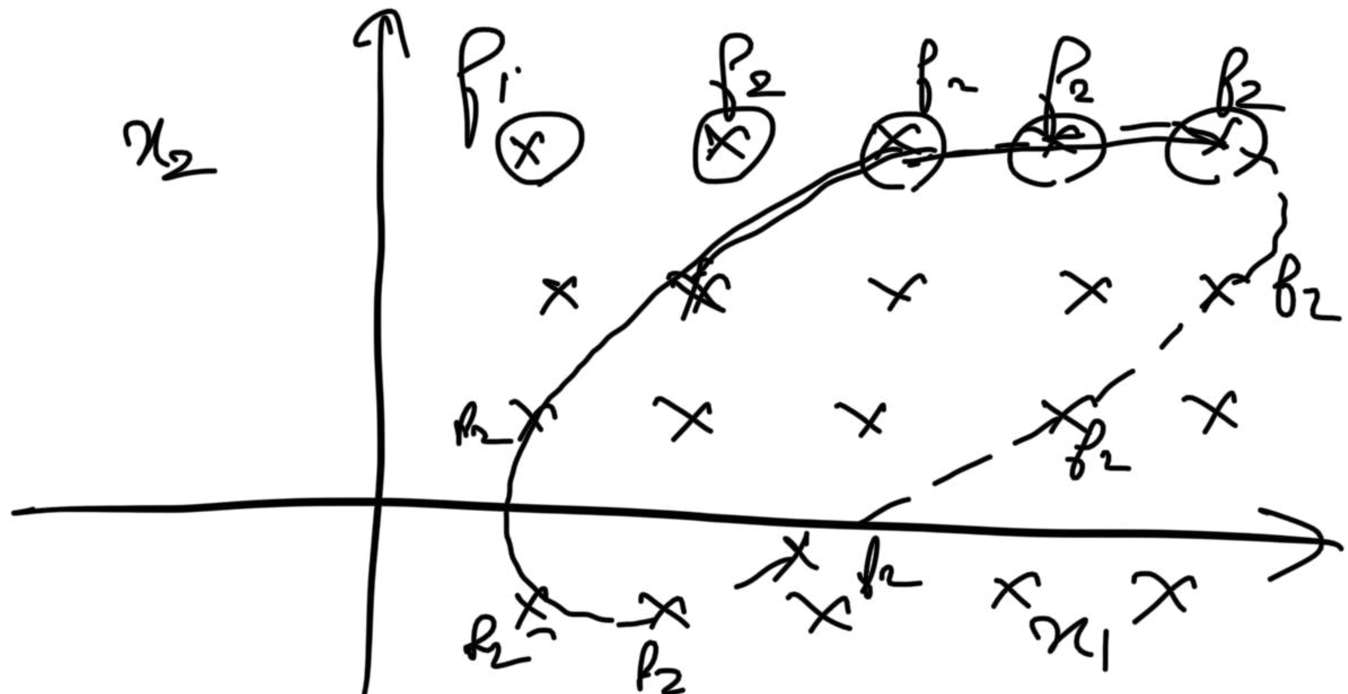
$$x_{k+1} = x_k - \nabla^2 f(x_k)^{-1} \nabla f(x_k)$$

first - Iterate

$$x_1 = x_0 - \nabla^2 f(x_0)^{-1} \nabla f(x_0)$$

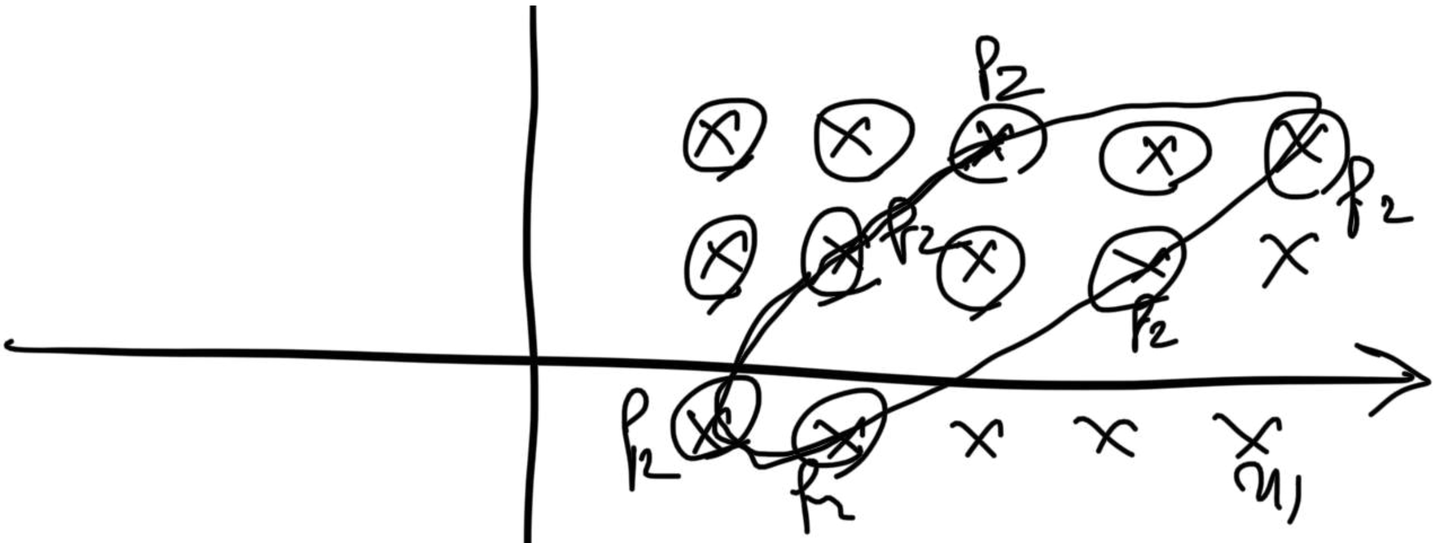
$$= \begin{bmatrix} -1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1602 & 400 \\ 400 & 200 \end{bmatrix}^{-1} \begin{bmatrix} -804 \\ -400 \end{bmatrix}$$

$$\boxed{x_1 = \begin{Bmatrix} -0.9950 \\ 0.9900 \end{Bmatrix}}$$



$f(x_1, x_2)$.

x_2



$$f(x_1, x_2) = \frac{1}{2} x^T A x - b^T x$$