Tutorial 4

29 January 2022 11:00

$f: \mathbb{R}^n \to \mathbb{R}$

$A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}$

$f(x) = \frac{1}{2}x^T A x + b^T x + c$

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f(x)}{\partial x_1} \\ \vdots \\ \frac{\partial f(x)}{\partial x_n} \end{bmatrix}$$

$$f(x) = \frac{1}{2}(x_1, \ldots\ldots x_n) \underbrace{\begin{bmatrix} \sum_{j=1}^{n} a_{1j} x_j \\ \vdots \\ \sum_{j=1}^{n} a_{nj} x_j \end{bmatrix}}_{Ax} + \underbrace{\sum_{i=1}^{n} b_i x_i}_{} + c$$

$$\frac{\partial f(x)}{\partial x_1} = a_{11}x_1 + \frac{1}{2}\sum_{j=2}^{n} x_j a_{1j} + \underbrace{\frac{1}{2}\sum_{j=2}^{n} a_{j1} x_j}_{} + b_1$$

$$= a_{11}x_1 + \frac{1}{2}\sum_{j=2}^{n} x_j (a_{1j} + a_{j1}) + b_1$$

$$= 2a_{1j}$$

$$= \sum_{j=1}^{n} a_{1j} x_j + b_1$$

$$= (Ax)_1 + b_1$$

$$\nabla f(x) = \begin{bmatrix} (Ax)_1 + b_1 \\ \vdots \\ (Ax)_n + b_n \end{bmatrix} = Ax + b$$

$$\frac{1}{2} x_1 \times \sum_{j=1}^{n} a_{1j} x_j$$

$$= \frac{1}{2} a_{11} x_1^2 + \frac{1}{2}\sum_{j=2}^{n} x_1 x_j a_{1j}$$

$$\frac{1}{2} x_2 \sum_{j=1}^{n} a_{2j} x_j$$

$$= \frac{1}{2} a_{21} x_2 x_1 + \ldots\ldots$$

$$\frac{\partial f(x)}{\partial x_1} = a_{11}x_1 + \frac{1}{2}\sum_{j=2}^{n} x_j (a_{1j} + a_{j1}) + b_1$$

$$= \frac{1}{2}\left(\sum_{j=1}^{n} a_{1j} x_j + \sum_{j=1}^{n} a_{j1} x_j\right) + b_1$$

$$= \frac{1}{2}\left((Ax)_1 + (A^T x)_1\right) + b_1$$

$B = A^T$

$$\sum a_{j1} x_j$$

$$= \frac{1}{2}\left((Ax)_1 + (A^T x)_1\right) + b_1$$

$$\sum_{j=1}^{n} a_{j1} x_j = \sum_{j=1}^{m} b_{1j} x_j = (Bx)_1 = (A^T x)_1$$

$$\nabla f(x) = \frac{1}{2}(Ax + A^T x) + b$$

$$\nabla^2 f(x) = \frac{1}{2}(A + A^T)$$

$$= A \text{ if } A \text{ is symmetric}$$

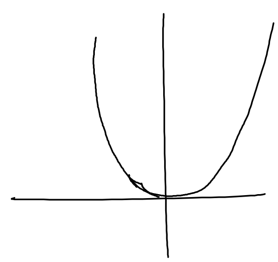
Q $f : S \to \mathbb{R}$

x^* is strict local minimum $\overset{?}{\Rightarrow}$ $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*)$ is p.d. (is not correct)

Let f be a fn. of single variable ($S \subseteq \mathbb{R}$)

$f(x) = x^4$

$x^* = 0$ a strict local min.? ✓



$f'(0) = 0$

$f''(0) = 0$ (not positive definite)

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* $A = \begin{bmatrix} 0 & 9 \\ 9 & 0 \end{bmatrix}$

$\lambda_1 + \lambda_2 = 0$

$\lambda_1 \lambda_2 = -81$

$(\lambda_1 = 9, \ \lambda_2 = -9)$

neither positive-semi definite nor negative semi definite.

✓ 0, 0, -81 - not positive semidefinite

A is negative-semi definite if all the principle minors of k^{th} order

A is negative-semidefinite if all the pri...

are

① ≤ 0 (non positive) if k is odd

② ≥ 0 (non-negative) if k is even

Example in the above example

principal minors of 1^{st} order $= 0$

principal minor of 2^{nd} order $= -81$

A is not negative semidefinite

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

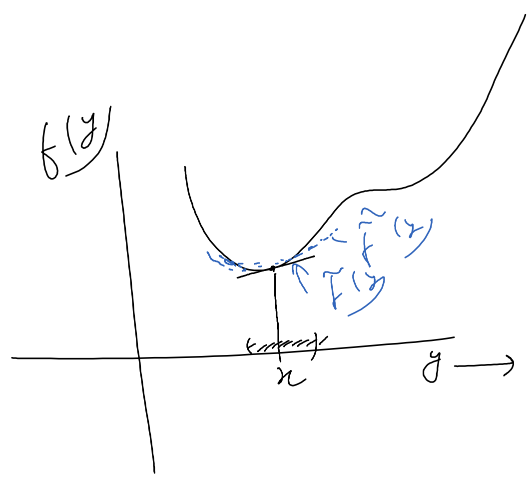
A is negative definite if all the leading principle minors of k^{th} order are

(i) negative if k is odd

(ii) positive if k is even

Taylor's approximations

Fix $x \in S$ s.t. f is differentiable at x



$$f(y) \approx \tilde{f}(y) = \boxed{f(x) + \nabla f(x)^T (y-x)}$$

f is twice differentiable at x

$$f(y) \approx \tilde{\tilde{f}}(y) = f(x) + \nabla f(x)^T (y-x) + \frac{1}{2}(y-x)^T \nabla^2 f(x) (y-x)$$

$= f(x) + \ldots$

quadratic in y

Tutorial 3
Q 4

$f(x) = 3x^2 + 5x - 4$

identify candidates for local minima, compare f at all to get the global minimum

Tutorial 2
Q 4

$f(x) = e^{\sum_{i=1}^{n} i x_i}$

take f to be differentiable

Tutorial 1
Q 5

$x_n = \underbrace{n^2 \sin\left(\frac{1}{n}\right) \log\left(1+\frac{1}{n}\right)}$

does $\lim_{n\to\infty} x_n$ exist?

$y_n = n^2 \log\left(1+\frac{1}{n}\right) =$

does $\lim_{n\to\infty} y_n$ exist?

$\frac{\log\left(1+\frac{1}{n}\right)}{\frac{1}{n^2}}$

$n\sin\left(\frac{1}{n}\right) \times n \log\left(1+\frac{1}{n}\right)$

$\lim_{n\to\infty} \frac{\sin 1/n}{1/n} = \lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1$

$\lim_{n\to\infty} \frac{\log(1+1/n)}{1/n}$

$= \lim_{x\to 0} \frac{\log(1+x)}{x}$

$= \lim_{x\to 0} \frac{\frac{1}{1+x}}{1} = 1$

$$\lim_{n\to\infty} x_n = 1$$

Taylor's expansion

$$n^2\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3} - \dots\right)\sin\left(\frac{1}{n}\right)$$

$$= \left(n-\frac{1}{2}+\frac{1}{3n} - \dots\right)\sin\frac{1}{n}$$

$$= n\sin\frac{1}{n} - \underbrace{\frac{1}{2}\sin\frac{1}{n}}_{\downarrow 0} + \underbrace{\frac{\sin 1/n}{3n}}_{\downarrow 0} - \dots$$

$$n\left(\frac{1}{n}-\frac{1}{n^3 3!} + \dots\right)$$

$$1 = \frac{1}{3n^2} + \dots$$

$$\lim_{n\to\infty}\sum_{k=1}^{\infty} x_k(n) \overset{?}{=} \sum_{k=1}^{\infty}\underbrace{\lim_{n\to\infty} x_k(n)}_{=0} = \sum_{k=1}^{\infty} 0 = 0$$

$$\lim_{n\to\infty} n\sin\left(\frac{1}{n}\right) \overset{?}{=} \lim_{x\to 0}\frac{1}{x}\sin(x)$$

L'Hospital

$$= \lim_{y\to\infty} y\sin\left(\frac{1}{y}\right)$$

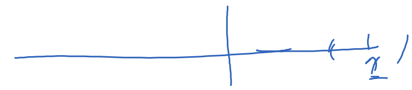
replace y with $\frac{1}{x}$

N_ϵ is such that

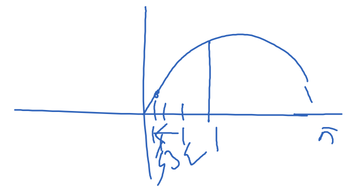
$$1+\epsilon \geq n^2\sin\left(\frac{1}{n}\right)\log\left(1+\frac{1}{n}\right) \geq 1-\epsilon \quad \forall n \geq N_\epsilon$$

$.99$

$1.01 > .99$



10^5 $\quad 10^5+1 \quad 10^5+2$



$\sin\left(\frac{1}{n}\right)$ $\quad 0 < ?$