Tutorial April 16th

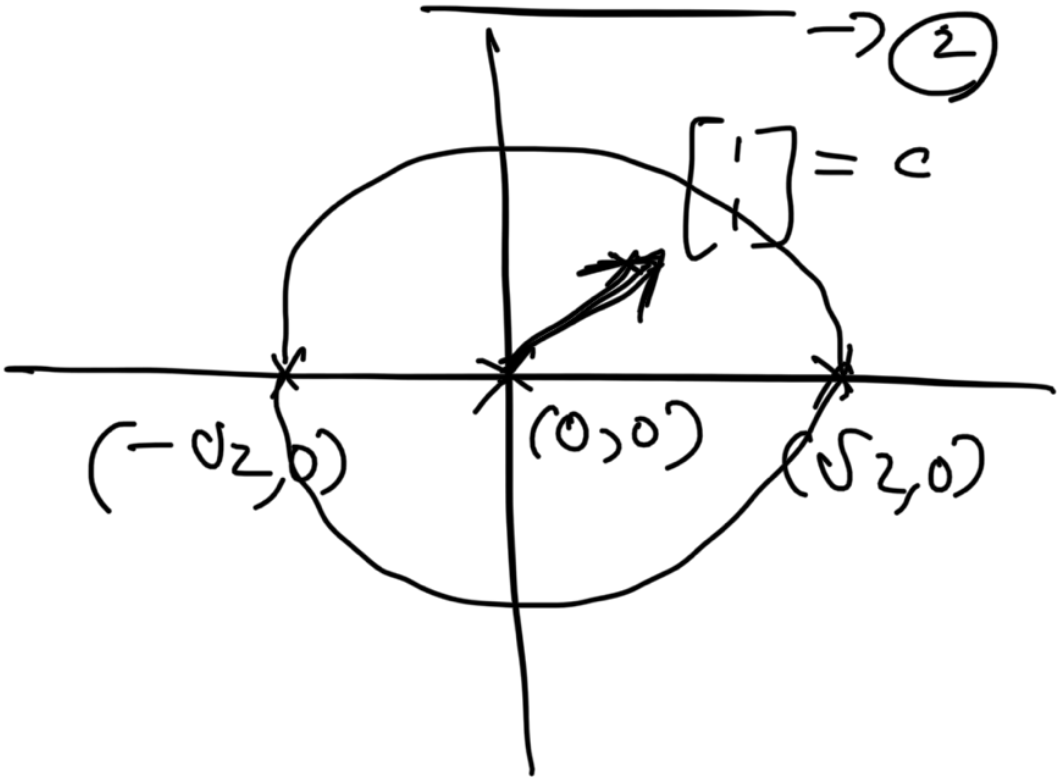
① Duality in L.P explained using one example

(primal) → Dual-variable

problem-set - 7

Q4 $\max_{x} \quad x_1 + x_2 \rightarrow (1)$

s.t $\quad x_1^2 + x_2^2 - 2 = 0 \rightarrow (2)$



$\min_{x} \quad -x_1 - x_2$

s.t $\quad x_1^2 + x_2^2 - 2 = 0$

$\min_{x} \; h(x)$

$h(x) = -x_1 - x_2 + c_k (x_1^2 + x_2^2 - 2)^2$

First-order-condition

$$\nabla h(x) = 0$$

$$\begin{bmatrix} -1 + 2c_k(x_1^2 + x_2^2 - 2)(2x_1) \\ -1 + 2c_k(x_1^2 + x_2^2 - 2)(2x_2) \end{bmatrix} = 0$$

$$4c_k x_1(x_1^2 + x_2^2 - 2) = 1$$

$$4c_k x_2(x_1^2 + x_2^2 - 2) = 1$$

$$4x_1(x_1^2 + x_2^2 - 2) = 1/c_k$$

$$4x_2(x_1^2 + x_2^2 - 2) = 1/c_k$$

$$c_k \to \infty \qquad k$$

$$4x_1(x_1^2 + x_2^2 - 2) = 0 \to 1$$

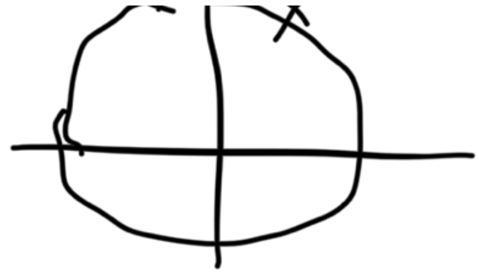
$$4x_2(x_1^2 + x_2^2 - 2) = 0$$

$x_1 = 0$ | or $(x_1^2 + x_2^2 - 2 = 0)$

$x_2(x_2^2 - 2) = 0$ | $\begin{Bmatrix} x_1 = \sqrt{2}\cos\theta \\ x_2 = \sqrt{2}\sin\theta \end{Bmatrix}$

$x_2 = 0 \quad x_2 = \pm\sqrt{2}$

$\begin{pmatrix}0\\0\end{pmatrix}$ $\begin{pmatrix}0\\\sqrt{2}\end{pmatrix}$ $\begin{pmatrix}0\\-\sqrt{2}\end{pmatrix}$



not feasible (feasible)

(second-order-condition)

$$\begin{cases} -1 + 4C_k x_1(x_1^2+x_2^2-2) = 0 \\ -1 + 4C_k x_2(x_1^2+x_2^2-2) = 0 \end{cases}$$

$$\nabla^2 h(x) = \begin{bmatrix} 4C_k(3x_1^2+x_2^2-2) & 8C_k x_1 x_2 \\ 8C_k x_1 x_2 & 4C_k(x_1^2+3x_2^2-2) \end{bmatrix}$$

$$\nabla^2 h\left(x^* = \begin{pmatrix}0\\\sqrt{2}\end{pmatrix}\right)$$

$$= \begin{bmatrix} 4C_k(0+2-2) & 0 \\ 0 & 4C_k(0+12-2) \end{bmatrix}$$

$$= \begin{bmatrix} 0=\lambda_1 & 0 \\ 0 & 40C_k=\lambda_2 \end{bmatrix} \Rightarrow (P.S.D)$$

necessary condition for minima is satisfied.

$\begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix}$ is a candidate for local minima

$\begin{pmatrix} 0 \\ \pm\sqrt{2} \end{pmatrix}$

$\begin{pmatrix} \sqrt{2}\cos\theta \\ \sqrt{2}\sin\theta \end{pmatrix}$

$$\begin{bmatrix} 4C_R(3\times 2\cos^2\theta + 2\sin^2\theta - 2) & 16\sin\theta\cos\theta \\ 16\sin\theta\cos\theta & 4C_R(2\cos^2\theta + 3\times 2\sin^2\theta - 2) \end{bmatrix}$$

$$\begin{bmatrix} 4C_R(\cancel{2}+4\cos^2\theta - \cancel{2}) & 16\sin\theta\cos\theta \\ 16\sin\theta\cos\theta & 8C_R(\cancel{1}+2\sin^2\theta - \cancel{1}) \end{bmatrix}$$

$$\begin{bmatrix} 16\cos^2\theta\, C_R & 16\sin\theta\cos\theta \\ 16\sin\theta\cos\theta & 16C_R\sin^2\theta \end{bmatrix}$$

$1/ \quad 0$

$\underline{16 \cos^2\theta \, C_k} > 0$ (P.D)

$\begin{pmatrix} \sqrt{2}\cos\theta \\ \sqrt{2}\sin\theta \end{pmatrix}$

$16^2 C_k^2 \sin^2\theta \cos^2\theta - 16^2 \sin^2\theta \cos^2\theta$

$16^2 \sin^2\theta \cos^2\theta \, (C_k^2 - 1) > 0$

$\cos^2\theta > 0$

$\sin^2\theta \cos^2\theta > 0$

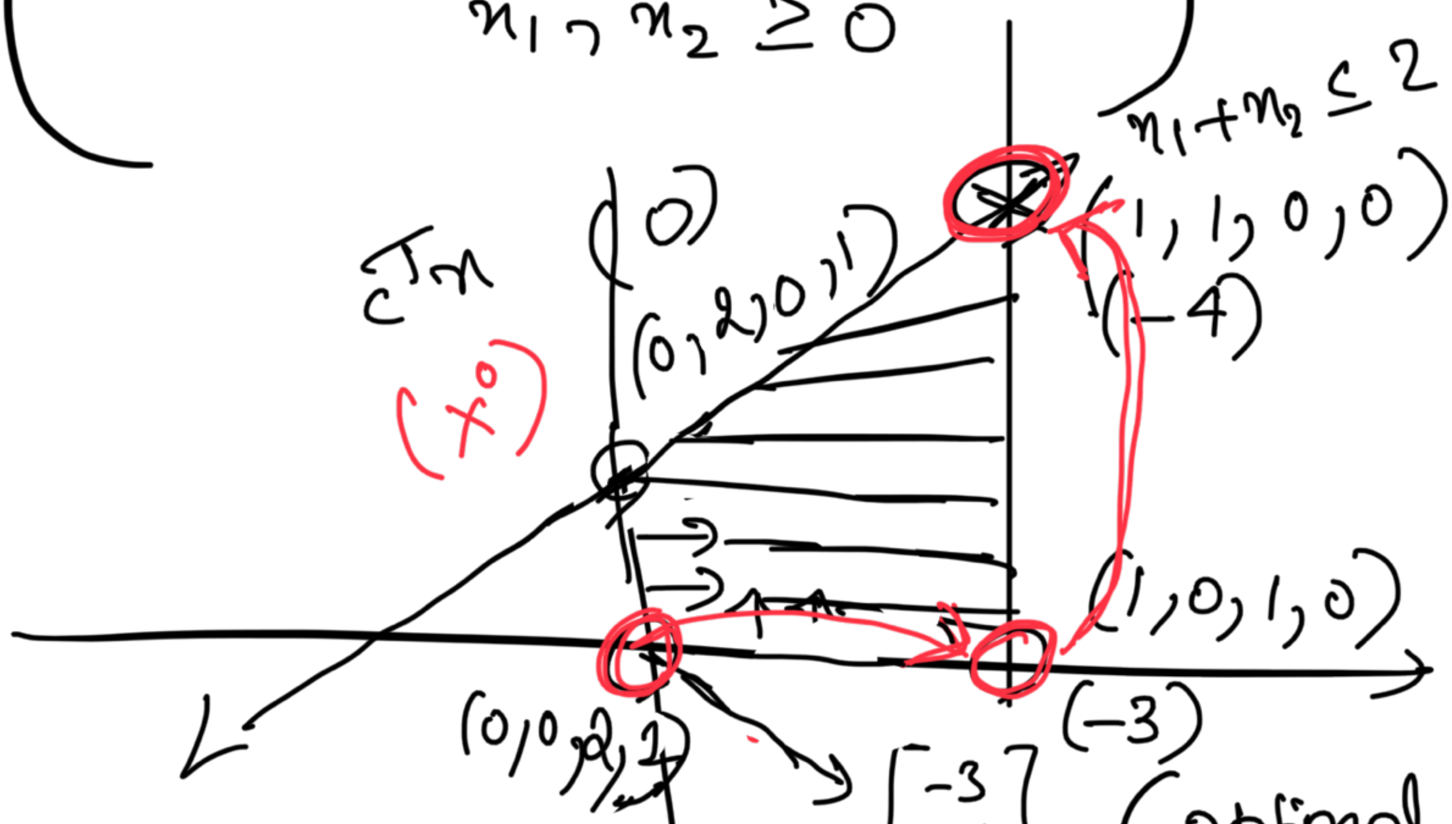
$\begin{pmatrix} \sqrt{2}\cos\theta \\ \sqrt{2}\sin\theta \end{pmatrix}$ (will be a local minima)

(first-order
second-order
sufficient
condition)

<u>L.P</u>

$$\left\{ \begin{aligned} \min_{x} \quad & -3x_1 - x_2 \\ \text{s.t} \quad & x_1 + x_2 \le 2 \\ & x_1 \le 1 \end{aligned} \right\}$$

$x_1, x_2 \geq 0$



(optimal soln to L.P → optimal basic feasible soln)

$$\min_{x} \quad -3x_1 - x_2$$

$$\text{s.t.} \quad x_1 + x_2 + u_1 = 2$$

$$x_1 + u_2 = 1$$

$$x_1, x_2 \geq 0$$

Initial Tableu

$I_{2\times 2}$ (basic variable u_1, u_2)

	x_1	x_2	u_1	u_2	b	
	1	1	1	0	2	2/1
	$\boxed{1}$	0	0	1	1	(1/1)
$-c^T$	3	1	0	0		

(0, 0, 2, 1)

(relative cost vector)

$x_1 \to$ enter the basis and Sol-

U_2 leave the

$R_1 \to R_1 - R_2$

$x_1 \to$ basic	x_2	$U_1 \to$ basic	U_2	
0	1	1	-1	1
1	0	0	1	1
3	1	0	0	

(1, 0, 1, 0)

$R_3 \to R_3 - 3R_2$

	x_1	x_2	U_1	U_2		
	0	$\boxed{1}$	1	-1	1	(1/1)
	1	0	0	1	1	(1/0)
$-c^T$	0	1	0	-3	-3	

x_2 will enter the basis and

U_1 will leave the basis

	x_1	x_2	U_1	U_2	
	0	1	1	-1	1
	1	0	0	1	1
c^T	0	0	-1	-2	4

$$x^* = (1, 1, 0, 0)$$

$$\begin{bmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 = 1 \\ x_2 = 1 \\ u_1 = 0 \\ u_2 = 0 \end{bmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Lecture - 26

$$\begin{array}{c|cc|c} & I & B D^{-1} & B^{-1} b \\ \hline & 0 & -\underbrace{(C_D^T - C_B^T B^{-1} D)} & (C_B^T B^{-1} b) \end{array}$$

$$y^* = -(C_D^T - C_B^T B^{-1} D)$$

$$y^* = (-1, -2)$$

$$b^T y \qquad b = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$b^T y = \begin{pmatrix} 2 \\ 1 \end{pmatrix}^T \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

$$= -2 - 2 = -4$$

$$\Longrightarrow$$

$(1, 1, 0, 0)$

$$c^T x = (-3 \quad -1) \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = -4$$

$$\max \; -2x_1 - x_2$$

$$\text{s.t} \quad x_1 + x_2 \geq 3$$

$$x_1 \geq 1$$

$$x_1, x_2 \geq 0$$

$\min c^T x$

s.t $Ax \geq b$

$x \geq 0$

$$\min_{x_1, x_2} \left[\begin{array}{l} 2x_1 + x_2 \\ \text{s.t} \quad x_1 + x_2 \geq 3 \\ \quad x_1 \geq 1 \\ \quad x_1, x_2 \geq 0 \end{array} \right]$$

S.f

$$\min_{x_1, x_2} \; 2x_1 + x_2$$

$c = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$-c = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$

$$x_1 + x_2 - U_1 = 3$$

$$x_1 - U_2 = 1$$

$\lambda_1, \lambda_2, v_1, v_2 \geq 0$

$$
\begin{array}{c|cccc|c}
 & \lambda_1 & \lambda_2 & v_1 & v_2 & b \\
\hline
 & 1 & 1 & -1 & 0 & 3 \\
 & 1 & 0 & 0 & -1 & 1 \\
\hline
-c^T & 2 & 1 & 0 & 0 &
\end{array}
$$

$$
\begin{bmatrix} 0 & \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \end{bmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}
$$

$v_1 = -3$
$v_2 = -1$ $\rightarrow$ not feasible

$x_1 = 1$, $x_2 = 2$, $\lambda_1 = 0$, $\lambda_2 = 0$

$$
\begin{array}{cccc|c}
\lambda_1 & \lambda_2 & v_1 = 0 & v_2 = 0 & \\
\hline
1 & 1 & -1 & 0 & 3 \\
1 & 0 & 0 & -1 & 1 \\
\hline
-2 & -1 & 0 & 0 & -4
\end{array}
$$

λ_1 (Basic) λ_2

$R_1 \rightarrow R_1 - R_2$

$$
\begin{array}{cccc|c}
0 & 1 & -1 & 0 & 2 \\
1 & 0 & 0 & -1 & 1
\end{array}
$$

$$\left[\begin{array}{cccc|c} \underbrace{2 \quad -1} & 0 & 0 & & -4 \end{array}\right]$$

$$r_j = c_j - \sum_{i=1}^{m} a_{ij} c_i$$

0	1	-1	0	2
1	0	0	-1	1
$r_1 = 0$	$r_2 = 0$	$r_3 = ?$ $\cancel{2}$ -2	$r_4 = ?$ -1	

$$r_3 = c_3 - \sum_{i=1}^{2} a_{i3} c_i$$

$$= 0 - \{(-1) \times -2 + 0 \times (\ )\}$$

$$r_3 = -+2$$

$$r_4 = c_4 - \sum_{i=1}^{2} a_{i4} c_i$$

$$= 0 - \{0 + -1 \times (-1)\}$$

$$= -1$$

$\lambda_1, \lambda_2, v_1, v_2 = ($

$$\begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ v_1 = 0 \\ v_2 = 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$x_p = \begin{pmatrix} \begin{pmatrix} -1 \\ -2 \end{pmatrix} \\ 0 \\ 0 \end{pmatrix} y^*$$

$2\lambda_1 + \lambda_2$

$-2 - 2 = -4$ //

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solved problem Levenberg example 3 chapter-3

$c^T x$

s.t $Ax = b$

$x \geq 0$

$c^T x$

$Ax \geq b$

$-Ax \geq -b$

$$\min_x \quad 2x_1 - 4x_2 - 7x_3 - x_4 - 5x_5$$

$$x_1 = x_2 + 2x_3 + x_4 + 2x_5 - 7$$

$$\min_{x} \quad \underbrace{2(x_2+2x_3+x_4+2x_5-7)}$$

$$\underbrace{-4x_2 - 7x_3 - x_4 - 5x_5}$$

s.t

$$-(x_2+2x_3+x_4+2x_5-7)$$

$$+2x_2+3x_3+x_4+x_5 = 6$$

$$\Longleftrightarrow$$

$$-(x_2+2x_3+x_4+2x_5-7)$$

$$+x_2+x_3+2x_4+x_5 = 4$$

$$x_2 \geq 0, \; x_3 \geq 0, \; x_4 \geq 0,$$

$$x_5 \geq 0$$

$$\hookrightarrow -(x_2+2x_3+x_4+2x_5-7)$$

$$+2x_2+3x_3+x_4+x_5 \geq 6$$

$$-U_1$$

$$x_2+2x_3+x_4+2x_5-7$$

$$-2x_2-3x_3-x_4-x_5 \geq -6$$

$$-U_2$$

	x_2	x_3	x_4	x_5	U_1	U_2	U_3

problemset -7

QN.8 : $x_{k+1} = P_S(x_k - g_k^+)$

$g_k^+ \in \partial f$

$$\min_{x} \quad \|x\|_1$$

$$\text{s.t } \underset{m \times n}{A} \underset{n \times 1}{x} = \underset{m \times 1}{b}$$

(under-det)

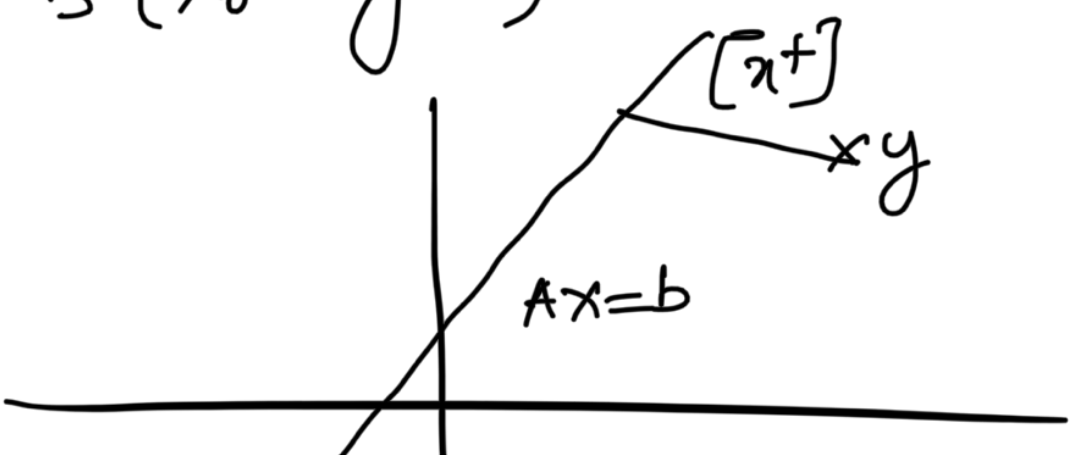
(classic problem in c.s)

$$x_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} \alpha_1 \in [-1, 1] \\ \alpha_2 \in [-1, 1] \\ \alpha_3 \in [-1, 1] \end{pmatrix}$$

$$x_1 = x_0 - g(0)$$

$$P_S(x_0 - g(0))$$



$$\min_{x^+} \frac{1}{2}\|x^+ - y\|_2^2$$
$$\text{s.t. } Ax^+ = b$$

$$\mathcal{L}(x^+, \lambda) = \frac{1}{2}\|x^+ - y\|_2^2 + \lambda^T(Ax^+ - b)$$

first-order optimality $\left\{ \begin{array}{l} a^T x^+ \\ \nabla_{x^+}(a^T x^+) \\ = a \end{array} \right.$

$$\nabla_{x^+}\mathcal{L}(x^+, \lambda) = \frac{2}{2}(x^+ - y) + A^T\lambda = 0$$

$$y = x^+ + A^T\lambda$$

$$\nabla_{\lambda}\mathcal{L}(x^+, \lambda) \Rightarrow Ax^+ = b$$

$$A(y - A^T\lambda) = b$$

$$Ay - AA^T\lambda = b$$

$A \in \mathbb{R}^{m \times n}$

$$\lambda = (AA^T)^{-1}(Ay - b)$$

$m \leq n$

$A \rightarrow$ full rank

$(AA^T)^{-1}$ exists

$$\boxed{x^+ = y - A^T(AA^T)^{-1}(Ay - b)}$$

$x^{(0)}$ $\quad$ $g^{(0)}$

$$x^{(k+1)} = P_S(x^0 - g^{(0)})$$

$= [\;]$

$$\left[\ \right] = y$$

$x^{(1)}$