Tutorial-11

Saturday, 26 March 2022 11:01 AM

$f(x) = x_1 - 2x_2$

(Q) Solve min $f(x)$

sub $1 + x_1 - x_2^2 \geq 0$

$x_2 \geq 0$

Constraint

using Barrier methods

$x_1 = 0$

$x_2 = 1$

min $f(x)$

st $g_j(x) \leq 0 \quad \forall j$

$$\boxed{\begin{aligned} g_1(x) &= x_2^2 - x_1 - 1 \\ g_2(x) &= -x_2 \end{aligned}}$$

Barrier function

$$\sum_{i=1}^{m} - \ln(-g_i(x)) = B(x)$$

Step (1)

Choose a seq $\{\epsilon_k\}$ decreasing and $\lim_{k \to \infty} \epsilon_k = 0$

min $f(x) + \epsilon_k B(x)$

$\nabla_x^2 \hat{L}(x, \epsilon_k) \Big|_{(x_1, x_2)} \geq 0$ min

$$f(x) + \epsilon_k B(x) \triangleq \hat{L}(x, \epsilon_k)$$

$f(x) = x_1 - 2x_2$ | $B(x) = -\ln(1 + x_1 - x_2^2) - \ln(x_2)$

$$\left.\begin{aligned} g_1(x) &= x_2^2 - x_1 - 1 \\ g_2(x) &= -x_2 \end{aligned}\right\}$$

$$\boxed{\hat{L}(x, \epsilon_k) = x_1 - 2x_2 + \epsilon_k\left(-\ln(1 + x_1 - x_2^2) - \ln(x_2)\right)}$$

$$\nabla_x \hat{L}(x, \epsilon_k) = \begin{bmatrix} 1 - \dfrac{\epsilon_k}{1 + x_1 - x_2^2} \\ -2 + 2\epsilon_k \dfrac{x_2}{1 + x_1 - x_2^2} - \dfrac{\epsilon_k}{x_2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{matrix} (1) \\ (2) \end{matrix}$$

$$\nabla_x \hat{\mathcal{L}}(x, \epsilon_k) = \begin{bmatrix} 1 - \dfrac{\epsilon_k}{1 + x_1 - x_2^2} \\ \\ -2 + 2\epsilon_k \dfrac{x_2}{1 + x_1 - x_2^2} - \dfrac{\epsilon_k}{x_2} \end{bmatrix} = \begin{bmatrix} 0 \\ \\ 0 \end{bmatrix} \begin{matrix} — ① \\ \\ — ② \end{matrix}$$

$$\epsilon_k = 1 + x_1 - x_2^2 \quad \text{From ①} \quad — ③$$

$$-2 + 2\frac{\epsilon_k x_2}{\epsilon_k} - \frac{\epsilon_k}{x_2} = 0 \quad — \text{sub ③ in ②}$$

$$x_2^2 - x_2 - \frac{\epsilon_k}{2} = 0$$

$$x_2 = \frac{1 \mp \sqrt{1 + 2\epsilon_k}}{2}, \quad x_2 = \frac{1 - \sqrt{1+2\epsilon_k}}{2}$$

$$\lim_{\epsilon_k \to 0}$$

$$\epsilon_k = 1 + x_1 - \left(\frac{1 - \sqrt{1+2\epsilon_k}}{2}\right)^2$$

$$(x_1, x_2) = \left(\frac{6\epsilon_k - 2\sqrt{1+2\epsilon_k} - 3}{2}, \frac{1 - \sqrt{1+2\epsilon_k}}{2}\right)$$

$$x_1 = \epsilon_k - 1 + \left(\frac{1 - \sqrt{1+2\epsilon_k}}{2}\right)^2$$

$$(x_1, x_2) \quad \lim_{\epsilon_k \to 0} x_1(\epsilon_k) = x_1^* \quad \lim_{\epsilon_k \to 0} x_2(\epsilon_k) = x_2^*$$

$$x_1 = \frac{4\epsilon_k - 4 + 1 + (1 + 2\epsilon_k) - 2\sqrt{1+2\epsilon_k}}{2}$$

$$= \left(\frac{6\epsilon_k + 2\sqrt{1+2\epsilon_k} - 3}{2}, \frac{1 + \sqrt{1+2\epsilon_k}}{2}\right) \quad x_2 = ''$$

$$= \frac{6\epsilon_k - 2\sqrt{1+2\epsilon_k} - 3}{2}$$

$$x_1 =$$

$$\epsilon_k = 1 + x_1 - \left(\frac{1 + \sqrt{1+2\epsilon_k}}{2}\right)^2$$

$$x_1 = \epsilon_k - 1 + \frac{1 + 1 + 2\epsilon_k + 2\sqrt{1+2\epsilon_k}}{4}$$

$$= \left(4\epsilon_k - 4 + 2 + 2\epsilon_k + 2\sqrt{1+2\epsilon_k}\right)$$

(x_1, x_2) and ∇

Penalty method

Q $\min \frac{1}{2}[(x_1-3)^2 + (x_2-2)^2]$

st $\begin{cases} -x_1 + x_2 \le 0 \\ \boxed{x_1 + x_2 \le 1} \\ -x_2 \le 0 \end{cases}$

$x_1 + x_2 = 1$

algo

$\min f(x)$ st $g_j(x) \le 0$

$\min f(x)$ st $g_i(x) = 0$

$\frac{\epsilon_k}{2} = c_k \uparrow$ choose a seq $\{c_k\}$ st $c_k \uparrow \infty$

$$\min \frac{1}{2}[(x_1-3)^2 + (x_2-2)^2] + \frac{\epsilon_k}{2}[\max(0, -x_1+x_2))^2] + \frac{\epsilon_k}{2}[\max(0, x_1+x_2-1)]^2 + \frac{\epsilon_k}{2}[\max(0, -x_2))^2]$$

$\min f(x) + c_k \sum_{j=1}^{r} ((g_j(x))^+)^2$

$(g_j(x))^+ = \begin{cases} g_j(x); & g_j(x) \ge 0 \\ 0; & \text{elsewhere} \end{cases}$

$\max(0, g_j(x))$

$x^0 = (3, 2)$

$\min \frac{1}{2}[(x_1-3)^2 + (x_2-2)^2] \Rightarrow x_1^*, x_2^*$

$$\min \frac{1}{2}[(x_1-3)^2 + (x_2-2)^2] + 0 + \frac{\epsilon_k}{2}[(x_1+x_2-1)^2] + 0$$

$$\min \frac{1}{2}[(x_1-3)^2 + (x_2-2)^2] + \frac{\epsilon_k}{2}[(x_1+x_2-1)^2] = \hat{\mathcal{L}}$$

$$\nabla_x \hat{\mathcal{L}} = \begin{bmatrix} (x_1-3) + \epsilon(x_1+x_2-1) \\ (x_2-3) + \epsilon(x_1+x_2-1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x_1(1+\epsilon) + x_2(\epsilon) = 3+\epsilon$$

$$x_2(1+\epsilon) + x_1(\epsilon) = 2+\epsilon$$

$$\begin{bmatrix} (1+\epsilon) & \epsilon \\ \epsilon & (1+\epsilon) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3+\epsilon \\ 2+\epsilon \end{bmatrix}$$

$$A \quad X = B$$

$\epsilon_k \to \infty$

$$X \begin{cases} x_1 = \dfrac{2\epsilon_k + 3}{2\epsilon_k + 1} \Rightarrow \dfrac{2 + 3/\epsilon_k}{2 + 1/\epsilon_k} = 1 \\ x_2 = \dfrac{2}{2\epsilon_k + 1} \Rightarrow \dfrac{2/\epsilon_k}{2 + 1/\epsilon_k} = 0 \end{cases}$$

$$(x_1^*, x_2^*) = \underline{\underline{(1, 0)}}$$

TUTORIAL - 11 :

Today.

1. discuss a problem

$$\left\{ \begin{array}{l} \min\limits_{x \in X} f(x) \\ \text{s.t } h(x) = 0 \end{array} \right\}$$

Lagrangian & compare with a penalty.

2. Another - example for the penalty - fn method
3. Barrier - function }
4. show some basic

(1) $\min f(x) = \{ x_1 + x_1 x_2 + 3x_2^2 \}$

s.t $g(x) = x_1 + 2x_2 - 3 = 0$

$$f(x) = \frac{1}{2} (x_1 \;\; x_2) \begin{pmatrix} 0 & 1 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$+ (1 \;\; 0) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \frac{1}{2} x^T A x + b^T x$$

$f(x)$ is convex or not.

$$\nabla^2_{xx} f = A = \begin{pmatrix} 0 & 1 \\ 1 & 6 \end{pmatrix}$$

$$\begin{vmatrix} -\lambda & 1 \\ 1 & 6-\lambda \end{vmatrix} = 0$$

$$\lambda(\lambda - 6) - 1 = 0$$

$$\lambda^2 - 6\lambda - 1 = 0$$

$$\lambda = \frac{6 \pm \sqrt{36+4}}{2},$$

$$(\lambda = + , -)$$

$$\mathcal{L}(x, \lambda) = x_1 + x_1 x_2 + 3x_2^2 + \lambda(x_1 + 2x_2 - 3)$$

$\nabla_x \mathcal{L}(x, \lambda) = 0$ (first-order necessary cond)

$$\begin{Bmatrix} 1 + x_2 - \lambda \\ x_1 + 6x_2 - 2\lambda \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$x_2 = \lambda - 1$$

$$x_1 = 2\lambda - 6x_2$$

$$x_1 + 2x_2 - 3 = 0$$

$$2\lambda - 6(\lambda - 1) + 2(\lambda - 1) - 3 = 0$$

$$2\lambda - 6\lambda + 6 + 2\lambda - 2 - 3 = 0$$

$$-2\lambda + 1 = 0$$

$$\boxed{\lambda = 1/2}$$

$$\boxed{\begin{aligned} x_2^* &= -1/2 \\ x_1^* &= 1 - 6 \times -1/2 = 4 \end{aligned}}$$

<u>Regularity</u>: $x_1 + 2x_2 - 3 = 0$

$$\boxed{x_1^* + 2x_2^* - 3 = 0}$$

{feasible}

$$\nabla g(x) = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$$

$$\nabla g(x^*) = \{\ 1\ \ 2$$

$$\begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$$

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second-order-necessary-condition

$$y^T \nabla_{xx}^2 \mathcal{L}\, y \geq 0$$

$$\forall y \in V(x)$$

$$V(x) = \{ y \mid y^T \nabla g(x) = 0 \}$$

$$V(x) = \left\{ y \mid y^T \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} = 0 \right\}$$

$$y_1 + 2y_2 = 0$$

$$y_1 = -2y_2$$

$$\begin{pmatrix} y_2 = a \\ y_1 = -2a \end{pmatrix}$$

$$V(x) = \left\{ \begin{pmatrix} -2a \\ a \end{pmatrix} \right\}$$

$$\nabla_{xx}^2 \mathcal{L}(x, \lambda) = \frac{1}{} \times 2 \begin{pmatrix} 0 & 1 \end{pmatrix}$$

$$2 \quad \begin{pmatrix} 1 & 6 \end{pmatrix}$$

$$+ \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 6 \end{pmatrix}$$

$y^T \nabla_{xx} \mathcal{L}\, y$

$$\begin{pmatrix} -2a \\ a \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} -2a \\ a \end{pmatrix}$$

$$\begin{pmatrix} -2a & a \end{pmatrix} \begin{pmatrix} a \\ -2a + 6a \end{pmatrix}$$

$$-2a^2 + \left(-2a^2 + 6a^2 \right.$$

$$\left(2a^2 \geq 0 \right)$$

Second-order neces satisfied

$\left(\left(x_1^*, x_2^* \right) \right.$ is a candidate for local minima$)$

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penalty-function

choose a sequence c_k

as $\lim_{k \to \infty} c_k \to \infty$

$$\operatorname*{arg\,min}_{x \in \mathbb{R}^2} \; x_1 + x_1 x_2 + 3x_2^2 + c_k \left((x_1 + 2x_2 - 3)^2 \right) = h(x)$$

unconstrained problem

first-order-necessary-

$$\nabla h(x) = \begin{Bmatrix} 1 + x_2 + c_k (2)(x_1 + 2x_2 - 3) \\ x_1 + 6x_2 + 4c_k (x_1 + 2x_2 - 3) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$2c_k x_1 + (1 + 4c_k) x_2 + 1 - 6c_k = 0$$

$$(1 + 4c_k) x_1 + (6 + 8c_k) x_2 - 12c_k = 0$$

x_1, x_2 ?

$$\left[\begin{array}{cc|c} 2Ck & 1+4Ck & 6Ck-1 \\ 1+4Ck & 6+8Ck & 12Ck \end{array}\right]$$

$$R_2 \to R_2 - 2R_1$$

$$\cancel{R_1 \to R_1}$$

$$\begin{bmatrix} 2Ck & 1+4Ck & 6Ck-1 \\ 1 & 4 & 2 \end{bmatrix}$$

$$R_1 \to R_1 - 2Ck\,R_2$$

$$\left[\begin{array}{cc|c} 0 & 1-4Ck & 2Ck-1 \\ 1 & 4 & 2 \end{array}\right]$$

$$(1-4Ck)\,x_2 = (2Ck-1)$$

$$x_2 = \frac{(2Ck-1)}{1-4Ck}$$

$$= 2 - \frac{1}{Ck}$$

$$\lim_{c_k \to \infty} \frac{2 - \frac{1}{c_k}}{\frac{1}{c_k} - 4} = \frac{2}{-4} = -\frac{1}{2}$$

$$\frac{1}{c_k} - 4$$

$$\boxed{x_2^* = -\frac{1}{2}}$$

$$\boxed{x_1^* = 4}$$

(Q2) Q $\min \frac{x^T x}{2} \rightarrow$

s.t $Ax = b$ → underdetermined system

$A \in \mathbb{R}^{m \times n}$

$(m < n)$

Aside

$\min \|x\|_1$

s.t $\underbrace{Ax = b}_{\text{underdet}}$

compressed sensing

?

$$\mathcal{L}(x,\lambda) = \frac{x^T x}{2} + \lambda^T\{Ax - b\}$$

$$\nabla_x \mathcal{L}(x,\lambda) = \frac{1}{2} \times 2x + \nabla_x \underbrace{(\lambda^T A)x} + 0$$

$$\nabla_x (a^T x) = a$$

$$a = A^T \lambda$$

$$\nabla_x \mathcal{L}(x,\lambda) = \underbrace{x + A^T \lambda} = 0$$

$$x = -A^T \lambda$$

$$Ax = b$$

$$A(-A^T \lambda) = b$$

A is full rank.

$$AA^T \in \mathbb{R}^{(m \times m)}$$

(grammian. matrices)

$$\boxed{\lambda = -(AA^T)^{-1} b}$$

$$x = -A^T(-(AA^T)^{-1} b)$$

$$\boxed{x^* = A^T(AA^T)^{-1} b}$$

C

$$h(x) = \arg\min_{x \in \mathbb{R}^n} \frac{x^T x}{2} + \sum_{j=1}^{m} \alpha_j (a_j^T x - b_j)^2$$

$$A = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

→ unconstrained - problem

$\alpha_j \to$ increasing

$$\nabla h(x) = \frac{1}{2}(2x) + \sum_{j=1}^{m} \alpha_j (2 (a_j^T x - b_j) a_j)$$

$$= x + 2A^T \alpha (Ax - b) = 0$$

$$= (I + 2A^T A \alpha) x - 2\alpha A^T b = 0$$

$$(I + 2A^T A \alpha) x = 2\alpha A^T b$$

$\alpha (\uparrow) \{\underline{\text{increasing}}\}$

→ deno.

(Q3) Barrier function

$$\min_{x \in X} (x+1)^2$$

s.t. $x \geq 0$

ε_k = (decreasing sequence) $\left(\frac{1}{2}\right)^k$

$$h(x) = \min_{x \in X} \underbrace{(x+1)^2 - \frac{1}{2^k} \ln(x)}$$

$x^{(0)} = \left(\quad \right)$

$\frac{d\,h(x)}{dx}$ = (first-order necess

$\cancel{\frac{d^2 h(x)}{dx^2} = \ldots}$

x^*

$\alpha^k = \frac{1}{2^k}$

$k = 1, 2, \ldots$

$\frac{1}{2} > \frac{1}{4} > \frac{1}{8}$

x^* $\{x^*\}$