E1 251-O: Tutorial Questions

Linear and Non-linear Optimization

April 1, 2022

1. Given the constraints
   $h_1 : \mathbb{R}^2 \to \mathbb{R}$, $h_1(x) = (x_1 - 1)^2 + x_2^2 = 1$
   $h_2 : \mathbb{R}^2 \to \mathbb{R}$, $h_2(x) = (x_1 + 1)^2 + x_2^2 = 1$.
   Is $x = (0, 0)^T$ a regular point?

2. For $x \in \mathbb{R}^2$ consider the constraints,
   $x_1 \geq 0$.
   $x_2 \geq 0$.
   $x_2 - (x_1 - 1)^2 \leq 0$.
   Show that $x_1 = 1$ and $x_2 = 0$ is feasible but not regular point.

3. Given $x \in \mathbb{R}^2$ solve the following optimization problem using Lagrangian first and second order conditions.
$$\begin{aligned} \min_x \quad & (x_1 - 1)^2 + (x_2 - 1)^2 \\ \text{s.t.} \quad & x_1^2 + x_2^2 - 1 = 0 \end{aligned}$$

4. Find the stationary points for the function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x) = x_1 x_2$, given the constraint $h : \mathbb{R}^2 \to \mathbb{R}$, $h(x) = x_1 + x_2 = 6$.

5. Find the stationary points for the function $f : \mathbb{R}^2 \to \mathbb{R}$, $f : C x_1^{\alpha} x_2^{(1-\alpha)}$, given the constraint $h : \mathbb{R}^2 \to \mathbb{R}$, $h(x) = a x_1 + b x_2 = m$, where $C, a, b, \alpha \in \mathbb{R}$ are constants.

6. Solve the following optimization problem using Lagrangian multiplier method. Where $A \in \mathbb{R}^{m \times n} (m < n)$ is full rank and $x, y \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$.
$$\begin{aligned} \min_y \quad & \tfrac{1}{2} ||x - y||^2 \\ \text{s.t.} \quad & Ay = b \end{aligned}$$

7. Find the local minima and maxima for the constrained optimization problem.
$$\begin{aligned} \min \quad & x_1 + x_2 \\ \text{s.t.} \quad & x_1^2 + x_2^2 = 1 \end{aligned}$$

8. Solve
$$\begin{aligned} \max_x \quad & 14\, x_1^2 - x_1 + 6x_2 - x_2^2 + 7 \\ \text{sub to} \quad & \\ & x_1 + x_2 \leq 2 \\ & x_1 + 2x_2 \leq 3 \end{aligned}$$

9. (a) Find the Dual of the following LP problem,

$$\begin{array}{ll} \min\limits_{x \in \mathbb{R}^n} & c^T x \\ \text{sub to} & \\ & \text{Ax} = b \\ & A_1\, x \leq b_1 \end{array}$$

Where $A, A_1 \in \mathbb{R}^{m \times n}$ and $b, b_1 \in \mathbb{R}^m$.

(b) For $x \in \mathbb{R}^4$ Find the dual of the following LP and also write down the KKT conditions for the following problem.

$$\begin{array}{ll} \max\limits_{x} & 18x_1 + 12x_2 + 2x_3 + 6x_4 \\ \text{sub to} & \\ & 3x_1 + x_2 - 2x_3 + x_4 = 2 \\ & x_1 + 3x_2 - x_4 = 2 \\ & x_1 \geq 0; x_2 \geq 0; x_3 \geq 0; x_4 \geq 0. \end{array}$$

10. Consider the primal problem (P), where A is anti-symmetric ($A^T = -A$),

$$\begin{array}{ll} \min\limits_{x} & c^T x \\ \text{sub to} & \\ & \text{Ax} \geq -c \\ & x \geq 0 \end{array}$$

Show that (P) and its dual are same.

11. Consider the problem,

$$\begin{array}{ll} \min\limits_{x} & -\sum_{i=1}^{n} \log(\alpha_i + x_i) \\ \text{sub to} & \\ & \sum_{i=1}^{n} x_i = 1 \\ & x_i \geq 0 \end{array}$$

Where $\alpha_i > 0$ are parameters. Using the KKT conditions find the solutions.

12. Consider the Quadratic program,

$$\begin{array}{ll} \min\limits_{x} & \frac{1}{2} x^T H x + c^T x \\ \text{sub to} & \\ & Ax \geq b \end{array}$$

where $H \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix. Find the dual of the above problem.