TUTORIAL :

Today:

1. discuss a non-linear problem and show how the C-G algorithm works
2. Demo's (2) (Last)
   C-G (QM)
   { rosen-brock } (Gen Non-linear problem)
3. problemset - 6 (QN-8)
   QN-(Last QN)

(Q1.) Midsem-1 (QN-5) { Newton's method }

$f: \mathbb{R}^2 \to \mathbb{R}$

$$f(x) = \min_{x \in \mathbb{R}^2} \left\{ x_1^4 + 2x_1^2 x_2^2 + x_2^4 \right\}$$

$$x^0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$x_{k+1} = x_k + \alpha_k d_k$$

$$x_1 = x_0 + \alpha_0 d_0$$

$$\nabla f(x) = \begin{Bmatrix} 4x_1^3 + 4x_1 x_2^2 \\ 4x_1^2 x_2 + 4x_2^3 \end{Bmatrix}$$

$$\nabla f(x^0) = \begin{Bmatrix} 40 \\ 20 \end{Bmatrix}$$

$$d^{(0)} = -\nabla f(x^0) = \begin{pmatrix} -40 \\ -20 \end{pmatrix}$$

$$\alpha_0 = \arg\min_{\alpha \in \mathbb{R}^+} f(x^0 + \alpha d^0)$$

$$= \underset{\alpha \in \mathbb{R}^+}{\arg\min} \; f\left(\begin{pmatrix}2\\1\end{pmatrix} + \alpha\begin{pmatrix}-40\\-20\end{pmatrix}\right)$$

$$g(\alpha) = \underset{\alpha \in \mathbb{R}^+}{\arg\min} \; (2-40\alpha)^4 + 2(2-40\alpha)^2(1-20\alpha)^2 + (1-20\alpha)^4$$

first-order-necess

$$\frac{dg(\alpha)}{d\alpha} = 0$$

$$25(1-20\alpha) = 0$$

$$\boxed{\alpha^* = \frac{1}{20}}$$

$$x^{(1)} = \begin{pmatrix}2\\1\end{pmatrix} + \frac{1}{20}\begin{pmatrix}-40\\-20\end{pmatrix}$$

$$= \begin{pmatrix}2\\1\end{pmatrix} + \begin{pmatrix}-2\\-1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

C-G- got to minima in exactly one-step.

$$x^0 = \begin{pmatrix}a\\b\end{pmatrix} \qquad a, b \in \mathbb{R}$$

$$\boxed{\alpha^* = -\frac{1}{4(b^2-a^2)}} \rightarrow \left\{\text{can check it out}\right\}$$

Non-linear-CG method

(Levenberg)

a ∫

start x^0 $d^0 = -\nabla f(x^0)$

$$k \geq 0, \quad x_{k+1} = x_k + \alpha_k \boxed{d_k} \qquad d_k = -\nabla f(x_k)$$

$$\alpha_k = \arg\min_{\alpha \in \mathbb{R}^+} f(x^k + \alpha d_k)$$

$$d_{k+1} = -\nabla f(x_{k+1}) + \beta_{k+1} d_k$$

$$\beta_{k+1} = \frac{\nabla f(x_{k+1})^T \nabla f(x_{k+1})}{\nabla f(x^k)^T \nabla f(x_k)}$$

$$\left\{ f : \mathbb{R}^n \to \mathbb{R} \right\}$$

$$x_{k+1} = x_k + \alpha_k \{d_k\} = \{-\nabla f(x_k)\}$$

$$\alpha_k = \arg\min_{\alpha \in \mathbb{R}^+} f(x_k + \alpha_k d_k)$$

$$f(x) = \left\{ \frac{1}{2} x^T A x - b^T x + c \right\} \qquad A \in \text{symmetric}$$

$$\nabla f = \{Ax - b\} \qquad d_k = -\nabla f = b - Ax$$

$$\alpha_k = \arg\min_{\alpha \in \mathbb{R}^+} f(x_k + \alpha d_k)$$

$$= g(\alpha) = \frac{1}{2}(x_k + \alpha d_k)^T A (x_k + \alpha d_k) - b^T (x_k + \alpha d_k) + c$$

$$\left\{ \frac{d g(\alpha)}{d\alpha} = 0 \right\}$$

$$f : \mathbb{R}^2 \to \mathbb{R}$$

$$\{ f \ldots \ln(x_1) \} \}$$

$$\alpha$$

show - Demos C-G works (QP)

(N-L?)

(Q2) Last QN problem-set-6

{ Find the Dual } $H \to \{\text{invertible}\}$

$$\min_{x} \ \frac{1}{2} x^T H x + c^T x$$
$$\text{s.t.} \quad Ax \geq b$$

$$\mathcal{L}(x, \mu) = \frac{1}{2} x^T H x + c^T x + \mu^T (b - Ax)$$

$$D(\mu) = \min_{x} \ \frac{1}{2} x^T H x + c^T x + \mu^T (b - Ax)$$

$$Hx + c - A^T \mu = 0$$

$$x = H^{-1}(A^T \mu - c)$$

Dual problem

$$\max_{\mu \geq 0} \ \frac{1}{2} (\mu^T A - c^T) H^{-1} H H^{-1} (A^T \mu - c) + c^T H^{-1} A^T \mu - c^T H^{-1} c + \mu^T b - \mu^T A H^{-1} (A^T \mu) + \mu^T A H^{-1} c$$

$$\max_{\mu \geq 0} \ -\frac{1}{2} \mu^T (A H^{-1} A^T) \mu + \mu^T (b + A H^{-1} c) - \frac{1}{2} c^T H^{-1} c$$

$$\max_{\mu \geq 0} -\frac{1}{2} \mu^T (AH^{-1}A^T)\mu + \mu^T(b + AH^{-1}c)$$

Q8.) problem - sheet - 6 :-

solve $\min_{x \in \mathbb{R}^2} 14x_1^2 - x_1 + 6x_2 - x_2^2 + 7$

$$\left.\begin{aligned} \text{s.t } x_1 + x_2 &\leq 2 \\ x_1 + 2x_2 &\leq 3 \end{aligned}\right\}$$

primal problem (P)

$$\mathcal{L}(x, \mu_1, \mu_2) = 14x_1^2 - x_1 + 6x_2 - x_2^2 + 7 + \mu_1(x_1 + x_2 - 2) + \mu_2(x_1 + 2x_2 - 3), \quad \mu_1 \geq 0, \mu_2 \geq 0$$

{ strong - Duality concept in QP }

primal problem obje

Dual problem objective.

{ P.S.D }

→ { strong - Duality hold } ✓

$$f(x) = 14x_1^2 - x_1 + 6x_2 - x_2^2 + 7$$

$$= \frac{1}{2}(x_1 \; x_2)\begin{bmatrix} 28 = \lambda_1 & 0 \\ 0 & -2 = \lambda_2 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (-1 \;\; 6)^T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + 7$$

$$= \frac{1}{2} x^T A x + c^T x + b$$

$\{$ A is neither PSD nor N.S.D $\}$

$\{ f$ is not convex nor concave $\}$

$\nabla_x \mathcal{L}(x, \mu) = 0$ (first - order condition) (Lagrange - Duality theorem)

$$\begin{Bmatrix} 28x_1 - 1 + \mu_1 + \mu_2 \\ -2x_2 + 6 + \mu_1 + 2\mu_2 \end{Bmatrix} = 0$$

$$\left.\begin{aligned} x_1 &= \frac{1 - \mu_1 - \mu_2}{28} \\ x_2 &= \frac{+6 + \mu_1 + 2\mu_2}{2} \end{aligned}\right\}$$

$\{$ complementary - slackness condition $\}$

$$\mu_1 \left(g_1(x_1(\mu), x_2(\mu)) \right) = 0$$

$$\mu_1 \left\{ \underbrace{\frac{1 - \mu_1 - \mu_2}{28} + \frac{6 + \mu_1 + 2\mu_2}{2} - 2} \right\} = 0$$

$$\mu_2 \left(\underbrace{\frac{1 - \mu_1 - \mu_2}{28} + 6 + \mu_1 + 2\mu_2 - 3} \right) = 0$$

μ_1, μ_2 $(0, 0)$

$(\mu_1 = 0)$ $\quad$ $(\mu_2 = 0)$

$$\left\{ \frac{1-\mu_2}{28} + 6 + 2\mu_2 - 3 = 0 \right\}$$

$\mu_2 = \_\_$

$$\left[\begin{array}{l} \frac{1-\mu_1-\mu_2}{28} + \frac{6+\mu_1+2\mu_2}{2} - 2 = 0 \\ \frac{1-\mu_1-\mu_2}{28} + 6+\mu_1+2\mu_2 - 3 = 0 \end{array} \right]$$

$Ax = b$ $\{(\mu_1, \mu_2)\}$

$(\mu_1, \mu_2 = ?)$

① $\boxed{\mu_1 = 0, \mu_2 = 0}$ (Lecture - 16 page no - 5)

(not satisfy constraints)

② $\boxed{\mu_1 = \frac{-1429}{14}, \mu_2 = \frac{6\cancel{7}3}{14}}$

(violated $\mu_1 \geq 0$)

③ $\boxed{\mu_1 = 0, \mu_2 = -1.545}$ (violated $\mu_2 \geq 0$)

④ $\boxed{\mu_2 = 0, \mu_1 = -2.2307}$ (violated $\mu_1 \geq 0$)

(no candidate for minima)

sufficiency condition for global.

(μ_1, μ_2)

(x_1, x_2)

(μ_1^*, μ_2^*) $f(x, \mu_1^*, \mu_2^*)$

(μ_1, μ_2)

$$\frac{\mathcal{L}\left(x^*_{\mu_1, \mu_2}\right) = \left\{ \min_{x} \mathcal{L}(x, \mu_1^*, \mu_2^*) \right\}}{\mathcal{L}(x^*}$$ sufficient

$\mu_1^* = 0, \quad \mu_2^* = 0$

$$\left\{ \begin{array}{l} g_1(x^*) \le 0 \\ g(x^*) \le 0 \end{array} \right\} \quad (x^*, \mu^*) = \{\text{ideal pair}\}$$

$\{$primal - ~~obje~~ctive - fn value$\}$

Dual problem

$$D(\lambda) = \arg \min_{x \in \mathbb{R}^2} 14x_1^2 - x_1 + 6x_2 - x_2^2 + 7 + \mu_1(x_1 + x_2 - 2) + \mu_2(x_1 + 2x_2 - 3$$

$$x_1 = \frac{1 - \mu_1 - \mu_2}{28}$$

$$x_2 = \frac{\mu_1 + 2\mu_2 + 6}{2}$$

Dual problem

$$\max_{\mu_1, \mu_2 \ge 0} \left\{ 14\left(\frac{1 - \mu_1 - \mu_2}{28}\right)^2 - \left(\frac{1 - \mu_1 - \mu_2}{28}\right) + 6\left(\frac{\mu_1 + 2\mu_2 + 6}{2}\right) - \left(\frac{\mu_1 + 2\mu_2 + 6}{2}\right)^2 + 7 + \mu_1\left(\left(\frac{1 - \mu_1 - \mu_2}{28}\right) + \left(\frac{\mu_1 + 2\mu_2 + 6}{2}\right) - 2\right) \right.$$

$$+ u_2\left(\frac{1 - u_1 - u_2}{28} + u_1 + 2u_2 \right\}\Bigg)$$

first.
second.

Dual-problem. objective- function.

value

Tutorial-10

Saturday, 19 March 2022 11:02 AM

① $f(x) = -\ln(x) + x^2$. Use conjugate gradient method to find the optimal sol. let $x^0 = 1$

$$\nabla f(x) = -\frac{1}{x} + 2x = 0 \Rightarrow -1 + 2x^2 = 0$$

$$\Rightarrow 2x^2 = 1$$

$$x^{k+1} = x^k + \alpha_k d^k \qquad \Rightarrow x = \frac{1}{\sqrt{2}} \text{ (optimal)}$$

$$d^0 = -\nabla f(x^0) \qquad \nabla f(x^0) = -1 + 2 = 1$$

$$d^0 = -1$$

$$x^1 = 1 + \alpha_0(-1) = 1 - \alpha_0 \quad \text{optim line search}$$

$$f(x^1) = g_{x^1}(\alpha_0) = -\ln(1-\alpha_0) + (1-\alpha_0)^2$$

$$g'_{x^1}(\alpha_0) = \frac{1}{1-\alpha_0} - 2(1-\alpha_0) = 0$$

$$= -1 + 2(1-\alpha_0)^2 = 0$$

$$\Rightarrow 1 - \alpha_0 = \frac{1}{\sqrt{2}}$$

$$= \alpha_0 = 1 - \frac{1}{\sqrt{2}}$$

$$x^1 = 1 - 1 + \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

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⑨ ⓐ Dual of the LP prob

$$\min_{x \in \mathbb{R}^n} c^T x$$

$$Ax = b$$

$$A_1 x \le b_1$$

$$\min c^T x \quad \text{st } Ax \le b, \; x \ge 0$$

$$\min c^T x \quad \text{st } Ax = b, \; x \ge 0$$

$$\lambda \in \mathbb{R}^m; \; \mu \in \mathbb{R}^m$$

$A_1 x \le b_1 \}\}$

$c \in \mathbb{R}^n$; A and $A_1 \in \mathbb{R}^{m \times n}$; b and $b_1 \in \mathbb{R}^m$.

$$\mathcal{L}(x, \lambda, \mu) = c^T x + \lambda^T (Ax - b) + \mu^T (A_1 x - b_1)$$

$$= \underbrace{(c^T + \lambda^T A + \mu^T A_1)}\boxed{x} - \lambda^T b - \mu^T b_1$$

Dual fun :-

$$D(\lambda, \mu) = \min_{x \in \mathbb{R}} \mathcal{L}(x, \lambda, \mu)$$

$\nabla_x \mathcal{L}(x, \lambda, \mu) = 0$

$= c^T + \lambda^T A + \mu^T A_1 = 0$

$$= \begin{cases} -\lambda^T b - \mu^T b_1 & c^T + \lambda^T A + \mu^T A_1 = 0. \\ -\infty & \text{otherwise} \end{cases}$$

Dual

$$\max_{\lambda, \mu} D(\lambda, \mu) \quad \mu \ge 0$$

$$= \max \; -\lambda^T b - \mu^T b_1$$

$$\mu \ge 0$$

$$c + A^T \lambda + A_1^T \mu = 0$$

dual problem

(b) $x \in \mathbb{R}^4$

$$\min \; 18x_1 + 12x_2 + 2x_3 + 6x_4$$

subto $3x_1 + x_2 - 2x_3 + x_4 = 2$

$x_1 + 3x_2 - x_4 = 2$

$x_1 \ge 0$; $x_2 \ge 0$; $x_3 \ge 0$; $x_4 \ge 0$

Dual

$\min \; c^T x$

$Ax = b$

$A_1 x \le b_1$

$$c = \begin{bmatrix} 18 \\ 12 \\ 2 \\ 6 \end{bmatrix} \qquad A = \begin{bmatrix} 3 & 1 & -2 & 1 \\ 1 & 3 & 0 & -1 \end{bmatrix}$$

2×4

$$b = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$\Gamma \lambda_1 \,]$

$$\lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \qquad b = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

\#eqn \#variables

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix} \qquad A_1 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}_{4\times 4}$$

$$\max \quad -\lambda^T b - \cancel{\mu^T b_1}^{\to 0}$$

$$\mu \geq 0$$

$$c + A^T\lambda + A_1^T \mu = 0$$

$$\Rightarrow \quad \max \quad -\begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$\mu_1 \geq 0; \quad \mu_2 \geq 0; \quad \mu_3 \geq 0; \quad \mu_4 \geq 0 \quad \text{— (1)}$$

$$\begin{bmatrix} 18 \\ 12 \\ 2 \\ 6 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 1 & 3 \\ -2 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} - \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix} = 0 \quad \text{— (2)}$$

$$\Rightarrow \quad \max \quad -2\lambda_1 - 2\lambda_2$$

$$\left.\begin{aligned} 18 + 3\lambda_1 + \lambda_2 - \mu_1 &= 0 \\ 12 + \lambda_1 + 3\lambda_2 - \mu_2 &= 0 \\ 2 - 2\lambda_1 - \mu_3 &= 0 \\ 6 + \lambda_1 - \lambda_2 - \mu_4 &= 0 \end{aligned}\right\} \quad \mu_1 \geq 0; \mu_2 \geq 0; \mu_3 \geq 0; \mu_4 \geq 0$$

$$\Rightarrow \quad \max \quad -2\lambda_1 - 2\lambda_2$$

sub

$$\left.\begin{aligned} 18 + 3\lambda_1 + \lambda_2 &\geq 0 \\ 12 + \lambda_1 + 3\lambda_2 &\geq 0 \\ 2 - 2\lambda_1 &\geq 0 \\ 6 + \lambda_1 - \lambda_2 &\geq 0 \end{aligned}\right\} \quad \text{Dual}$$

KKT - conditions :-

$$\min_{x} \quad 18x_1 + 12x_2 + 2x_3 + 6x_4$$

$$3x_1 + x_2 - 2x_3 + x_4 = 2$$

$$x_1 + 3x_2 - x_4 = 2$$

$$x_1 \geq 0; \quad x_2 \geq 0, \quad x_3 \geq 0; \quad x_4 \geq 0$$

allaction, x_1, x_2, x_3, x_4, (x_1, x_2), (x_1, x_3),

$$\mathcal{L}(x, \lambda, \mu) = 18x_1 + 12x_2 + 2x_3 + 6x_4 + \lambda_1(3x_1 + x_2 - 2x_3 + x_4 - 2)$$

$$+ \lambda_2(x_1 + 3x_2 - x_4 - 2) - \sum_{i=1}^{4} \mu_i x_i$$

$$\nabla_x \mathcal{L}(x, \lambda, \mu) = \begin{bmatrix} 18 + 3\lambda_1 + \lambda_2 - \mu_1 \\ 12 + \lambda_1 + 3\lambda_2 - \mu_2 \\ 2 - 2\lambda_1 - \mu_3 \\ 6 + \lambda_1 - \lambda_2 - \mu_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

complen slack

$$\mu_i x_i = 0 \quad \forall i = 1, 2, 3, 4$$

$$\mu_1 = 0 = \mu_2 = \mu_3 = \mu_4$$

① feasiability Conditions

② $\nabla_x \mathcal{L}(x, \lambda, \mu) = 0 \quad ; \mu \geq 0 \rightarrow \rightarrow (\quad)$

③ complementary slackness $\Rightarrow$ case