Tutorial 2

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Open sets

$X = \mathbb{R}$

$A = [a, b)$

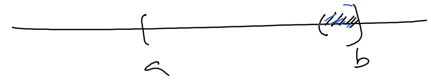
* A is open in A?

$X = A$

$\forall x \in A$

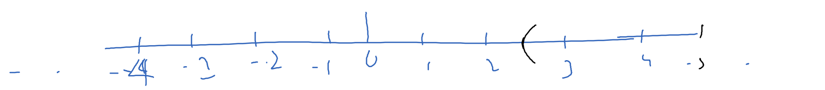
$B_\epsilon(x) = \{ y \in X (= A) : \|y - x\|_2 < \epsilon \} \subseteq A$

$B_\epsilon(b) = \{ y \in [a, b) : \|y - b\|_2 < \epsilon \}$



Example of a ball

$X = \mathbb{Z}$



$B_{2.5}(5) = \{ y \in \mathbb{Z} : \|y - 5\|_2 < 2.5 \}$

$= \{3, 4, 5, 6, 7\}$

$B_{0.5}(5) = \{5\}$

$f : \mathbb{R}^n \to \mathbb{R}$

- $f(x) = \|x\|_2^2$ is a convex fn
- $f(x) = \|x\|_2$ is not a convex fn
- If $f : \mathbb{R}^n \to \mathbb{R}$ is convex then so are
  1. $f(x + a)$ $\forall a \in \mathbb{R}^n$
  2. $g(f(x))$ where g is an increasing fn.

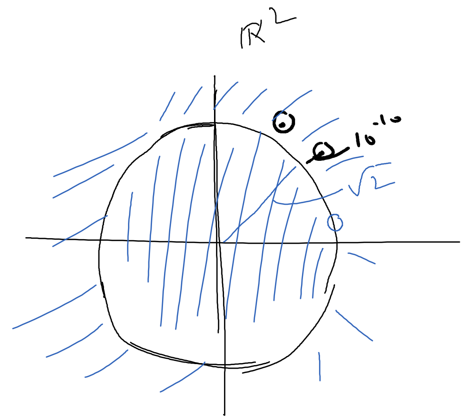
* If $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}$ are convex fns so is $f_1 + f_2 : \mathbb{R}^n \to \mathbb{R}$

Hence $\left(\|x - x^0\|_2^2\right)^{3/2} + \left(\|x - x^1\|_2^2\right)^{5/2}$

$(a^b)^c = a^{bc}$

i.e. $\|x - x^0\|_2^3 + \|x - x^1\|^5$ is convex

- open set
- closed set : A is closed if A^c is open
- compact set : A is compact if it is closed and bounded.



Examples $\quad X = \mathbb{R}^2$

$$A = \{ x \in \mathbb{R}^2 : \underbrace{x_1^2 + x_2^2}_{=\|x\|_2^2} \neq 2 \}$$

is A open or closed or both or none?

$$A = \underbrace{\{ x \in \mathbb{R}^2 : \|x\|_2^2 < 2 \}}_{A_1} \cup \underbrace{\{ x \in \mathbb{R}^2 : \|x\|_2^2 > 2 \}}_{A_2}$$

- A_1, A_2 are open $\Rightarrow A_1 \cup A_2$ is also open

$A^c = \{ x \in \mathbb{R}^2 : \|x\|_2^2 = 2 \}$ is not open

$\Rightarrow A$ is not closed

Given x

- def^n of open set depends on def^n of ball which in turn depends on def^n of distance

- default: distance is defined via 2-norm $\boxed{d(x, y) = \|x - y\|_2}$

Example $\quad X = \mathbb{R}^2$

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

$\epsilon = 0.5$

$$B_\epsilon(x) = \{ y \in \mathbb{R}^2 : d(y, x) < \epsilon \} = \{x\}$$

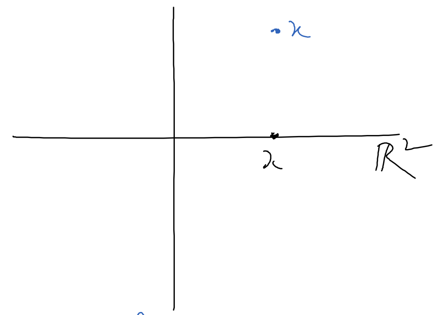
$$B_\epsilon(x) = \{ y \in \mathbb{R}^n : \|y - x\|_2 < \epsilon \}$$

$\epsilon = 1.5$

$$B_\epsilon(x) = \mathbb{R}^2$$

$\epsilon = 1.5$

$B_\epsilon(x) = \mathbb{R}^2$



Is $A = \{1\}$ open?

$B_{9/10}(1) = \{1\} \subseteq A$

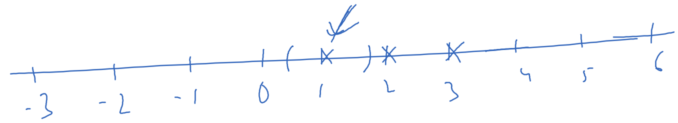
$\Rightarrow A$ is open

Example space: $X = \mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \pm 4, \ldots\}$

$d(x,y) = \|x-y\|_2$

$A = \{1, 2, 3\}$

Is A open?



$B_{7/10}(1) = \{1\} \subseteq A$ ✓

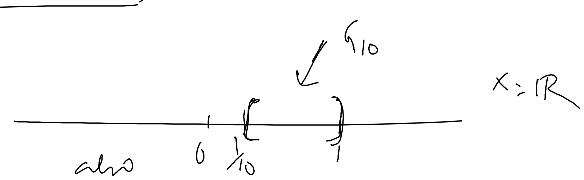
$B_{7/10}(1) = \{y \in \mathbb{Z} : \|y-1\| < 7/10\}$

$\Rightarrow A$ is open

Example $X = \mathbb{R}$

$G_n = [\frac{1}{n}, 1]$ $n \geq 1$ $(n \in \mathbb{Z}_{++})$

Is G_n open or closed?



G_n is closed $\forall n \geq 1$

$\Rightarrow G_1 = \{1\}$ is also closed

$G_2^c = \underbrace{(-\infty, \frac{1}{2})}_{\text{open}} \cup \underbrace{(1, \infty)}_{\text{open}}$ is open

$G_n^c = (-\infty, \frac{1}{n}) \cup \underbrace{(1, \infty)}_{\text{open}}$

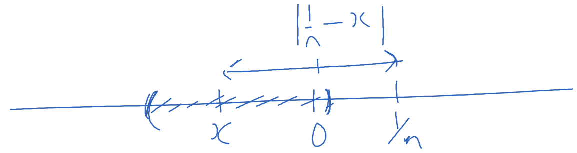
take any $x \in (-\infty, \frac{1}{n})$

then

$B_{\ldots}(x) \subseteq (-\infty, \frac{1}{n})$

then

$B_{\frac{|\frac{1}{n}-x|}{2}}(x) \subseteq (-\infty, \frac{1}{n})$ – this implies that $(-\infty, \frac{1}{n})$ is open.



Continuity space $X = \mathbb{R}^n$, $A \subseteq X$

$f : A \longrightarrow B$

f is called continuous at $x \in A$ if $\lim_{n\to\infty} f(x_n) \longrightarrow f(x)$ $\forall$ x_n such that $x_n \longrightarrow x$

Examples (1) $X = \mathbb{R}$, $A = \mathbb{R}$

$f(x) = e^x$

Is f continuous at $x = 1$?

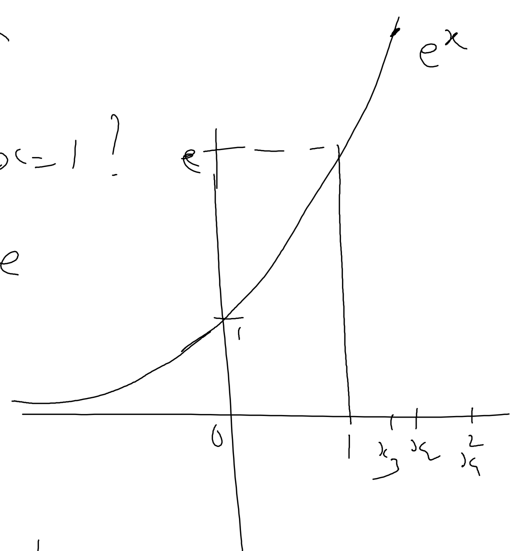
For this if $x_n \longrightarrow 1 \Rightarrow e^{x_n} \longrightarrow e$

$x_n = 1 + \frac{1}{n}$ $\forall n \geq 1$

$x_n \to 1$? ✓

$e^{x_n} \longrightarrow e$? ✓

$\Rightarrow$ f is continuous at 1



* f is called continuous in A if it is continuous at all $x \in A$.

Example $X = \mathbb{R}^2$

$$f(x) = \begin{cases} \frac{x_1 x_2}{x_1^2 + x_2^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Is f continuous at $x = 0$

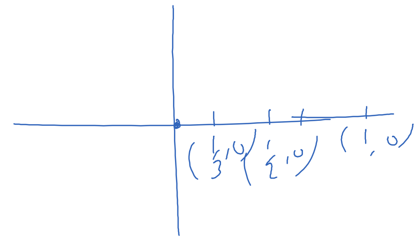
If $x_n \longrightarrow 0 \Longrightarrow f(x_n) \longrightarrow f(0) = 0$

If $\underset{\uparrow}{x_n} \longrightarrow 0 \implies f(x_n) \longrightarrow f(0) = 0$

nth term in a sequence

Check $x_n = \left(\frac{1}{n}, 0\right) =$

$$f(x_n) = \frac{\frac{1}{n} \times 0}{\left(\frac{1}{n}\right)^2 + 0} = 0$$



$\implies x_n \longrightarrow 0$, also $f(x_n) \longrightarrow 0$

Can we say that f is cts at 0?

$\implies$

Now consider $x_n = \left(\frac{1}{n}, \frac{1}{n}\right)$

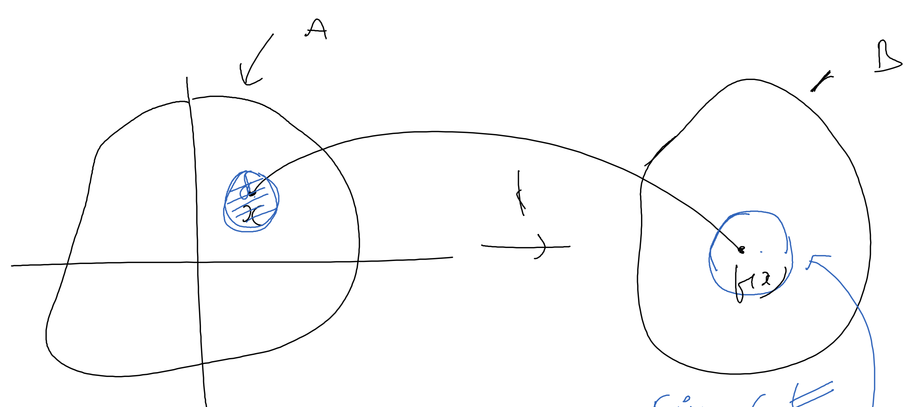
$$f(x_n) = \frac{1}{2}$$

$x_n \to 0$ but $f(x_n) \not\to f(0)$

$\implies f$ is not continuous at 0

Def^n 2 $\quad X = \mathbb{R}^n \ (n=2)$

$f: A \longrightarrow B$



Fix ϵ

- consider $B_\epsilon(f(x))$
- Can we find $\delta > 0$ s.t. $f(B_\delta(x)) \subseteq B_\epsilon(f(x))$!

If yes, we say that f is conti at x

f is continuous at x, if $\forall \epsilon > 0$ there exists a δ s.t.

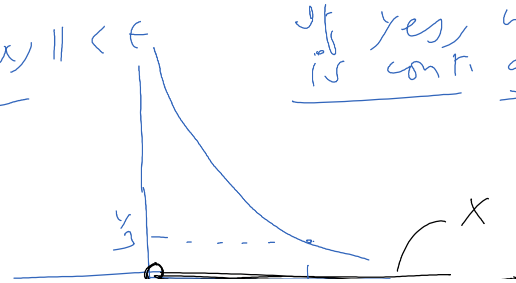
$f(B_\delta(x)) \subseteq B_\epsilon(f(x))$

- equivalently,

$\|y - x\| < \delta \implies \|f(y) - f(x)\| < \epsilon$

Example $X = \mathbb{R}_{++}$

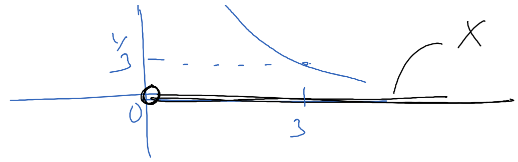
$f(x) = 1$



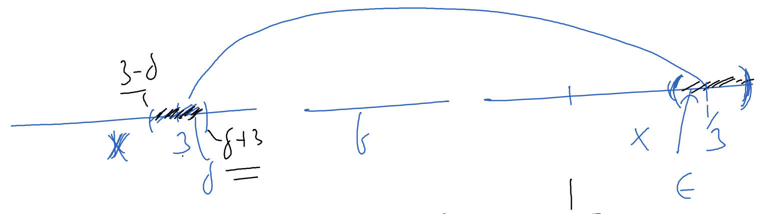
Example $X = \mathbb{R}_{++}$

$f(x) = \frac{1}{x}$

is f continuous at $x = 3$



$f((3-\delta, 3+\delta)) \subseteq (\frac{1}{3}-\epsilon, \frac{1}{3}+\epsilon)$



$(\epsilon < \frac{1}{3})$

$$\frac{1}{3+\delta} > \frac{1}{3} - \epsilon \iff 3+\delta < \frac{1}{\frac{1}{3}-\epsilon}$$

$$\delta < \frac{1}{\frac{1}{3}-\epsilon} - 3$$

$$\frac{1}{3-\delta} < \frac{1}{3} + \epsilon \iff 3-\delta > \frac{1}{\frac{1}{3}+\delta}$$

$$\delta < 3 - \frac{1}{\frac{1}{3}+\delta}$$

set $\delta = \frac{1}{2} \min \left\{ \frac{1}{\frac{1}{3}-\epsilon} - 3,\ 3 - \frac{1}{\frac{1}{3}+\delta} \right\}$

Derivatives

$X = \mathbb{R}^n$

$f : X \longrightarrow \mathbb{R}$

Functions of single variables

$S \subseteq \mathbb{R} \quad f : S \longrightarrow \mathbb{R}$

- assume: f is continuous at $x \in S$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

is called derivative of f at x, provided the limit exists.

- if the limit exists, f is called differentiable at x

- Example $f : \mathbb{R} \longrightarrow \mathbb{R}$

  - $f(x) = \sin x$

$df(x)$

— Examples
— $f(x) = \sin x$
$f'(x) = \cos x$

$\frac{dy(x)}{dx} =: f'(x)$

— $f(x) = x^2 e^x$
$f'(x) = x^2 e^x + 2x e^x$

* f is called differentiable on S if f is differentiable at each $x \in S$.

* f is called "continuously differentiable" on S if $f'(x)$ exists and is continuous at all $x \in S$

Suppose $f : S \longrightarrow \mathbb{R}$ is continuously differentiable on S
$f' : S \longrightarrow \mathbb{R}$ is continuous

f' is differentiable at x if
$$\lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} \text{ exists}$$

— this is called 2nd derivative, $f''(x)$, of f at x.

— If $f''(x)$ exists, f is called twice differentiable at x.

— Can similarly define nth order derivative.

— Recall the def^n of
Convex set

Ex $X = \mathbb{R}$
$A = (2, 3)$ is convex
$B = (2, 4]$ is convex
$C = [1, 2) \cup [3, 4)$ is not convex

— only convex sets in $\mathbb{R}$ are the intervals

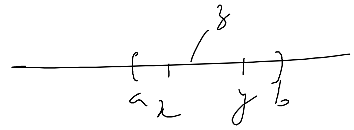
Consider: $f : S \longrightarrow \mathbb{R}$, $S = [a, b]$ $a < b$

Assume f is differentiable on S, consider $x, y \in [a, b]$ (say
$x < y$) there exists a point $z \in (x, y)$ s.t.

Assume f is differentiable on S, consider x, y ... $y > x$). Then there exists a point $z \in [x, y]$ s.t.

$$\boxed{f(y) = f(x) + f'(z)(y-x)}$$

(Mean Value theorem)

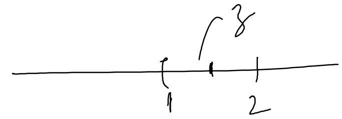


Example

$f(x) = 1 + x^2$

consider $x = 1$, $y = 2$

$f(1) = 2$, $f(2) = 5$



$\forall x$

$f'(x) = 2x$

$5 = 2 + f'(z)(1)$

$f'(z) = 3$

$z = 3/2$

Ex $f(x) = x^2 e^x$

$x = 1$, $y = 2$

try to find a z that follows from the MVT!