Tutorial 12

02 April 2022 11:31

$\min f(x)$
s.t. $h(x) = 0$

Stationary pt

x^* $\nabla f(x^*) = 0$
$h(x^*) = 0$

$f(x) + \sum_{i=1}^{m} \lambda_i h_i(x)$

$\nabla f(x) + \sum_{i=1}^{m} \lambda_i \nabla h_i(x)$

Lagrangian $L(x, \lambda) = f(x) + \lambda^T h(x)$

x^* star. point of
$\exists \lambda^*$ s.t. $\boxed{\nabla L(x^*, \lambda^*) = 0}$

$h(x^*) = 0$ ✓
$\nabla f(x^*) + \nabla h(x^*) \lambda^* = 0$

$h_1(x) = 0$
$h_2(x) = 0$
$h_m(x) = 0$

$$\begin{bmatrix} \nabla h_1(x) & \cdots & \nabla h_m(x) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{bmatrix}$$

KKT conditions

$\min f(x)$
s.t. $h(x) = 0$ $h_1(x) = 0, \ldots, h_m(x) = 0$
$g(x) \leq 0$ $g_1(x) \leq 0, \ldots, g_r(x) \leq 0$

f, h_i, g_j are continuously differentiable

necessary condition
x^* is regular with respect to h_i's and active g_j's. x^* is a local minimum only if there exist λ and μ s.t.

$\nabla_x L(x^*, \lambda, \mu) = 0$
$\mu_j \geq 0 \quad \forall j$
$\mu_j = 0 \quad \forall j \notin A(x^*)$

1st order

$h(x^*) = 0$
$g(x^*) \leq 0$

2nd order

$y^T \nabla_{xx} L(x^*, \lambda, \mu) y > 0$

$\forall y$ s.t. $y^T \nabla h_i(x^*) = 0 \quad \forall i$
$y^T \nabla g_j(x^*) = 0 \quad \forall j \in A(x^*)$

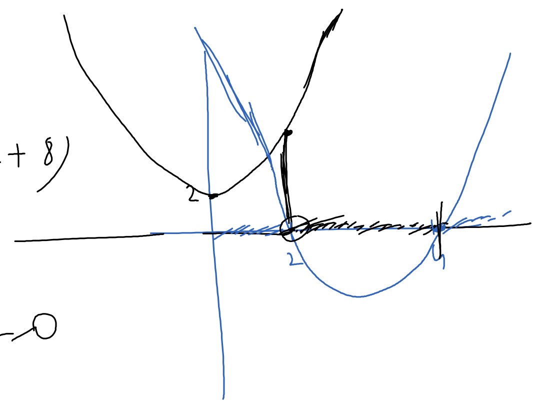
$\nabla h_i(x^*)$, $i = 1, \ldots, m$
& $\nabla g_j(x^*)$, $j \in A(x^*)$

$y^T \nabla g_j(x^*) = 0 \quad \forall j \in \cancel{A(x^*)}$ & $\nabla g_j(x^*),\ j \in \underbrace{\cancel{A(x^*)}}_{\text{Active constraints}}$

— linearly ind.

Q.3

$$\min_x \ x^2+1$$
$$\text{s.t.} \ (x-2)(x-4) \le 0$$



$$L(x,\mu) = x^2+1 + \mu(x^2-6x+8)$$

$$\nabla_x L(x,\mu) = 0$$

$$2x + 2\mu x - 6\mu = 0$$

$$x = \frac{6\mu}{2+2\mu} = \frac{3\mu}{1+\mu}$$

$$\boxed{\text{if } g(x) \neq 0 \Rightarrow \mu = 0}$$

$$\mu g(x) = 0$$

$$\mu\left(\frac{3\mu}{1+\mu} - 2\right)\left(\frac{3\mu}{1+\mu} - 4\right) = 0$$

either $\mu = 0, 2, -4$

at $\mu = 0$, $x = 0$

at $\mu = 3$ $x = 2$

Second order conditions

$$\nabla^2_{xx} L(x,\mu) = 2+2\mu \ge 6 \quad \text{at } \mu = 2$$

Q.4

$$L(x,\mu) = -14x_1 + x_1^2 - 6x_2 + x_2^2 - 7 + \mu_1(x_1+x_2-2) + \mu_2(x_1+2x_2-3)$$

Q

$$\nabla_x L(x,\mu) = \begin{bmatrix} -14 + 2x_1 + \mu_1 + \mu_2 \\ -6 + 2x_2 + \mu_1 + 2\mu_2 \end{bmatrix}$$

$$\nabla_x L(x,\mu) = \begin{bmatrix} -14 + 2x_1 + \mu_1 + \mu_2 \\ -6 + 2x_2 + \mu_1 + 2\mu_2 \end{bmatrix}$$

$$\nabla^2_{xx} L(x,\mu) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$\lambda_1 + \lambda_2 = 4$

$\lambda_1 \lambda_2 = 4$

$\cancel{x}\ 2$

Q.5 $\min c^T x$

s.t. $Ax = b$

$x \le 0$

Lagrangian $L(x,\lambda,\mu) = c^T x + \lambda^T (Ax - b) + \mu^T x$

$\mu^T I = \mu^T$

$$D(\lambda,\mu) = \min_x c^T x + \lambda^T (Ax - b) + \mu^T x$$

$$= \min_x (c^T + \lambda^T A + \mu^T) x - \lambda^T b$$

$$= \begin{cases} -\lambda^T b & \text{if } c^T + \lambda^T A + \mu^T = 0 \\ -\infty & \text{otherwise} \end{cases}$$

$\min_{x \le 0} c^T x$

s.t. $Ax = b$

$$L(x,\lambda) = c^T x + \lambda^T (Ax - b)$$

$$D(\lambda) = \min_{x \le 0} (\underbrace{c^T + \lambda^T A}_{z^T}) x - \lambda^T b$$

$$= \begin{cases} -\lambda^T b & c + A^T \lambda \le 0 \\ -\infty & \text{otherwise} \end{cases}$$

$$\min_{x \le 0} z^T x = \min_{x \le 0} \sum z_i x_i$$

Dual $\max D(\lambda)$

$\max -\lambda^T b$

$\boxed{c + A^T \lambda \le 0}$

$$\min_{x \le 0} \beta x = \begin{cases} -\infty & \text{if } \beta > 0 \\ & \text{if } \beta < 0 \end{cases}$$

0 if $\mu = 0$

Dual $\max_{\mu \geq 0} D(\mu, \lambda)$

$$\max_{\mu \geq 0} \; -\lambda^T b \quad \text{s.t.} \quad c + A^T\lambda + \mu = 0$$

$$\max \; -\lambda^T b \quad \text{s.t.} \quad c + A^T\lambda + \mu = 0, \quad \mu \geq 0$$

$$\max \; -\lambda^T b \quad c + A^T\lambda \leq 0$$

Tutorial 6

Q.6 $A \in \mathbb{R}^{m \times n}$ $(m < n)$

$\text{rank}(A) = m$

$\Rightarrow \text{rank}(AA^T) = m$

Q.11

$$\min_x \; -\sum_{i=1}^{n} \log(\alpha_i + x_i)$$

subs. $\sum_{i=1}^{n} x_i = 1$ $\qquad \alpha_i > 0$

$x_i \geq 0$

Lagrangian $L(x, \lambda, \mu) = -\sum_{i=1}^{n} \log(\alpha_i + x_i) + \lambda\left(\sum_{i=1}^{n} x_i - 1\right) - \sum \mu_i x_i$

$\nabla_x L(x, \lambda, \mu) = 0$

$$-\frac{1}{\alpha_i + x_i} + \lambda - \mu_i = 0 \quad \text{—— (1)}$$

$$\mu_i x_i = 0 \quad \text{—— (2)}$$

From (1) $\alpha_i + x_i = \dfrac{1}{\lambda - \mu_i}$

Suppose $\alpha_i \neq 0 \Rightarrow x_i = \frac{1}{\lambda} - \alpha_i$ (because $x_i = 0$)

i.e. $x_i + \alpha_i = \frac{1}{\lambda}$

— we can assume that $\alpha_i \uparrow i$

argue that if $x_i = 0$, $x_j = 0 \ \forall j > i$

if $x_i = 0$

$\alpha_i = \frac{1}{\lambda - \mu_i} \geq \frac{1}{\lambda}$

if $x_j \neq 0$

$x_j = \frac{1}{\lambda} - \alpha_j \leq \alpha_i - \alpha_j \leq 0$

(contradiction)

$$\sum_{i=1}^{i^*} x_i = 1 \quad \text{— (1)}$$

$x_i + \alpha_i = \frac{1}{\lambda} \quad \forall i = 1, \dots i^*$

$x_i = -\alpha_i + \frac{1}{\lambda} \quad \forall i = 1, \dots i^*$

$$\sum_{i=1}^{i^*} \left(\frac{1}{\lambda} - \alpha_i\right) = 1$$

$x_i + \alpha_i = \theta$

$\theta > \alpha_i$

$$i^*/\lambda - \sum_{i=1}^{i^*} \alpha_i = 1$$

$$\boxed{1/\lambda = \frac{1 + \sum_{i=1}^{i^*} \alpha_i}{i^*} \geq \alpha_{i^*}}$$

X $x_i = 0 \ \forall i$

$x_1 \neq 0 \quad x_j = 0 \quad \forall j > 1 \qquad x_1 = 1 = \frac{1}{\lambda} - \alpha_1$

$x_1, x_2 \neq 0, \quad x_j = 0 \quad \forall j > 2$

$x_i \neq 0 \quad \forall i = 1, \dots n$

$$1 + \sum_{i=1}^{i^*} \alpha_i > i^* \alpha_{i^*}$$

$$\boxed{1 > \sum_{i=1}^{i^*} (\alpha_{i^*} - \alpha_i)}$$

R.H.S $i^* = 1 \Rightarrow$ R.H.S $= 0$

$i^* = 2 \Rightarrow$ R.H.S $= (\alpha_2 - \alpha_1)$

$i^* = 3 \Rightarrow$ R.H.S $= (\alpha_3 - \alpha_1) + (\alpha_3 - \alpha_2)$

$i^* = 2 \Rightarrow$ R.H.S

$i^* = 3 \Rightarrow \text{R.H.S} = (\alpha_3 - \alpha_1) + (\alpha_3 - \alpha_2)$

$i^* = 4 \Rightarrow \text{R.H.S} = (\alpha_4 - \alpha_1) + (\alpha_4 - \alpha_2) + (\alpha_4 - \alpha_3)$

$$\boxed{i^* = \max\left\{ j : \sum_{i=1}^{j} (\alpha_j - \alpha_i) < 1 \right\}}$$

Proof that the above i^* is the correct choice.

Consider $j > i^*$ then $\sum_{i=1}^{j} (\alpha_j - \alpha_i) \geq 1$ — (a)

suppose $\bar{x}$ where $\bar{x}_i > 0 \ \forall \ i \leq j$ and $\bar{x}_i = 0 \ \forall i > j$ is an optimal solution. Then

$$\sum_{i=1}^{n} \bar{x}_i = 1$$

$$\Rightarrow \sum_{i=1}^{j} \bar{x}_i = 1$$

$$\Rightarrow \sum_{i=1}^{j} \left(\frac{1}{\lambda} - \alpha_i\right) = 1$$

$$\Rightarrow \sum_{i=1}^{j} \left(\frac{1}{\lambda} - \alpha_i\right) \leq \sum_{i=1}^{j} (\alpha_j - \alpha_i) \qquad \text{(from (a))}$$

$$\Rightarrow \frac{1}{\lambda} - \alpha_j \leq 0$$

$$\Rightarrow x_j \leq 0$$

$\Rightarrow x_j = 0$ which is a contradiction

Now consider $j < i^*$ then $\sum_{i=1}^{j+1} (\alpha_{j+1} - \alpha_i) < 1$ — (b)

suppose $\bar{x}$ where $\bar{x}_i > 0 \ \forall \ i \leq j$ and $\bar{x}_i = 0 \ \forall i > j$ is an optimal solution. Then

$$-\frac{1}{\alpha_{j+1} + \underbrace{\bar{x}_{j+1}}_{=0}} + \lambda - \underbrace{\mu_{j+1}}_{\geq 0} = 0$$

$$\Rightarrow -\frac{1}{\alpha_{j+1}} + \lambda \geq 0$$

$$\Rightarrow \alpha_{j+1} \geq \frac{1}{\lambda} \quad \text{— (c)}$$

$\Rightarrow \alpha_{j+1} = \lambda$ —— ⓒ

Further $\sum_{i=1}^{j} \bar{x}_i = 1$

$\Rightarrow \sum_{i=1}^{j} \left(\frac{1}{\lambda} - \alpha_i\right) = 1$

$\Rightarrow \sum_{i=1}^{j} \left(\alpha_{j+1} - \alpha_i\right) \geq 1$ (from ⓒ)

$\Rightarrow \sum_{i=1}^{j+1} \left(\alpha_{j+1} - \alpha_i\right) \geq 1$ which contradicts Ⓑ.

Hence for the optimal solution x^*, $x^*_i > 0 \ \forall\ i \leq i^*$ and $x^*_i = 0 \ \forall\ i > i^*$.