Tutorial 9

12 March 2022 11:00

Recap

$$\min_{x \in S} f(x) \quad \text{s.t. } h(x) = 0, \; g(x) \le 0$$

$S \subseteq \mathbb{R}^n$, $h: \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $g: \mathbb{R}^n \longrightarrow \mathbb{R}^r$

(Primal problem - may be non convex)

Lagrangian $L: S \times \mathbb{R}^m \times \mathbb{R}^r_+ \longrightarrow \mathbb{R}$

$$L(x, \lambda, \mu) = f(x) + \lambda^T h(x) + \mu^T g(x)$$

Dual function

$$D(\lambda, \mu) = \min_{x \in S} L(x, \lambda, \mu)$$

(concave always)

Dual Problem $\max_{\mu \ge 0} D(\lambda, \mu)$ (convex-optimization problem)

* If $\bar{x}$ is primal feasible, ($\bar{x} \in S$, $h(\bar{x}) = 0$, $g(\bar{x}) \le 0$), $(\bar{\lambda}, \bar{\mu})$ are dual feasible ($\bar{\mu} \ge 0$), and

$$f(\bar{x}) = D(\bar{\lambda}, \bar{\mu})$$

then $\bar{x}$ is primal optimal and $(\bar{\lambda}, \bar{\mu})$ are dual optimal.

* $x^*, (\lambda^*, \mu^*)$ are an (primal) optimal - Lagrange multiplier pair if and only if

$x^* \in S$, $h(x^*) = 0$, $g(x^*) \le 0$ } primal feasibility

$\mu^* \ge 0$ } Dual feasibility

${\mu^*}^T g(x^*) = 0$ } complementary slackness

$x^* = \arg\min_{x \in S} L(x, \lambda^*, \mu^*)$ } Lagrangian optimality

Linear programs (LP)

- the objective as well as the constraints are affine.

(P) $\max_{x \in \mathbb{R}^n} c^T x$ s.t. $Ax \le b$, -

$$(P)\quad \begin{array}{|ll} \max\limits_{x \in \mathbb{R}^n} & c'x \\ \text{s.t.} & Ax \le b, - \\ & x \ge 0 \quad - n \text{ constraints} \end{array}$$

$$A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}_{m \times n} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \le \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

(m - affine constraints).

$(g(x) := Ax - b)$

Dual of (P)

Lagrangian

$$L(x, \mu, \nu) = -c^T x + \mu^T (Ax - b) - \nu^T x \;;\; \mu \in \mathbb{R}^m_+,\ \nu \in \mathbb{R}^n_+$$

$$= \left(-c^T + \mu^T A - \nu^T\right) x - \mu^T b$$

Lagrange multipliers : $(\mu_1, \ldots, \mu_m, \nu_1, \ldots, \nu_n)$

Dual

$$D(\mu, \nu) = \min_{x \in \mathbb{R}^n} \left(-c^T + \mu^T A - \nu^T\right) x - \mu^T b$$

Earlier $\sum_{i=1}^{m} \mu_i\, g_i(x)$

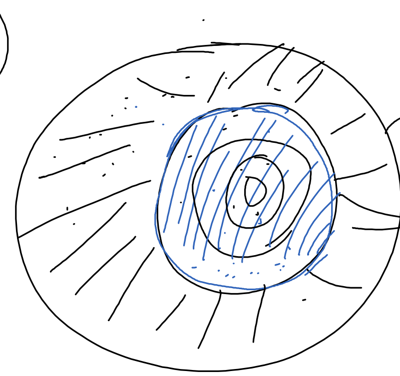
$$= \begin{cases} -\mu^T b & -c^T + \mu^T A - \nu^T = 0 \\ -\infty & -c^T + \mu^T A - \nu^T \neq 0 \\ \text{(doesn't exist)} & \end{cases}$$

i^{th} component

$$[\; - \quad \gtrless 0 \; - \;] \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ x_i \neq 0 \\ \vdots \\ 0 \end{bmatrix}$$

Dual Problem

$$\begin{array}{|l} \max \; D(\mu, \nu) \\ \mu \ge 0,\ \nu \ge 0 \end{array}$$



$\mathbb{R}_+^{m+n}$

$\{(\mu, \nu) :\ \mu \ge 0, \nu \ge 0,\ -c^T + \mu^T A - \nu^T = 0\}$

Equivalent Problem

$$\begin{array}{|ll} \max\limits_{\mu \ge 0,\, \nu \ge 0} & -\mu^T b \\ \text{s.t.} & -c^T + \mu^T A = \nu^T \end{array}$$

$-c^T + \mu^T A - \nu^T = 0$

which is equivalent to

which is equivalent to

$$\max_{\mu \geq 0} -\mu^T b$$
$$\text{s.t. } -c^T + \mu^T A \geq 0$$

$\mu^T A \geq c^T$

Dual Problem (D)

$$\min \; b^T \mu$$
$$\text{s.t. } A^T \mu \geq c$$
$$\mu \geq 0$$

$$\begin{bmatrix} — a_1^T — \\ \vdots \\ — a_n^T — \end{bmatrix} \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_m \end{bmatrix}$$

Primal

$$\max \; c^T x$$
$$\text{s.t. } Ax \leq b$$
$$x \geq 0$$

Dual

$$\min \; b^T \mu$$
$$\text{s.t. } A^T \mu \geq c$$
$$\mu \geq 0$$

(D)

- strong duality holds

if x^* is primal optimal and μ^* is dual optimal

$$c^T x^* = b^T \mu^*$$

|| weak duality would imply

$$c^T x^* \leq b^T \mu^*$$

$(b^T \mu^* - c^T x^* =: \text{duality gap})$

- duality gap is zero

- complementary slackness

$$\mu_i^* ((Ax^*)_i - b_i) = 0 \quad \forall \; i = 1, \dots m$$

(1)

(if $\mu_i^* > 0 \Rightarrow (Ax^* - b)_i = 0$)

$$(\underbrace{\mu_1^*}_{\geq 0} \dots \underbrace{\mu_m^*}_{\geq 0}) \begin{bmatrix} \underbrace{(Ax^*)_1 - b_1}_{\leq 0} \\ \vdots \\ \underbrace{(Ax^*)_m - b_m}_{\leq 0} \end{bmatrix}$$

(This implies

$$\sum_{i=1}^m \mu_i^* ((Ax)_i - b_i) = 0$$

i.e. $\boxed{\mu^{*T} (Ax - b) = 0}$

i.e. $\mu^{*T}(Ax-b)=0$

(2) $\nu_i^* x_i^* = 0 \quad \forall i=1,\dots n$

If $\nu_i^* > 0 \Rightarrow x_i^* = 0$

i.e. $(A^T\mu^*)_i - c_i > 0 \Rightarrow x_i^* = 0$

$$\left((A^T\mu^*)_i - c_i\right) x_i^* = 0 \quad \forall i=1,\dots n$$

$$\sum_{i=1}^{n}\left((A^T\mu^*)_i - c_i\right) x_i^* = 0$$

$$\boxed{x^{*T}(A^T\mu^* - c) = 0}$$

Example

$$\begin{aligned} \max \quad & 2x_1 + x_2 \\ \text{s.t.} \quad & x_1 + x_2 \le 4 \\ & -x_1 + x_2 \ge -2 \\ & x_1 \ge 0,\ x_2 \ge 0 \end{aligned}$$

$$\begin{aligned} \max \quad & 2x_1 + x_2 \\ \text{s.t.} \quad & x_1 + x_2 \le 4 \\ & x_1 - x_2 \le 2 \\ & x_1 \ge 0,\ x_2 \ge 0 \end{aligned}$$

$Ax \le b$

Dual

$$\begin{aligned} \min \quad & 4\mu_1 + 2\mu_2 \\ \text{s.t.} \quad & \mu_1 + \mu_2 \ge 2 \\ & \mu_1 - \mu_2 \ge 1 \\ & \mu_1 \ge 0,\ \mu_2 \ge 0 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^T$$

$a_1 \quad a_2$

Exercise Primal

$$\begin{aligned} \min \quad & b^T x \\ & A^T x \ge c \\ & x \ge 0 \end{aligned}$$

show that its dual is

$$\begin{aligned} \max \quad & c^T \mu \\ & A\mu \le b \\ & \mu \ge 0 \end{aligned}$$

Example max $c^T x$ $x \in \mathbb{R}^n$

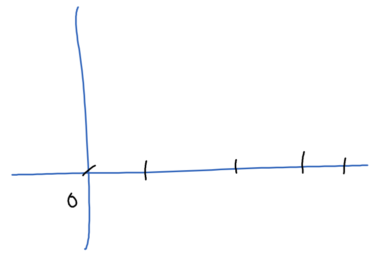
s.t. $a^T x \le b$ — (single constraint)

$x \ge 0$. (a is column vector, b is a scalar) a^T $[\ \ ]_n \le b$

dual min $b\mu$

s.t. $a_i \mu \ge c_i$ $\forall i = 1, \dots, n$ } n constraints

$\mu \ge 0$



Equivalently

min $b\mu$:

(Assume all $a_i \ne 0$)

$\mu \ge \frac{c_i}{a_i}$ if $a_i > 0$

$\mu \le \frac{c_i}{a_i}$ if $a_i < 0$

$\mu \ge 0$

Ex (a) max $2x_1 - 3x_2 + 4x_3$

s.t. $3x_1 - 4x_2 + 5x_3 \le 6$

$x_1 \ge 0$

$x_2 \ge 0$

$x_3 \ge 0$

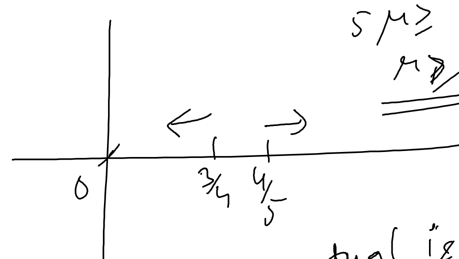
dual min 6μ

s.t. $3\mu \ge 2$ = $\mu \ge \frac{2}{3}$

$-4\mu \ge -3$ $\mu \le \frac{3}{4}$

$5\mu \ge 4$ $\mu \ge \frac{4}{5}$

$\mu \ge 0$ $\mu \ge 0$ //



dual is infeasible

$\Rightarrow$ primal objective is unbounded

(b) max $2x_1 + 4x_2$

s.t. $3x_1 + 5x_2 \le 6$

$x_1 \ge 0$

$x_2 \ge 0$

dual min 6μ ||

$i\xi \geq 0$

dual min 6μ

s.t. $\mu \geq 2/3$

$\mu \geq 4/5$

$\mu \geq 0$

$\mu^* = 4/5$

dual opt. value $= 6 \times 4/5 = 24/5$

Primal opt. value $= 24/5$

Primal opt. soln

$x_1 (\underbrace{\mu - 2/3}_{\neq 0}) = 0$

$\Rightarrow x_1 = 0$

$\underset{\neq 0}{\mu} (\underbrace{3x_1 + 5x_2 - 6}_{= 0}) = 0$

$x_2 = 6/5$ //

Next lecture: Barrier function method
Penalty function method

Termtest 2: 26th March