E1 215-O: Tutorial Questions

Linear and Non-linear Optimization

January 22, 2022

1. Find the largest open subset $E \in \mathbb{R}^2$ in which following functions are continuously differentiable, and write down the derivatives of these functions.
   (a) $f : \mathbb{R}^2 \backslash \{\underline{0}\} \to \mathbb{R}$, $f(x) = \frac{-x_1}{||x||^3}$
   (b) $f : [-1, \infty) \times [-1, \infty) \to \mathbb{R}$, $f(x) = \sqrt{x_1 + 1} - \sqrt{x_2 + 1}$
2. Let $f : \mathbb{R}\backslash\{3\} \to \mathbb{R}$, $f(x) = (x - 3)^{-2}$.
   (a) Find a $c \in (4, 5)$ such that $f'(c) = \frac{f(4)-f(5)}{4-5}$.
   (b) Find a $c \in (2, 4)$ such that $f'(c) = \frac{f(4)-f(2)}{4-2}$. Does it contradict mean value theorem?
3. Compute the gradient for the following functions $f : \mathbb{R}^n \to \mathbb{R}$
   (a) $f(x) = a^T x$
   (b) $f(x) = ||x||^2$
   (c) $f(x) = x^T Ax$, when A is symmetric
   (d) $f(x) = x^T Ax$, when A is not symmetric
4. Consider the function $f : \mathbb{R}^5 \to \mathbb{R}$, $f(x) = e^{\sum_{i=1}^{5} ix_i}$
   (a) Calculate the total derivative and Hessian of f at $x = \underline{0}$
   (b) Determine the first and second order Taylor polynomials of f at $x = \underline{0}$
   (c) Optional: Evaluate the second order Taylor polynomial at (0.1,0,0,0,0) and compare it with the actual value, do the same for x = (10,0,0,0,0), comment.
5. Calculate the Hessian for the following functions at the indicated point a
   (a) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x) = \frac{1}{x_1^2+x_2^2+1}$ at $a = \underline{0}$
   (b) $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x) = \cos(x_1)\sin(x_2)$ at $a = (\pi/4, \pi/3)$
   (c) $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x) = e^{2x_1-3x_2}\sin(5x_3)$ at $a = \underline{0}$
6. Consider a function $h : \mathbb{R}^n \to \mathbb{R}^n$,where h $= \begin{bmatrix} h_1(x) \\ h_2(x) \\ . \\ h_n(x) \end{bmatrix}$. Compute the total derivative of the function $f(x) = h(x)^T h(x)$. Assume that f is continuously differentiable.

7. If $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = \sin(xy)$ and $x = s + t$, $y = s^2 + t^2$

   (a) Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial x}{\partial s}$, $\frac{\partial x}{\partial t}$, $\frac{\partial y}{\partial s}$ and $\frac{\partial y}{\partial t}$.

   (b) Find the partial derivatives $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$. Do you see a relation between them? state the relation.

8. Monty, the housefly, finds himself caught in the oven at point $(0, 0, 1)$. The temperature at the points in the oven is given by the function, $T : \mathbb{R}^3 \to \mathbb{R}$.

$$T(x) = 10(x_1 e^{-x_2^2} + x_3 e^{-x_1^2})$$

   Where the units are in Celsius. If Monty begins to move towards the point $(2, 3, 1)$ at what rate (in deg/cm) does he find the temperature changing?

9. Find the lim inf and lim sup for the following sequences.

   (a) $x_n = (-1)^n \times \frac{(n+5)}{n}$.

   (b) $x_n = 1^n + (-1)^n$.

10. Find limit infimum for the following functions.

   (a) $f : \mathbb{R} \to [-1, 1]$, $f(t) = \sin(t)$ for $t \to \infty$

   (b) $f : \mathbb{R} \to [-1, 1]$, $f(t) = -\sin(t)$ for $t \to \infty$

   (c) $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$, $f(t) = \frac{\log(|c|)}{t}$ for $t \to \infty$ and c is a constant.

11. Find limit supremum for the following functions.

   (a) $f : \mathbb{R}\backslash\{0\} \to [-1, 1]$, $f(t) = \cos(\frac{1}{t})$ for $t \to \infty$

   (b) $f : \mathbb{R} \to \mathbb{R}$, $f(t) = e^{\sin(t)}$ for $t \to \infty$.