E1 251-O: Test-1

Linear and Non-linear Optimization

February 11, 2022

9:30 - 11:30 am

Instructions:

Answer any 5 questions. Each question carries 6 marks.

Scan all the answers as a single pdf file and upload it.

1. (a) For what values of λ the matrix $A = \begin{bmatrix} 2 & \lambda & -1 \\ \lambda & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$ is positive definite. (3-marks)

   (b) Are the following sets open or closed or both or none?

      i. $A = \{1, 2, 3\}$ in $\mathbb{Z}$. (1-mark)

      ii. $A = [0, 1)$ in $X = \{x \in \mathbb{R} : x \geq 0\}$. (1-mark)

      iii. $A = \cap_{i=1}^{\infty} [1, 1 + \frac{1}{n})$ in $\mathbb{R}$. (1-mark)

2. (a) Consider $f : (0, \infty) \to \mathbb{R}$ defined as

$$f(x) = \frac{1}{x^2}.$$

      i. Is $f(x)$ Lipschitz continuous? (1-mark)

      ii. Write second-order Taylor's series approximation of f around $x = 2$. (2-marks)

   (b) Consider $g : [-1, 1] \times [-1, 1] \to \mathbb{R}$ defined as

$$g(x) = x_1^3 + 2x_2^2 + 3x_1x_2^2.$$

      i. Obtain a number M such that

$$|g(x) - g(y)| \leq M||x - y||$$

      for all $x, y \in [-1, 1] \times [-1, 1]$. (2-marks)

      ii. What is the direction derivative of g at x in the direction u. (1-mark)

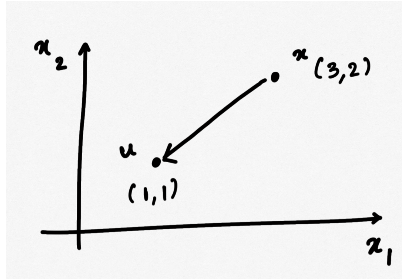

Figure 1: 2(b)(ii)

3. (a) Find the supremum and infimum of $A = \{\frac{n+1}{n+2} : n = 2, 3, 4, ...\}$. (2-marks)
   (b) Find $\inf_{x \in (0,\infty)} \left(\frac{1}{x} - \sin x\right)$. (2-marks)
   (c) Consider $a_n = 3 + \frac{(-1)^n n}{n+8}$, $n \geq 1$. Find $\liminf_{n\to\infty} a_n$ and $\limsup_{n\to\infty} a_n$. (2-marks)

4. Find all local minima and local maxima of $f : \mathbb{R}^2 \to \mathbb{R}$ where
   (a) $f(x) = x_1^2 + x_1x_2 + x_2^2 - 2x_1 - x_2$. (3-marks)
   (b) $f(x) = -x_1^3 + 2x_1x_2 + x_2^2 + x_1$. (3-marks)

5. (a) Are the following sets convex?
      i. $A = \{x \in \mathbb{R}^2 : x_1x_2 \geq 1\}$. (1-mark)
      ii. $A = \{x \in \mathbb{R}^2 : x_2 \geq e^{x_1}\}$. (1-mark)
   (b) Consider $f : (0, \infty) \to \mathbb{R}$
   $$f(x) = x \log x.$$
   Is f strictly convex? Is it strongly convex? (2-marks)
   (c) Consider
   $$f(x) = 10x_1^{1/3}x_2^{1/2} \forall x \in \{y \in \mathbb{R}^2 : y_1 > 0, y_2 > 0\}.$$
   Is f convex or concave or none? Is it strictly convex or strictly concave? (2-marks)

6. Can you infer local/global maxima/minima of the following functions just based on the first order condition? What can you infer?
   (a) $f : \mathbb{R}^3 \to \mathbb{R}, f(x) = 1 - x_1^2 - x_2^2 - x_3^2$. (3 marks)
   (b) $f : \mathbb{R}^2 \to \mathbb{R}, f(x) = e^{3x_1} - 3x_1 + 4x_2^2 - 1$. (3 marks)