# Homework #4 Cheat Sheet

CS231

# 1 Language of Booleans and Integers

## 1.1 Syntax

$$
\begin{array}{lcl}
\texttt{t} & ::= & \texttt{true} \mid \texttt{false} \mid \texttt{if t then t else t} \\
 & \mid & \texttt{n} \mid \texttt{t + t} \mid \texttt{t > t} \\
\texttt{n} & ::= & \text{integer constant} \\
\texttt{v} & ::= & \texttt{true} \mid \texttt{false} \mid \texttt{n} \\
\texttt{T} & ::= & \texttt{Bool} \mid \texttt{Int}
\end{array}
$$

## 1.2 Small-Step Operational Semantics

$$\frac{}{\texttt{if true then } \texttt{t}_2 \texttt{ else } \texttt{t}_3 \longrightarrow \texttt{t}_2} \quad \text{(E-IfTrue)}$$

$$\frac{}{\texttt{if false then } \texttt{t}_2 \texttt{ else } \texttt{t}_3 \longrightarrow \texttt{t}_3} \quad \text{(E-IfFalse)}$$

$$\frac{\texttt{t}_1 \longrightarrow \texttt{t}_1'}{\texttt{if } \texttt{t}_1 \texttt{ then } \texttt{t}_2 \texttt{ else } \texttt{t}_3 \longrightarrow \texttt{if } \texttt{t}_1' \texttt{ then } \texttt{t}_2 \texttt{ else } \texttt{t}_3} \quad \text{(E-If)}$$

$$\frac{\texttt{t}_1 \longrightarrow \texttt{t}_1'}{\texttt{t}_1 \texttt{ + } \texttt{t}_2 \longrightarrow \texttt{t}_1' \texttt{ + } \texttt{t}_2} \quad \text{(E-Plus1)}$$

$$\frac{\texttt{t}_2 \longrightarrow \texttt{t}_2'}{\texttt{v}_1 \texttt{ + } \texttt{t}_2 \longrightarrow \texttt{v}_1 \texttt{ + } \texttt{t}_2'} \quad \text{(E-Plus2)}$$

$$\frac{\texttt{n} = \texttt{n}_1 \;[[+]]\; \texttt{n}_2}{\texttt{n}_1 \texttt{ + } \texttt{n}_2 \longrightarrow \texttt{n}} \quad \text{(E-PlusRed)}$$

$$\frac{\texttt{t}_1 \longrightarrow \texttt{t}_1'}{\texttt{t}_1 \texttt{ > } \texttt{t}_2 \longrightarrow \texttt{t}_1' \texttt{ > } \texttt{t}_2} \quad \text{(E-GT1)}$$

$$\frac{\texttt{t}_2 \longrightarrow \texttt{t}_2'}{\texttt{v}_1 \texttt{ > } \texttt{t}_2 \longrightarrow \texttt{v}_1 \texttt{ > } \texttt{t}_2'} \quad \text{(E-GT2)}$$

$$\frac{\texttt{v} = \texttt{n}_1 \;[[>]]\; \texttt{n}_2}{\texttt{n}_1 \texttt{ > } \texttt{n}_2 \longrightarrow \texttt{v}} \quad \text{(E-GTRed)}$$

## 1.3 Static Type System

$$\frac{}{\texttt{true} : \texttt{Bool}} \quad \text{(T-True)} \qquad\qquad \frac{}{\texttt{false} : \texttt{Bool}} \quad \text{(T-False)}$$

$$\frac{\mathtt{t}_1 : \mathtt{Bool} \qquad \mathtt{t}_2 : \mathtt{T} \qquad \mathtt{t}_3 : \mathtt{T}}{\mathtt{if}\ \mathtt{t}_1\ \mathtt{then}\ \mathtt{t}_2\ \mathtt{else}\ \mathtt{t}_3 : \mathtt{T}} \tag{T-If}$$

$$\frac{}{\mathtt{n} : \mathtt{Int}} \tag{T-Num}$$

$$\frac{\mathtt{t}_1 : \mathtt{Int} \qquad \mathtt{t}_2 : \mathtt{Int}}{\mathtt{t}_1\ +\ \mathtt{t}_2 : \mathtt{Int}} \tag{T-Plus}$$

$$\frac{\mathtt{t}_1 : \mathtt{Int} \qquad \mathtt{t}_2 : \mathtt{Int}}{\mathtt{t}_1\ >\ \mathtt{t}_2 : \mathtt{Bool}} \tag{T-GT}$$

# 2 Simply-Typed Lambda Calculus

## 2.1 Syntax

$$\begin{array}{lll} \mathtt{t} & ::= & \mathtt{x} \mid \mathtt{function}\ \mathtt{x{:}T} \to \mathtt{t} \mid \mathtt{t}\ \mathtt{t} \\ \mathtt{v} & ::= & \mathtt{function}\ \mathtt{x{:}T} \to \mathtt{t} \\ \mathtt{T} & ::= & \mathtt{T}_1 \to \mathtt{T}_2 \end{array}$$

## 2.2 Substitution

$[\mathtt{x} \mapsto \mathtt{v}]\mathtt{x} = \mathtt{v}$
$[\mathtt{x} \mapsto \mathtt{v}]\mathtt{x}' = \mathtt{x}'$, where $\mathtt{x} \neq \mathtt{x}'$
$[\mathtt{x} \mapsto \mathtt{v}]\mathtt{function}\ \mathtt{x{:}T} \to \mathtt{t}_0 = \mathtt{function}\ \mathtt{x{:}T} \to \mathtt{t}_0$
$[\mathtt{x} \mapsto \mathtt{v}]\mathtt{function}\ \mathtt{x}_0\mathtt{:T} \to \mathtt{t}_0 = \mathtt{function}\ \mathtt{x}_0\mathtt{:T} \to [\mathtt{x} \mapsto \mathtt{v}]\mathtt{t}_0$, where $\mathtt{x} \neq \mathtt{x}_0$
$[\mathtt{x} \mapsto \mathtt{v}]\mathtt{t}_1\ \mathtt{t}_2 = [\mathtt{x} \mapsto \mathtt{v}]\mathtt{t}_1\ [\mathtt{x} \mapsto \mathtt{v}]\mathtt{t}_2$

## 2.3 Small-Step Operational Semantics

$$\frac{}{((\mathtt{function}\ \mathtt{x{:}T} \to \mathtt{t})\ \mathtt{v}) \longrightarrow [\mathtt{x} \mapsto \mathtt{v}]\mathtt{t}} \tag{E-AppBeta}$$

$$\frac{\mathtt{t}_1 \longrightarrow \mathtt{t}_1'}{\mathtt{t}_1\ \mathtt{t}_2 \longrightarrow \mathtt{t}_1'\ \mathtt{t}_2} \tag{E-App1}$$

$$\frac{\mathtt{t}_2 \longrightarrow \mathtt{t}_2'}{\mathtt{v}_1\ \mathtt{t}_2 \longrightarrow \mathtt{v}_1\ \mathtt{t}_2'} \tag{E-App2}$$

## 2.4 Static Type System

$\Gamma$ is a finite function from variable names to types.

$$\frac{\Gamma(\mathtt{x}) = \mathtt{T}}{\Gamma \vdash \mathtt{x} : \mathtt{T}} \tag{T-Var}$$

$$\frac{\Gamma, \mathtt{x{:}T}_1 \vdash \mathtt{t} : \mathtt{T}_2}{\Gamma \vdash \mathtt{function}\ \mathtt{x{:}T}_1\ \mathtt{->}\ \mathtt{t} : \mathtt{T}_1 \to \mathtt{T}_2} \tag{T-Fun}$$

$$\frac{\Gamma \vdash \mathtt{t}_1 : \mathtt{T}_2 \to \mathtt{T} \qquad \Gamma \vdash \mathtt{t}_2 : \mathtt{T}_2}{\Gamma \vdash \mathtt{t}_1\ \mathtt{t}_2 : \mathtt{T}} \tag{T-App}$$

# 3 Extensions

We augment our language with a unit value, pairs, tagged unions, `let`, `letrec`.

## 3.1 Syntax

```
t  ::=  () | (t,t) | fst t | snd t
     |  left t | right t | (match t with left x -> t | right x -> t)
     |  let x=t in t | letrec x=v in t
v  ::=  () | (v,v) | left v | right v
T  ::=  Unit | T ∧ T | T ∨ T
```

## 3.2 Small-Step Operational Semantics

$$\frac{\mathtt{t}_1 \longrightarrow \mathtt{t}_1'}{(\mathtt{t}_1,\mathtt{t}_2) \longrightarrow (\mathtt{t}_1',\mathtt{t}_2)} \quad \text{(E-PAIR1)} \qquad \frac{\mathtt{t}_2 \longrightarrow \mathtt{t}_2'}{(\mathtt{v}_1,\mathtt{t}_2) \longrightarrow (\mathtt{v}_1,\mathtt{t}_2')} \quad \text{(E-PAIR2)}$$

$$\frac{\mathtt{t} \longrightarrow \mathtt{t}'}{\mathtt{fst\ t} \longrightarrow \mathtt{fst\ t}'} \quad \text{(E-FST)} \qquad \frac{}{\mathtt{fst\ (v_1,v_2)} \longrightarrow \mathtt{v}_1} \quad \text{(E-FSTRED)}$$

$$\frac{\mathtt{t} \longrightarrow \mathtt{t}'}{\mathtt{snd\ t} \longrightarrow \mathtt{snd\ t}'} \quad \text{(E-SND)} \qquad \frac{}{\mathtt{snd\ (v_1,v_2)} \longrightarrow \mathtt{v}_2} \quad \text{(E-SNDRED)}$$

$$\frac{\mathtt{t} \longrightarrow \mathtt{t}'}{\mathtt{left\ t} \longrightarrow \mathtt{left\ t}'} \quad \text{(E-LEFT)} \qquad \frac{\mathtt{t} \longrightarrow \mathtt{t}'}{\mathtt{right\ t} \longrightarrow \mathtt{right\ t}'} \quad \text{(E-RIGHT)}$$

$$\frac{\mathtt{t} \longrightarrow \mathtt{t}'}{\mathtt{match\ t\ with\ left\ x_1\ {->}\ t_1\ |\ right\ x_2\ {->}\ t_2} \longrightarrow \mathtt{match\ t'\ with\ left\ x_1\ {->}\ t_1\ |\ right\ x_2\ {->}\ t_2}} \quad \text{(E-MATCH)}$$

$$\frac{}{\mathtt{match\ left\ v\ with\ left\ x_1\ {->}\ t_1\ |\ right\ x_2\ {->}\ t_2} \longrightarrow [\mathtt{x}_1 \mapsto \mathtt{v}]\mathtt{t}_1} \quad \text{(E-MATCHLEFT)}$$

$$\frac{}{\mathtt{match\ right\ v\ with\ left\ x_1\ {->}\ t_1\ |\ right\ x_2\ {->}\ t_2} \longrightarrow [\mathtt{x}_2 \mapsto \mathtt{v}]\mathtt{t}_2} \quad \text{(E-MATCHRIGHT)}$$

$$\frac{\mathtt{t}_1 \longrightarrow \mathtt{t}_1'}{\mathtt{let\ x{=}t_1\ in\ t_2} \longrightarrow \mathtt{let\ x{=}t_1'\ in\ t_2}} \quad \text{(E-LET)} \qquad \frac{}{\mathtt{let\ x{=}v\ in\ t} \longrightarrow [\mathtt{x} \mapsto \mathtt{v}]\mathtt{t}} \quad \text{(E-LETRED)}$$

$$\frac{}{\mathtt{letrec\ x{=}v\ in\ t} \longrightarrow \mathtt{let\ x{=}}[\mathtt{x} \mapsto \mathtt{letrec\ x{=}v\ in\ x}]\mathtt{v\ in\ t}} \quad \text{(E-LETREC)}$$

## 3.3 Static Type System

$$\frac{}{\Gamma \vdash \mathtt{()} : \mathtt{Unit}} \quad \text{(T-UNIT)}$$

$$\frac{\Gamma \vdash \mathtt{t}_1 : \mathtt{T}_1 \qquad \Gamma \vdash \mathtt{t}_2 : \mathtt{T}_2}{\Gamma \vdash (\mathtt{t}_1,\mathtt{t}_2) : \mathtt{T}_1 \wedge \mathtt{T}_2} \quad \text{(T-PAIR)}$$

$$\frac{\Gamma \vdash \mathtt{t} : \mathtt{T}_1 \wedge \mathtt{T}_2}{\Gamma \vdash \mathtt{fst\ t} : \mathtt{T}_1} \quad \text{(T-FST)} \qquad \frac{\Gamma \vdash \mathtt{t} : \mathtt{T}_1 \wedge \mathtt{T}_2}{\Gamma \vdash \mathtt{snd\ t} : \mathtt{T}_2} \quad \text{(T-SND)}$$

$$\frac{\Gamma \vdash \texttt{t : T}_1}{\Gamma \vdash \texttt{left t : T}_1 \lor \texttt{T}_2} \quad \text{(T-LEFT)} \qquad \frac{\Gamma \vdash \texttt{t : T}_2}{\Gamma \vdash \texttt{right t : T}_1 \lor \texttt{T}_2} \quad \text{(T-RIGHT)}$$

$$\frac{\Gamma \vdash \texttt{t : T}_1 \lor \texttt{T}_2 \qquad \Gamma, \texttt{x}_1\texttt{:T}_1 \vdash \texttt{t}_1 \texttt{ : T} \qquad \Gamma, \texttt{x}_2\texttt{:T}_2 \vdash \texttt{t}_2 \texttt{ : T}}{\Gamma \vdash \texttt{match t with left x}_1 \texttt{ -> t}_1 \texttt{ | right x}_2 \texttt{ -> t}_2 \texttt{ : T}} \quad \text{(T-MATCH)}$$

$$\frac{\Gamma \vdash \texttt{t}_1 \texttt{ : T}_1 \qquad \Gamma, \texttt{x:T}_1 \vdash \texttt{t}_2 \texttt{ : T}}{\Gamma \vdash \texttt{let x=t}_1 \texttt{ in t}_2 \texttt{ : T}} \quad \text{(T-LET)} \qquad \frac{\Gamma, \texttt{x:T}_1 \vdash \texttt{v}_1 \texttt{ : T}_1 \qquad \Gamma, \texttt{x:T}_1 \vdash \texttt{t}_2 \texttt{ : T}}{\Gamma \vdash \texttt{letrec x=v}_1 \texttt{ in t}_2 \texttt{ : T}} \quad \text{(T-LETREC)}$$

# 4 System F (Polymorphic Lambda Calculus)

We extend the Simply-Typed Lambda Calculus from Section 2 to support explicit polymorphism.

## 4.1 Syntax

```
t  ::=  ... | function X → t | t T
v  ::=  ... | function X → t
T  ::=  ... | X | ∀X.T
```

## 4.2 Type Substitution

$[\texttt{X} \mapsto \texttt{T}]\texttt{x} = \texttt{x}$
$[\texttt{X} \mapsto \texttt{T}]\texttt{function x:T}_0 \texttt{ -> t}_0 = \texttt{function x:}[\texttt{X} \mapsto \texttt{T}]\texttt{T}_0 \texttt{ -> } [\texttt{X} \mapsto \texttt{T}]\texttt{t}_0$
$[\texttt{X} \mapsto \texttt{T}]\texttt{t}_1 \texttt{ t}_2 = [\texttt{X} \mapsto \texttt{T}]\texttt{t}_1 [\texttt{X} \mapsto \texttt{T}]\texttt{t}_2$
$[\texttt{X} \mapsto \texttt{T}]\texttt{function X -> t} = \texttt{function X -> t}$
$[\texttt{X} \mapsto \texttt{T}]\texttt{function X}' \texttt{ -> t} = \texttt{function X}' \texttt{ -> } [\texttt{X} \mapsto \texttt{T}]\texttt{t}$, where $\texttt{X} \neq \texttt{X}'$
$[\texttt{X} \mapsto \texttt{T}]\texttt{t}_1 \texttt{ t}_2 = [\texttt{X} \mapsto \texttt{T}]\texttt{t}_1 [\texttt{X} \mapsto \texttt{T}]\texttt{t}_2$
$[\texttt{X} \mapsto \texttt{T}]\texttt{t}_1 \texttt{ T}_2 = [\texttt{X} \mapsto \texttt{T}]\texttt{t}_1 [\texttt{X} \mapsto \texttt{T}]\texttt{T}_2$

$[\texttt{X} \mapsto \texttt{T}]\texttt{X} = \texttt{T}$
$[\texttt{X} \mapsto \texttt{T}]\texttt{X}' = \texttt{X}'$, where $\texttt{X} \neq \texttt{X}'$
$[\texttt{X} \mapsto \texttt{T}]\texttt{T}_1 \to \texttt{T}_2 = [\texttt{X} \mapsto \texttt{T}]\texttt{T}_1 \to [\texttt{X} \mapsto \texttt{T}]\texttt{T}_2$
$[\texttt{X} \mapsto \texttt{T}]\forall\texttt{X.T}_0 = \forall\texttt{X.T}_0$
$[\texttt{X} \mapsto \texttt{T}]\forall\texttt{X}'\texttt{.T}_0 = \forall\texttt{X}'\texttt{.}[\texttt{X} \mapsto \texttt{T}]\texttt{T}_0$, where $\texttt{X} \neq \texttt{X}'$

## 4.3 Small-Step Operational Semantics

$$\frac{}{\texttt{((function X} \to \texttt{t) T)} \longrightarrow [\texttt{X} \mapsto \texttt{T}]\texttt{t}} \quad \text{(E-TAPPBETA)}$$

$$\frac{\texttt{t}_1 \longrightarrow \texttt{t}_1'}{\texttt{t}_1 \texttt{ T}_2 \longrightarrow \texttt{t}_1' \texttt{ T}_2} \quad \text{(E-TAPP1)}$$

## 4.4 Static Type System

$$\frac{\Gamma \vdash \texttt{t : T}}{\Gamma \vdash \texttt{function X -> t : } \forall\texttt{X.T}} \quad \text{(T-TFUN)}$$

$$\frac{\Gamma \vdash \texttt{t}_1 \texttt{ : } \forall\texttt{X.T}}{\Gamma \vdash \texttt{t}_1 \texttt{ T}_2 \texttt{ : } [\texttt{X} \mapsto \texttt{T}_2]\texttt{T}} \quad \text{(T-TAPP)}$$

# 5 Mutable References

## 5.1 Syntax

$$
\begin{array}{lll}
\texttt{t} & ::= & \texttt{ref t} \mid \texttt{!t} \mid \texttt{t := t} \mid \texttt{l} \mid \texttt{()} \\
\texttt{v} & ::= & \texttt{l} \mid \texttt{()} \\
\texttt{l} & ::= & \text{memory locations} \\
\texttt{T} & ::= & \texttt{Unit} \mid \texttt{Ref T}
\end{array}
$$

## 5.2 Small-Step Operational Semantics

$\mu$ is a finite function from memory locations to values.

$$\frac{<\texttt{t}_1, \mu> \longrightarrow <\texttt{t}_1', \mu'>}{<\texttt{ref t}_1, \mu> \longrightarrow <\texttt{ref t}_1', \mu'>} \ (\text{E-Ref})$$

$$\frac{\texttt{l} \notin \text{dom}(\mu)}{<\texttt{ref v}, \mu> \longrightarrow <\texttt{l}, \mu \cup \{(\texttt{l},\texttt{v})\}>} \ (\text{E-RefV})$$

$$\frac{<\texttt{t}_1, \mu> \longrightarrow <\texttt{t}_1', \mu'>}{<\texttt{!t}_1, \mu> \longrightarrow <\texttt{!t}_1', \mu'>} \ (\text{E-Deref})$$

$$\frac{\mu(\texttt{l})=\texttt{v}}{<\texttt{!l}, \mu> \longrightarrow <\texttt{v}, \mu>} \ (\text{E-DerefLoc})$$

$$\frac{<\texttt{t}_1, \mu> \longrightarrow <\texttt{t}_1', \mu'>}{<\texttt{t}_1 \texttt{ := t}_2, \mu> \longrightarrow <\texttt{t}_1' \texttt{ := t}_2, \mu'>} \ (\text{E-Assign1})$$

$$\frac{<\texttt{t}_2, \mu> \longrightarrow <\texttt{t}_2', \mu'>}{<\texttt{v}_1 \texttt{ := t}_2, \mu> \longrightarrow <\texttt{v}_1 \texttt{ := t}_2', \mu'>} \ (\text{E-Assign2})$$

$$\frac{}{<\texttt{l := v}, \mu> \longrightarrow <\texttt{()}, [\texttt{l} \mapsto \texttt{v}]\mu>} \ (\text{E-Assign})$$

## 5.3 Type System

$\Sigma$ is a finite function from memory locations to types.

$$\frac{}{\Gamma;\Sigma \vdash \texttt{()} : \texttt{Unit}} \ (\text{T-Unit})$$

$$\frac{\Sigma(\texttt{l}) = \texttt{T}}{\Gamma;\Sigma \vdash \texttt{l} : \texttt{Ref T}} \ (\text{T-Loc})$$

$$\frac{\Gamma;\Sigma \vdash \texttt{t}_1 : \texttt{T}_1}{\Gamma;\Sigma \vdash \texttt{ref t}_1 : \texttt{Ref T}_1} \ (\text{T-Ref})$$

$$\frac{\Gamma;\Sigma \vdash \texttt{t} : \texttt{Ref T}}{\Gamma;\Sigma \vdash \texttt{!t} : \texttt{T}} \ (\text{T-Deref})$$

$$\frac{\Gamma;\Sigma \vdash \texttt{t}_1 : \texttt{Ref T} \qquad \Gamma;\Sigma \vdash \texttt{t}_2 : \texttt{T}}{\Gamma;\Sigma \vdash \texttt{t}_1 \texttt{ := t}_2 : \texttt{Unit}} \ (\text{T-Assign})$$