Exercise set 1

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Practical information

- Only problem 2 and 3 counts for the grade. But the first one may help you to solve the other two.
- Your answer can be handed in as a pdf-file generated from LaTeX, as a Jupyter notebook, or as scanned handwritten text (but this is only an option if at least one student in the group can write nice and clearly readable).
- Information about *how to* hand in the exercise can be found on the wiki-page.

In this exercise you will:

- Derive useful results for the solution of the 1-dimensional Poisson equation.
- Prove convergence in the ∞-norm.
- Confirm the theoretical results by numerical experiments.
- Handle different boundary conditions.

Introduction

Given the model problem (the 1-dimensional Poisson problem)

$$-u_{xx} = f(x), \qquad u(0) = \alpha, \qquad u(1) = \beta. \tag{1}$$

NB! Notice the minus-sign.

The problem is solved by a finite difference scheme, using central differences: For a given M, define the grid by

$$h = 1/M, \qquad x_m = mh, \quad m = 0, 1, \dots, M$$

and the scheme is

$$\frac{-U_{m+1} + 2U_m - U_{m-1}}{h^2} = f(x_m), \qquad m = 1, 2, \dots, M-1 \tag{2}$$

or in matrix form as

$$\frac{1}{h^2}\begin{bmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_{M-1} \end{bmatrix} = \begin{bmatrix} f(x_1) + \alpha/h^2 \\ f(x_2) \\ \vdots \\ f(x_{M-1}) + \beta/h^2 \end{bmatrix} \qquad \text{or simply} \qquad A_h\vec{U} = \vec{F}. \tag{3}$$

The scheme is implemented in the enclosed code file.

The exercises

$\boxed{1}$ *0 pts* , This exercise is given as a preparation for the two others. It should be done, but not handed in.

First of all, make yourself familiar with the code, and make sure you understand it.

a) Check that the implementation is correct:
As a test problem we have used

$$-u_{xx} = f(x), \qquad u(0) = 0, \qquad u(1) = 1.$$

Solve this problem by the difference scheme, using $M = 10$ (already implemented, so just run the code).

We expect the the scheme to solve this problem exact (why?), so let us check if this is true:

Find the exact solution of the problem, and compare the the exact and the numerical solution, for instance by making a plot of the error.

b) Construct a test problem:
Test the code and measure the numerical error on a problem on your own liking. A dirty trick to make test problems is to choose the solution, and choose $f(x)$ and boundary conditions from that. Do not make your problem too simple.

Solve your test problem for $M = 10$, $M = 20$ and $M = 40$. Measure the error in the function 2-norm and the max norm (see the note, p. 16). Use the results to get an idea of the order of the method.

c) Convergence plots:
Assume that the error of some numerical scheme (not necessarily from this course) depends on a parameter h. If the method is of order p, we expect

$$\|e_h\| \approx Ch^p.$$

which, by taking the logarithm of each side becomes

$$\log \|e_h\| = p \log h + \log C.$$

So the logarithm of the error as a function of the logartihm of h will be a straight line with slope p.

To measure p, and to present the results graphically, the procedure is as follows: Solve a test problem for several values of h (usually $h = h_0/2^i$, $i = 0, 1, 2, \ldots,$ for some initial stepsize h_0). Measure the corresponding errors. Plot the error vs. the stepsize using a loglog plot, which should produce a straight line. The order p can be estimated by using a least square approximation to the straight line. Hint: Use `polyfit` for this.

As an example: Measure the order of the following approximations to $u_x(x_0)$, with $u(x) = \sin(x)$, $x_0 = \pi/4$ using $h = 0.1, 0.05, \cdots, 0.1/2^P$, for some appropriate value of P, for instance $P = 10$.

$$u_x(x_0) \approx \begin{cases} \frac{u(x_0+h)-u(x_0)}{h} & a) \\ \frac{u(x_0+h)-u(x_0-h)}{2h} & b) \\ \frac{-u(x_0+2h)+6u(x_0+h)-3u(x_0)-2u(x_0-h)}{6h} & c) \end{cases} \tag{4}$$

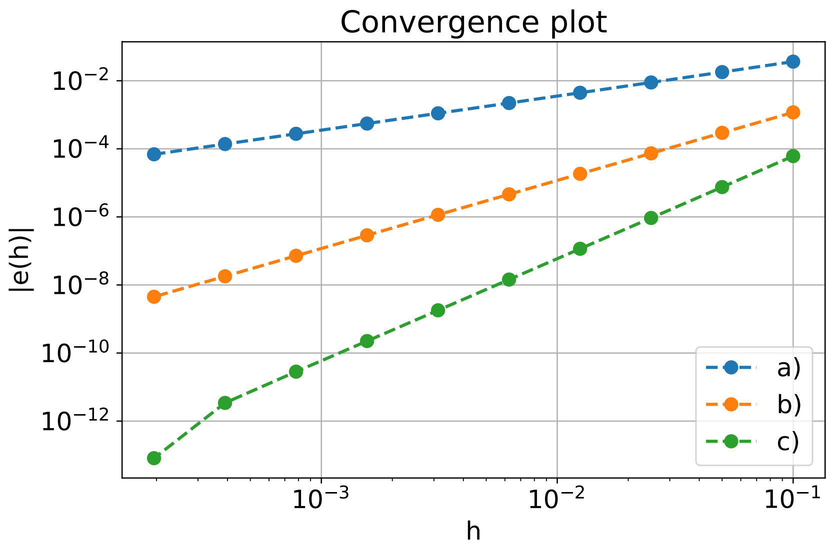

Figure 1: Convergence plot for the schemes in (4)

A convergence plot for these examples is given in Figure 1. You are encouraged to estimate the order from the measurements yourself, the results should be close to 1, 2 and 3 respectively.

$\boxed{2}$ *3pts* This should be handed in. Parts of the theoretical questions have been roughly covered in the lectures, and in this exercise, you are supposed to work out the details!

Given the model problem

$$-u_{xx} = f(x), \qquad u(0) = u(1) = 0. \tag{5}$$

We will in the following assume that $f \in C(0,1)$. Further, $f > 0$ means that $f > 0$ for all $x \in (0,1)$. Let the Green function be defined as

$$G(x,y) = \begin{cases} y(1-x) & 0 \le y \le x \\ x(1-y) & x \le y \le 1 \end{cases} \tag{6}$$

Let $\Omega = [0,1]\times[0,1]$ be the unit square, let $\partial\Omega$ be its boundary, and convince yourself that the Green function satisfies:

(a) $G \in C(\Omega)$.
(b) $G(x,y) = G(y,x)$
(c) $G = 0$ on $\partial\Omega$.
(d) For a fixed y, G is a piecewise linear function in x. And vice versa.
(e) $G > 0$ on $\Omega\backslash\partial\Omega$.

a) Prove that the solution of (5) is given by

$$u(x) = \int_0^1 G(x,y)f(y)dy, \tag{7}$$

Use this formula to find the solutions of (5) for $f = 1$ and $f = \sin(\pi x)$.

b) Use the result to prove the following statements:
- If $f > 0$ then $u > 0$.
- $\|u\|_\infty \le \frac{1}{8}\|f\|_\infty$.

where $\|f\|_\infty = \max_{x\in[0,1]} |f(x)|$, and similar for u, similar for matrices.

In the next points, these results will be transfered to the discrete case. We will use the notation $\vec{u} > 0$ when all elements of $\vec{u}$ are positive.

Consider the numerical scheme given by (3), with $\alpha = \beta = 0$. From the note we know the the local error is given by

$$\tau_m = -\frac{u(x_m + h) - 2u(x_m) + u(x_m - h)}{h^2} - f(x_m) = -\frac{1}{12}u_{4x}(\xi_m), \tag{8}$$

where $\xi_m \in (x_m, x_m + h)$.

Notice that:

(a) A_h is invertible.
(b) If $f \in \mathbb{P}_n$ then $u \in \mathbb{P}_{n+2}$.
(c) If $f \in \mathbb{P}_1$ then $U_m = u(x_m)$ for $m = 0, 1, 2, \ldots, M$.

Define $G_h \in \mathbb{R}^{(M-1)\times(M-1)}$ as the symmetric matrix with elements

$$(G_h)_{m,n} = hG(x_m, x_n), \qquad m = 1, 2, \ldots, M-1.$$

Obviously, $G_h > 0$.

c) Prove that

$$A_h G_h = I_{M-1} \qquad \text{so} \qquad A_h^{-1} = G_h.$$

Hint: Notice that row m of the matrix $A_h G_h$ is nothing but central difference scheme applied to the points of the function $G(x, x_n)$, and that $G(x, x_n)$ is piecewise linear.

d) Prove that

$$\|A_h^{-1}\|_\infty \le \frac{1}{8}.$$

Hint: Consider the numerical solution of the model problem when $f = 1$, and use the information given and/or derived above.

The error in each grid point is given by $e_m = u(x_m) - U_m$. By taking the difference of (8) and (2) we get the relation

$$\frac{-e_{m+1} + 2e_m - e_{m-1}}{h^2} = \tau_m, \qquad m = 1, 2, \ldots.$$

e) Finally, find an expression for the error of the form

$$\|e_h\|_\infty \le Ch^p\|u_{rx}\|_\infty$$

(that is, find C, p and r.)

Express your convergence result in terms of a theorem.

NB! The result will also hold for Dirichlet boundary conditions different from 0.

f) Confirm your convergence result numerically:

Solve (1) numerically by the finite difference scheme for the following test problems:

1 $f(x) = \sin(\pi x)$, $u(0) = u(1) = 0$ with exact solution $u(x) = \sin(\pi(x))/\pi^2$.
2 $f(x) = 1/x$, $u(0) = 0, u(1) = 1$ with the exact solution $u(x) = -x\ln(x) + x$.
3 A problem you have chosen yourself (specify it in your hand-in).

Make a convergence plot as described in the preparatory exercise (one plot for all cases), and measure the order in each case. Do not use too small stepsizes, it may take a lot of time for the code to excecute.

Hand in the convergence plot and the estimated order in each case.

$\boxed{3}$ *2pts* Consider the problem with mixed boundary conditions at the right boundary

$$-u_{xx} = \sin(\pi x), \qquad u(0) = 0, \qquad u_x(1) + u(1) = 1 \tag{9}$$

and solve this by the central difference scheme for all inner points, as before.

As discussed in the lecture, the derivative at the right boundary can approximated by

$$u_x(1) = \begin{cases} \frac{u_M - u_{M-1}}{h} + \mathcal{O}(h) \\ \frac{3u_M - 4u_{M-1} + u_{M-2}}{2h} + \mathcal{O}(h^2) \\ \frac{u_{M+1} - u_{M-1}}{2h} + \mathcal{O}(h^2) \end{cases}$$

How will the equation for the boundary point M be for each of these approximations?

Modify the code used previously in order to solve this problem, using each of the three approximations in the boundary point. Make a convergence plot comparing the three methods, and estimate the error. Comment on the result.